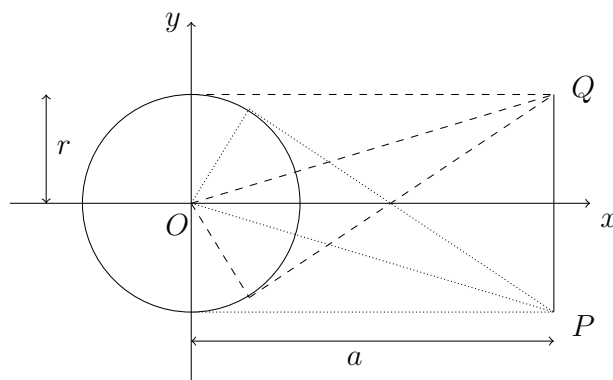


1. On 1 July 2019, Hong Kong protesters occupied the Legislative Council chamber. They have left words on a pillar 「是你教我們和平遊行是沒用的」 (translation: it was you who made us realize that peaceful marches did not work). The words were written on the curved surface of the pillar which cannot be all read from a single observation point (ie. one cannot read/understand the whole message with a fixed/stubborn view point). (See below a photo from Jeffie Lam's Twitter: stated therein pic via Martin Lam).



Consider the 2-D plan view. Let the pillar be the circle centered at the origin with radius r . Suppose someone is walking in constant speed from point $P(a, -r)$ to point $Q(a, r)$. From any point between P and Q looking at the pillar, the viewing angle made between the two tangents to the circle is changing. For example, at the end points P and Q , the angle takes its minimum value, whereas, at the middle $(a, 0)$, the angle takes its maximum value. Let the distance between the observation point to the center of the circle be μ and the distance between the observation point to the point tangent of the circle be b , and let y be the y -coordinate of the observation point along PQ , thus, $-r \leq y \leq r$. Let the angle made from the observation point to the center of the circle to each side of the tangent to the circle be θ .

(a) Explain why that

(i) $\mu \sin(\theta) = r$, [2 points]

(ii) $\mu^2 = b^2 + r^2$, [2 points]

(iii) $\tan(\theta) = \frac{r}{b} = \frac{r}{\sqrt{\mu^2 - r^2}}$, and [2 points]

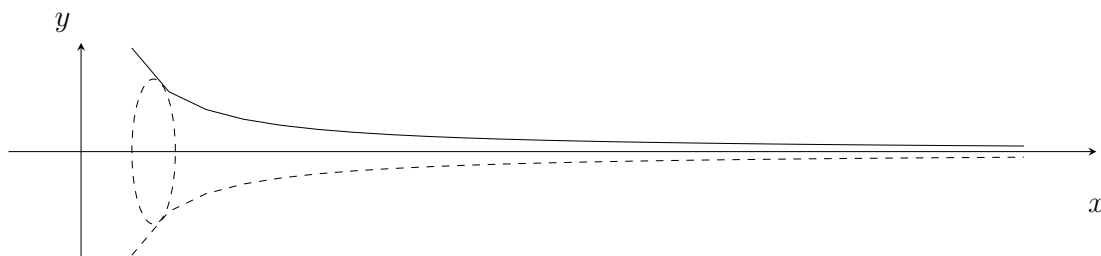
(iv) $y^2 + a^2 = \mu^2$. [2 points]

(b) Find $\frac{d\theta}{d\mu}$ in terms of r and μ only (i.e. not in terms of θ). [10 points]

(c) Find $\frac{d\theta}{dy}$ in terms of y , r , and a only. [7 points]

This question is written by the Subject Lecturer Dr. Joseph Lee. It does not represent the political position of The Department of Applied Mathematics.

2. Undoubtedly, the song 願榮光歸香港 (*Glory to Hong Kong*) is now one of the most frequently sung song by the people of Hong Kong during large-scale public events. It has been widely adopted as the anthem of the Hong Kong protests ever since its inception at the end of August 2019, with some even regarding it as the *national anthem of Hong Kong*. The song has also been played or sung all around the world by people supporting the cause. In its lyrics, there is a line 迷霧裡最遠處吹來號角聲 (translation: in the troubling mist of disarray brought the sound of a **horn** from afar). In mathematics, there is a famous geometric figure named Gabriel's Horn 加俾額爾號角 (Archangel Gabriel who blows the horn to announce Judgment Day), or Torricelli's trumpet 托里拆利小號 named after an Italian physicist and mathematician Evangelista Torricelli 1608-1647. The horn is formed by rotating $y = f(x) = \frac{1}{x}$ for $x \geq 1$ along the x -axis.



- (a) Find the volume of the Gabriel's Horn. [7 points]
- (b) The surface area of an object generated by rotating part of the non-negative graph of $y = f(x)$ along the x -axis for $a \leq x \leq b$ is given by

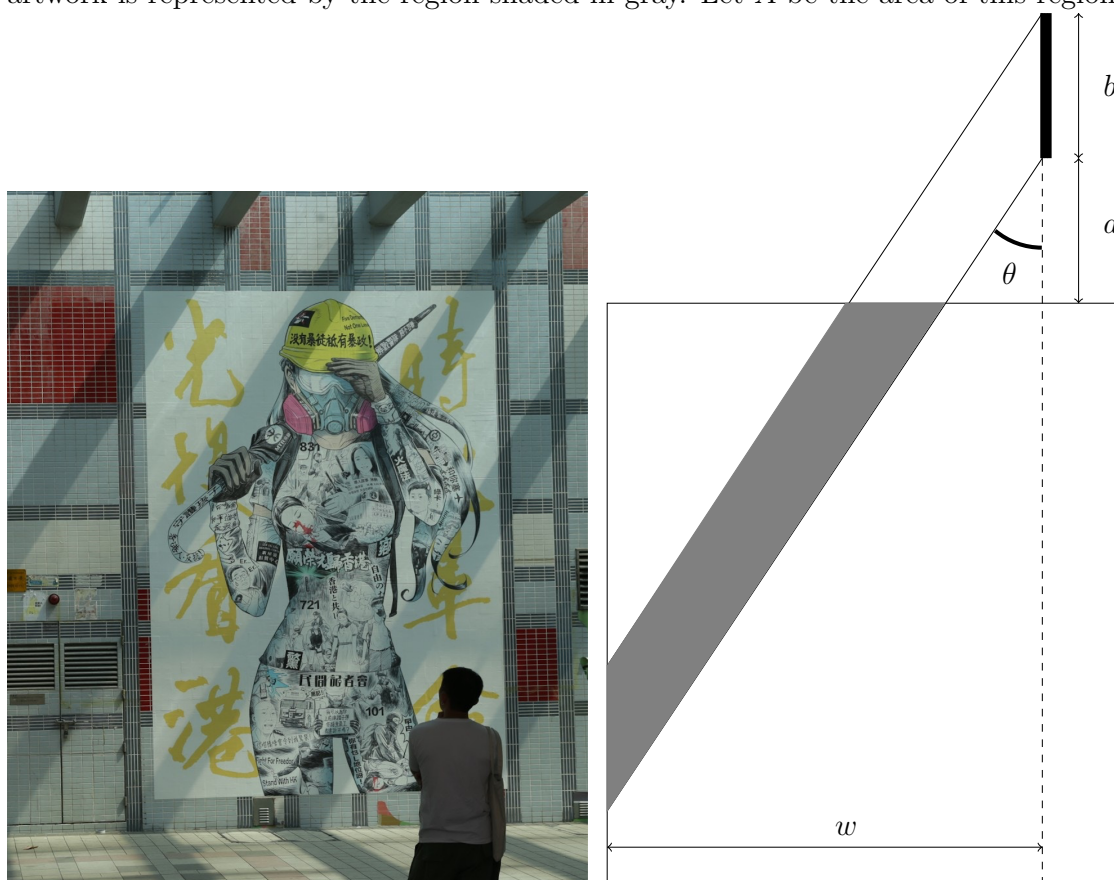
$$A = 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2} dx.$$

Show that the surface area of the Gabriel's Horn is infinite. [8 points]

(This is known as the Painter's Paradox, that a finite volume of paint can *fill-up* the Horn, to paint the infinite inner surface).

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3. On 26 Oct 2019, a huge poster of 3m high appeared on the *Kwai Fong Lennon Tunnel* 「葵芳連儂隧道」. The artwork depicted the symbolic image of the Hong Kong Lady Liberty (香港民主女神), in which her clothing was covered by detailed drawings of many recent monumental events that had happened in Hong Kong since June 2019 (see below a photo taken by 黃偉民 published in *HK01*). During daytime at the location, and if there is a reasonably strong amount of sunlight coming through the ceiling, shadows of the perpendicular columns above the artwork would be cast onto the artwork. Consider a simplified version of the scenario as shown in the figure below. The shadow cast onto the artwork is represented by the region shaded in gray. Let A be the area of this region.



- (a) Explain why $A = wb - \left(\frac{b^2 + 2ab}{2}\right) \tan(\theta)$, where θ is in the range for which the shadow is fully crossing only the top and the left side of the artwork. [10 points]
- (b) The sunlight angle θ is changing with respect to time. Within the same range of θ (excluding end points), find $\frac{dA}{dt}$. Express your answer in terms of $\frac{d\theta}{dt}$, a , b and θ only. [10 points]

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