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On NCP-Functions*

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Abstract. In this paper we reformulate several NCP-functions for the nonlinear complementarity problem (NCP) from their merit function forms and study some important properties of these NCP-functions. We point out that some of these NCP-functions have all the nice properties investigated by Chen, Chen and Kanzow [2] for a modified Fischer-Burmeister function, while some other NCP-functions may lose one or several of these properties. We also provide a modified normal map and a smoothing technique to overcome the limitation of these NCP-functions. A numerical comparison for the behaviour of various NCP-functions is provided.

Keywords: nonlinear complementarity problem, NCP-function, normal map, smoothing

Dedication: (A tribute to Olvi from Liqun Qi)
Olvi was one of my teachers at University of Wisconsin-Madison during my Ph.D. study. He had always encouraged me to work on optimization theory and algorithms, on solution of optimization problems, instead of abstract analysis. Olvi is always helpful and friendly. Olvi is like an evergreen tree; he is always energetic and active. I am very proud to have him as my teacher.

1. Introduction

Consider the nonlinear complementarity problem (NCP for abbreviation) : Find an $x \in \Re^n$ such that

$$x \geq 0, \quad F(x) \geq 0, \quad x^T F(x) = 0, \tag{1}$$

where F maps from $\Re^n$ to $\Re^n$ and will be assumed to be continuously differentiable in this paper. The NCP has received a lot of attention due to its various applications in operations research, economic equilibrium, and engineering design [17, 25, 11].

A popular way to solve the NCP is via an NCP-function $\phi : \Re^2 \to \Re$:

$$\phi(a,b) = 0 \Longleftrightarrow a, b \geq 0, \quad ab = 0$$

to reformulate the NCP as (nonsmooth) equations. We refer the reader to [15] for an up-to-date review on NCP-functions. Here we only list several NCP-functions which we will focus on:

$$\begin{aligned}\phi_1(a,b) &= \min(a,b),\\ \phi_2(a,b) &= \sqrt{a^2+b^2} - (a+b),\end{aligned}$$

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$$\begin{aligned}
\phi_3(a,b) &= \sqrt{\{[\phi_2(a,b)]_+\}^2 + \alpha[(ab)_+]^2}, \quad \alpha > 0,\\
\phi_4(a,b) &= \phi_2(a,b) - \alpha a_+ b_+, \quad \alpha > 0,\\
\phi_5(a,b) &= \sqrt{[\phi_2(a,b)]^2 + \alpha(a_+ b_+)^2}, \quad \alpha > 0,\\
\phi_6(a,b) &= \sqrt{[\phi_2(a,b)]^2 + \alpha[(ab)_+]^4}, \quad \alpha > 0,\\
\phi_7(a,b) &= \sqrt{[\phi_2(a,b)]^2 + \alpha[(ab)_+]^2}, \quad \alpha > 0,
\end{aligned}$$

where for any $v \in \Re^m, m \geq 1$, $(v_+)_i = \max\{0, v_i\}, i = 1, \cdots, m$. The function ϕ_1 is usually labelled as a natural residual or "min" function. Its piecewise linear structure is very favourable. However, its square $(\phi_1(\cdot))^2$ is not continuously differentiable, which makes the generalized Newton direction not necessarily a descent direction. We will not study ϕ_1 in this paper. The function ϕ_2 is called the Fischer-Burmeister function [13] and has been well studied. Among its many nice properties the Fischer-Burmeister function has the feature that its square $(\phi_2(\cdot))^2$ is continuously differentiable [19]. A recent study of Fischer-Burmeister function with various hybrid techniques is included in [7]. The function ϕ_3 is reformulated from a merit function studied in [21], which was based on earlier results of [34] and [22]. The function ϕ_4 is called a penalized Fischer-Burmeister function in [2] and a regularized Fischer-Burmeister function in [33]. Chen, Chen and Kanzow [2] have studied several nice properties of ϕ_4 and have reported very encouraging numerical results based on ϕ_4 for solving nonlinear complementarity problems. The function ϕ_5 is a variant of ϕ_4. The function ϕ_6 is reformulated from a merit function in [35] while ϕ_7 is a variant of ϕ_6. It is noted that the function ϕ_2 in $\phi_3 - \phi_7$ can be replaced by a piecewise rational NCP-function introduced in [26]. Note that for $i = 2, 3, 4, 5, 6, 7$, we have

$$\phi_i(a,b) \equiv \phi_2(a,b)$$

for all $(a,b) \in N_-$, where

$$N_- = \{(a,b) |\ ab \leq 0\}.$$

So functions ϕ_i for $i = 2, 3, 4, 5, 6, 7$, are only different in the first or third quadrant. For $i = 3, 5, 6, 7$, we may also define

$$\bar{\phi}_i(a,b) = \begin{cases} \phi_i(a,b) \equiv \phi_2(a,b) & \text{if } (a,b) \in N_-,\\ -\phi_i(a,b) & \text{if } a \geq 0, b \geq 0,\\ \phi_i(a,b) & \text{if } a \leq 0, b \leq 0. \end{cases}$$

Then $\bar{\phi}_i$ have the same properties for ϕ_i discussed in this paper, for $i = 3, 5, 6, 7$.

The concept of semismoothness was originally introduced by Mifflin [24] for functionals. Convex functions, smooth functions, and piecewise linear functions are examples of semismooth functions. The composition of semismooth functions is still a semismooth function [24]. In [29], Qi and J. Sun extended the definition of semismooth functions to $\Phi : \Re^n \to \Re^m$. It has been proved in [29] that Φ is semismooth at x if and only if all its component functions are.

LEMMA 1 *All $\phi_i, i \in \{2, 3, 4, 5, 6, 7\}$ are strongly semismooth functions and all $(\phi_i)^2, i \in \{2, 3, 4, 5, 6, 7\}$ are continuously differentiable.*

Proof: The strong semismoothness of ϕ_2 and ϕ_4 have been proved in [27] and [2], respectively. For the continuous differentiability of $(\phi_2)^2$, $(\phi_3)^2$ and $(\phi_4)^2$, see [19], [21] and [2], respectively. By noticing that $\sqrt{(\cdot)^2 + (\cdot)^2}$ and $(\cdot)_+$ are strongly semismooth functions and that the composition of strongly semismooth functions is a strongly semismooth function [14, Theorem 19], we obtain that $\phi_i, i \in \{3, 5, 6, 7\}$ are also strongly semismooth functions. Since $(\phi_2)^2$ is continuously differentiable, it is trivial to prove that all $(\phi_i)^2, i \in \{5, 6, 7\}$ are continuously differentiable. ∎

Define $H : \Re^n \to \Re^n$ by

$$H_i(x) = \phi(x_i, F_i(x)), \quad i = 1, 2, \cdots, n, \tag{2}$$

where ϕ is an NCP-function. Then to solve the NCP is equivalent to find a root of $H(x) = 0$. By using different NCP-functions we get various versions of equations $H(x) = 0$. Define

$$f(x) = \frac{1}{2}\|H(x)\|^2. \tag{3}$$

We use $\|\cdot\|$ for the 2-norm in this paper.

THEOREM 1 *For any $i \in \{2, 3, 4, 5, 6, 7\}$, the corresponding map H constructed via $\phi = \phi_i$ is semismooth on $\Re^n$. If F' is locally Lipschitz continuous around a point $x \in \Re^n$, then H is strongly semismooth at x.*

Proof: The conclusions for H constructed via ϕ_2 and ϕ_4 have been proved in [14] and [2], respectively. By Theorem 5 in [24] and Lemma 1 we obtain that for each $i \in \{3, 5, 6, 7\}$, all $H_j, j \in \{1, 2, \cdots, n\}$ are semismooth at any $x \in \Re^n$, and so, H is semismooth on $\Re^n$. Under the assumption that F' is locally Lipschitz continuous around a point $x \in \Re^n$, by Theorem 19 in [14] and Lemma 1, we get that H is strongly semismooth at x. ∎

We need the following definitions concerning matrices and functions.

Definition 1. A matrix $W \in \Re^{n \times n}$ is called a

- P_0-matrix if each of its principal minors is nonnegative;
- P-matrix if each of its principal minors is positive.

Obviously a positive semidefinite matrix is a P_0-matrix and a positive definite matrix is a P-matrix.

Definition 2. A function $F : \Re^n \to \Re^n$ is called a

- P_0-function on a set $D \subseteq \Re^n$ if, for every $x \in D$ and $y \in D$ with $x \neq y$, there is an index i such that

$$x_i \neq y_i, \quad (x_i - y_i)(F_i(x) - F_i(y)) \geq 0;$$

- P-function on a set $D \subseteq \Re^n$ if, for every $x \in D$ and $y \in D$ with $x \neq y$, there is an index i such that

$$x_i \neq y_i, \quad (x_i - y_i)(F_i(x) - F_i(y)) > 0;$$

- uniform P-function on a set $D \subseteq \Re^n$ if there exists a positive constant μ such that, for every $x \in D$ and $y \in D$, there is an index i such that

$$(x_i - y_i)(F_i(x) - F_i(y)) \geq \mu \|x - y\|^2;$$

- monotone function on a set $D \subseteq \Re^n$ if, for every $x \in D$ and $y \in D$,

$$(x - y)^T (F(x) - F(y)) \geq 0;$$

- strongly monotone function on a set $D \subseteq \Re^n$ if there exists a positive constant μ such that, for every $x \in D$ and $y \in D$,

$$(x - y)^T (F(x) - F(y)) \geq \mu \|x - y\|^2.$$

It is known that every strongly monotone function is a uniform P-function and every monotone function is a P_0-function. Furthermore, the Jacobian of a continuously differentiable P_0-function (uniform P-function) at a point is a P_0-matrix (P-matrix).

The organization of this paper is as follows. In the next section we prove that if F is monotone on $\Re^n$ and there exists a strictly feasible point, i.e., there exists $x \in \Re^n$ such that $x > 0, F(x) > 0$, the merit function f, constructed via ϕ_5, ϕ_6, ϕ_7 as well as via ϕ_3 and ϕ_4, has bounded level sets. In Section 3 we prove that if $x \in \Re^n$ is a stationary point of f and $F'(x)$ is a P_0-matrix, then x is a solution of the NCP if f is constructed via ϕ_5, ϕ_6, ϕ_7 as well as via ϕ_2, ϕ_3 and ϕ_4. We also give an example to show that this property may not hold for two NCP-functions closely related to those discussed here. In Section 4 we prove that if the NCP is R-regular at a solution x^*, then all the generalized Jacobian $V \in H(x^*)$ are nonsingular with H constructed via ϕ_5, ϕ_6, ϕ_7 as well as via ϕ_2 and ϕ_4. We also point out that a similar result does not hold for ϕ_3. In Section 5 we give a modified normal map in order to overcome the limitation of all the NCP-functions and discuss a smoothing technique. We also point out why some formulas are not recommended. We present some numerical experiments for various NCP-functions in Section 6 and make final remarks in Section 7.

2. Level Sets Conditions

For any $c \in \Re$ and any function $g : \Re^n \to \Re$, define

$$L_c g = \{x \in \Re^n |\ g(x) \leq c\}. \tag{4}$$

Assumption 1. [2, Condition 3.8] For any sequence $\{x^k\}$ such that

$$\|x^k\| \to \infty, \ \limsup_{k\to\infty} \|[-x^k]_+\| < \infty, \ \limsup_{k\to\infty} \|[-F(x^k)]_+\| < \infty,$$

it holds

$$\max_i [x_i^k]_+ [F_i(x^k)]_+ \to \infty.$$

It was proved in [2, Proposition 3.10] that Assumption 1 holds if either F is a monotone function with a strictly feasible point or F is an R_0-function.

THEOREM 2 *Suppose that Assumption 1 holds and that f is constructed via $\phi_i, i \in \{3,4,5,6,7\}$. Then for any $c \geq 0$, $L_c f$ is bounded.*

Proof: For $\phi_i, i \in \{3,4\}$, the conclusion has been proved in [21, Theorem 4.2] and [2, Theorem 3.9], respectively. For $\phi_i, i \in \{5,6,7\}$, the proof is similar to that of [2, Theorem 3.9]. We omit the detail. ■

Remark. The conclusion of Theorem 2 is not true for ϕ_1 and ϕ_2 as was shown in [21] by the example with $F(x) \equiv 1, x \in \Re$.

3. Stationary Point Conditions

For any locally Lipschitz continuous function $\Phi : \Re^n \to \Re^m$, let $\partial\Phi(x)$ denote Clarke's generalized Jacobian of Φ at $x \in \Re^n$ [5].

First we study the structure of $\partial H_j(x), j \in \{1,2,\cdots,n\}$. Let e_j denote the jth unit row vector of $\Re^n$, $j \in \{1,2,\cdots,n\}$.

LEMMA 2 *Suppose that F is continuously differentiable and H and f are defined by (2) and (3) respectively, where ϕ is one of $\phi_i, i \in \{2,3,4,5,6,7\}$. Then for any $j \in \{1,2,\cdots,n\}$ and $V \in \partial H_j(x)$, there exist scalars β and γ such that $V = \beta e_j + \gamma F_j'(x)$, where β and γ satisfy*

(i) $\beta\gamma > 0$ *if* $H_j(x) \neq 0$;

(ii) $\beta \neq 0, \gamma = 0$ *if* $H_j(x) = 0$ *and* $x_j = 0, F_j(x) > 0$;

(iii) $\beta = 0, \gamma \neq 0$ *if* $H_j(x) = 0$ *and* $x_j > 0, F_j(x) = 0$;

(iv) $\beta\gamma \geq 0$ *and* $\beta + \gamma \neq 0$ *if* $H_j(x) = 0$, $x_j = F_j(x) = 0$, *and* $\phi = \phi_i, i \in \{2,4,5,6,7\}$ *(Note that $i \neq 3$.)*

Proof: For $\phi = \phi_i, i \in \{2,4\}$, the conclusions of this lemma have been proved in [27] and [2], respectively. For the other cases, we can get the results by computing $\partial H_j(x)$ directly and using the facts that

$$|a_+b_+|, |(ab)_+| \leq \max(|a|,|b|)|\phi_1(a,b)| \leq (2+\sqrt{2}) \max(|a|,|b|)|\phi_2(a,b)|, \quad (5)$$

where the last inequality comes from [34]. For example, suppose that $\phi = \phi_7$. Then

$$H_j(x) = \sqrt{[\sqrt{x_j^2 + F_j(x)^2} - (x_j + F_j(x))]^2 + \alpha[(x_j F_j(x))_+]^2}.$$

For any $x \in \Re^n$ such that $H_j(x) \neq 0$, H_j is continuously differentiable at x and

$$H_j'(x) = \beta e_j + \gamma F_j'(x),$$

where

$$\beta = \frac{\phi_2(x_j, F_j(x))\left[\frac{x_j}{\sqrt{x_j^2+F_j(x)^2}} - 1\right] + \alpha[x_j F_j(x)]_+ F_j(x)}{H_j(x)}, \tag{6}$$

$$\gamma = \frac{\phi_2(x_j, F_j(x))\left[\frac{F_j(x)}{\sqrt{x_j^2+F_j(x)^2}} - 1\right] + \alpha[x_j F_j(x)]_+ x_j}{H_j(x)}. \tag{7}$$

Then we get (i). When $H_j(x) = 0$, by taking limits in (6) and (7) and noticing of (5), we get (ii), (iii), and (iv), respectively. We omit the detail of the proof for $\phi = \phi_i, i \in \{3, 5, 6\}$. ■

Remark. In the above lemma we exclude ϕ_3 for conclusion (iv). This suggests that (iv) of lemma 2 may no hold for ϕ_3. In fact, this can be shown clearly by letting $F(x) = x, x \in \Re$. Then by taking $\phi = \phi_3$ we have (by setting $\alpha = 1$)

$$H(x) = \sqrt{[(\sqrt{2x^2} - 2x)_+]^2 + [(x^2)_+]^2}, \quad x \in \Re.$$

For any $x > 0$, $H(x) = x^2$ and $H'(x) = 2x$. Consequently, $0 \in \partial H(0)$ and 0 cannot be expressed as in (iv) of Lemma 2.

By Lemma 1, f is continuously differentiable on $\Re^n$ if ϕ is one of $\phi_i, i \in \{2,3,4,5,6,7\}$.

THEOREM 3 *Suppose that ϕ is chosen from $\phi_i, i \in \{2, 3, 4, 5, 6, 7\}$. Suppose that $x \in \Re^n$ is a stationary point of f, i.e., $\nabla f(x) = 0$. If $F'(x)$ is a P_0-matrix, then x is a solution point of the NCP.*

Proof: For $\phi = \phi_i, i \in \{2, 3, 4\}$, the conclusion of this lemma has been proved in [10], [21] and [2], respectively. The proof for other NCP-functions is similar to that of [10, Theorem 4.1] in regard of (i) of Lemma 2. Again, we omit the detail. ■

Remark. It can be verified that the function $\phi : \Re^2 \to \Re$ defined by

$$\phi(a, b) = \sqrt{[\phi_2(a, b)]^2 + \alpha(ab)^2}, \quad \alpha > 0 \tag{8}$$

is an NCP-function and its square $(\phi(\cdot))^2$ is continuously differentiable. However, for this NCP-function, conclusion (i) of Lemma 2 may not hold. To show this let $F(x) = (x - 0.5)^2 + c$, where $x \in \Re$ and c is any solution of

$$\theta(c) := \left(1 + c + \frac{c^2}{2}\right) - \frac{1}{\sqrt{1+4c^2}}(1 + c + 2c^2) = 0$$

in $(-\infty, 0)$. Such a c exists because $\theta(-1) < 0$ and $\theta(-10) > 0$. Then we have (by setting $\alpha = 1$)

$$H(x) = \sqrt{\phi_2(x, (x-0.5)^2 + c)^2 + \{x[(x-0.5)^2 + c]\}^2}, \quad x \in \Re.$$

By direct computation, $F'(0.5) = 0$ and $H'(0.5) = [\theta(c)/H(0.5)] \cdot 1 + 0 = 0$. Thus, (i) of Lemma 2 does not hold for the NCP-function defined in (8) and $x = 0.5$ is only a local solution of $\min f(x)$ instead of a global solution, i.e., a solution of the NCP, even if $F'(0.5) = 0$ is a P_0-matrix. Due to the same reason, the function $\phi : \Re^2 \to \Re$ defined by

$$\phi(a, b) = \phi_2(a, b) - \alpha(ab)_+, \quad \alpha > 0 \tag{9}$$

is also not recommended though ϕ is also strongly semismooth and its square $(\phi(\cdot))^2$ is continuously differentiable.

4. Nonsingularity Conditions

In this section we study under what conditions the generalized Jacobians of H are nonsingular at a solution point x^* of the NCP. Define

$$\begin{aligned}
\mathcal{I} &= \{j|\ x_j^* > 0,\ F_j(x^*) = 0\},\\
\mathcal{J} &= \{j|\ x_j^* = 0,\ F_j(x^*) = 0\},\\
\mathcal{K} &= \{j|\ x_j^* = 0,\ F_j(x^*) > 0\}.
\end{aligned}$$

Let $W := F'(x^*)$. Then the NCP is said to be R-regular at x^* if $W_{\mathcal{I}\mathcal{I}}$ is nonsingular and its Schur complement

$$W_{\mathcal{J}\mathcal{J}} - W_{\mathcal{J}\mathcal{I}} W_{\mathcal{I}\mathcal{I}}^{-1} W_{\mathcal{I}\mathcal{J}}$$

is a P-matrix [10]. R-regularity is equivalent to Robinson's strong regularity [30].

THEOREM 4 *Suppose that the NCP is R-regular at x^* and ϕ is any one of $\phi_i, i \in \{2, 4, 5, 6, 7\}$. Then all $V \in \partial H(x^*)$ are nonsingular.*

Proof: The conclusion of this theorem for ϕ_2 and ϕ_4 has been proved in [10] and [2], respectively. We consider the rest cases.

According to Lemma 2, any $V \in \partial H(x^*)$ can be expressed as

$$V = D_\beta + D_\gamma F'(x^*),$$

where D_β and D_γ are some diagonal matrices satisfying

(i) $(D_\beta)_{jj} = 0$ and $(D_\gamma)_{jj} \neq 0$ if $j \in \mathcal{I}$;

(ii) $(D_\beta)_{jj}(D_\gamma)_{jj} \geq 0$ and $(D_\beta)_{jj} + (D_\gamma)_{jj} \neq 0$ if $j \in \mathcal{J}$;

(iii) $(D_\beta)_{jj} \neq 0$ and $(D_\gamma)_{jj} = 0$ if $j \in \mathcal{K}$.

Then by using standard analysis (e.g., [10, Proposition 3.2]), we can prove that V is nonsingular. ■

Remark. In regard of the above discussions, ϕ_5, ϕ_6 and ϕ_7 have all the important properties as ϕ_4 has while the conclusion of Theorem 4 does not hold for ϕ_3. This may explain why no superlinearly convergent methods have been constructed for solving the NCP based on ϕ_3 and $(\phi_3)^2$. Another NCP-function, which has received a lot of attention, is the implicit Lagrangian [23]

$$h(a,b) = ab + \frac{\gamma}{2}\{[(a-\gamma^{-1}b)_+]^2 - a^2 + [(b-\gamma^{-1}a)_+]^2 - b^2\}, \quad \gamma \in (0,1). \tag{10}$$

Since $h(a,b) \geq 0$ for all $a, b \in \Re$ [23], we can define an NCP-function $\phi : \Re^2 \to \Re$ by

$$\phi(a,b) = \sqrt{h(a,b)} \tag{11}$$

and its modifications as in Section 1. For example, we can define

$$\phi(a,b) = \sqrt{h(a,b) + \alpha[(ab)_+]^2}, \quad \alpha > 0. \tag{12}$$

However, two observations prevent us from doing further on these NCP-functions. Firstly, it is not clear whether ϕ defined in (11) is semismooth or not. Secondly, the example given in Section 5 of [33] suggests that the R-regularity may not be sufficient for guaranteeing the superlinear convergence of generalized Newton methods based on (11) or (12).

There are many available methods for solving $H(x) = 0$. For example, see [6] for a line search model and [18] for a trust region model.

5. Modified Normal Map and Smoothing Techniques

In the above sections we have assumed that F is well defined on the whole space $\Re^n$. This, however, may not be satisfied for a few problems [11]. On the other hand, even if F has definition on $\Re^n$, some important properties of F may be lost outside $\Re^n_+$, in particular, the monotonicity of Karush-Kuhn-Tucker systems for convex programming problems with nonlinear constraints may not hold. To handle this problem, one may try to add constraints $x \geq 0$ to the minimization problem considered in the above sections. This, however, constitutes of a constrained optimization problem. Instead of doing so, in this section we will discuss a modified normal map and its smoothing forms.

It is well known (see, e.g., [31]) that to solve the NCP is also equivalent to find a root of the following normal equation:

$$M(z) := F(z_+) + z - z_+ = 0. \tag{13}$$

M is called the normal map in the literature [31]. Here we will discuss a modification of M. Let $G : \Re^n \to \Re^n$ be defined by

$$G(z) := M(z) + \alpha z_+ \bullet [F(z_+)]_+, \quad \alpha > 0, \tag{14}$$

where for $u, v \in \Re^n$, $u \bullet v = (u_1 v_1 \ \cdots \ u_n v_n)^T$. By noting that if $z \in \Re^n$ is a solution of $M(z) = 0$ then $F(z_+) \geq 0$ and $z_+ \bullet F(z_+) = 0$ and that if $z \in \Re^n$ is a solution of $G(z) = 0$ then $z_+ \bullet [F(z_+)]_+ = 0$ we can verify that to solve $M(z) = 0$ is equivalent to solve $G(z) = 0$. Based on the same idea we can also verify that to solve $M(z) = 0$ is equivalent to solve $N(z) = 0$ with $N : \Re^n \to \Re^n$ defined by

$$N(z) := M(z) + \alpha z_+ \bullet F(z_+), \quad \alpha > 0. \tag{15}$$

However, this modification is not recommended by the following observation. Let $F(x) = x - 2, x \in \Re$. Then N is differentiable at $z = 1/2$ with $N'(z) = 0$ (by setting $\alpha = 1$). This means that even if F is strongly monotone a local solution of $\|N(z)\|^2$ may not be a solution of $M(z) = 0$.

Define the merit function $g : \Re^n \to \Re$ by

$$g(z) = \frac{1}{2}\|G(z)\|^2. \tag{16}$$

THEOREM 5 *If F is monotone on $\Re^n_+$ and there exists a strictly feasible point $\bar{z}$, i.e., $\bar{z} > 0$ and $F(\bar{z}) > 0$, then $L_c g$ is bounded for any $c \geq 0$.*

Proof: Suppose that the conclusion is false. Then there exist a constant $c \geq 0$ and a sequence $\{z^k\}$ such that $g(z^k) \leq c$ but $\|z^k\| \to \infty$ as $k \to \infty$. Since F is monotone on $\Re^n_+$ and $\bar{z} > 0$, we have

$$[F(z^k_+) - F(\bar{z})]^T (z^k_+ - \bar{z}) \geq 0. \tag{17}$$

Define

$$I^\infty = \{i | \ z^k_i \text{ is unbounded}\}.$$

Then I^∞ is nonempty because $\|z^k\| \to \infty$ as $k \to \infty$. By passing to a subsequence, we may assume that for any $i \in I^\infty$, $|z^k_i| \to \infty$ as $k \to \infty$. Define

$$\begin{aligned} I^\infty_+ &= \{i \in I^\infty | \ z^k_i \to +\infty\}, \\ I^\infty_- &= \{i \in I^\infty | \ z^k_i \to -\infty\}. \end{aligned}$$

Again, by passing to a subsequence, we may assume that

$$I^\infty_+ \cup I^\infty_- = I^\infty.$$

Let

$$\bar{I}^\infty = \{1, 2, \cdots, n\} \backslash I^\infty.$$

Since $z^k \in L_c g$,

$$\|F(z^k_+) + z^k - z^k_+ + \alpha z^k_+ \bullet [F(z^k_+)]_+\|^2 \leq 2c.$$

Then, we have

(i) $i \in I_+^\infty$, $F_i(z_+^k) \le \frac{1}{2} F_i(\bar z)$ for all k sufficiently large;

(ii) $i \in I_-^\infty$, $F_i(z_+^k) \to +\infty$ as $k \to +\infty$;

(iii) $i \in \bar I^\infty$, $|F_i(z_+^k)|$ is bounded.

Thus, from (17), for all k sufficiently large,

$$\sum_{i\in I_+^\infty} [\frac{1}{2}F_i(\bar z) - F_i(\bar z)](z_i^k - \bar z_i) + \sum_{i \in I_-^\infty} [F_i(z_+^k) - F_i(\bar z)](0 - \bar z_i) \ge - \sum_{i \in \bar I^\infty} [F_i(z_i^k) - F_i(\bar z)][(z_+^k)_i - \bar z_i],$$

which is impossible because the left-hand-side tends to $-\infty$ while the right-hand-side is bounded. This completes the proof. ■

Remark. The conclusion of Theorem 5 does not hold if the mapping G is replaced by the normal map M. This can be seen clearly by assuming $F(x) \equiv 1, x \in \Re$.

Comparing to Theorem 2, we can see that the benefit of using G is that we only need to assume F to be monotone on $\Re_+^n$ instead of on $\Re^n$. On the other hand, the disadvantage of G is that the merit function defined in (16) is not continuously differentiable on $\Re^n$. However, this shortcoming can be overcome by using some smoothing functions to approximate z_+ and $[F(z_+)]_+$. Though we can use any one of the smoothing functions described in [4], we are particularly interested in using the Chen-Harker-Kanzow-Smale (CHKS) function $\psi : \Re^2 \to \Re$ [3, 20, 32]:

$$\psi(\mu, w) = \frac{\sqrt{w^2 + 4\mu^2} + w}{2}, \quad (\mu, w) \in \Re^2. \tag{18}$$

Among many of its nice properties the CHKS function has the feature

$$\psi(\mu, w) > 0 \quad \text{for any } w \in \Re,\ \mu \neq 0. \tag{19}$$

For any $u \in \Re^n$ and $x \in \Re^n$, define $p(u, x)$ by

$$p_i(u, x) = \psi(u_i, x_i), \quad i \in \{1, 2, \cdots, n\}. \tag{20}$$

Then, from (19), for any $x \in \Re^n$ if $u > 0$,

$$p(u, x) > 0.$$

Define $E : \Re^{2n} \to \Re^n$ by

$$E(u, z) = F(p(u, z)) + z - p(u, z) + \alpha p(u, z) \bullet p(u, F(p(u, z))), \quad \alpha > 0 \tag{21}$$

and $\Phi : \Re^{2n} \to \Re^{2n}$ by

$$\Phi(u, z) := \begin{pmatrix} u \\ E(u, z) \end{pmatrix}, \quad \alpha > 0. \tag{22}$$

THEOREM 6 *Suppose that F is continuously differentiable on an open set containing $\Re^n_+$. Then*

(i) $z \in \Re^n$ *is a solution of* $M(z) = 0$ *if and only if* $(0, z) \in \Re^n \times \Re^n$ *is a solution of* $\Phi(u, z) = 0$*;*

(ii) Φ *is continuously differentiable on* $\Re^n_{++} \times \Re^n$*, where* $\Re^n_{++}$ *is the interior part of* $\Re^n_+$*;*

(iii) Φ *is semismooth on* $\Re^{2n}$*, and if* F' *is locally Lipschitz continuous at* $p(u, z)$*,* $(u, z) \in \Re^n \times \Re^n$*, then* Φ *is strongly semismooth at* (u, z)*.*

Proof: By noting that for any $x \in \Re^n$, $p(0, x) = x_+$ and that $z \in \Re^n$ is a solution of $M(z) = 0$ if and only if $z \in \Re^n$ is a solution of $G(z) = 0$, we get (i). Since p is continuously differentiable at any $(u, x) \in \Re^n_{++} \times \Re^n$, we get (ii). For (iii) we refer to Proposition 3.2 and Theorem 3.1 of [28] for a similar discussion. ■

Denote $f : \Re^{2n} \to \Re$ by

$$f(u, z) = \frac{1}{2}\|\Phi(u, z)\|^2, \quad (u, z) \in \Re^n \times \Re^n.$$

THEOREM 7 *If F is monotone on $\Re^n_+$ and there exists a strictly feasible point $\bar{z}$, i.e., $\bar{z} > 0$ and $F(\bar{z}) > 0$, then $L_c f$ is bounded for any $c \geq 0$.*

Proof: Suppose that the conclusion is false. Then there exist a constant $c \geq 0$ and a sequence $\{(u^k, z^k)\}$ such that $f(u^k, z^k) \leq c$ but $\|(u^k, z^k)\| \to \infty$ as $k \to \infty$. Since $\|u^k\|^2 \leq 2f(u^k, z^k) \leq 2c$, we have that $\|z^k\| \to \infty$ as $k \to \infty$. By using the facts that F is monotone on $\Re^n_+$, $p(0, \bar{z}) = \bar{z} > 0$ and $p(u^k, z^k) \in \Re^n_+$, we have

$$[F(p(u^k, z^k)) - F(p(0, \bar{z}))]^T [p(u^k, z^k) - p(0, \bar{z})] \geq 0. \tag{23}$$

Define

$$I^\infty = \{i |\ z_i^k \text{ is unbounded}\}.$$

Then I^∞ is nonempty because $\|z^k\| \to \infty$ as $k \to \infty$. By passing to a subsequence, we may assume that for any $i \in I^\infty$, $|z_i^k| \to \infty$ as $k \to \infty$. Define

$$I^\infty_+ = \{i \in I^\infty |\ z_i^k \to +\infty\},$$
$$I^\infty_- = \{i \in I^\infty |\ z_i^k \to -\infty\}.$$

Again, by passing to a subsequence, we may assume that

$$I^\infty_+ \cup I^\infty_- = I^\infty.$$

Let

$$\bar{I}^\infty = \{1, 2, \cdots, n\} \backslash I^\infty.$$

Since $(u^k, z^k) \in L_c f$,

$$\|F(p(u^k, z^k)) + z^k - p(u^k, z^k) + \alpha p(u^k, z^k) \bullet p(u^k, F(p(u^k, z^k)))\|^2 \leq 2c.$$

Then, by combining with the properties of ψ:

(i) $|\psi(\mu, w) - w_+| \le \mu$ for all $\mu, w \in \Re$;

(ii) $\psi(\mu, w) \ge w$ for all $\mu \in \Re$;

(iii) $\psi(\mu, w) \to 0$ as $w \to -\infty$ for any μ in a bounded set,

we have

(i) $i \in I_+^\infty$, $F_i(p(u^k, z^k)) \le \frac{1}{2} F_i(p(0, \bar{z}))$ for all k sufficiently large;

(ii) $i \in I_-^\infty$, $F_i(p(u^k, z^k)) \to +\infty$ as $k \to +\infty$;

(iii) $i \in \bar{I}^\infty$, $|F_i(p(u^k, z^k))|$ is bounded.

Similar to that of Theorem 5, we can complete our proof. Details are omitted here. ■

THEOREM 8 *Suppose that F is continuously differentiable on an open set containing $\Re^n_+$. If for some $(u, z) \in \Re^n_{++} \times \Re^n$, $F'(p(u, z))$ is a P_0-matrix, then $\Phi'(u, z)$ is nonsingular.*

Proof: By considering of (22) we only need to prove that $\partial E(u, z)/\partial z$ is nonsingular under the assumptions. By direct computation and noticing of that for any $y \in \Re^n$ and each $i \in \{1, 2, \cdots, n\}$, $(\partial p_i(u, y)/\partial y)_j = 0$ if $j \ne i$ and $j \in \{1, 2, \cdots, n\}$, we have

$$\begin{aligned} \partial E(u, z)/\partial z &= F'(p(u, z))\partial p(u, z)/\partial z + I - \partial p(u, z)/\partial z \\ &\quad + \alpha C F'(p(u, z))\partial p(u, z)/\partial z + \alpha D \end{aligned}$$

for some diagonal matrices C and D satisfying

$$\begin{aligned} C_{ii} &= p_i(u, z) \left(\partial p_i(u, y)/\partial y|_{y = F(p(u,z))} \right)_i, \\ D_{ii} &= p_i(u, F(p(u, z)))(\partial p_i(u, z)/\partial z)_i, \end{aligned}$$

where $i \in \{1, 2, \cdots, n\}$. By noting that for any $u > 0$ and $y \in \Re^n$, $(\partial p_i(u, y)/\partial y)_i \in (0, 1)$ and $p_i(u, y) > 0$, $i \in \{1, 2, \cdots, n\}$, we obtain the conclusion from Theorem 3.3 of [3]. ■

THEOREM 9 *Suppose that F is continuously differentiable on an open set containing $\Re^n_+$ and that $z^* \in \Re^n$ is a solution of (13). If the NCP is R-regular at $[z^*]_+$, then all $V \in \partial \Phi(0, z^*)$ are nonsingular.*

Proof: Let

$$V = \begin{pmatrix} V_1 & V_2 \\ V_3 & V_4 \end{pmatrix}$$

be an element of $\partial \Phi(0, z^*)$, where $V_1, \cdots, V_4 \in \Re^{n \times n}$. Then $V_1 = I$, $V_2 = 0$. Therefore, to prove that V is nonsingular is equivalent to prove that V_4 is nonsingular. By inspecting the structure of Φ, we can write V_4 as

$$V_4 = (I + \alpha \mathrm{diag}(p(0, z^*))W)F'(p(0, z^*))U + I - U + \alpha \mathrm{diag}(p(0, F(p(0, z^*))))U,$$

where W and U are two diagonal matrices with the i-th diagonal entries given by

$$W_{ii} = \begin{cases} 1 & \text{if } F_i(p(0,z^*)) > 0 \\ \varepsilon \in [0,1] & \text{if } F_i(p(0,z^*)) = 0 \\ 0 & \text{if } F_i(p(0,z^*)) < 0 \end{cases}$$

and

$$U_{ii} = \begin{cases} 1 & \text{if } z_i^* > 0 \\ \varepsilon \in [0,1] & \text{if } z_i^* = 0 \\ 0 & \text{if } z_i^* < 0 \end{cases},$$

respectively. Let $\mathcal{I}$, $\mathcal{J}$ and $\mathcal{K}$ be the index sets defined in Section 4 at solution point $x^* := [z^*]_+ = p(0,z^*)$ of the NCP. Then

$$\begin{aligned} \mathcal{I} &= \{i|\ z_i^* > 0\}, \\ \mathcal{J} &= \{i|\ z_i^* = 0\}, \\ \mathcal{K} &= \{i|\ z_i^* < 0\}. \end{aligned}$$

Thus,

$$\alpha \mathrm{diag}(p(0, F(p(0,z^*))))U = 0$$

and

$$V_4 = (I + \alpha \mathrm{diag}(p(0,z^*))W)F'(p(0,z^*))U + I - U.$$

Let $D = I + \alpha \mathrm{diag}(p(0,z^*))W$. Then D is a diagonal matrix with all diagonal entries positive. Let

$$V' := (D^{-1}V_4)^T = U(F'(p(0,z^*)))^T + (I-U)D^{-1}.$$

By [10, Proposition 3.2]), we can prove that V', and so V_4, is nonsingular under the assumption of R-regularity. We complete our proof. ∎

So far, we have demonstrated several important properties of the reformulated function Φ. The remaining task is to design a suitable algorithm to find a solution of $\Phi(u,z) = 0$ such that during the process of the iteration $u > 0$ is kept while global and locally superlinear convergence can still be achieved under suitable assumptions. Such an algorithm with other nice features has already been constructed in [28]. We can apply the method in [28] to solve $\Phi(u,z) = 0$ directly by noticing of Theorems 6–9. See [36] for numerical performance on Φ with an additional regularization technique.

Note that we may also define $G : \Re^n \to \Re^n$ by

$$G_i(z) :\ = \ \sqrt{(M_i(z))^2 + \alpha\{[z_i F_i(z_+)]_+\}^2}, \quad \alpha > 0, \quad i = 1,2,\cdots,n \tag{24}$$

or

$$G_i(z) :\ = \ \sqrt{(M_i(z))^2 + \alpha\{(z_i)_+[F_i(z_+)]_+\}^2}, \quad \alpha > 0, \quad i = 1,2,\cdots,n \tag{25}$$

and its smoothing counterpart. However, the smoothing function of such defined G may lose some favourable properties, in particular that described in Theorem 8.

6. Numerical Experiments

In this section we present some numerical experiments for the NCP-functions ϕ_2–ϕ_7 using the test complementarity problems from GAMS and MCP libraries [1, 8, 12]. We used the damped generalized Newton method introduced in [6] for solving complementarity problems:

Step 0. Given $x^0 \in \Re^n$, $\beta > 0$, $p > 2$, $\rho, \sigma \in (0, 1/2)$. $k := 0$.

Step 1. Select an element $V_k \in \partial_B H(x^k)$ and solve

$$H(x^k) + V_k d = 0. \tag{26}$$

Let d^k be the solution of (26) if it is solvable. If (26) is unsolvable or if the condition

$$\nabla f(x^k)^T d^k \leq -\beta \|d^k\|^p$$

is not satisfied, let $d^k = -\nabla f(x^k)$.

Step 2. Let m_k be the smallest nonnegative integer m such that

$$f(x^k + \rho^m d^k) - f(x^k) \leq -\sigma \rho^m \nabla f(x^k)^T d^k.$$

Set $t_k = \rho^{m_k}$ and $x^{k+1} = x^k + t_k d^k$.

Step 3. Replace k by $k+1$ and go to Step 1.

The algorithm was implemented in Matlab and run on a SUN Sparc Server 3002. Instead of a monotone line search we used a nonmonotone version as described in [9], which was originally due to Grippo, Lampariello and Lucidi [16] and can be stated as follows. Let $\ell \geq 1$ be a pre-specified constant and $\ell_k \geq 1$ be an integer which is adjusted at each iteration k. Calculate a steplength $t_k > 0$ satisfying the nonmonotone Armijo-rule

$$f(x^k + t_k d^k) \leq \mathcal{W}_k + \sigma t_k \nabla f(x^k)^T d^k, \tag{27}$$

where $\mathcal{W}_k := \max\{f(x^j) | j = k + 1 - \ell_k, \ldots, k\}$ denotes the maximal function value of f over the last ℓ_k iterations. Note that $\ell_k = 1$ corresponds to the monotone Armijo-rule. In the implementation, we used the following adjustment of ℓ_k:

1. Set $\ell_k = 1$ for $k = 0, 1$, i.e. start the algorithm using the monotone Armijo-rule for the first two steps.

2. $\ell_{k+1} = \min\{\ell_k + 1, \ell\}$ at all remaining iterations ($\ell = 8$ in our implementation).

Throughout the computational experiments the starting points are provided by GAMS or MCP libraries. The parameters used in the algorithm were $\rho = 0.5$, $\beta = 10^{-10}$, $p = 2.1$ and $\sigma = 10^{-4}$. The parameter α used in the implementation was $\alpha = (0.05)^2$ for ϕ_3, ϕ_5 and ϕ_7, $\alpha = 0.05$ for ϕ_2 and $\alpha = (0.05)^4$ for ϕ_6. The value of α used for ϕ_2 is close to the

Table 1. Numerical results for MCPLIB problems

	ϕ_2		ϕ_3		ϕ_4		ϕ_5		ϕ_6		ϕ_7	
problem	Ir	Fev	Ir	Fev	Ir	Fev	Ir	Fev	Ir	Fev	Ir	Fev
bertsekas(1)	31	256	29	246	34	264	31	254	31	256	31	254
bertsekas(2)	32	247	13	22	27	142	31	192	25	108	31	192
bertsekas(3)	31	241	24	25	27	34	27	28	46	59	27	28
billups	253	3820	176	2055	253	3820	253	3820	253	3820	215	2813
colvdual(1)	43	108	11	14	12	15	12	15	15	16	12	15
colvdual(2)	299	818	F	-	31	46	F	-	F	-	F	-
colvnlp(1)	18	43	11	14	12	15	12	15	15	16	12	15
colvnlp(2)	14	18	10	12	12	15	13	16	18	27	13	16
cycle	3	5	3	5	4	6	4	6	6	7	4	6
explcp	19	31	19	31	19	31	19	31	19	31	19	31
hanskoop(1)	10	20	10	20	10	20	10	20	10	20	10	20
hanskoop(2)	10	20	11	21	10	20	10	20	10	20	10	20
hanskoop(3)	9	18	7	15	9	18	9	18	9	18	10	20
hanskoop(4)	9	12	9	12	9	12	9	12	9	12	9	18
hanskoop(5)	13	27	10	20	14	26	14	26	13	27	9	12
josephy(1)	7	10	8	9	7	8	7	8	7	10	14	26
josephy(2)	7	12	6	7	6	8	6	8	7	12	7	8
josephy(3)	37	48	16	17	15	16	15	16	29	30	6	8
josephy(4)	5	6	5	6	5	6	5	6	5	6	15	16
josephy(5)	4	5	4	5	4	5	4	5	4	5	5	6
josephy(6)	7	10	7	8	6	9	7	9	6	12	4	5
kojshin(1)	F	-	7	9	7	10	7	10	9	11	7	9
kojshin(2)	8	15	8	19	7	13	6	9	5	8	7	10
kojshin(3)	8	10	24	25	15	16	15	16	28	29	6	9
kojshin(4)	4	5	4	5	4	5	4	5	4	5	15	16
kojshin(5)	5	7	5	7	5	7	5	7	5	7	4	5
kojshin(6)	5	7	F		5	7	7	8	6	8	5	7
mathinum(1)	4	5	4	5	6	9	4	5	4	5	7	8
mathinum(2)	5	6	5	6	5	6	5	6	5	6	4	5

Table 2. (continued) Numerical results for MCPLIB problems

	ϕ_2		ϕ_3		ϕ_4		ϕ_5		ϕ_6		ϕ_7	
problem	Ir	Fev	Ir	Fev	Ir	Fev	Ir	Fev	Ir	Fev	Ir	Fev
mathinum(3)	11	34	6	8	9	13	8	14	9	22	5	6
mathinum(4)	7	8	5	6	6	7	7	8	6	7	8	14
mathisum(1)	5	7	3	4	5	7	5	7	5	7	5	7
mathisum(2)	7	8	6	7	6	7	6	7	6	7	6	7
mathisum(3)	8	9	8	11	8	9	8	9	8	10	8	9
mathisum(4)	6	7	6	7	6	7	6	7	6	7	6	7
nash(1)	8	9	8	9	8	9	8	9	8	9	8	9
nash(2)	11	25	9	10	10	11	10	11	22	23	10	11
pgvon105(1)	47	106	F	-	45	93	39	87	46	97	39	87
pgvon105(2)	DV	-	DV	-	DV	-	DV	-	DV	-	DV	-
pgvon105(3)	F	-	LF	-	F		F	-	F	-	F	-
pgvon106	197	2646	101	631	LF	-	33	68	36	76	33	68
powell(1)	10	12	9	11	10	12	10	12	10	12	10	12
powell(2)	12	15	12	15	12	15	12	15	12	15	12	15
powell(3)	12	29	9	23	12	26	12	26	13	27	12	26
powell(4)	12	13	26	41	12	13	12	13	12	13	12	13
scarfanum(1)	11	14	12	13	11	13	11	14	11	14	11	14
scarfanum(2)	12	21	12	13	12	17	11	16	12	21	11	16
scarfanum(3)	11	14	10	11	11	14	10	13	11	14	10	13
scarfasum(1)	8	10	7	8	7	9	8	10	8	10	8	10
scarfasum(2)	11	19	9	12	11	14	11	17	11	19	11	17
scarfasum(3)	12	18	9	12	11	14	12	17	12	18	12	17
scarfbnum(1)	20	30	21	29	21	23	22	26	20	30	22	26
scarfbnum(2)	12	37	29	69	13	31	14	35	23	41	13	32
scarfbsum(1)	24	37	17	28	17	27	11	13	17	33	11	13
scarfbsum(2)	32	93	18	36	19	35	18	38	22	44	18	38
sppe(1)	8	10	8	10	8	10	8	10	9	11	8	10
sppe(2)	6	7	8	9	6	7	6	7	9	10	6	7
tobin(1)	10	13	8	11	8	11	8	11	9	13	8	11
tobin(2)	8	10	15	19	9	13	9	12	13	27	9	12

Table 3. Numerical results for GAMSLIB problems

	ϕ_2		ϕ_3		ϕ_4		ϕ_5		ϕ_6		ϕ_7	
problem	Ir	Fev	Ir	Fev	Ir	Fev	Ir	Fev	Ir	Fev	Ir	Fev
cafemge	10	18	9	11	9	11	10	12	10	14	10	12
dmcmge	F	-	25	70	31	102	34	124	42	142	34	124
etamge	19	46	9	12	17	26	19	37	19	46	19	37
hansmcp	11	17	12	18	15	27	17	21	19	25	17	21
harkmcp	14	20	12	15	12	15	11	14	12	25	11	14
kehomge	11	14	9	10	10	11	11	12	12	16	11	12
mr5mcp	10	14	8	9	9	11	11	13	15	26	11	13
nsmge	12	16	8	10	16	19	12	15	F	-	12	15
oligomcp	6	7	6	7	6	7	6	7	6	7	6	7
scarfmge	11	13	9	10	11	12	11	12	11	12	11	12
scarfmcp	9	12	8	9	11	12	8	11	9	12	8	11
transmcp	11	18	11	18	11	18	12	19	10	16	12	19
unstmge	9	11	8	9	9	11	9	11	9	11	9	11
vonthmcp	F	-	LF	-	34	123	F	-	F	-	40	162
vonthmge	F	-	F	-	F	-	LF	-	33	69	F	-

one used in [2] (0.05 versus $0.05/0.95$). The choices of value of α for other functions are based on the same order of scaling. See [33] for an explanation about the scaling.

The iteration of the algorithm is stopped if either

$$f(x^k) \leq 10^{-12} \qquad \text{or} \qquad \|\nabla f(x^k)\| \leq 10^{-10}$$

or if either

– the number of iterations exceeds 300,

– the number of line search steps exceeds 80.

Finally we note that in our algorithm we assume that F is well defined everywhere, whereas there are a few examples in the GAMS and MCP libraries where the function F may be not defined outside of $\Re^n_+$ or even on the boundary of $\Re^n_+$. To avoid this problem partially our implementation used the following heuristic technique introduced in [9]: Let t denote a stepsize for which inequality (27) shall be tested. Before testing check whether $F(x^k + td^k)$ is well-defined or not. If $F(x^k + td^k)$ is not well-defined then set $t := t/2$ and check again. Repeat this process until F is well defined or the limit of 80 line search steps is exceeded. In the first case continue with the nonmonotone Armijo line search. Otherwise the algorithm stops. This is equivalent to taking $f(x) = \infty$ for all points x where $F(x)$ is not defined.

The numerical results are summarized in Tables 1–2 for the MCPLIB complementarity problems and in Table 3 for the GAMS complementarity problems. In these tables the first column gives the name of the problem, Ir denotes the number of iterations (F means after the maximum number of iteration steps (300) a solution has not been found under the specified accuracy, LF means that the maximum number of line search steps (80) was exceeded). Fev denotes the number of evaluations of the function F. Ir is equal to the number of evaluations of the Jacobian $F'(x)$ and the number of subproblems (26) or systems of linear equations solved. In the "problem" column of Tables 1–2, the number after each problem specifies which starting point from the library is used. In the "Ir" column of Table 2, DV means that the starting point is not in the domain of function or Jacobian.

Tables 1–2 and 3 show that for every NCP-function the algorithm was able to solve most complementarity problems in GAMS and MCP libraries. More precisely, there are four (4) failures for ϕ_4 and ϕ_7, five (5) failures for ϕ_5 and ϕ_6 and six (6) failures for ϕ_2 and ϕ_3 (for `Billups` problem, where there is a local solution which is not a global solution, we observed that the iteration sequence was not trapped in the local solution and it was always the case that the global solution was found for all merit functions considered. This might be attributed to the use of nonmonotone line search. Of course, the nonmonotone line search itself cannot always guarantee the iteration sequence to escape from local solutions.) However, it is not our intention to declare a winner at this stage because the scaled parameter α can affect the behavior of the corresponding algorithm. Moreover, the numerical results reported for a squared smoothing Newton method in [36] indicates that when the iterate x is far away from the solution set of the problem, a large value of α is favorable while when the iterate x is near the solution set, a small value of α is recommended. More study on the choice of different merit functions and different choices of α is necessary.

7. Final Remarks

In this paper we have reformulated several NCP-functions and discussed their main properties. In particular, we showed that the NCP-function ϕ_3 may introduce singularity issue for resulted nonsmooth equations while ϕ_5, ϕ_6 and ϕ_7 have all the discussed properties as ϕ_4 has. We believe that the research done here can deepen the understanding on NCP-functions. We also presented a modified normal map by just requiring F to be monotone on $\Re^n_+$ instead of on $\Re^n$ to get a global result. The modified normal map and its smoothing forms need further investigation in regard of their nice numerical performance reported in [36].

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