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Digital redesign via the generalised bilinear transformation

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A new controller discretisation approach, the generalised bilinear transformation (GBT), is proposed in Zhang, G., Chen, T., and Chen X. (2007a). Given an analog controller K, GBT generates a class of digital controllers K_{gbt} parameterised by a real number $\alpha \in (-\infty, \infty)$. A geometric interpretation of GBT is first presented. Second, when the original analog feedback system is stable, two methods are proposed to find the value of the parameter α which provides an upper bound of sampling periods guaranteeing closed-loop stability of the resulting sampled-data system. Finally, several examples, namely, an IIR digital filter, an example studied in Rattan, K.S. (1984), 'Digitization of Existing Continuous Control Systems,' *IEEE Transactions on Automatic Control*, 29, 282–306, and Keller, J.P., and Anderson, B.D.O. (1992), 'A New Approach to the Discretisation of Continuous-time Controllers,' *IEEE Transaction on Automatic Control*, 37, 214–223, and an H_∞ control problem investigated in Chen, T., and Francis, B. (1995), *Optimal Sampled-Data Control Systems*, London: Springer, are used to demonstrate the strength of our discretisation approach. These examples show that GBT is able to retain the simplicity of the emulation methods such as the Tustin method, and simultaneously sustain closed-loop performance even at slow sampling.

Keywords: controller discretisation; generalised bilinear transformation (GBT); closed-loop stability; optimisation

1. Introduction

The methods of designing digital controllers can be roughly classified into four categories. The first category is the so-called *discrete design*, where a digital controller is designed for the discrete-time counterpart of the original analogue plant. Therefore, inter-sample ripple may be a problem in this design method (Chen and Francis 1995). The intent to cope with inter-sample behaviour motivates the other extreme (the second category): the *sampled-data* (*SD*) *design*, namely, design based on continuous-time specifications. Although this design method can deal with inter-sample behaviour perfectly, its inherent design complexity hinders its practical applicability (Wittenmark, Åström and Årzén 2002). The third category, the *emulation method*, can be regarded as a method in between the two methods aforementioned. The emulation method consists of two steps. The first step is to design a (fictitious) continuous-time controller: we call this step the *design* step. The second step is to discretise the continuous-time controller to obtain a discrete-time controller. We call this step the *implementation* step. There are lots of methods supporting controller design in the continuous-time domain. However, there are merely several controller discretisation approaches in the implementation step, the Tustin's method, ZOH equivalent, the Euler method and the matched pole-zero method being the most popular ones (Kowalczuk 1993; Franklin, Powell and E-Naeini 2002). Therefore, the emulation method usually goes like this: first design in the continuous-time domain, then try several controller discretisation approaches such as those listed above. If the closed-loop performance is satisfactory, the job is done; otherwise, go back to redesign in the continuous-time domain. In other words, the emulation method is essentially an *open-loop* approach and only works well at fast sampling. With the aim of improving this situation, a new controller discretisation – the generalised bilinear transformation (GBT) is proposed and studied in Zhang, Chen and Chen (2007a) and Zhang, Chen and Chen (2007b).

Generalised bilinear transformation provides a class of controller approximations parameterised by a parameter $\alpha \in (-\infty, \infty)$. With the aid of GBT, the emulation method now works in the following way: first, an analogue controller is designed in the continuous-time domain to fulfill pre-specified performance requirement; second, an optimisation problem is solved over $\alpha \in (-\infty, \infty)$ and the sampling period h to find

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a desired digital controller by taking *closed-loop requirement* into account. It is worthwhile to discuss a bit about the choice of the sampling period h. Hardware constraints will provide a lower bound on h; nevertheless, the designer can still choose h in an interval to guarantee the closed-loop requirements of the control system such as bandwidth and phase degradation. Given this, we assume in the article that the sampling period will be chosen based on the closed-loop requirement. Moreover, as is illustrated by examples, a good choice of α can allow relatively slow sampling while still sustaining acceptable closed-loop performance. Clearly, this will reduce technical limitations imposed on the sampling period; in other words, when it is preferred to use slow sampling, our design method will have an advantage. Nevertheless, if the sampling period h is pre-specified, then the optimisation problem reduces to an optimisation problem of only one variable, namely α. Consequently, GBT is a *closed-loop* digital redesign method. Because only two variables, namely α and h, are involved at most in the associated optimisation problems, they can be very easily solved via many standard methods (see the discussion below Theorem 3.5 as well as Remark 4 in § 3 for details). In this manner, the emulation method becomes a systematic method and the search for an *optimal* digital controller is made possible. One may doubt if the resulting digital controller obtained via GBT works satisfactorily. This article will show that design via GBT is indeed effective in various contexts, ranging from internal stability, digital filter design, to H_∞ control, etc. It may even provide almost the same control performance as the SD design. Therefore, it is reasonable to say that GBT is able to retain the simplicity of the emulation methods such as the Tustin's method, and simultaneously sustain closed-loop performance even at slow sampling.

Clearly, GBT belongs to the fourth category of discretisation methods: optimisation-based controller discretisation techniques which have been extensively investigated in the literature (e.g. Keller and Anderson 1992; Markazi and Hori 1995; Rafee, Chen and Malik 1997; Shieh, Wang and Tsai 1998; Anderson, Chongsrid, Limebeer and Hara 1999; Rosenvasser, Polyakov and Lampe 1999; Rosenwasser, Polyakov and Lampe 1999; Rosenwasser and Lampe 2000; Hwang, Chang and Hwang 2003). In each of these approaches, given an analogue controller, an optimisation problem is solved to produce a digital controller which is optimal in a certain sense such as H_2 or H_∞. These methods normally give good performance, but are at the cost of a large computational load. More importantly, in each case, only one controller is obtained. Since every control system operates undoubtedly under a variety of uncertainties, it is often desirable to adjust control law to cope with real situations if necessary (Jelali 2006), i.e. it is better to have an *adaptive* controller. The parameter α in GBT adds an extra degree of freedom to the control system concerned, therefore making the closed-loop system indeed adaptive. Hence, the real-time adjustment of α possibly leads to better control. The authors believe this will be one of the major advantages of GBT in the discretisation of analogue controllers. Another advantage of GBT is that it allows slow sampling. It has been shown in Chen and Francis (1995), Rabbath, Hori and Lechevin (2004), and Zhang et al. (2007a) that as the sampling period h goes to zero, the performance of sampled-data systems converges to that of the original continuous-time systems. However, instead of concentrating on the behaviour as $h \to 0$, in practice it is more desirable to express system performance as a function of h. To the best knowledge of the authors, except the plant-input mapping (PIM) hybrid-type closed-loop discretisation approach which can guarantee closed-loop stability for almost all non-pathological sampling periods (Markazi and Hori 1995, 2003), the reference Keller and Anderson (1992) is the only source where an upper bound of h is derived rigourously to guarantee closed-loop stability. A well-known example was employed to illustrate that the proposed approach guarantees closed-loop stability at relatively slow sampling. This example is re-studied in Example 4.2 which shows that our method gives a better upper bound than that given in Keller and Anderson (1992). Clearly, in order to make full use of GBT, two problems should be solved: one is a complete sampled-data control theory for GBT; the other is the associated optimisation over α and h. This article focusses on the first problem. The second problem, which is a one (or at most two)-dimensional optimisation problem, can be easily solved by many existing techniques (this problem is briefly discussed below the proof of Theorem 3.5).

Several techniques similar to GBT have already been proposed in the literature. A new approximation of an integral is discussed in Al-Alaoui (1993) which is a special case of GBT when $\alpha = 1/8$. A technique based on compensated integration is investigated in Chen and Liu (2001) which contains a parameter n. When $n = \infty$, the bilinear transformation is recovered. A study closely related to Chen and Liu (2001) can be found in Liu, Shou and Chen (2006). In Liu and Chen (2005) a modified bilinear integrations algorithm (Equation (9)) is proposed based on the modulated sine function. The proposed method is equivalent to GBT when α is restricted to the interval $[0.5, \infty)$. A technique identical to that in Zhang et al. (2007a) is studied in Sekara (2006) where the parameter is restricted to the interval $[0, 1]$. However, it is shown by Examples 4.1–4.3 that, by allowing $\alpha \in (-\infty, \infty)$, better performance could be achieved. Moreover, although all these papers demonstrated the effectiveness of their methods via numerical examples, none of them ever studied analytically the problem of closed-loop stability, which is a very important problem in any control system. The main purpose of this article is to establish several analytical results for *closed-loop stability* of the resulting sampled-data control system when GBT is applied. Therefore, the results in this article can also provide some theoretic ground for all these related methods. The remaining parts of this article are organised as follows. § 2 introduces the GBT from a *geometrical* point of view. § 3 investigates the stability of sampled-data systems designed based on GBT. Assume that the original analogue feedback system is stable, two methods are proposed sequentially to find the suitable parameter α which gives upper bound of the sampling periods guaranteeing the closed-loop stability of the resulting sampled-data system. § 4 contains applications of GBT. Three examples are discussed, the first being an IIR digital filter design, the second being a re-study of an example investigated in Rattan (1984), Keller and Anderson 1992), and the third being an H_∞ control of a system studied extensively in Chen and Francis (1995). By comparing GBT with existing methods via these examples, effectiveness of GBT is demonstrated. § 5 consists of some concluding remarks.

Throughout this article the following notations are adopted. The norm symbol $\|\cdot\|$ represents the Euclidean norm if it is on a vector or its largest singular value if on a matrix; $\|\cdot\|_{\ell_p}$ is the ℓ_p norm if applied to a vector and ℓ_p-induced norm if applied to a system. Following the convention, for a continuous-time controller K, $K(s)$ is used to denote its transfer function. A state-space realisation of K is denoted (A_K, B_K, C_K, D_K) and we also define

$$\left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right] := D_K + C_K(sI - A_K)^{-1}B_K.$$

The discrete-time case can be treated in a similar way.

2. The generalised bilinear transformation

The (traditional) bilinear transformation is motivated by considering the trapezoidal approximation of an integrator. Given an integrator $1/s$ with input u and output y, the trapezoidal approximation of

$$y(kh+h) = y(kh) + \int_{kh}^{kh+h} u(\tau)\mathrm{d}\tau \tag{1}$$

is

$$y(kh+h) = y(kh) + h\frac{u(kh+h)+u(kh)}{2},$$

that is, the integral is approximated using the average value of $u(kh+h)$ and $u(kh)$. Now we approximate the integral in Equation (1) using some other combination of $u(kh+h)$ and $u(kh)$. More specifically, consider

$$y(kh+h) = y(kh) + h[\alpha u(kh+h) + (1-\alpha)u(kh)], \tag{2}$$

where the real number $\alpha \in (-\infty, \infty)$. The transfer function of Equation (2) is (in z transform)

$$\frac{y(z)}{u(z)} = h\frac{\alpha z + (1-\alpha)}{z-1}.$$

This motivates us to introduce the following GBT

$$\frac{1}{s} = h\frac{\alpha z + (1-\alpha)}{z-1},$$

that is,

$$s = \frac{1}{h}\frac{z-1}{\alpha z + (1-\alpha)}. \tag{3}$$

Therefore, under the GBT, a finite-dimensional linear time-invariant (FDLTI) continuous-time transfer function $K(s)$ is mapped to an FDLTI discrete-time transfer function $K_{\mathrm{gbt}}(z)$, where

$$K_{\mathrm{gbt}}(z) := K\left(\frac{1}{h}\frac{z-1}{\alpha z + (1-\alpha)}\right) \tag{4}$$

with $\alpha \in (-\infty, \infty)$. In terms of state-space data, bring in a *minimal* realisation of $K(s)$, namely, (A_K, B_K, C_K, D_K), it is straightforward to derive that $K_{\mathrm{gbt}}(z)$ has a state-space model $(A_{K_{\mathrm{gbt}}}, B_{K_{\mathrm{gbt}}}, C_{K_{\mathrm{gbt}}}, D_{K_{\mathrm{gbt}}})$, where

$$\begin{aligned} A_{K_{\mathrm{gbt}}} &= (I - \alpha h A_K)^{-1}[I + (1-\alpha)hA_K], \\ B_{K_{\mathrm{gbt}}} &= (I - \alpha h A)^{-1} h B_K, \\ C_{K_{\mathrm{gbt}}} &= C_K(I - \alpha h A_K)^{-1}, \quad D_{K_{\mathrm{gbt}}} = D_K + \alpha C_K B_{K_{\mathrm{gbt}}}. \end{aligned} \tag{5}$$

Remark 1: When $\alpha = 0$, $1/2$ and 1, $K_{\mathrm{gbt}}(z)$ is the forward difference (Euler method), the Tustin's method and backward difference approximations of $K(s)$, respectively. These approximations are illustrated geometrically in Figure 1. In each case, the integral equals the area enclosed by the curve, the two vertical dotted lines and the horizontal axis. In (a), (b) and (c), the approximations equal the area enclosed by the three dotted lines and the horizontal axis. However, in (d), the approximation equals the difference between the positive area (above the horizontal axis) and the negative area (under the horizontal axis).

Remark 2: The parameter α is restricted to be in the interval $[0, 1]$ in Zhang et al. (2007a). It will be shown that performance could be improved significantly by allowing $\alpha \in (-\infty, \infty)$.

3. Stability

The purpose of this section is to study the stability of sampled-data systems implemented via GBT. Two methods are proposed sequentially to find the suitable parameter α which yields an upper bound of sampling periods that guarantee the closed-loop stability of the resulting sampled-data system.

3.1 *Basic set-up*

Consider the continuous-time feedback control system Σ_1 as shown in Figure 2. Suppose that both the plant G and the controller K are FDLTI. Furthermore, assume that G is strictly proper and K is proper. Bring in minimal state-space realisations:

$$G(s) = \left[\begin{array}{c|c} A_G & B_G \\ \hline C_G & 0 \end{array}\right], \quad K(s) = \left[\begin{array}{c|c} A_K & B_K \\ \hline C_K & D_K \end{array}\right]. \tag{6}$$

Let the state variables of G and K be x_G and x_K, respectively. Moreover, define their compound $x_{G,K} := [x_G^T x_K^T]^T$, where the superscript T is the matrix transpose operation. The state $x_{G,K}$ is called *internally stable* if for every initial time t_0 and initial state $x_{G,K}(t_0)$, we have

$$x_{G,K}(t) \to 0, \quad \text{as } t \to \infty.$$

Under the assumption of *minimal* realisation, it is well known that internal stability is equivalent to the stability of the closed-loop A-matrix

$$A_{\text{cl}} = \begin{bmatrix} A_G + B_G D_K C_G & B_G C_K \\ B_K C_G & A_K \end{bmatrix},$$

that is, all its eigenvalues have negative real parts. From now on we identify internal stability of Σ_1 with that of matrix A_{cl}. Clearly, that A_{cl} is stable is also equivalent to that system Σ_1 is input–output stable in the sense of L_p induced norm for every $p \in [1, \infty]$ (interested readers can refer to the reference Chen and Francis (1995)).

Now digitise the analogue system Σ_1 by replacing K with $HK_{\text{gbt}}S$, where H is a zero-order hold, K_{gbt} is an approximation of K via GBT for some $\alpha \in (-\infty, \infty)$ and S is an ideal sampler. Denote the resulting sampled-data system by Σ_2 as shown in Figure 3. In what follows we will investigate internal stability of system Σ_2. To do so, internal stability of a certain discrete-time system should be discussed first. Let $G_{\text{zoh}} = SGH$, i.e. G_{zoh} is the zero-order hold equivalent of G. Then the closed-loop A-matrix of the discrete-time system consisting of G_{zoh} and K_{gbt} is given by

$$A_{\text{cl},d} = \begin{bmatrix} A_{G_{\text{zoh}}} + B_{G_{\text{zoh}}} D_{K_{\text{gbt}}} C_G & B_{G_{\text{zoh}}} C_{K_{\text{gbt}}} \\ B_{K_{\text{gbt}}} C_G & A_{K_{\text{gbt}}} \end{bmatrix}. \tag{7}$$

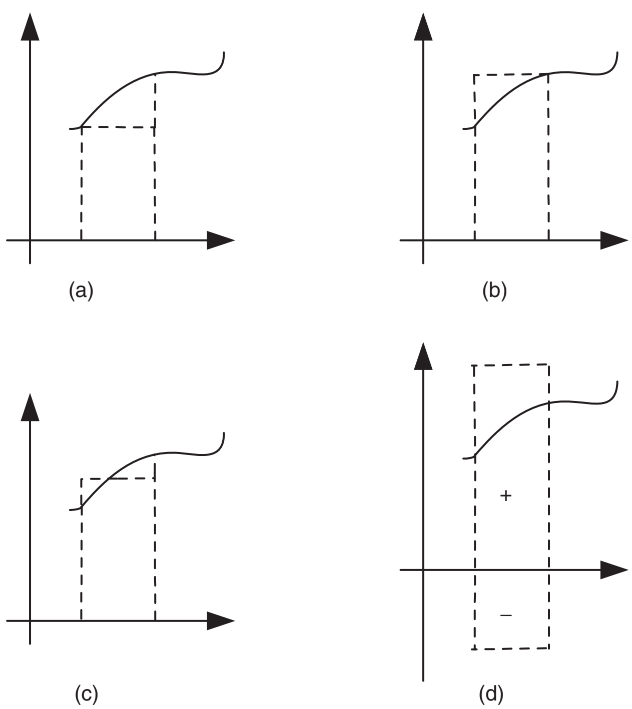

Figure 1. (a): Forward difference (Euler), (b): Backward difference, (c): Tustin's, (d): GBT with $\alpha > 1$.

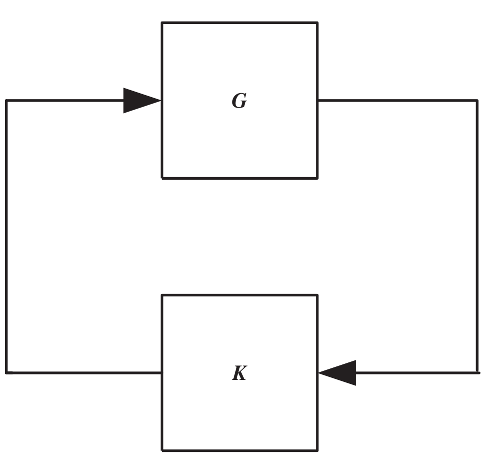

Figure 2. Analog system Σ_1.

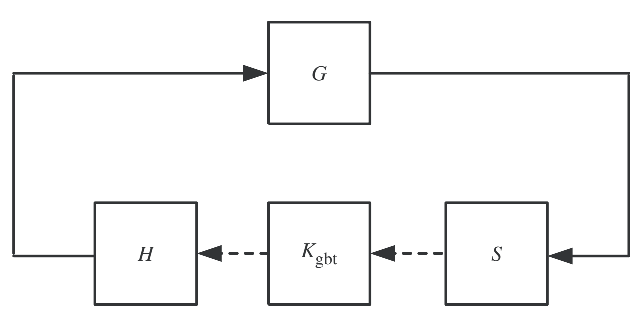

Figure 3. Sampled-data system Σ_2.

For convenience, denote this discrete-time system by Σ_{2d}. Clearly, the internal stability of Σ_{2d} is equivalent to the stability of the matrix $A_{\mathrm{cl},d}$.

3.2 *The first approach*

In this section, we formulate an optimisation problem whose solution provides an upper bound of sampling periods which guarantee closed-loop stability of sampled-data systems when analogue controllers are implemented via GBT.

The following result is used in the proof of Theorem 3.2:

Proposition 3.1: *Let $A \in \mathbb{C}^{n\times n}$ and $\epsilon > 0$ be given. Then there is a consistent matrix norm $\|\cdot\|_{sp}$ such that*

$$\|I + hA\|_{sp} \le \rho(I + hA) + \epsilon h, \tag{8}$$

where h is an arbitrary positive real number.

Proof: This result is a variant of Lemma 5.6.10 in Horn and Johnson (1986) (p. 297, chap. 5). According to the Schur triangularisation theorem (Theorem 2.3.1, Horn and Johnson 1986), there exists a unitary matrix U and an upper triangular matrix Ξ such that $A = U^* \, \Xi U$. Define $Q_t := \mathrm{diag}(t, t^2, \ldots, t^n)$. It is easy to show that

$$Q_t \Xi Q_t^{-1} = \begin{bmatrix} \lambda_1 & t^{-1}d_{12} & t^{-2}d_{13} & \cdots & t^{-(n-2)}d_{1(n-1)} & t^{-(n-1)}d_{1n} \\ 0 & \lambda_2 & t^{-1}d_{23} & \cdots & t^{-(n-3)}d_{2(n-1)} & t^{-(n-2)}d_{2n} \\ 0 & 0 & \lambda_3 & \cdots & t^{-(n-4)}d_{3(n-1)} & t^{-(n-3)}d_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \lambda_{n-1} & t^{-1}d_{(n-1)n} \\ 0 & 0 & 0 & 0 & 0 & \lambda_n \end{bmatrix}.$$

Consequently, for $t > 0$ large enough, the sum of all the absolute values of the off-diagonal entries of $Q_t \Xi Q_t^{-1}$ is less than the pre-specified scalar ϵ. As a result,

$$\left\|I + hQ_t \Xi Q_t^{-1}\right\|_1 \le \rho(I + hA) + \epsilon h \tag{9}$$

for any $h > 0$. In view of this, define a matrix norm $\|\cdot\|_{\mathrm{sp}}$ by

$$\|V\|_{sp} := \left\|Q_t U V U^* Q_t^{-1}\right\|_1 = \left\|(Q_t U)V(Q_t U)^{-1}\right\|_1 \tag{10}$$

for any $V \in \mathbb{C}^{n\times n}$. Then, it can be readily shown that

$$\begin{aligned}\|I + hA\|_{sp} &= \left\|I + hQ_t UU^* \Xi UU^* Q_t^{-1}\right\|_1 \\ &= \left\|I + hQ_t \Xi Q_t^{-1}\right\|_1 \le \rho(I + hA) + \epsilon h.\end{aligned}$$

Recall that a matrix norm $\|\cdot\|$ defined on $\mathbb{C}^{n\times n}$ is said to be consistent if the inequality

$$\|AB\| \le \|A\| \cdot \|B\| \tag{11}$$

holds for all pairs of matrices $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times n}$. Next, we show that $\|\cdot\|_{\mathrm{sp}}$ is a consistent matrix norm. Note that for any two matrices V_1, $V_2 \in \mathbb{C}^{n\times n}$,

$$\begin{aligned}\|V_1 V_2\|_{sp} &= \left\|(Q_t U)V_1 V_2 (Q_t U)^{-1}\right\|_1 \\ &= \left\|(Q_t U)V_1 (Q_t U)^{-1}(Q_t U)V_2 (Q_t U)^{-1}\right\|_1 \\ &\le \left\|(Q_t U)V_1 (Q_t U)^{-1}\right\|_1 \left\|(Q_t U)V_2 (Q_t U)^{-1}\right\|_1 \\ &= \|V_1\|_{sp} \|V_1\|_{sp}.\end{aligned}$$

Consequently, $\|\cdot\|_{\mathrm{sp}}$ is a consistent matrix norm. □

We are ready to establish a result concerning an upper bound of sampling periods for the discrete-time system Σ_{2d}.

Theorem 3.2: *Suppose that the original continuous-time system Σ_1 is internally stable. Then for every $h \in (0, \delta)$, there exists a real number α such that system Σ_{2d} is internally stable too, where the positive scalar δ is given in* (19) *which is to be made explicit in the proof below.*

Proof: Define

$$\Lambda := \max\left\{\left|\mathrm{Re}(\lambda)\right| : \lambda \text{ is an eigenvalue of } A_{\mathrm{cl}}\right\}.$$

Then according to Proposition 3.1, for every $h \in (0, 1/\Lambda)$ and any $\epsilon > 0$ independent of h, there exists a consistent matrix norm $\|\cdot\|_{\mathrm{sp}}$ such that

$$\|I + hA_{\mathrm{cl}}\|_{sp} \le \rho(I + hA_{\mathrm{cl}}) + \epsilon h.$$

In what follows, we confine ϵ to be $\varepsilon \in (0, \Gamma/4)$.

Define a positive scalar Γ as

$$\Gamma := \min\left\{\left|\mathrm{Re}(\lambda)\right| : \lambda \text{ is an eigenvalue of } A_{\mathrm{cl}}\right\}.$$

Let $a_i + jb_i$ be any eigenvalue of A_{cl} where $j = \sqrt{-1}$. Then there is a number $\delta_2 > 0$ such that

$$\frac{a_i + h\left(a_i^2 + b_i^2\right)}{\sqrt{(1 + ha_i)^2 + (hb_i)^2}} + \frac{3}{4}\Gamma < 0, \quad \forall h \in (0, \delta_2). \tag{12}$$

Next we find the maximal δ_2 such that

$$\frac{3}{4}\Gamma < \frac{-a-h\left(a^2+b^2\right)}{\sqrt{(1+ha)^2+(hb)^2}}. \tag{13}$$

First of all, it is required that

$$-a-h\left(a^2+b^2\right)>0,$$

that is,

$$h < \frac{-a}{a^2+b^2}. \tag{14}$$

Under (14), inequality (13) reduces to

$$\frac{9}{16}\Gamma^2\left[(1+ha)^2+(hb)^2\right] < a^2+2a\left(a^2+b^2\right)h + \left(a^2+b^2\right)^2h^2,$$

that is,

$$\left(a^2+b^2\right)\left(a^2+b^2-\frac{9}{16}\Gamma^2\right)h^2+2a\left(a^2+b^2-\frac{9}{16}\Gamma^2\right)h + a^2-\frac{9}{16}\Gamma^2>0.$$

Notice that

$$\begin{aligned}\Upsilon &:= \left[2a\left(a^2+b^2-\frac{9}{16}\Gamma^2\right)\right]^2 - 4\left(a^2+b^2\right)\left(a^2+b^2-\frac{9}{16}\Gamma^2\right)\left(a^2-\frac{9}{16}\Gamma^2\right)\\ &= 4\left(a^2+b^2-\frac{9}{16}\Gamma^2\right)\left[a^2\left(a^2+b^2-\frac{9}{16}\Gamma^2\right) - \left(a^2+b^2\right)\left(a^2-\frac{9}{16}\Gamma^2\right)\right]\\ &= 4\left(a^2+b^2-\frac{9}{16}\Gamma^2\right)\frac{9}{16}b^2\Gamma^2\\ &> 0.\end{aligned}$$

As a result,

$$h < \frac{-2a\left(a^2+b^2-\frac{9}{16}\Gamma^2\right)h-\sqrt{\Upsilon}}{2(a^2+b^2)\left(a^2+b^2-\frac{9}{16}\Gamma^2\right)} = \frac{-a}{a^2+b^2} - \frac{3|b|\Gamma}{4(a^2+b^2)\sqrt{a^2+b^2-\frac{9}{16}\Gamma^2}} \tag{15}$$

Combining (14) and (15) yields

$$h < \frac{-a}{a^2+b^2} - \frac{3|b|\Gamma}{4(a^2+b^2)\sqrt{a^2+b^2-\frac{9}{16}\Gamma^2}}.$$

Consequently, the maximal δ_2 is

$$\delta_2 = \min_{a_i+jb_i\in A_{\rm cl}}\left\{\frac{-a_i}{a_i^2+b_i^2} - \frac{3|b_i|\Gamma}{4\left(a_i^2+b_i^2\right)\sqrt{a_i^2+b_i^2-\frac{9}{16}\Gamma^2}}\right\}. \tag{16}$$

Observe that

$$A_{{\rm cl},d} = I + hA_{\rm cl} + h^2\Delta(h,\alpha).$$

Thus

$$\Delta(h,\alpha) = \frac{1}{h^2}\left(A_{{\rm cl},d} - I - hA_{\rm cl}\right).$$

Define

$$\delta_3 := \min\left\{\delta_2, \frac{1}{\Lambda}\right\},$$

and

$$\eta(\alpha) := \max_{h\in(0,\delta_3)}\left\|\Delta(h,\alpha)\right\|_{sp}.$$

Then

$$\left\|\Delta(h,\alpha)\right\|_{sp} \le \eta(\alpha).$$

Minimise $\eta(\alpha)$ to find an optimal α^*, namely,

$$\alpha^* = \min_{\alpha}\eta(\alpha) = \min_{\alpha}\max_{h\in(0,\delta_3)}\left\|\Delta(h,\alpha)\right\|_{sp}. \tag{17}$$

Denote

$$\eta^* = \eta(\alpha^*), \tag{18}$$

and define

$$\delta := \min\left\{\frac{\Gamma}{4\eta^*}, \delta_3\right\}. \tag{19}$$

We are now ready to show that when α^* is used as the parameter in GBT, then for every $h\in(0,\delta)$, the system Σ_{2d} is internally stable, i.e. the matrix $A_{{\rm cl},d}$ is stable.

Clearly, it suffices to show that

$$\rho(A_{{\rm cl},d}) < \|A_{{\rm cl},d}\|_{sp} \le \|I+hA_{\rm cl}\|_{sp} + h^2\left\|\Delta(h,\alpha^*)\right\|_{sp} \le 1. \tag{20}$$

Notice that

$$\begin{aligned}&\|I+hA_{\rm cl}\|_{sp}+h^2\|\Delta\|_{sp}\\ &\quad\le \rho(I+hA_{\rm cl})+\epsilon h+h^2\eta^*, \quad \forall h\in(0,\delta).\end{aligned}$$

As a result, it is sufficient to show that

$$\rho(I+hA_{\rm cl})+\epsilon h+h^2\eta^* \le 1, \quad h\in(0,\delta). \tag{21}$$

In what follows we show that

$$\sqrt{(1+ha_i)^2+(hb_i)^2}+\epsilon h+h^2\eta^* \le 1, \quad h\in(0,\delta). \tag{22}$$

Define a function

$$g_i(h) := \sqrt{(1+ha_i)^2+(hb_i)^2} + \epsilon h + h^2\eta^*, \quad h \in [0, \delta).$$

Clearly, $g_i(0) = 1$. It is straightforward to show that the derivative of $g_i(h)$ with respect to h satisfies

$$\begin{aligned}\frac{\mathrm{d}g_i(h)}{\mathrm{d}h} &= \epsilon + 2h\eta^* + \frac{a_i + h\left(a_i^2+b_i^2\right)}{\sqrt{(1+ha_i)^2+(hb_i)^2}} \\ &\le \frac{\Gamma}{4} + 2\eta^*\frac{\Gamma}{4\eta^*} + \frac{a_i + h\left(a_i^2+b_i^2\right)}{\sqrt{(1+ha_i)^2+(hb_i)^2}} \\ &= \frac{3\Gamma}{4} + \frac{a_i + h\left(a_i^2+b_i^2\right)}{\sqrt{(1+ha_i)^2+(hb_i)^2}}.\end{aligned}$$

Combining this and inequality (12) yields

$$\frac{\mathrm{d}g_i(h)}{\mathrm{d}h} < 0, \quad h \in (0, \delta).$$

As a result, inequality (22) holds. Inequalities (21) and (20) are thereby established. □

Remark 1: The min–max optimisation problem defined in (17) should be solved in order to get a good upper bound δ. Fortunately, since only two variables are involved, it is not difficult to solve this optimisation problem.

Armed with Theorem 3.2, we are ready to establish a result concerning the upper bound of sampling periods for the internal stability of the sampled-data system Σ_2. Notice that Σ_2 is a hybrid system, so the first problem is to define its state. We adopt the definition in Chen and Francis (1995, section 11.1). Let $x_{K_{\mathrm{gbt}}}$ be the state of the discrete-time system K_{gbt}. Define

$$x_{\mathrm{sd}}(t) := \begin{bmatrix} x_G \\ x_{K_{\mathrm{gbt}}} \end{bmatrix}, \quad kh \le t < (k+1)h. \tag{23}$$

After defining the state of Σ_2, we are able to define its internal stability. We will adopt the definition in Chen and Francis (1995, section 11.1), that is, the sampled-data system Σ_2 is said to be *internally stable* if for every initial t_0, $0 \le t_0 < h$, and initial state $x_{\mathrm{sd}}(t_0)$, we have $x_{\mathrm{sd}}(t) \to 0$ as $t \to \infty$.

We have the following result concerning the internal stability of Σ_2.

Theorem 3.3: *Suppose that the original continuous-time system Σ_1 is internally stable. Then for every $h \in (0, \delta)$, there exists a real number α such that system Σ_2 is internally stable, where the positive scalar δ is given in Equation* (19).

Proof: Suppose that the original continuous-time system Σ_1 is internally stable. Then according to Theorem 3.2, for every $h \in (0, \delta)$, there exists a real number α such that system Σ_{2d} is internally stable too, where the positive scalar δ is given in (19). Consequently, by Theorem 11.1 in Chen and Francis (1995), the sampled-data system Σ_2 is internally stable too. □

Remark 2: The upper bound of h obtained in this way might be somewhat conservative, as illustrated by the following example.

Example 3.4: Consider the following system composed of G and K given below:

$$G(s) = \frac{1}{s} = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right], \quad K(s) = \frac{-1}{s+a} = \left[\begin{array}{c|c} -a & 1 \\ \hline -1 & 0 \end{array}\right].$$

Via positive feedback, one has

$$A_{\mathrm{cl}} = \begin{bmatrix} 0 & -1 \\ 1 & -a \end{bmatrix}. \tag{24}$$

Applying the step-invariant transformation to G and GBT to K, respectively, yields

$$G_{\mathrm{zoh}}(z) = \left[\begin{array}{c|c} 1 & h \\ \hline 1 & 0 \end{array}\right], \quad K_{\mathrm{gbt}}(z) = \left[\begin{array}{c|c} 1 - \dfrac{ha}{1+\alpha ha} & \dfrac{h}{1+\alpha ha} \\ \hline \dfrac{-1}{1+\alpha ha} & \dfrac{-\alpha h}{1+\alpha ha} \end{array}\right].$$

Therefore, the closed-loop A-matrix is

$$A_{\mathrm{cl},d} = \begin{bmatrix} 1 - \dfrac{\alpha h^2}{1+\alpha ha} & -\dfrac{h}{1+\alpha ha} \\ \dfrac{h}{1+\alpha ha} & 1 - \dfrac{ha}{1+\alpha ha} \end{bmatrix}.$$

It is easy to show that

$$A_{\mathrm{cl},d} - (I + hA_{\mathrm{cl}}) = \frac{\alpha h^2}{1+\alpha ha}\begin{bmatrix} -1 & a \\ -a & a^2 \end{bmatrix}.$$

Therefore,

$$\Delta(h,\alpha) = \frac{1}{h^2}\left(A_{\mathrm{cl},d} - I - hA_{\mathrm{cl}}\right) = \frac{-\alpha}{1+\alpha ha}\begin{bmatrix} 1 & a \\ -a & a^2 \end{bmatrix}. \tag{25}$$

Next, fix $a = \sqrt{2}$. Then by Equation (24), the closed-loop poles of the original continuous-time system are

$$s_{1,2} = \frac{-\sqrt{2} \pm \sqrt{\left(\sqrt{2}\right)^2 - 4}}{2} = \frac{-\sqrt{2} \pm j\sqrt{2}}{2}.$$

Thus,

$$\Gamma = \Lambda = \frac{\sqrt{2}}{2},$$

$$\frac{-a_i}{a_i^2 + b_i^2} = \frac{\sqrt{2}}{2},$$

$$\frac{3|b_i|\Gamma}{4\left(a_i^2 + b_i^2\right)\sqrt{a_i^2 + b_i^2 - \frac{9}{16}\Gamma^2}} = \frac{3 \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2}}{4 \times \sqrt{1 - \frac{9}{16} \times \left(\frac{\sqrt{2}}{2}\right)^2}} = \frac{3\sqrt{2}}{2\sqrt{23}}.$$

According to (16),

$$\delta_2 = \frac{\sqrt{2}}{2} - \frac{3\sqrt{2}}{2\sqrt{23}} = \frac{\sqrt{2}}{2}\left(1 - \frac{3}{\sqrt{23}}\right) \approx 0.2648.$$

This bound is rather conservative. In fact, when $\alpha = 0.75$, for any $h \in (0, 4.565)$, the closed loop is internally stable.

3.3 *The second approach*

Example 3.4 reveals the conservativeness of Theorem 1. In this section we will propose a more general result.

As previously discussed, the stability of the closed-loop system is equivalent to

$$\rho\left(I + hA_{\rm cl} + h^2\Delta\right) < 1. \tag{26}$$

Let $x + jy$ be any eigenvalue of the matrix $A_{\rm cl} + h\Delta$. Then closed-loop stability requires that

$$\left|1 + h(x + jy)\right| < 1,$$

that is,

$$(1 + hx)^2 + (hy)^2 < 1,$$

$$h < \frac{-2x}{x^2 + y^2}. \tag{27}$$

Therefore, the problem of closed-loop stability reduces to an optimisation problem:

$$\max_{h,\alpha} \min_{x+jy\in\lambda(A_{\rm cl}+h\Delta)} \frac{-2x}{x^2 + y^2}. \tag{28}$$

Remark 3: In this formulation, $I + hA_{\rm cl}$ may not be stable necessarily.

Note that

$$A_{{\rm cl},d} = I + hA_{\rm cl} + h^2\Delta,$$

that is,

$$A_{\rm cl} + h\Delta = \frac{1}{h}\left(A_{{\rm cl},d} - I\right).$$

Therefore, (28) is equivalent to

$$\max_{h,\alpha} \min_{x+jy\in\lambda\left(\frac{1}{h}\left(A_{{\rm cl},d}-I\right)\right)} \frac{-2x}{x^2 + y^2},$$

that is,

$$\max_{h,\alpha} \min_{x+jy\in\lambda\left(A_{{\rm cl},d}-I\right)} \frac{-2hx}{x^2 + y^2}. \tag{29}$$

However,

$$x + jy \in \lambda\left(A_{{\rm cl},d} - I\right)$$

is equivalent to

$$(1 + x) + jy \in \lambda\left(A_{{\rm cl},d}\right);$$

so, (29) is equivalent to

$$\max_{h,\alpha} \min_{x+jy\in\lambda\left(A_{{\rm cl},d}\right)} f(h,\alpha) = 2h\frac{(1 - x)}{(1 - x)^2 + y^2}. \tag{30}$$

Theorem 3.5: *Suppose that the preceding optimisation is optimal at* (h^*, α^*). *If*

$$h^* < f(h^*, \alpha^*), \tag{31}$$

then system Σ_2 *is stable for the sampling period* h^* *when* α^* *is adopted.*

Proof: When inequality (31) holds, for any eigenvalue $x + jy$ of $A_{{\rm cl},d}$,

$$h^* < 2h^* \frac{(1 - x)}{(1 - x)^2 + y^2}. \tag{32}$$

Hence, one has

$$\frac{(1 - x)}{(1 - x)^2 + y^2} > \tfrac{1}{2}, \tag{33}$$

which is equivalent to

$$x^2 + y^2 < 1.$$

Therefore, the closed-loop system is stable. When α^* is adopted, h^* guarantees closed-loop stability. □

The optimisation problem defined in Equation (30) is a two-dimensional optimisation problem. In what follows we will discuss how to solve it. First, we discuss the sampling period h. On the one hand, hardware constraints provide a lower bound on h; on the other hand, to guarantee satisfactory closed-loop control performance, we cannot sample too slowly. Therefore, we can assume that h is in the interval $[\underline{h}, \bar{h}]$.

Second, numerical experience tells us that when α is too big or too small, closed-loop performance is usually bad. As a result, we confine α to the interval $[\underline{\alpha}, \bar{\alpha}]$. Third, according to Equations (7) and (5), we must guarantee that the matrix $I - \alpha h A_K$ is invertible to make the matrix $A_{\mathrm{cl},d}$ in Equation (7) properly defined. Because A_K is fixed, the following equation

$$\alpha h A_K = I$$

will divide the $a-h$ plane into several pieces. Clearly, the number of pieces is at most the dimension of A_K. Therefore, the rectangle $[\underline{h}, \bar{h}] \times [\underline{\alpha}, \bar{\alpha}]$ will contain several subsets. Within each subset a two-dimensional optimisation problem (30) is to be solved. Since only two variables, α and h, are involved, many standard optimisation methods (e.g., the differential evolution algorithms (Storn and Price 1997; Hwang et al. 2003)) can be employed to find optimal α and h. After finding all of them over all subsets, the global one can be obtained. In what follows, we formalise the procedure:

Step 1. Determine the rectangle $[\underline{h}, \bar{h}] \times [\underline{\alpha}, \bar{\alpha}]$.

Step 2. Find all eigenvalues λ_i of matrix A_K. Divide the rectangle $[\underline{h}, \bar{h}] \times [\underline{\alpha}, \bar{\alpha}]$ using the curves $\alpha h = \lambda_i^{-1}$ if λ_i is real. Denote the obtained subsets by Γ_j.

Step 3. Within each Γ_j, solve the two-dimensional optimisation problem (30) to obtain optimal (α_j, h_j). Differential evolution algorithms (Storn and Price 1997) can be employed to get 'global' optimal solution in each Γ_j with fair good probability. (Theorem 3.2 ensures the existence of at least one such (α_j, h_j).)

Step 4. Find the global optimal (α^*, h^*) over all (α_j, h_j).

Remark 4: The above procedure is to find the upper bound of the sampling period h within which the closed loop is internally stable. The procedure can be easily extended to other performance specifications such as optimal H_2 and H_∞ performance. In each of these cases, an optimal analogue controller K is first designed, then GBT is applied to obtain a digital controller K_{gbt} which contains parameters h and α. After that lifting technique can be used to convert the resulting sampled-data system to an equivalent discrete-time system, and finally the optimal h and α can be found by solving an optimisation problem of two variables. In fact, Example 4.3 is studied following this procedure.

Next we re-study the preceding example under this new framework. We will show that when $a = \sqrt{2}$, $\alpha = (3/4)$, an upper bound of h can be found to be 4.5523, which is quite close to the optimal value 4.565.

According to (24) and (25),

$$hA_{\mathrm{cl}} + h^2\Delta = A_{\mathrm{cl},d} - I = \frac{h}{1+\alpha ha}\begin{bmatrix} -\alpha h & -1 \\ 1 & -a \end{bmatrix}.$$

Accordingly,

$$A_{\mathrm{cl}} + h\Delta = \frac{1}{1+\alpha ha}\begin{bmatrix} -\alpha h & -1 \\ 1 & -a \end{bmatrix}.$$

It is easy to show the eigenvalues of $A_{\mathrm{cl}} + h\Delta$ are

$$\frac{1}{1+\alpha ha}\,\frac{-(\alpha h + a) \pm \sqrt{(\alpha h - a)^2 - 4}}{2}. \tag{34}$$

In what follows we discuss two cases.

Case 3.6: $(\alpha h - a)^2 - 4 < 0$, i.e.

$$-(2-a) < \alpha h < 2 + a. \tag{35}$$

Hence, the real part x and the imaginary part y of an eigenvalue $x + jy$ are

$$x = \frac{-(\alpha h + a)}{2(1+\alpha ha)}, \quad y = \frac{\sqrt{4-(\alpha h - a)^2}}{2(1+\alpha ha)},$$

respectively. Therefore,

$$\begin{aligned} x^2 + y^2 &= \frac{(\alpha h + a)^2 + 4 - (\alpha h - a)^2}{4(1+\alpha ha)^2} \\ &= \frac{4\alpha ha + 4}{4(1+\alpha ha)^2} = \frac{1}{1+\alpha ha}, \end{aligned}$$

and

$$\frac{-2x}{x^2+y^2} = \frac{-2\dfrac{-(\alpha h + a)}{2(1+\alpha ha)}}{\dfrac{1}{1+\alpha ha}} = \alpha h + a. \tag{36}$$

Define

$$\mu = \alpha h.$$

Then (35) and (36) become

$$\begin{aligned} -(2-a) < \mu < 2 + a. \qquad (37)\\ \frac{-2x}{x^2+y^2} = \mu + a. \end{aligned}$$

Consequently, the maximal value is achieved by solving (37) and

$$h \le \mu + a = \alpha h + a. \tag{38}$$

When $a > 0$, $\alpha > 1$, (38) is satisfied for all h. And (37) yields

$$h < \frac{2+a}{\alpha}.$$

When $a > 0$, $\alpha < 0$, (38) gives

$$h < \frac{a}{1-\alpha}.$$

And (37) yields

$$h < \frac{-(2-a)}{\alpha}.$$

When $a > 0$, $0 < \alpha < 1$, (37) yields

$$\frac{-(2-a)}{\alpha} < h < \frac{2+a}{\alpha}.$$

Equation (38) gives

$$h < \frac{a}{1-\alpha}.$$

In particular, when $a = \sqrt{2}$, $\alpha = (3/4)$, the minimal h is

$$\min\left\{\frac{2+a}{\alpha}, \frac{a}{1-\alpha}\right\} \approx 4.5523, \tag{39}$$

which is quite close to the value given in Example 1.

Case 3.7: $(\alpha h - a)^2 - 4 \geq 0$, i.e.

$$\alpha h \geq 2 + a \quad \text{or} \quad \alpha h \leq -2 + a.$$

In this case, the imaginary part

$$y = 0.$$

There are two real roots, which are

$$x_1 = \frac{1}{1+\alpha h a}\frac{-(\alpha h + a) + \sqrt{(\alpha h - a)^2 - 4}}{2} \quad \text{and}$$
$$x_2 = \frac{1}{1+\alpha h a}\frac{-(\alpha h + a) - \sqrt{(\alpha h - a)^2 - 4}}{2}.$$

Equation (28) becomes

$$\max_{h,\alpha}\left\{\frac{-2}{x_1}, \frac{-2}{x_2}\right\}. \tag{40}$$

We will not go into detail of this optimisation problem. However, according to (39), it is clear that the upper bound of h guaranteeing closed-loop stability is no less than 4.5523.

3.4 *A special case*

When the analogue controller K is stable, we have the following result:

Theorem 3.8: *Assume that the continuous-time controller K is stable. Let $\bar{h}$ be a solution to the following optimisation problem*

$$\begin{aligned} &\max_h \rho\left(\lim_{\alpha\to\infty} A_{G_{\text{zoh}}} + B_{G_{\text{zoh}}} D_{K_{\text{gbt}}} C_G\right) \\ &\textit{subject to}: \rho\left(\lim_{\alpha\to\infty} A_{G_{\text{zoh}}} + B_{G_{\text{zoh}}} D_{K_{\text{gbt}}} C_G\right) < 1. \end{aligned} \tag{41}$$

Then for each $h \leq \bar{h}$, the closed-loop system Σ_2 is stable for sufficiently large α.

Proof: When the original continuous-time controller K is stable, i.e. all the eigenvalues of A_K reside on the left half of the complex plane, it is easy to show that, as $\alpha \to +\infty$,

$$B_{G_{\text{zoh}}} C_{K_{\text{gbt}}} \to 0,$$

and

$$B_{K_{\text{gbt}}} C_G \to 0.$$

Moreover, all the eigenvalues of $A_{K_{\text{gbt}}}$ approach the point $1 + 0i$ on the complex plane from within the unit circle. Clearly, this process is independent of h. Finally, it is not hard to show that

$$\begin{aligned} &A_{G_{\text{zoh}}} + B_{G_{\text{zoh}}} D_{K_{\text{gbt}}} C_G \\ &\quad = e^{A_G h} + \int_0^h e^{A_G \tau} B_G \mathrm{d}\tau D_K C_G \\ &\qquad + \alpha h \int_0^h e^{A_G \tau} B_G \mathrm{d}\tau C_K\left[\alpha h (I - \alpha h A_K)^{-1}\right] B_K C_G. \end{aligned}$$

It is easy to show that when $\alpha \to \infty$, the term

$$\alpha h \int_0^h e^{A_G \tau} B_G \mathrm{d}\tau C_K\left[\alpha h (I - \alpha h A_K)^{-1}\right] B_K C_G$$

approaches a constant matrix. For example, if A_K is given by

$$A_K = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix},$$

then

$$\alpha h (I - \alpha h A_K)^{-1} = \begin{bmatrix} \frac{\alpha h}{1-\alpha h \lambda} & \frac{\alpha h}{(1-\alpha h\lambda)^2} & \frac{\alpha h}{(1-\alpha h\lambda)^3} & \frac{\alpha h}{(1-\alpha h\lambda)^4} \\ 0 & \frac{\alpha h}{1-\alpha h \lambda} & \frac{\alpha h}{(1-\alpha h\lambda)^2} & \frac{\alpha h}{(1-\alpha h\lambda)^3} \\ 0 & 0 & \frac{\alpha h}{1-\alpha h \lambda} & \frac{\alpha h}{(1-\alpha h\lambda)^2} \\ 0 & 0 & 0 & \frac{\alpha h}{1-\alpha h \lambda} \end{bmatrix}.$$

Consequently, as $\alpha \to \infty$,

$$\alpha h(I - \alpha h A_K)^{-1} \to \begin{bmatrix} -\frac{1}{\lambda} & & & \\ & -\frac{1}{\lambda} & & \\ & & -\frac{1}{\lambda} & \\ & & & -\frac{1}{\lambda} \end{bmatrix},$$

which is a constant matrix. Therefore, closed-loop stability can be studied based on the lower-order matrix $\lim_{\alpha\to\infty} A_{G_{zoh}} + B_{G_{zoh}} D_{K_{gbt}} C_G$. Clearly, if $\bar{h}$ is a solution to the optimisation problem (41), then for each $h \le \bar{h}$, the closed-loop system is stable. □

Remark 5: If all the eigenvalues of K are outside the unit circle, then a similar procedure can be developed for $\alpha \to -\infty$.

4. Examples

It has been shown in Zhang et al. (2007b) that if the original analogue system is step-tracking, when GBT is applied to obtain a digital controller, the resulting sampled-data is also step-tracking. In this section, three examples are studied. The first is an IIR digital filter design, the second is a re-study of an example investigated in Rattan (1984) and Keller and Anderson (1992), and the third is an H_∞ control of a system studied in Chen and Francis (1995). By comparing GBT with existing methods via these examples, effectiveness of GBT is demonstrated.

Example 4.1: (IIR digital filter design, example 15.1 Rorabaugh 1993) Use the GBT to obtain an IIR filter from a second-order Butterworth analogue filter with a 3-dB cutoff frequency of 3 KHz. The sampling rate for the digital filter is 30,000 samples per second. Clearly, the second-order analogue IIR filter is given by

$$K(s) = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2},$$

where $\omega_c = 6000\pi$. Following Chen and Francis (1995) (section 3.5), define error in the frequency domain:

$$\text{error}(\omega) := \left|K(j\omega) - r(j\omega)K_{\text{gbt}}\left(e^{j\omega h}\right)\right|, \qquad (42)$$

where

$$r(j\omega) = \frac{1 - e^{j\omega h}}{j\omega h},$$

as defined in Equation (3.7) in Chen and Francis (1995). We plot error(ω), as shown in Figure 4. Clearly, when $\alpha = 0.8$, GBT is superior to the Tustin's method ($\alpha = 0.5$). That is, GBT gives a better approximation of the filter $K(s)$ than the Tustin's method.

Example 4.2: (Step-tracking, Rattan's example, Rattan (1984); Keller and Anderson (1992)) Consider the analogue system as shown in Figure 5, where r is a step input and the transfer functions of the plant G and the controller K are given by:

$$G(s) = \frac{10}{s(s+1)}, \quad K(s) = \frac{0.416s + 1}{0.139s + 1}. \qquad (43)$$

Discretising K using GBT yields a sampled-data system as shown in Figure 6. It is proved in Zhang et al. (2007b) that step-tracking is preserved if the sampled-data system in Figure 6 is stable. This example is first studied in Rattan (1984). It is reported that common digitisation methods yield either non-stabilising controllers or systems with very poor closed-loop performance. A digital re-design approach, first proposed in Keller and Anderson (1992) and further studied in Anderson et al. (1999) and Lechevin, Rabbath and Dufour (2005), could produce controllers guaranteeing satisfactory closed-loop performance

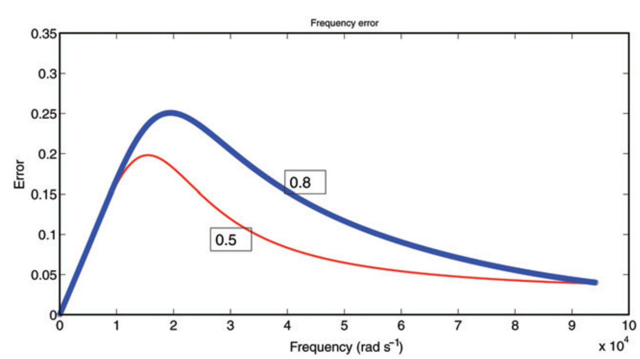

Figure 4. Errors in the frequency domain.

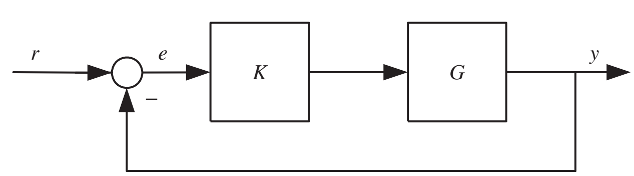

Figure 5. Unity-feedback analogue system Σ_3.

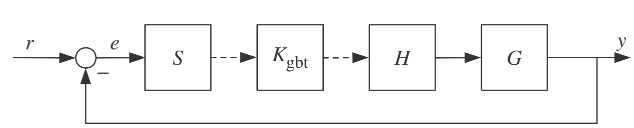

Figure 6. Sampled-data system Σ_4 via GBT.

even when the sampling period h is large. In this part, we will compare it with GBT to demonstrate the strength of the latter.

Now we compare unit step response of three discretisation approaches (Figure 7), where the underlying sampling period $h = 0.157$. In Figure 7, the solid line is the continuous-time step response; the dashed line is the step response using the method in Keller and Anderson (1992); the dashdot line is the step response using the Tustin's method; the dotted line is the step response using GBT with ($\alpha = -0.2$). Clearly, the performance using the Tustin's method is unacceptable, while GBT gives as good a performance as that proposed in Keller and Anderson (1992). In Liu and Chen (2005), a modified bilinear integrations algorithm is proposed based on the modulated sine function. The proposed method is equivalent to GBT when $\alpha \in [0.5, \infty)$. This example shows that by extending the range of α, better control performance can be obtained.

When $h > 0.42$, the plant is not stabilisable via controllers designed using the method proposed in Keller and Anderson (1992). However, for $h \in (0, 0.578)$, it can be stabilised by controllers designed via GBT. Moreover, controllers obtained based on the method in Keller and Anderson (1992) are of the second-order, while those via GBT are of the first-order.

Example 4.3: (H_∞ control, Chen and Francis (1995)) In this example, a sampled-data H_∞ control problem discussed in Chen and Francis (1995) is re-studied. Consider the continuous-time system as shown in Figure 8, where

$$G(s) = \frac{20 - s}{(s + 0.01)(s + 20)}, \quad F(s) = \frac{1}{(0.5/\pi)s + 1},$$

$$W(s) = \frac{1}{((2.5/\pi)s + 1)^2}, \quad k_1 = k_2 = 0.01.$$

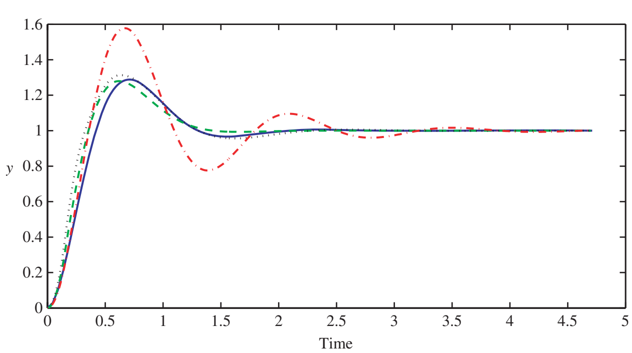

Figure 7. Step response: Continuous (solid); Keller and Anderson (dashed); GBT (dotted); Tustin (dashdot).

This example was studied extensively in Chen and Francis (1995). The following stabilising analogue controller K is designed:

$$K(s) = \frac{\left\{ \begin{array}{l} 1.4261 \times 10^5 (s + 20)(s + 6.2832) \\ \times (s + 3.9436)(s + 0.01) \end{array} \right\}}{\left\{ \begin{array}{l} (s + 631.69)(s + 159.56)(s + 39.230) \\ \times (s + 1.3212)(s + 1.1876) \end{array} \right\}}. \quad (44)$$

In fact, the analogue controller K given in Equation (44) is obtained by minimising the H_∞ norm of the transfer function

$$T_{zw} : \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \longmapsto \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

and achieves $\|T_{zw}\|_{\ell_2} = 0.0813$ (Chen and Francis 1995; pp. 340–341). Implement the analogue controller K digitally to get a sampled-data system as shown in Figure 9. In what follows, we shall address the following problems:

- Can controllers designed via GBT stabilise the closed-loop control system in Figure 9 even when the sampling period is large?
- Pertaining to Figure 9, define $\gamma_{\text{opt}}(h) := \inf\{\|T_{zw}\|_{\ell_2} : K_d \text{ is stablising}\}$. Can we choose values of α such that the sampled-data system has a similar (even the same) value of $\gamma_{\text{opt}}(h)$ as that obtained via the SD design?

Note that k_1 is set to be zero in Figure 9 to guarantee finiteness of $\gamma_{\text{opt}}(h)$, as explained in Example 13.8.1 in Chen and Francis (1995).

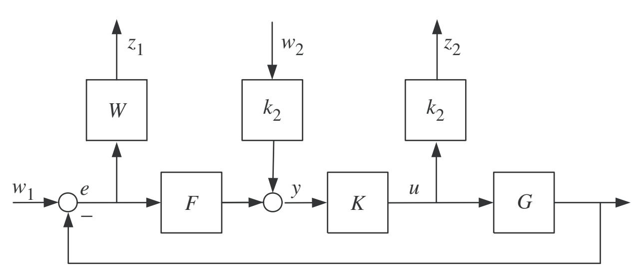

Figure 8. Analogue feedback system.

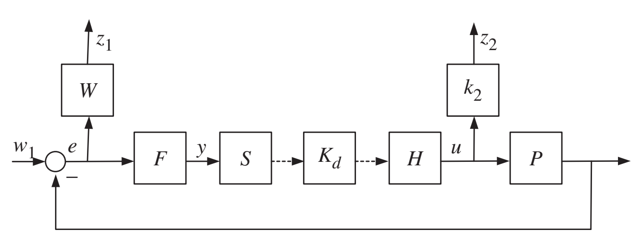

Figure 9. Sampled-data feedback system.

Now we address the first problem. As has been done before, let $A_{\mathrm{cl},d}$ be the A-matrix of the closed-loop system in Figure 9. Define

$$d(A_{\mathrm{cl},d}) := \frac{1}{h}(\rho(A_{\mathrm{cl},d}) - 1).$$

Then $A_{\mathrm{cl},d}$ is stable if and only if $d(A_{\mathrm{cl},d}) < 0$. Next we plot $d(A_{\mathrm{cl},d})$ *versus* h. According to Figure 10, the sampled-data system is stable for every $h \in (0, 0.02)$ when K_d is the zero-order hold equivalent of K; while it is stable for every $h \in (0, 0.45)$ when K_d is obtained via the bilinear transformation. However, when $\alpha = 17$,

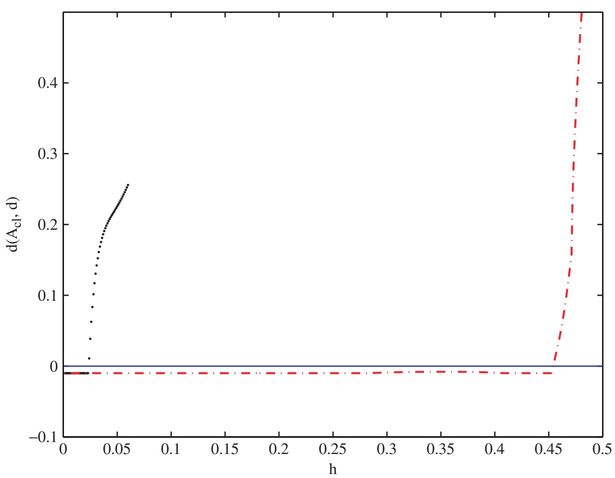

Figure 10. ZOH (dotted), the bilinear transformation (dashdot).

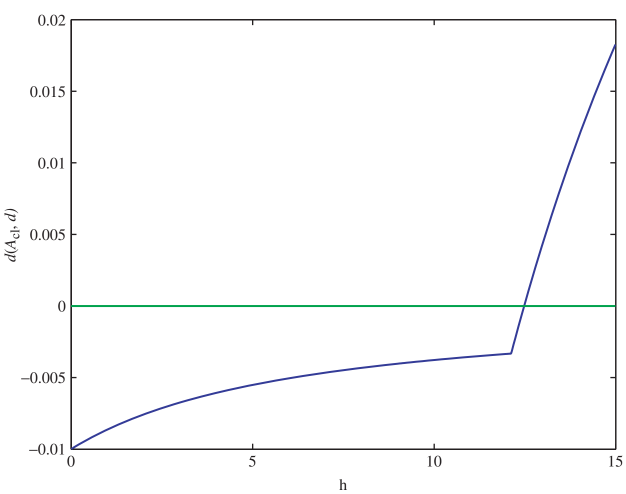

Figure 11. The GBT with $\alpha = 17$ (solid).

Table 1. Comparison of GBT and SD design.

h	0.05	0.113	0.33
SD design	0.1	0.2	0.5
GBT	0.11	0.21	0.5301
(α)	(0.63, 0.77)	(14.5, 17)	(12, 18)

Figure 11 shows that the sampled-data system is stable for every $h \in (0, 12.4)$, a much larger range. In fact, because the controller K is stable, according to Theorem 3.8, it can be shown that the closed-loop system is stable even when $h = 17.2$. Therefore, when α is appropriately chosen, a sampled-data control obtained via GBT may still be stable even in very slow sampling.

In what follows, we study the second problem. For some values of h, we compare the corresponding values of $\gamma_{\mathrm{opt}}(h)$ for K_d obtained via GBT and the SD design, respectively. The result is outlined in Table 1.

Table 1 tells us the discrepancy between the emulation method via GBT and the SD design is very small. Hence, it is arguable that the emulation method based on GBT can achieve quite similar performance as that via the SD design.

5. Conclusions

In this article we have studied a new controller discretisation approach, namely, the GBT. GBT provides a class of digital approximations of an analogue controller, thus optimal discretisation is rendered possible. We have studied international stability of sampled-data systems obtained via GBT. Numerical examples demonstrate the effectiveness of the controller re-design approach via GBT.

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