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Solvability of monotone tensor complementarity problems

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Abstract The tensor complementarity problem is a special instance in the class of nonlinear complementarity problems, which has many applications in multi-person noncooperative games, hypergraph clustering problems and traffic equilibrium problems. Two most important research issues are how to identify the solvability and how to solve such a problem via analyzing the structure of the involved tensor. In this paper, based on the concept of monotone mappings, we introduce a new class of structured tensors and the corresponding monotone tensor complementarity problem. We show that the solution set of the monotone tensor complementarity problem is nonempty and compact under the feasibility assumption. Moreover, a necessary and sufficient condition for ensuring the feasibility is given via analyzing the structure of the involved tensor. Based on the Huber function, we propose a regularized smoothing Newton method to solve the monotone tensor complementarity problem and establish its global convergence. Under some mild assumptions, we show that the proposed algorithm is superlinearly convergent. Preliminary numerical results indicate that the proposed algorithm is very promising.

Keywords tensor complementarity problem, Huber function, monotone, smoothing Newton method, superlinear convergence

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1 Introduction

The finite-dimensional complementarity problem has been studied extensively due to its plenty of practical applications [11,16,17]. When the involved function is nonlinear, it is called a nonlinear complementarity problem (NCP). It is well known that various special types of functions play important roles in the studies of NCPs. In recent years, various tensors with special structures have been studied widely [33,35]. In 2015, as an application of structured tensors, a class of NCPs with the involved function being defined by a tensor, which is called the tensor complementarity problem (TCP), was used in [40], and it was studied initially by Song and Qi [41] and Che et al. [3]. Since then, the TCP has attracted much attention and obtained rapid development from theory to solution methods and applications. The state-of-the-art studies for the TCP and related models have been summarized in the recent survey papers [21,22,34]. Very

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recently, some generalizations of the TCP, such as the stochastic TCP [4, 10, 29], the tensor variational inequality [45], the polynomial complementarity problem [13] and the weakly homogeneous variational inequality [14, 27], have been studied.

TCPs have many applications in multi-person noncooperative games [7, 20], hypergraph clustering problems and traffic equilibrium problems [34]. Therefore, it is necessary to establish the theory and solution methods for the TCP via analyzing the special structure of the involved tensor. Since the TCP is a subclass of the NCP, the theory and solution methods for NCPs are applicable to TCPs, if the required conditions are satisfied. However, since the function involved in the TCP is a special class of polynomials defined by a tensor, one may expect to obtain stronger theoretical results and more effective methods by making use of the structure of the involved tensor. In the theoretical studies of the NCP, the monotonicity properties play important roles [11, 16, 17]. It is well known that the NCP has no more than one solution when the involved function is strictly monotone on $\mathbb{R}^n_+$; the NCP is possibly unsolvable even if it is feasible when the involved function is monotone on $\mathbb{R}^n_+$. A natural question is how about the solvability of the TCP under the same condition. This is a significant research issue to be considered in this paper.

How to design efficient solution methods for such TCPs, particularly for large-scale ones, is also very important. It is well known that the smoothing Newton-type algorithm is a class of effective methods for solving NCPs (see [5, 6, 8, 12, 18, 23, 30, 36] and [37, 43, 47, 49]). Although such solution methods in the literature have been used to solve the TCP [7, 20], the global convergence has not been studied in details yet. In this paper, we present a smoothing Newton-type algorithm to solve the TCP. Based on the Huber function [24], which is widely used in many fields such as statistic optimization and compression sensing [2, 46], we reformulate the TCP as a system of smoothing equations. We apply a Newton-type method to solve the system of smoothing equations at each iteration by ensuring the smoothing parameter to tend to zero so that a solution to the TCP can be found. We show that the solution set of the monotone TCP is nonempty and compact if it is feasible. Moreover, the global and superlinear convergence of the proposed algorithm is also established.

The basic contribution of this paper is as follows:

- Based on the concept of monotone mappings, we introduce the *strictly positive semidefinite* tensor on $\mathbb{R}^n_+$ and the corresponding *monotone* TCP.
- We show that the solution set of the monotone TCP is nonempty and compact under the feasibility assumption. This result is stronger than that of the NCP. A necessary and sufficient condition for ensuring the feasibility is given based on the structure of the involved tensor.
- We construct a novel Newton-type algorithm for solving such TCPs via the Huber function and the normal equation.

The rest of this paper is organized as follows. In Section 2, we recall some basic symbols, definitions and conclusions. In Section 3, we introduce a new class of structured tensors and investigate the solvability of the corresponding TCP. In Section 4, we present a smoothing Newton method to solve such a TCP and establish the global convergence in Section 5. Numerical results are reported in Section 6 and some conclusions are given in Section 7.

2 Preliminaries

In this section, we recall some basic concepts and results, which are useful for our subsequent analysis. Throughout this paper, for any given positive integer n, we use $[n]$ to denote the set $\{1,\dots,n\}$. $\mathcal{A}=(a_{i_1\cdots i_m})$ with $a_{i_1\cdots i_m}\in\mathbb{R}$ for any $i_j\in[n]$ and $j\in[m]$ is called an m-th-order n-dimensional real tensor, and we denote the set of all the m-th-order n-dimensional real tensors by $\mathbb{R}^{[m,n]}$. We denote scalars, vectors and tensors by lowercase letters, bold lowercase letters and calligraphic letters, respectively, for example, a, $\boldsymbol{a}$ and $\mathcal{A}$ correspondingly. For any $\mathcal{A}\in\mathbb{R}^{[m,n]}$ and $\boldsymbol{x}=(x_1,\dots,x_n)^{\mathrm{T}}\in\mathbb{R}^n$, $\|\boldsymbol{x}\|$ denotes the 2-norm, $\boldsymbol{x}_+=\max\{\boldsymbol{x},\boldsymbol{0}\}\in\mathbb{R}^n$, and $\mathcal{A}\boldsymbol{x}^{m-1}\in\mathbb{R}^n$ is a column vector defined by

$$(\mathcal{A}\boldsymbol{x}^{m-1})_i=\sum_{i_2,\dots,i_m=1}^{n}a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m},\quad \forall\, i\in[n].$$

For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, let $\boldsymbol{x} \geqslant (>) \, \boldsymbol{y}$ mean $x_i \geqslant (>) \, y_i$ for any $i \in [n]$. In addition, $\mathbf{0}$ is the vector of all zeros, $\boldsymbol{e}$ is the vector of all ones, $\mathbb{R}^n_{++} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} > \mathbf{0}\}$ and $\mathbb{R}^n_{+} = \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \geqslant \mathbf{0}\}$. We also define $\mathbb{R}_{++} = \{x \in \mathbb{R} \mid x > 0\}$.

Given a function $F : \mathbb{R}^n_+ \to \mathbb{R}^n$, then the NCP (denoted by NCP(F)) is to find a point $\boldsymbol{x} \in \mathbb{R}^n$ such that

$$\boldsymbol{x} \geqslant \mathbf{0}, \quad F(\boldsymbol{x}) \geqslant \mathbf{0}, \quad \boldsymbol{x}^{\mathrm{T}} F(\boldsymbol{x}) = 0.$$

When $F(\boldsymbol{x}) = \mathcal{A}\boldsymbol{x}^{m-1} + \boldsymbol{q}$ with given $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and $\boldsymbol{q} \in \mathbb{R}^n$, NCP($F$) reduces to the TCP (denoted by TCP($\mathcal{A}, \boldsymbol{q}$)), which consists in finding a point $\boldsymbol{x} \in \mathbb{R}^n$ such that

$$\boldsymbol{x} \geqslant \mathbf{0}, \quad \mathcal{A}\boldsymbol{x}^{m-1} + \boldsymbol{q} \geqslant \mathbf{0}, \quad \boldsymbol{x}^{\mathrm{T}}(\mathcal{A}\boldsymbol{x}^{m-1} + \boldsymbol{q}) = 0. \tag{2.1}$$

The TCP($\mathcal{A}, \boldsymbol{q}$) (2.1) arises in many real applications from multi-person noncooperative games, hypergraph clustering problems and traffic equilibrium problems [22].

In the theoretical studies of the NCP, some special types of functions play important roles [11, 16, 17]. The following two classes of functions will be used in this paper.

Definition 2.1. A mapping $F : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is said to be

(a) monotone on Ω, if for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$, $(\boldsymbol{x} - \boldsymbol{y})^{\mathrm{T}}(F(\boldsymbol{x}) - F(\boldsymbol{y})) \geqslant 0$;

(b) strictly monotone on Ω, if for all $\boldsymbol{x}, \boldsymbol{y} \in \Omega$ with $\boldsymbol{x} \neq \boldsymbol{y}$, $(\boldsymbol{x} - \boldsymbol{y})^{\mathrm{T}}(F(\boldsymbol{x}) - F(\boldsymbol{y})) > 0$.

If F is a continuously differentiable function defined on an open convex set, we have the following connection between the above monotonicity properties and the positive semidefiniteness of the Jacobian matrices JF of F [11, Proposition 2.3.2].

Proposition 2.2. *Let $F : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable on an open convex set Ω. The following statements are valid*:

(a) *F is monotone on Ω if and only if $JF(\boldsymbol{x})$ is positive semidefinite for all $\boldsymbol{x} \in \Omega$, i.e.,*

$$\forall\, \boldsymbol{x} \in \Omega, \quad \boldsymbol{d}^{\mathrm{T}} JF(\boldsymbol{x})\boldsymbol{d} \geqslant 0, \quad \forall\, \boldsymbol{d} \in \mathbb{R}^n.$$

(b) *F is strictly monotone on Ω if $JF(\boldsymbol{x})$ is positive definite for all $\boldsymbol{x} \in \Omega$, i.e.,*

$$\forall\, \boldsymbol{x} \in \Omega, \quad \boldsymbol{d}^{\mathrm{T}} JF(\boldsymbol{x})\boldsymbol{d} > 0, \quad \forall\, \boldsymbol{d} \in \mathbb{R}^n, \quad \boldsymbol{d} \neq \mathbf{0}.$$

Obviously, every strictly monotone function is a monotone function. With the above definition, we have the following proposition for NCP(F) (see, e.g., [17, Proposition 3.2] and [11, 16]).

Proposition 2.3. *If F is strictly monotone on $\mathbb{R}^n_+$, then* NCP(F) *has no more than one solution. If F is monotone on $\mathbb{R}^n_+$, then* NCP(F) *is possibly unsolvable even if it is feasible, i.e., there exists an $\hat{\boldsymbol{x}} \geqslant \mathbf{0}$ such that $F(\hat{\boldsymbol{x}}) \geqslant \mathbf{0}$.*

Recently, many classes of structured tensors are introduced, and the related properties are studied [1, 15, 33, 35, 42, 45, 48]. Among them, there is a class of structured tensors connected with the monotonicity properties [45, Definition 4.1]. In this paper, we need the following concepts of the structured tensors (see, e.g., [31, 32, 45]).

Definition 2.4. Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. Then $\mathcal{A}$ is said to be

(a) a positive semidefinite tensor if $\boldsymbol{x}^{\mathrm{T}}\mathcal{A}\boldsymbol{x}^{m-1} \geqslant 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$; a positive definite tensor if $\boldsymbol{x}^{\mathrm{T}}\mathcal{A}\boldsymbol{x}^{m-1} > 0$ for all $\boldsymbol{x} \in \mathbb{R}^n$ with $\boldsymbol{x} \neq \mathbf{0}$;

(b) a copositive tensor if $\boldsymbol{x}^{\mathrm{T}}\mathcal{A}\boldsymbol{x}^{m-1} \geqslant 0$ for all $\boldsymbol{x} \in \mathbb{R}^n_+$; a strictly copositive tensor if $\boldsymbol{x}^{\mathrm{T}}\mathcal{A}\boldsymbol{x}^{m-1} > 0$ for all $\boldsymbol{x} \in \mathbb{R}^n_+$ with $\boldsymbol{x} \neq \mathbf{0}$;

(c) a strictly positive definite tensor on Ω if $\mathcal{A}\boldsymbol{x}^{m-1}$ is strictly monotone on Ω, where $\Omega \subseteq \mathbb{R}^n$ and $\Omega \neq \emptyset$.

Obviously, when $\Omega = \mathbb{R}^n_+$, every strictly positive definite tensor must be a strictly copositive tensor; when $\Omega = \mathbb{R}^n$, every strictly positive definite tensor must be a positive definite tensor if m is even. However, if $m > 2$, a positive definite tensor is not necessarily a strictly positive definite tensor [45].

By virtue of [45, Theorem 4.3], we immediately obtain the following theorem.

Theorem 2.5. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a strictly positive definite tensor on $\mathbb{R}^n_+$. Then the* TCP$(\mathcal{A}, \boldsymbol{q})$ (2.1) *has a unique solution for any $\boldsymbol{q} \in \mathbb{R}^n$.*

The exceptional family of elements is a powerful tool to investigate the solvability of NCP(F) [11, 16, 25, 26]. In this paper, we use the following definition [25].

Definition 2.6. A set of points $\{\boldsymbol{x}^k\} \subset \mathbb{R}^n_+$ is an exceptional family of elements for the continuous function $F : \mathbb{R}^n_+ \to \mathbb{R}^n$, if $\|\boldsymbol{x}^k\| \to \infty$ as $k \to +\infty$, and for each $k > 0$, there exists a scalar $\mu_k > 0$ such that

$$\begin{cases} F_i(\boldsymbol{x}^k) = -\mu_k x_i^k, & \text{if } x_i^k > 0, \\ F_i(\boldsymbol{x}^k) \geqslant 0, & \text{if } x_i^k = 0. \end{cases}$$

About the relationship between the exceptional family of elements and the solution to NCP(F), we will use the following lemma whose proof can be found in [25].

Lemma 2.7. *For any continuous function $F : \mathbb{R}^n_+ \to \mathbb{R}^n$, there exists either a solution to* NCP(F) *or an exceptional family of elements for F.*

3 Monotone TCPs

In this section, we introduce a new class of structured tensors based on the concept of the monotone mapping given in Definition 2.1 and study the solvability of the corresponding TCP.

Definition 3.1. Let $\Omega \subseteq \mathbb{R}^n$ and $\Omega \neq \emptyset$. A tensor $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is said to be **strictly positive semidefinite** on Ω if $\mathcal{A}\boldsymbol{x}^{m-1}$ is monotone on Ω. We say that the TCP$(\mathcal{A}, \boldsymbol{q})$ (2.1) is a **monotone TCP** if the involved tensor $\mathcal{A}$ is strictly positive semidefinite on $\mathbb{R}^n_+$.

A basic question is whether or not there exists a strictly positive semidefinite tensor on some subset of $\mathbb{R}^n$. The following example gives a positive answer to this question.

Example 3.2. Let $\Omega = \mathbb{R}^2_+$, and $\mathcal{A} = (a_{ijk}) \in \mathbb{R}^{[3,2]}$ be defined as $a_{112} = 2$, $a_{211} = -1$ and the other elements be equal to zero. Then $\mathcal{A}$ is a strictly positive semidefinite tensor on Ω.

For any $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^2_+$, we define

$$G(\boldsymbol{x}) := \mathcal{A}\boldsymbol{x}^2 = \begin{pmatrix} 2x_1x_2 \\ -x_1^2 \end{pmatrix}.$$

By direct computation, we obtain that the Jacobian matrix of $G(\boldsymbol{x})$ is

$$JG(\boldsymbol{x}) = \begin{pmatrix} 2x_2 & 2x_1 \\ -2x_1 & 0 \end{pmatrix}.$$

It is easy to see that the matrix $JG(\boldsymbol{x})$ is positive semidefinite on $\mathbb{R}^2_+$. By Proposition 2.2(a) and Definition 3.1, we obtain that $\mathcal{A}$ is a strictly positive semidefinite tensor on $\mathbb{R}^2_+$.

Remark 3.3. Every strictly positive definite tensor on Ω must be a strictly positive semidefinite tensor on Ω. In addition, when $\Omega = \mathbb{R}^n_+$, every strictly positive semidefinite tensor must be a copositive tensor; when $\Omega = \mathbb{R}^n$, every strictly positive semidefinite tensor must be a positive semidefinite tensor if m is even.

However, if $m > 2$, a positive semidefinite tensor is not necessarily a strictly positive semidefinite tensor, which can be seen in the following example.

Example 3.4. Let $\mathcal{A} = (a_{ijkl}) \in \mathbb{R}^{[4,2]}$ be defined as $a_{1111} = a_{2222} = a_{2112} = 1$, $a_{1122} = -1$ and the others be equal to zero. Then $\mathcal{A}$ is a positive definite tensor but it is not a strictly positive semidefinite tensor on $\mathbb{R}^2$.

First, we show that $\mathcal{A}$ is a positive definite tensor. For any $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^2$, we define

$$G(\boldsymbol{x}) := \mathcal{A}\boldsymbol{x}^3 = \begin{pmatrix} x_1^3 - x_1x_2^2 \\ x_1^2x_2 + x_2^3 \end{pmatrix}. \tag{3.1}$$

It follows that for any $\boldsymbol{x} \in \mathbb{R}^2$ with $\boldsymbol{x} \neq \boldsymbol{0}$,

$$\boldsymbol{x}^{\mathrm{T}}\mathcal{A}\boldsymbol{x}^3 = x_1(x_1^3 - x_1x_2^2) + x_2(x_1^2x_2 + x_2^3) = x_1^4 + x_2^4 > 0.$$

Hence, $\mathcal{A}$ is a positive definite tensor from Definition 2.4(a).

Second, we show that $\mathcal{A}$ is not a strictly positive semidefinite tensor on $\mathbb{R}^2$. To this end, we compute the Jacobian matrix of $G(\boldsymbol{x})$ defined by (3.1), i.e.,

$$JG(\boldsymbol{x}) = \begin{pmatrix} 3x_1^2 - x_2^2 & -2x_1x_2 \\ 2x_1x_2 & 3x_2^2 \end{pmatrix}.$$

Taking $\hat{\boldsymbol{x}} = (0, 1)^{\mathrm{T}} \in \mathbb{R}^2$, we have

$$JG(\hat{\boldsymbol{x}}) = \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix}.$$

It is easy to see that $JG(\hat{\boldsymbol{x}})$ is not a positive semidefinite matrix. By Proposition 2.2(a) and Definition 3.1, $\mathcal{A}$ is not a strictly positive semidefinite tensor on $\mathbb{R}^2$.

We now investigate the solvability of the monotone $\mathrm{TCP}(\mathcal{A}, \boldsymbol{q})$ (2.1) for any given $\boldsymbol{q} \in \mathbb{R}^n$. It is easy to see that the following result holds.

Theorem 3.5. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a strictly positive semidefinite tensor on $\mathbb{R}^n_+$. Then the* $\mathrm{TCP}(\mathcal{A}, \boldsymbol{q})$ (2.1) *has at least one solution if $\boldsymbol{q} \geqslant \boldsymbol{0}$, and $\boldsymbol{0}$ is a unique solution if $\boldsymbol{q} > \boldsymbol{0}$.*

We can confirm that a monotone $\mathrm{TCP}(\mathcal{A}, \boldsymbol{q})$ is not necessarily solvable for any $\boldsymbol{q} \in \mathbb{R}^n$. Consider the following $\mathrm{TCP}(\mathcal{A}, \boldsymbol{q})$, where $\mathcal{A}$ is given in Example 3.2 and $\boldsymbol{q} = (q_1, q_2)^{\mathrm{T}} \in \mathbb{R}^2$, which consists in finding $\boldsymbol{x} = (x_1, x_2)^{\mathrm{T}} \in \mathbb{R}^2$ such that

$$\begin{cases} x_1 \geqslant 0, \\ x_2 \geqslant 0, \end{cases} \quad \begin{cases} 2x_1x_2 + q_1 \geqslant 0, \\ -x_1^2 + q_2 \geqslant 0, \end{cases} \quad \begin{cases} x_1(2x_1x_2 + q_1) = 0, \\ x_2(-x_1^2 + q_2) = 0. \end{cases} \tag{3.2}$$

From Theorem 3.5, the TCP (3.2) has at least one solution when $\boldsymbol{q} \geqslant \boldsymbol{0}$. For example, taking $q_1 = 0$ and $q_2 = 1$, we obtain that all $\boldsymbol{x} = (w, 0)^{\mathrm{T}}$ with $w \in (0, 1]$ are solutions. It is easy to see that the solutions to the TCP (3.2) are distributed into four cases as follows:

- When $q_1 \geqslant 0$ and $q_2 \geqslant 0$, it has at least one solution.
- When $q_1 \in \mathbb{R}$ and $q_2 < 0$, it has no solution.
- When $q_1 < 0$ and $q_2 = 0$, it has no solution.
- When $q_1 < 0$ and $q_2 > 0$, it has a unique solution $\boldsymbol{x}^* = (\sqrt{q_2}, \frac{-q_1}{2\sqrt{q_2}})^{\mathrm{T}}$.

Based on the above observations, we may ask a natural question: under what conditions is the monotone $\mathrm{TCP}(\mathcal{A}, \boldsymbol{q})$ solvable for any $\boldsymbol{q} \in \mathbb{R}^n$? In what follows, we present a sufficient condition to guarantee that the monotone $\mathrm{TCP}(\mathcal{A}, \boldsymbol{q})$ is solvable for any $\boldsymbol{q} \in \mathbb{R}^n$. For this aim, we need the following lemmas.

Lemma 3.6. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$. Then there exists a vector $\boldsymbol{u} \in \mathbb{R}^n_+$ such that $\mathcal{A}\boldsymbol{u}^{m-1} > \boldsymbol{0}$ if and only if for any $\boldsymbol{q} \in \mathbb{R}^n$, the* $\mathrm{TCP}(\mathcal{A}, \boldsymbol{q})$ *is feasible, i.e., there exists a $\boldsymbol{v} \geqslant \boldsymbol{0}$ such that $\mathcal{A}\boldsymbol{v}^{m-1} + \boldsymbol{q} \geqslant \boldsymbol{0}$.*

Proof. On the one hand, assume that the $\mathrm{TCP}(\mathcal{A}, \boldsymbol{q})$ is feasible for any $\boldsymbol{q} \in \mathbb{R}^n$. Then taking $\boldsymbol{q} < \boldsymbol{0}$, we know that there must exist a vector $\boldsymbol{u} \geqslant \boldsymbol{0}$ such that $\mathcal{A}\boldsymbol{u}^{m-1} + \boldsymbol{q} \geqslant 0 \Rightarrow \mathcal{A}\boldsymbol{u}^{m-1} \geqslant -\boldsymbol{q} > 0$, which implies that there exists a $\boldsymbol{u} \in \mathbb{R}^n_+$ such that $\mathcal{A}\boldsymbol{u}^{m-1} > \boldsymbol{0}$.

On the other hand, assume that there exists a $\boldsymbol{u} \in \mathbb{R}^n_+$ such that $\mathcal{A}\boldsymbol{u}^{m-1} > \boldsymbol{0}$. Then we have $\boldsymbol{u} \neq \boldsymbol{0}$ and for any given $\boldsymbol{q} \in \mathbb{R}^n$ we can find the desired vector $\boldsymbol{v}$. In fact, if $q_i < 0$ for some $i \in [n]$, then we take

$$t = \max_{i \in [n]} \left\{ \frac{-q_i}{(\mathcal{A}\boldsymbol{u}^{m-1})_i} \,\middle|\, q_i < 0 \right\} + 1, \quad \boldsymbol{v} = t^{\frac{1}{m-1}}\boldsymbol{u}.$$

Clearly, $\boldsymbol{v} \geqslant \mathbf{0}$, $\boldsymbol{v} \neq \mathbf{0}$ and $\mathcal{A}\boldsymbol{v}^{m-1} + \boldsymbol{q} > \mathbf{0}$. □

Lemma 3.7. *Let $F(\boldsymbol{x}) = \mathcal{A}\boldsymbol{x}^{m-1} + \boldsymbol{q}$ for any given $\mathcal{A} \in \mathbb{R}^{[m,n]}$ and $\boldsymbol{q} \in \mathbb{R}^n$. If there is a vector $\boldsymbol{u} \in \mathbb{R}^n_+$ such that $\mathcal{A}\boldsymbol{u}^{m-1} > \mathbf{0}$ and the tensor $\mathcal{A}$ is strictly positive semidefinite on $\mathbb{R}^n_+$, then for any sequence $\{\boldsymbol{x}^k\} \subset \mathbb{R}^n_+$ with $\|\boldsymbol{x}^k\| \to \infty$ as $k \to +\infty$, there exist k_0 and $\boldsymbol{y} \in \mathbb{R}^n_+$ with $\|\boldsymbol{y}\| < \|\boldsymbol{x}^{k_0}\|$ such that $(\boldsymbol{x}^{k_0} - \boldsymbol{y})^{\mathrm{T}} F(\boldsymbol{x}^{k_0}) > 0$.*

Proof. Since $\boldsymbol{u} \geqslant \mathbf{0}$ and $\mathcal{A}\boldsymbol{u}^{m-1} > \mathbf{0}$, by Lemma 3.6, there exists a nonzero vector $\boldsymbol{v} \geqslant \mathbf{0}$ such that $F(\boldsymbol{v}) > \mathbf{0}$ and $\boldsymbol{v}^{\mathrm{T}} F(\boldsymbol{v}) > 0$. Assume that there exists a sequence $\{\boldsymbol{x}^k\} \subset \mathbb{R}^n_+$ with $\|\boldsymbol{x}^k\| \to \infty$ as $k \to \infty$ such that

$$(\boldsymbol{x}^k - \boldsymbol{y})^{\mathrm{T}} F(\boldsymbol{x}^k) \leqslant 0, \quad \forall\, \boldsymbol{y} \in \mathbb{R}^n_+, \quad \|\boldsymbol{y}\| < \|\boldsymbol{x}^k\|, \quad \forall\, k > 0. \tag{3.3}$$

Since $\|\boldsymbol{x}^k\| \to \infty$ as $k \to \infty$, we have $\|\boldsymbol{v}\| < \|\boldsymbol{x}^k\|$ holds for sufficiently large k. It follows from (3.3) that $(\boldsymbol{x}^k - \boldsymbol{v})^{\mathrm{T}} F(\boldsymbol{x}^k) \leqslant 0$ for sufficiently large k. Since $\mathcal{A}$ is strictly positive semidefinite on $\mathbb{R}^n_+$, we have

$$(\boldsymbol{x}^k - \boldsymbol{v})^{\mathrm{T}} (F(\boldsymbol{x}^k) - F(\boldsymbol{v})) \geqslant 0.$$

Hence, we obtain that for sufficiently large k,

$$0 \geqslant (\boldsymbol{x}^k - \boldsymbol{v})^{\mathrm{T}} F(\boldsymbol{x}^k) \geqslant (\boldsymbol{x}^k - \boldsymbol{v})^{\mathrm{T}} F(\boldsymbol{v}) \Rightarrow 0 \leqslant (\boldsymbol{x}^k)^{\mathrm{T}} F(\boldsymbol{v}) \leqslant \boldsymbol{v}^{\mathrm{T}} F(\boldsymbol{v}), \tag{3.4}$$

which implies that the sequence $\{\boldsymbol{x}^k\}$ is bounded. This is a contradiction with $\|\boldsymbol{x}^k\| \to \infty$ as $k \to \infty$. Hence, the desired result holds. □

Let $\mathrm{SOL}(\mathcal{A}, \boldsymbol{q})$ denote the solution set of the $\mathrm{TCP}(\mathcal{A}, \boldsymbol{q})$ (2.1). Based on the above lemmas, one can show that $\mathrm{SOL}(\mathcal{A}, \boldsymbol{q})$ is bounded for the monotone $\mathrm{TCP}(\mathcal{A}, \boldsymbol{q})$.

Theorem 3.8. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ be a strictly positive semidefinite tensor on $\mathbb{R}^n_+$. If there exists a $\boldsymbol{u} \in \mathbb{R}^n_+$ such that $\mathcal{A}\boldsymbol{u}^{m-1} > \mathbf{0}$, then the solution set $\mathrm{SOL}(\mathcal{A}, \boldsymbol{q})$ is nonempty and compact for any $\boldsymbol{q} \in \mathbb{R}^n$.*

Proof. We first show that $\mathrm{SOL}(\mathcal{A}, \boldsymbol{q}) \neq \emptyset$. Suppose that the monotone $\mathrm{TCP}(\mathcal{A}, \boldsymbol{q})$ (2.1) has no solution. Then by Lemma 2.7, there exists an exceptional family of elements $\{\boldsymbol{x}^k\} \subset \mathbb{R}^n_+$ for $F(\boldsymbol{x}) = \mathcal{A}\boldsymbol{x}^{m-1} + \boldsymbol{q}$. Thus, we have $\|\boldsymbol{x}^k\| \to \infty$ as $k \to \infty$, and for each $k > 0$, there exists a scalar $\mu_k > 0$ such that $\boldsymbol{y}^k = F(\boldsymbol{x}^k) + \mu_k \boldsymbol{x}^k \geqslant \mathbf{0}$, $(\boldsymbol{x}^k)^{\mathrm{T}} \boldsymbol{y}^k = 0$ and

$$F_i(\boldsymbol{x}^k) = \begin{cases} y_i^k - \mu_k x_i^k, & \text{if } x_i^k > 0, \\ y_i^k, & \text{if } x_i^k = 0. \end{cases}$$

Hence, for any $\boldsymbol{z} \in \mathbb{R}^n_+$ and each $k > 0$, we have

$$\begin{aligned} (\boldsymbol{x}^k - \boldsymbol{z})^{\mathrm{T}} F(\boldsymbol{x}^k) &= \sum_{x_i^k > 0} (x_i^k - z_i)(y_i^k - \mu_k x_i^k) + \sum_{x_i^k = 0} (x_i^k - z_i) y_i^k \\ &= -\boldsymbol{z}^{\mathrm{T}} \boldsymbol{y}^k - \mu_k \sum_{x_i^k > 0} (x_i^k - z_i) x_i^k \leqslant \mu_k \sum_{x_i^k > 0} x_i^k (z_i - x_i^k) \\ &= \mu_k (\boldsymbol{x}^k)^{\mathrm{T}} (\boldsymbol{z} - \boldsymbol{x}^k) \leqslant \mu_k \|\boldsymbol{x}^k\| (\|\boldsymbol{z}\| - \|\boldsymbol{x}^k\|). \end{aligned} \tag{3.5}$$

On the other hand, by Lemma 3.7, there exist $\boldsymbol{z}^0 \in \mathbb{R}^n_+$ and $\boldsymbol{x}^{k_0}$ with $\|\boldsymbol{z}^0\| < \|\boldsymbol{x}^{k_0}\|$ such that $(\boldsymbol{x}^{k_0} - \boldsymbol{z}^0)^{\mathrm{T}} F(\boldsymbol{x}^{k_0}) > 0$. Thus, it follows from (3.5) that

$$0 < (\boldsymbol{x}^{k_0} - \boldsymbol{z}^0)^{\mathrm{T}} F(\boldsymbol{x}^{k_0}) \leqslant \mu_{k_0} \|\boldsymbol{x}^{k_0}\| (\|\boldsymbol{z}^0\| - \|\boldsymbol{x}^{k_0}\|) < 0,$$

which leads to a contradiction. Therefore, the $\mathrm{TCP}(\mathcal{A}, \boldsymbol{q})$ has at least one solution, i.e., $\mathrm{SOL}(\mathcal{A}, \boldsymbol{q}) \neq \emptyset$.

We now show that the solution set $\mathrm{SOL}(\mathcal{A}, \boldsymbol{q})$ is compact. Let $\{\boldsymbol{x}^k\} \subset \mathrm{SOL}(\mathcal{A}, \boldsymbol{q})$ be a sequence with $\boldsymbol{x}^k \to \boldsymbol{x}^*$ as $k \to \infty$. Then it follows that

$$\boldsymbol{x}^k \geqslant \mathbf{0}, \quad \mathcal{A}(\boldsymbol{x}^k)^{m-1} + \boldsymbol{q} \geqslant \mathbf{0}, \quad (\boldsymbol{x}^k)^{\mathrm{T}} (\mathcal{A}(\boldsymbol{x}^k)^{m-1} + \boldsymbol{q}) = 0.$$

Thus, letting $k \to \infty$, we obtain

$$\boldsymbol{x}^* \geqslant \boldsymbol{0}, \quad \mathcal{A}(\boldsymbol{x}^*)^{m-1} + \boldsymbol{q} \geqslant \boldsymbol{0}, \quad (\boldsymbol{x}^*)^{\mathrm{T}}(\mathcal{A}(\boldsymbol{x}^*)^{m-1} + \boldsymbol{q}) = 0,$$

i.e., $x^* \in \mathrm{SOL}(\mathcal{A}, \boldsymbol{q})$. Hence, the solution set $\mathrm{SOL}(\mathcal{A}, \boldsymbol{q})$ is closed.

Suppose that the solution set $\mathrm{SOL}(\mathcal{A}, \boldsymbol{q})$ is unbounded. Then there exists a sequence $\{\boldsymbol{x}^k\} \subset \mathrm{SOL}(\mathcal{A}, \boldsymbol{q})$ such that $\|\boldsymbol{x}^k\| \to \infty$ as $k \to \infty$. By Lemma 3.7, there exist $\boldsymbol{z}^0 \in \mathbb{R}^n_+$ and $\boldsymbol{x}^{k_0}$ with $\|\boldsymbol{z}^0\| < \|\boldsymbol{x}^{k_0}\|$ such that $(\boldsymbol{x}^{k_0} - \boldsymbol{z}^0)^{\mathrm{T}} F(\boldsymbol{x}^{k_0}) > 0$. Consequently, we obtain

$$0 < (\boldsymbol{x}^{k_0} - \boldsymbol{z}^0)^{\mathrm{T}} F(\boldsymbol{x}^{k_0}) = -(\boldsymbol{z}^0)^{\mathrm{T}} F(\boldsymbol{x}^{k_0}) \leqslant 0,$$

which leads to a contradiction. So the solution set $\mathrm{SOL}(\mathcal{A}, \boldsymbol{q})$ is bounded. Consequently, the solution set $\mathrm{SOL}(\mathcal{A}, \boldsymbol{q})$ is nonempty and compact for any $\boldsymbol{q} \in \mathbb{R}^n$. □

Remark 3.9. In the theory of NCPs, when the involved function F is monotone on $\mathbb{R}^n_+$, one cannot ensure that the corresponding NCP(F) is solvable even if it is feasible; when the involved function F is strictly monotone on $\mathbb{R}^n_+$, one can only obtain that the corresponding NCP(F) has no more than one solution (see Proposition 2.3). For the TCP($\mathcal{A}, \boldsymbol{q}$), the special case of NCP(F), when the involved tensor $\mathcal{A}$ is strictly positive semidefinite on $\mathbb{R}^n_+$ and hence the corresponding function $F(\boldsymbol{x}) = \mathcal{A}\boldsymbol{x}^{m-1} + \boldsymbol{q}$ is monotone on $\mathbb{R}^n_+$, we can obtain that the corresponding TCP($\mathcal{A}, \boldsymbol{q}$) is solvable if it is feasible for any $\boldsymbol{q} \in \mathbb{R}^n$, and the solution set is compact from Lemma 3.6 and Theorem 3.8. Furthermore, if the involved tensor $\mathcal{A}$ is strictly positive definite on $\mathbb{R}^n_+$ and hence the corresponding function $F(\boldsymbol{x}) = \mathcal{A}\boldsymbol{x}^{m-1} + \boldsymbol{q}$ is strictly monotone on $\mathbb{R}^n_+$, we can obtain that the corresponding TCP($\mathcal{A}, \boldsymbol{q}$) has a unique solution from Theorem 2.5.

4 A Newton-type algorithm for TCPs

In this section, based on the well-known Huber function, we propose a smoothing Newton method to solve the monotone TCP.

For the sake of convenience, we define $F(\boldsymbol{x}) = \mathcal{A}\boldsymbol{x}^{m-1} + \boldsymbol{q}$. Then the TCP($\mathcal{A}, \boldsymbol{q}$) (2.1) is equivalent to the following normal equation:

$$G(\boldsymbol{x}) := F(\boldsymbol{x}_+) + \boldsymbol{x} - \boldsymbol{x}_+ = \boldsymbol{0} \tag{4.1}$$

in the sense that if $\boldsymbol{x}^* \in \mathbb{R}^n$ is a solution to (4.1) then $\boldsymbol{x}^*_+$ is a solution to (2.1), and conversely if $\boldsymbol{x}^*$ is a solution to (2.1), then $\boldsymbol{x}^* - F(\boldsymbol{x}^*)$ is a solution to (4.1) [37, 39, 44].

We now propose a smoothing Newton method for solving the normal equation (4.1). The non-smoothness of $\boldsymbol{x}_+$ in (4.1) prevents a straightforward application of the classical Newton method to (4.1). Thus, we first focus on approximating the nonsmooth $\boldsymbol{x}_+$ by a smooth function. We recall the following Huber function [24]:

$$h_\gamma(t) = \begin{cases} \dfrac{t^2}{2\gamma}, & \text{if } |t| \leqslant \gamma, \\ |t| - \dfrac{\gamma}{2}, & \text{if } |t| > \gamma, \end{cases}$$

where $\gamma > 0$ is a given constant and $t \in \mathbb{R}$. The smaller the parameter γ of the Huber function is, the better the function approximates $|t|$. This function is quadratic for $|t| \leqslant \gamma$ and linear for $|t| > \gamma$. In addition, it is convex and first-order differentiable. Moreover, it has the nice sparsity property [2]. These features allow it to be widely used in many fields such as machine learning, statistic optimization and compressed sensing recently [2, 46]. Let $\mu > 0$ be a given constant. After considering these good properties of the so-called Huber function and $a_+ = \frac{|a|+a}{2}$ with $a_+ = \max\{a, 0\}$ for any $a \in \mathbb{R}$, we propose the following Huber-type smoothing function of a_+:

$$\phi(\mu,a)=\begin{cases}\dfrac{a^2}{4\mu}+\dfrac{a}{2}, & \text{if } |a|\leqslant\mu,\\ a-\dfrac{\mu}{4}, & \text{if } a>\mu,\\ -\dfrac{\mu}{4}, & \text{if } a<-\mu.\end{cases} \tag{4.2}$$

It is easy to see that $\phi(\mu,\cdot)$ is also convex and continuously differentiable with

$$\phi'(\mu,a)=\begin{cases}\dfrac{a}{2\mu}+\dfrac{1}{2}, & \text{if } |a|\leqslant\mu,\\ 1, & \text{if } a>\mu,\\ 0, & \text{if } a<-\mu.\end{cases} \tag{4.3}$$

Moreover, $\phi(\mu,a)$ has the following property:

$$|\phi(\mu,a)-a_+|\leqslant\frac{\mu}{4}, \tag{4.4}$$

which implies that the smaller the parameter μ of the function $\phi(\mu,a)$ is, the better the function approximates a_+.

Based on the smooth function $\phi(\mu,a)$ defined as (4.2), we define a smoothing function of $G(\boldsymbol{x})$ as

$$G(\mu,\boldsymbol{x})=F(\Phi(\mu,\boldsymbol{x}))+\boldsymbol{x}-\Phi(\mu,\boldsymbol{x}), \tag{4.5}$$

where $\Phi(\mu,\boldsymbol{x})\in\mathbb{R}^n$ is given by

$$\Phi(\mu,\boldsymbol{x})=\begin{pmatrix}\phi(\mu,x_1)\\ \vdots\\ \phi(\mu,x_n)\end{pmatrix}.$$

Clearly, $G(\cdot,\cdot)$ is continuous on $\mathbb{R}^{n+1}$ and is continuously differentiable on $\mathbb{R}_{++}\times\mathbb{R}^n$. Then by (4.4), to solve the TCP (2.1) is equivalent to solve

$$H(\mu,\boldsymbol{x}):=\begin{pmatrix}\mu\\ G(\mu,\boldsymbol{x})+\mu\boldsymbol{x}\end{pmatrix}=\mathbf{0}, \tag{4.6}$$

where $G(\mu,\boldsymbol{x})$ is defined by (4.5). Since H is continuously differentiable on $\mathbb{R}_{++}\times\mathbb{R}^n$ and may be nonsmooth on $0\times\mathbb{R}^n$, we can see (4.6) as a smoothing-nonsmooth reformulation of the TCP (2.1).

Let $H:\mathbb{R}_{++}\times\mathbb{R}^n\to\mathbb{R}^{n+1}$ be defined by (4.6). We discuss a method to find a solution to $H(\boldsymbol{z})=\mathbf{0}$, where $\boldsymbol{z}=(\mu,\boldsymbol{x})\in\mathbb{R}_{++}\times\mathbb{R}^n$. Since the Jacobian matrix of $H(\boldsymbol{z})$ plays an important role for the discussion, we now take a look at its structure. By straightforward computation, we obtain the following result based on (4.3).

Theorem 4.1. *Let $H(\boldsymbol{z})$ be defined by (4.6). Then $H(\boldsymbol{z})$ is continuously differentiable on $\mathbb{R}_{++}\times\mathbb{R}^n$ with its Jacobian matrix*

$$JH(\boldsymbol{z})=\begin{pmatrix}1 & \mathbf{0}^{\mathrm{T}}\\ JF(\Phi(\mu,\boldsymbol{x}))v(\boldsymbol{z})+\boldsymbol{x} & JF(\Phi(\mu,\boldsymbol{x}))D(\boldsymbol{z})+(\mu+1)I-D(\boldsymbol{z})\end{pmatrix}, \tag{4.7}$$

where $JF(\boldsymbol{x})\in\mathbb{R}^{n\times n}$ is the Jacobian matrix of $F(\boldsymbol{x})$, I is an $n\times n$ identity matrix, $D(\boldsymbol{z})\in\mathbb{R}^{n\times n}$ is a diagonal matrix with the i-th diagonal element $\phi'(\mu,x_i)$ defined by (4.3) for $i\in[n]$, and $v(\boldsymbol{z})\in\mathbb{R}^n$ is a vector with the i-th component

$$v(\boldsymbol{z})_i=\begin{cases}-\dfrac{x_i^2}{4\mu^2}, & \text{if } |x_i|\leqslant\mu,\\ -\dfrac{1}{4}, & \text{if } |x_i|>\mu,\end{cases}\quad i\in[n].$$

The following theorem shows that the Jacobian matrix $JH(\boldsymbol{z})$ of $H(\boldsymbol{z})$ is nonsingular if the tensor $\mathcal{A}$ is strictly positive semidefinite on $\mathbb{R}^n_+$.

Theorem 4.2. *Let the Jacobian matrix $JH(\boldsymbol{z})$ of $H(\boldsymbol{z})$ be given by (4.7). If the tensor $\mathcal{A}$ is strictly positive semidefinite on $\mathbb{R}^n_+$, then $JH(\boldsymbol{z})$ is nonsingular on $\mathbb{R}_{++} \times \mathbb{R}^n$.*

Proof. Since $\mathcal{A}$ is strictly positive semidefinite on $\mathbb{R}^n_+$, by Definition 3.1, $F(\boldsymbol{x})$ is monotone on $\mathbb{R}^n_+$ for any $\boldsymbol{q} \in \mathbb{R}^n$. From Proposition 2.2, the Jacobian matrix $JF(\boldsymbol{x})$ is positive semidefinite on $\mathbb{R}^n_+$. Hence, the matrix $JF(\Phi(\mu, \boldsymbol{x}))$ is positive semidefinite for any $\boldsymbol{x} \in \mathbb{R}^n$. It follows from (4.3) and $\mu > 0$ that for $i \in [n]$,

$$0 \leqslant D(\boldsymbol{z})_{ii} \leqslant 1, \quad 0 < \mu \leqslant (\mu + 1) - D(\boldsymbol{z})_{ii} \leqslant \mu + 1,$$

which implies that $D(\boldsymbol{z})$ is a positive semidefinite diagonal matrix and $(\mu + 1)I - D(\boldsymbol{z})$ is a positive definite matrix. Consequently, the matrix $JF(\Phi(\mu, \boldsymbol{x}))D(\boldsymbol{z}) + (\mu + 1)I - D(\boldsymbol{z})$ is nonsingular for any $\boldsymbol{x} \in \mathbb{R}^n$. Thus, the desired result holds. □

In what follows, we present a Newton-type algorithm to solve $H(\boldsymbol{z}) = \boldsymbol{0}$. Let $\gamma \in (0, 1)$. For any $\boldsymbol{z} = (\mu, \boldsymbol{x}) \in \mathbb{R}_{++} \times \mathbb{R}^n$, define

$$\beta(\boldsymbol{z}) = \gamma \|H(\boldsymbol{z})\| \min\{1, \|H(\boldsymbol{z})\|\}. \tag{4.8}$$

Then we have the following theorem.

Theorem 4.3. *Choose a scalar $\bar{\mu} > 0$ such that $\gamma\bar{\mu} < 1$. Let $\bar{\boldsymbol{u}} = (\bar{\mu}, \boldsymbol{0}) \in \mathbb{R}_{++} \times \mathbb{R}^n$. Then $H(\boldsymbol{z}) = \boldsymbol{0}$ if and only if $H(\boldsymbol{z}) = \beta(\boldsymbol{z})\bar{\boldsymbol{u}}$.*

Proof. Assume that $H(\boldsymbol{z}) = \boldsymbol{0}$. We have $\beta(\boldsymbol{z}) = 0$ from (4.8), and hence $\beta(\boldsymbol{z})\bar{\boldsymbol{u}} = \boldsymbol{0}$. Consequently, $H(\boldsymbol{z}) = \beta(\boldsymbol{z})\bar{\boldsymbol{u}}$. On the other hand, assume that $H(\boldsymbol{z}) = \beta(\boldsymbol{z})\bar{\boldsymbol{u}}$. Then we have

$$\|H(\boldsymbol{z})\|(1 - \gamma\bar{\mu} \min\{1, \|H(\boldsymbol{z})\|\}) = 0.$$

Note that $\gamma\bar{\mu} \min\{1, \|H(\boldsymbol{z})\|\} \leqslant \gamma\bar{\mu} < 1$. Hence, $H(\boldsymbol{z}) = \boldsymbol{0}$. □

In order to solve the TCP$(\mathcal{A}, \boldsymbol{q})$, based on (4.8) and Theorem 4.3, we present a Newton-type iteration method to solve $H(\boldsymbol{z}) = \boldsymbol{0}$ following the algorithmic scheme given in [37, 44].

Note that Algorithm 1 solves only one linear system of the equation (4.9) and performs only one Armijo-type line search (4.10). By Theorem 4.2, the equation (4.9) is solvable. The direction $\boldsymbol{d}^k$ computed in (4.9) is an approximated generalized Newton direction of H at $\boldsymbol{z}^k$ because $\beta(\boldsymbol{z}^k)\bar{\boldsymbol{u}}$ is introduced on the right-hand side of (4.9). Such an introduction can ensure that all μ_k's satisfy $\mu_k > 0$ and $\mu_k \geqslant \beta(\boldsymbol{z}^k)\mu_0$ (see Theorem 4.4). This plays an important role in proving the global convergence of Algorithm 1.

Algorithm 1 A Newton-type algorithm for the monotone TCP

Choose $\delta \in (0, 1)$, $\sigma \in (0, \frac{1}{2})$ and $\varepsilon > 0$. Let $\mu_0 \in \mathbb{R}_{++}$, $\boldsymbol{x}^0 \in \mathbb{R}^n$ be an arbitrary point and $\boldsymbol{z}^0 = (\mu_0, \boldsymbol{x}^0)$. Choose $\gamma \in (0, 1)$ such that $\gamma\|H(\boldsymbol{z}^0)\| < 1$ and $\gamma\mu_0 < 1$. Let $\bar{\boldsymbol{u}} = (\mu_0, \boldsymbol{0}) \in \mathbb{R}_{++} \times \mathbb{R}^n$. Set $k := 0$.

while $\|H(\boldsymbol{z}^k)\| > \varepsilon$ **do**

Compute $\boldsymbol{d}^k = (\boldsymbol{d}^k_\mu, \boldsymbol{d}^k_{\boldsymbol{x}}, \boldsymbol{d}^k_{\boldsymbol{y}})$ by solving

$$H(\boldsymbol{z}^k) + JH(\boldsymbol{z}^k)\boldsymbol{d} = \beta(\boldsymbol{z}^k)\bar{\boldsymbol{u}}. \tag{4.9}$$

Let m_k be the smallest nonnegative integer m satisfying

$$\|H(\boldsymbol{z}^k + \delta^m \boldsymbol{d}^k)\| \leqslant (1 - \sigma(1 - \gamma\mu_0)\theta\delta^m)\|H(\boldsymbol{z}^k)\|. \tag{4.10}$$

Let $\alpha_k = \delta^{m_k}$ and update $\boldsymbol{z}^{k+1} = \boldsymbol{z}^k + \alpha_k \boldsymbol{d}^k$. Set $k := k + 1$.

end while

The following theorem shows that Algorithm 1 is well defined.

Theorem 4.4. *Algorithm 1 is well defined for solving the* TCP$(\mathcal{A}, \boldsymbol{q})$ (2.1) *if the tensor $\mathcal{A}$ is strictly positive semidefinite on $\mathbb{R}^n_+$. If Algorithm 1 generates an infinite sequence $\{\boldsymbol{z}^k = (\mu_k, \boldsymbol{x}^k)\}$, then $\mu_k \in \mathbb{R}_{++}$ and $\mu_k \geqslant \beta(\boldsymbol{z}^k)\mu_0$ for all k.*

Proof. If $\mu_k > 0$, then the linear system of the equation (4.9) is solvable from Theorem 4.2 at the k-th iteration. From (4.9), we obtain

$$\boldsymbol{d}_\mu^k = -\mu_k + \beta(\boldsymbol{z}^k)\mu_0,$$

which together with $\mu_k > 0$ and (4.8) implies that for $\alpha \in (0, 1)$,

$$\mu_k + \alpha \boldsymbol{d}_\mu^k = (1-\alpha)\mu_k + \alpha\beta(\boldsymbol{z}^k)\mu_0 > 0.$$

Thus, by Theorem 4.1, H is continuously differentiable at $\boldsymbol{z}^k + \alpha\boldsymbol{d}^k$. For any $\alpha \in (0, 1)$, define

$$\boldsymbol{r}(\alpha) = H(\boldsymbol{z}^k + \alpha\boldsymbol{d}^k) - H(\boldsymbol{z}^k) - \alpha JH(\boldsymbol{z}^k)\boldsymbol{d}^k. \tag{4.11}$$

Then $\|\boldsymbol{r}(\alpha)\| = o(\alpha)$ when $\alpha > 0$ is sufficiently small. Consequently, combining (4.9), (4.8) and (4.11), we obtain that for sufficiently small $\alpha > 0$,

$$\begin{aligned}\|H(\boldsymbol{z}^k + \alpha\boldsymbol{d}^k)\| &\leqslant \|\boldsymbol{r}(\alpha)\| + (1-\alpha)\|H(\boldsymbol{z}^k)\| + \alpha\beta(\boldsymbol{z}^k)\mu_0 \\ &\leqslant (1-\alpha)\|H(\boldsymbol{z}^k)\| + \alpha\gamma\mu_0\|H(\boldsymbol{z}^k)\| + o(\alpha) \\ &= (1-\alpha(1-\gamma\mu_0))\|H(\boldsymbol{z}^k)\| + o(\alpha),\end{aligned}$$

which implies that there exists an $\hat{\alpha} \in (0, 1)$ such that

$$\|H(\boldsymbol{z}^k + \alpha\boldsymbol{d}^k)\| \leqslant (1 - \sigma(1-\gamma\mu_0)\alpha)\|H(\boldsymbol{z}^k)\|, \quad \forall\, \alpha \in (0, \hat{\alpha}].$$

This shows that the Armijo-type line search (4.10) is available at the k-th iteration.

On the other hand, by (4.9) and (4.10), we have $\alpha_k \in (0, 1)$ and

$$\mu_{k+1} = \mu_k + \alpha_k\boldsymbol{d}_\mu^k = (1-\alpha_k)\mu_k + \alpha_k\beta(\boldsymbol{z}^k)\mu_0 > 0.$$

Hence, it follows from $\mu_0 > 0$ that Algorithm 1 is well defined. Moreover, if it generates an infinite sequence $\{\boldsymbol{z}^k = (\mu_k, \boldsymbol{x}^k)\}$, then we have $\mu_k > 0$ for all k.

We now prove that $\mu_k \geqslant \beta(\boldsymbol{z}^k)\mu_0$ for all k by induction on k. Clearly, $\beta(\boldsymbol{z}^0) \leqslant \gamma\|H(\boldsymbol{z}^0)\| < 1$ from (4.8). Thus, we have $\mu_0 \geqslant \beta(\boldsymbol{z}^0)\mu_0$. Assume that $\mu_k \geqslant \beta(\boldsymbol{z}^k)\mu_0$. Then it follows from (4.9) that

$$\begin{aligned}\mu_{k+1} - \beta(\boldsymbol{z}^{k+1})\mu_0 &= (1-\alpha_k)\mu_k + \alpha_k\beta(\boldsymbol{z}^k)\mu_0 - \beta(\boldsymbol{z}^{k+1})\mu_0 \\ &\geqslant (\beta(\boldsymbol{z}^k) - \beta(\boldsymbol{z}^{k+1}))\mu_0.\end{aligned} \tag{4.12}$$

From (4.8), we have $\beta(\boldsymbol{z}^{k+1}) \leqslant \gamma\|H(\boldsymbol{z}^{k+1})\|$, $\beta(\boldsymbol{z}^{k+1}) \leqslant \gamma\|H(\boldsymbol{z}^{k+1})\|^2$ and

$$\beta(\boldsymbol{z}^k) = \begin{cases}\gamma\|H(\boldsymbol{z}^k)\|^2, & \text{if } \|H(\boldsymbol{z}^k)\| < 1, \\ \gamma\|H(\boldsymbol{z}^k)\|, & \text{otherwise.}\end{cases}$$

It follows from (4.10) that $\|H(\boldsymbol{z}^{k+1})\| \leqslant \|H(\boldsymbol{z}^k)\|$. Hence, by (4.12) we obtain $\mu_{k+1} \geqslant \beta(\boldsymbol{z}^{k+1})\mu_0$. So we complete the proof. □

From the proof of Theorem 4.4, it is not difficult to obtain that the sequence $\{\|H(\boldsymbol{z}^k)\|\}$ is monotonically nonincreasing and hence it is convergent. The following theorem proves that the sequence $\{\mu_k\}$ is also monotonically nonincreasing.

Theorem 4.5. *Suppose that $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is strictly positive semidefinite on $\mathbb{R}^n_+$ and $\{\boldsymbol{z}^k = (\mu_k, \boldsymbol{x}^k, \boldsymbol{y}^k)\}$ is an infinite sequence generated by Algorithm* 1. *Then for any $k \geqslant 0$,*

$$0 < \mu_{k+1} \leqslant \mu_k \leqslant \mu_0. \tag{4.13}$$

Proof. First, $\mu_0 > 0$. By Theorem 4.4, Algorithm 1 is well defined. Moreover, $\mu_k > 0$ and $\mu_k \geqslant \beta(\boldsymbol{z}^k)\mu_0$ for all $k \geqslant 0$. Hence, it follows from (4.9) and (4.10) that

$$\mu_{k+1} = \mu_k + \alpha_k\boldsymbol{d}_\mu^k = (1-\alpha_k)\mu_k + \alpha_k\beta(\boldsymbol{z}^k)\mu_0 \leqslant (1-\alpha_k)\mu_k + \alpha_k\mu_k = \mu_k.$$

Therefore, by the induction on k, (4.13) holds. □

5 Global and superlinear convergence analysis

In this section, we analyze the global convergence for Algorithm 1. For this purpose, we first establish the following lemmas based on Theorem 4.1 and [19, Lemma 2.4 and Theorem 3.1]. For any $(\mu, \boldsymbol{x}) \in \mathbb{R}_{++} \times \mathbb{R}^n$, define

$$Q(\mu, \boldsymbol{x}) = G(\mu, \boldsymbol{x}) + \mu \boldsymbol{x}. \tag{5.1}$$

Then it follows from the proof of Theorem 4.1 that the Jacobian matrix of $Q(\mu, \boldsymbol{x})$ is positive definite and hence $Q(\mu, \boldsymbol{x}) : \mathbb{R}_{++} \times \mathbb{R}^n \to \mathbb{R}^n$ is a strictly monotone mapping from Proposition 2.2. So we easily obtain the following results from Lemma 2.4 and Theorem 3.1 in [19], respectively. Because the proofs are very similar, we omit the proofs of Lemmas 5.1 and 5.2 here.

Lemma 5.1. *Suppose that $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is strictly positive semidefinite on $\mathbb{R}^n_+$ and $\hat{\mu}_1$ and $\hat{\mu}_2$ are two given positive numbers such that $\hat{\mu}_1 \leqslant \hat{\mu}_2$. Then for any sequence $\{(\mu_k, \boldsymbol{x}^k)\}$ with $\mu_k \in [\hat{\mu}_1, \hat{\mu}_2]$ and $\|\boldsymbol{x}^k\| \to \infty$, we have*

$$\lim_{k\to\infty} \|Q(\mu_k, \boldsymbol{x}^k)\| = \infty.$$

Lemma 5.2. *Suppose that $\{\mu_k\}$ and $\{\eta_k\}$ are two infinite sequences such that for each $k \geqslant 0$, $\mu_k > 0$ and $\eta_k \geqslant 0$ satisfying $\lim_{k\to\infty} \mu_k = 0$ and $\lim_{k\to\infty} \eta_k = 0$. For each $k \geqslant 0$, let $\boldsymbol{x}^k \in \mathbb{R}^n$ satisfy $\|Q(\mu_k, \boldsymbol{x}^k)\| \leqslant \eta_k$. If $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is strictly positive semidefinite on $\mathbb{R}^n_+$ and the solution set* $\mathrm{SOL}(\mathcal{A}, \boldsymbol{q})$ *of the* $\mathrm{TCP}(\mathcal{A}, \boldsymbol{q})$ (2.1) *is nonempty and bounded, then $\{\boldsymbol{x}^k\}$ remains bounded.*

The following theorem proves that each accumulation point of $\{\boldsymbol{z}^k\}$ generated by Algorithm 1 is a solution to $H(\boldsymbol{z}) = \boldsymbol{0}$.

Theorem 5.3. *Let $\mathcal{A} \in \mathbb{R}^{[m,n]}$ in the* $\mathrm{TCP}(\mathcal{A}, \boldsymbol{q})$ (2.1) *be strictly positive semidefinite on $\mathbb{R}^n_+$ and $\{\boldsymbol{z}^k = (\mu_k, \boldsymbol{x}^k)\}$ be an infinite sequence generated by Algorithm* 1*. Then we have*

$$\lim_{k\to\infty} \|H(\boldsymbol{z}^k)\| = 0, \quad \lim_{k\to\infty} \mu_k = 0. \tag{5.2}$$

Moreover, any accumulation point of $\{\boldsymbol{z}^k\}$ is a solution to $H(\boldsymbol{z}) = \boldsymbol{0}$.

Proof. By Theorem 4.4, the sequence $\{\boldsymbol{z}^k = (\mu_k, \boldsymbol{x}^k)\}$ satisfies $\mu_k > 0$ and $\mu_k \geqslant \beta(\boldsymbol{z}^k)\mu_0$ for all $k \geqslant 0$. By (4.10), we have $0 \leqslant \|H(\boldsymbol{z}^{k+1})\| \leqslant \|H(\boldsymbol{z}^k)\|$ for all $k \geqslant 0$. By Theorem 4.5, $0 < \mu_{k+1} \leqslant \mu_k \leqslant \mu_0$ for all $k \geqslant 0$. Hence, the sequences $\{\|H(\boldsymbol{z}^k)\|\}$, $\{\mu_k\}$ and $\{\beta(\boldsymbol{z}^k)\}$ are all monotonically nonincreasing and hence they are all convergent. Therefore, there exist $\tilde{h} \geqslant 0$, $\tilde{\mu} \geqslant 0$ and $\tilde{\beta} \geqslant 0$ such that $\|H(\boldsymbol{z}^k)\| \to \tilde{h}$, $\mu_k \to \tilde{\mu}$ and $\beta(\boldsymbol{z}^k) \to \tilde{\beta}$ as $k \to \infty$. Moreover, letting $k \to \infty$ in (4.8) and $\mu_k \leqslant \|H(\boldsymbol{z}^k)\|$, we obtain

$$\tilde{\beta} = \gamma \tilde{h} \min\{1, \tilde{h}\}, \quad 0 \leqslant \tilde{\mu} \leqslant \tilde{h}.$$

If $\tilde{h} = 0$, then we obtain the desired result. Suppose that $\tilde{h} > 0$. Then, $\tilde{\beta} > 0$. Since $\mu_k \geqslant \beta(\boldsymbol{z}^k)\mu_0$ for all $k \geqslant 0$, we have

$$\tilde{\beta}\mu_0 \leqslant \mu_k \leqslant \mu_0, \quad \forall\, k \geqslant 0.$$

Hence, by Lemma 5.1, the infinite sequence $\{\boldsymbol{x}^k\}$ must be bounded due to

$$\lim_{k\to\infty} \|Q(\mu_k, \boldsymbol{x}^k)\| = \lim_{k\to\infty} \sqrt{\|H(\boldsymbol{z}^k)\|^2 - \mu_k^2} = \sqrt{\tilde{h}^2 - \tilde{\mu}^2}.$$

Therefore, the infinite sequence $\{\boldsymbol{z}^k\}$ must be bounded. Then there exists at least one accumulation point $\boldsymbol{z}^* = (\mu^*, \boldsymbol{x}^*)$ of $\{\boldsymbol{z}^k\}$ such that $\tilde{\beta}\mu_0 \leqslant \mu^* \leqslant \mu_0$. By taking a subsequence if necessary, we may assume that $\{\boldsymbol{z}^k\}$ converges to $\boldsymbol{z}^*$. It is easy to see that

$$\tilde{h} = \|H(\boldsymbol{z}^*)\|, \quad \tilde{\beta} = \beta(\boldsymbol{z}^*), \quad \mu^* = \tilde{\mu}.$$

Hence, we have

$$\|H(\boldsymbol{z}^*)\| > 0, \quad 0 < \beta(\boldsymbol{z}^*) \leqslant \gamma \|H(\boldsymbol{z}^*)\|, \quad 0 < \beta(\boldsymbol{z}^*)\mu_0 \leqslant \mu^* \leqslant \mu_0. \tag{5.3}$$

Thus, by (4.10), we obtain

$$\|H(\boldsymbol{z}^{k+1})\| \leqslant (1-\sigma(1-\gamma\mu_0)\alpha_k)\|H(\boldsymbol{z}^k)\|,$$

which, together with $\sigma \in (0,1)$ and $\gamma\mu_0 < 1$, implies that

$$\lim_{k\to\infty} \alpha_k = 0.$$

It follows from the Armijo-type line search (4.10) that

$$\|H(\boldsymbol{z}^k + \alpha_k/\delta \boldsymbol{d}^k)\| > (1-\sigma(1-\gamma\mu_0)\alpha_k/\delta)\|H(\boldsymbol{z}^k)\|,$$

which, together with letting $k \to \infty$, yields

$$H(\boldsymbol{z}^*)^{\mathrm{T}} JH(\boldsymbol{z}^*)\boldsymbol{d}^* \geqslant -\sigma(1-\gamma\mu_0)\|H(\boldsymbol{z}^*)\|^2, \tag{5.4}$$

where $JH(\boldsymbol{z}^*)$ is well defined from Theorem 4.2, and hence $\boldsymbol{d}^*$ is the limit point of $\{\boldsymbol{d}^k\}$. Letting $k\to\infty$ in (4.9), we obtain

$$JH(\boldsymbol{z}^*)\boldsymbol{d}^* = -H(\boldsymbol{z}^*) + \beta(\boldsymbol{z}^*)\bar{\boldsymbol{u}},$$

which implies

$$\begin{aligned} H(\boldsymbol{z}^*)^{\mathrm{T}} JH(\boldsymbol{z}^*)\boldsymbol{d}^* &= -\|H(\boldsymbol{z}^*)\|^2 + \beta(\boldsymbol{z}^*)H(\boldsymbol{z}^*)^{\mathrm{T}}\bar{\boldsymbol{u}} \\ &\leqslant -\|H(\boldsymbol{z}^*)\|^2 + \beta(\boldsymbol{z}^*)\mu_0\|H(\boldsymbol{z}^*)\|, \end{aligned}$$

where the inequality holds from the Cauchy-Schwartz inequality and $\|\bar{\boldsymbol{u}}\| = \mu_0$. Invoking (5.4) and (5.3), we obtain

$$(1-\sigma(1-\gamma\mu_0))\|H(\boldsymbol{z}^*)\|^2 \leqslant \beta(\boldsymbol{z}^*)\mu_0\|H(\boldsymbol{z}^*)\| \leqslant \gamma\mu_0\|H(\boldsymbol{z}^*)\|^2,$$

which, together with $\|H(\boldsymbol{z}^*)\| > 0$, yields $(1-\sigma(1-\gamma\mu_0)) \leqslant \gamma\mu_0$, i.e., $(1-\delta)(1-\gamma\mu_0) \leqslant 0$. However, since $\delta \in (0,1)$ and $\gamma\mu_0 < 1$, we have $(1-\delta)(1-\gamma\mu_0) > 0$, which leads to a contradiction. Consequently, we must have $\tilde{h} = 0$ and hence (5.2) must hold. Moreover, if there exists an accumulation point $\hat{\boldsymbol{z}}$ of $\{\boldsymbol{z}^k\}$, then by the continuity, $\|H(\hat{\boldsymbol{z}})\| = 0$, i.e., $H(\hat{\boldsymbol{z}}) = \boldsymbol{0}$. So we complete the proof. □

Theorem 5.3 shows that any accumulation point of the infinite sequence $\{\boldsymbol{z}^k\}$ generated by Algorithm 1, if it exists, is a solution to $H(\boldsymbol{z}) = \boldsymbol{0}$. Moreover, let $\boldsymbol{z}^* = (\mu^*, \boldsymbol{x}^*)$ be an accumulation point of $\{\boldsymbol{z}^k\}$. Then $\boldsymbol{x}^*_+ \in \mathrm{SOL}(\mathcal{A}, \boldsymbol{q})$ from Theorem 5.3. An important question here is whether such an accumulation point exists or not. We answer this question by investigating under what conditions the infinite sequence $\{\boldsymbol{z}^k\}$ generated by Algorithm 1 is bounded. We prove that if the solution set $\mathrm{SOL}(\mathcal{A}, \boldsymbol{q})$ of the $\mathrm{TCP}(\mathcal{A}, \boldsymbol{q})$ (2.1) is nonempty and bounded, then such a sequence $\{\boldsymbol{z}^k\}$ must be bounded.

Theorem 5.4. *Suppose that $\mathcal{A} \in \mathbb{R}^{[m,n]}$ in the $\mathrm{TCP}(\mathcal{A}, \boldsymbol{q})$ (2.1) is strictly positive semidefinite on $\mathbb{R}^n_+$ and the solution set $\mathrm{SOL}(\mathcal{A}, \boldsymbol{q})$ is nonempty and bounded. Then the infinite sequence $\{\boldsymbol{z}^k = (\mu_k, \boldsymbol{x}^k)\}$ generated by Algorithm 1 is bounded and any accumulation point of $\{\boldsymbol{z}^k\}$ is a solution to $H(\boldsymbol{z}) = \boldsymbol{0}$.*

Proof. From Theorem 5.3, $\lim_{k\to\infty} \|H(\boldsymbol{z}^k)\| = 0$ and $\lim_{k\to\infty} \mu_k = 0$. Then

$$\lim_{k\to\infty} \|Q(\mu_k, \boldsymbol{x}^k)\| = \lim_{k\to\infty} \sqrt{\|H(\boldsymbol{z}^k)\|^2 - \mu_k^2} = 0.$$

By Theorem 4.4, $\mu_k > 0$ for all $k \geqslant 0$. Hence, it follows from Lemma 5.2 that $\{\boldsymbol{x}^k\}$ remains bounded. Consequently, $\{\boldsymbol{z}^k\}$ is bounded. So from Theorem 5.3, any accumulation point of $\{\boldsymbol{z}^k\}$ is a solution to $H(\boldsymbol{z}) = \boldsymbol{0}$. □

From Theorems 3.8 and 5.4, we immediately obtain the following result.

Corollary 5.5. *Suppose that $\mathcal{A} \in \mathbb{R}^{[m,n]}$ in the $\mathrm{TCP}(\mathcal{A}, \boldsymbol{q})$ (2.1) is strictly positive semidefinite on $\mathbb{R}^n_+$ and there exists a $\boldsymbol{u} \geqslant \boldsymbol{0}$ satisfying $\mathcal{A}\boldsymbol{u}^{m-1} > \boldsymbol{0}$. Then the infinite sequence $\{\boldsymbol{z}^k = (\mu_k, \boldsymbol{x}^k)\}$ generated by Algorithm 1 is bounded and any accumulation point of $\{\boldsymbol{z}^k\}$ is a solution to $H(\boldsymbol{z}) = \boldsymbol{0}$.*

In order to discuss the rate of convergence of Algorithm 1, we need the concept of semismoothness, which was originally introduced by Mifflin [28] for functions and extended by Qi and Sun [38] for vector-valued functions. Convex functions, smooth functions and piecewise linear functions are examples of semismooth functions. The composition of semismooth functions is still a semismooth function [28]. A vector-valued function is semismooth if and only if all its component functions are [38].

A locally Lipschitzian function $\Psi : \mathbb{R}^l \to \mathbb{R}^m$ has a generalized Jacobian $\partial\Psi(\boldsymbol{x})$ as defined in [9]. If it is semismooth at $\boldsymbol{x}$, then $\Psi'(\boldsymbol{x};\boldsymbol{h})$, the directional derivative of Ψ at $\boldsymbol{x}$ in the direction $\boldsymbol{h}$, exists for any $\boldsymbol{h} \in \mathbb{R}^l$. The following lemma given in [38] shows the relationship between them.

Lemma 5.6. *Suppose that $\Psi : \mathbb{R}^l \to \mathbb{R}^m$ is a locally Lipschitzian function and is semismooth at $\boldsymbol{x}$. Then for any $V \in \partial\Psi(\boldsymbol{x}+\boldsymbol{h})$ and $\boldsymbol{h} \to \boldsymbol{0}$,*

$$\|V\boldsymbol{h} - \Psi'(\boldsymbol{x};\boldsymbol{h})\| = o(\|\boldsymbol{h}\|), \quad \|\Psi(\boldsymbol{x}+\boldsymbol{h}) - \Psi(\boldsymbol{x}) - \Psi'(\boldsymbol{x};\boldsymbol{h})\| = o(\|\boldsymbol{h}\|).$$

We also need the following result given in [44, Theorem 2.2].

Lemma 5.7. *For any $\boldsymbol{x} \in \mathbb{R}^n$, all $S \in \partial\boldsymbol{x}_+$ are symmetric, positive semidefinite and $\|S\| \leqslant 1$.*

Let Q be defined by (5.1). Since F is locally Lipschitz continuous and $\boldsymbol{x}_+$ is semismooth, by Lemma 5.7, $Q(0,\boldsymbol{x})$ has a generalized Jacobian $\partial Q(0,\boldsymbol{x})$ in the sense of Clarke [9], and $V \in \partial Q(0,\boldsymbol{x})$ can be written as

$$V = JF(\boldsymbol{x}_+)S + I - S, \quad S \in \partial\boldsymbol{x}_+. \tag{5.5}$$

The following theorem shows that Algorithm 1 has superlinear convergence under mild conditions.

Theorem 5.8. *Suppose that the assumption in Theorem 5.4 is satisfied and $\boldsymbol{z}^* = (0,\boldsymbol{x}^*)$ is an accumulation point of the infinite sequence $\{\boldsymbol{z}^k = (\mu_k,\boldsymbol{x}^k)\}$ generated by Algorithm 1. If all $V \in \partial Q(\boldsymbol{z}^*)$ are nonsingular, then the whole sequence $\{\boldsymbol{z}^k\}$ converges to $\boldsymbol{z}^*$ superlinearly, i.e.,*

$$\|\boldsymbol{z}^{k+1} - \boldsymbol{z}^*\| = o(\|\boldsymbol{z}^k - \boldsymbol{z}^*\|), \quad \mu_{k+1} = o(\mu_k).$$

Proof. It is not difficult to see that Φ is continuously differentiable at any $\boldsymbol{z} = (\mu,\boldsymbol{x}) \in \mathbb{R}_{++} \times \mathbb{R}^n$. Since the function a_+ is semismooth on $\mathbb{R}$, the function Q is semismooth. By Lemma 5.6 and following the proof of [43, Theorem 5.1], we can complete our proof. □

Remark 5.9. In Theorem 5.8, we assume that all $V \in \partial Q(\boldsymbol{z}^*)$ are nonsingular in order to obtain a high-order convergent result. It follows from (5.5) that $V = I - S[I - JF(\boldsymbol{x}^*_+)]$ with $S \in \partial\boldsymbol{x}^*_+$ given in Lemma 5.7. By [44, Theorem 4.3], this assumption is satisfied if $JF(\boldsymbol{x}^*_+)$ is positive definite. There are many structured tensors such as the M-tensor [48] such that $F(\boldsymbol{x}) = \mathcal{A}\boldsymbol{x}^{m-1} + \boldsymbol{q}$ satisfies the condition.

6 Numerical results

In this section, we present some numerical experiments for Algorithm 1 to solve the TCP$(\mathcal{A},\boldsymbol{q})$ (2.1). All the codes were written by using MATLAB R2017b. The numerical experiments were done on a computer with an Intel(R) Core(TM) i7-7700 CPU (3.60 GHz) and RAM of 16.0 GB.

In our experiments, we set the parameters $\varepsilon = 10^{-12}$, $\delta = 0.75$, $\sigma = 0.25$, $\gamma = 0.99$ and $\mu_0 = 0.01$ in Algorithm 1. The termination criterion of Algorithm 1 is that the stop condition $\|H(\boldsymbol{z}^k)\| \leqslant \varepsilon$ is satisfied in 100 iterations. We choose $\boldsymbol{x}^0 = \boldsymbol{e}$ as the starting point in all the tested problems. We give $\mathcal{A}$ in the following tested numerical examples. Different vectors $\boldsymbol{q} \in \mathbb{R}^n$ are used in our experiments for the first 5 tested problems. The 6-th tested problem is an application of the TCP in multi-person noncooperative games, which is to illustrate the application of the TCP.

All the numerical results of the first 5 tested problems are reported in the following tables, where $\boldsymbol{q}$ denotes the vector $\boldsymbol{q}$ used in the tested problems, $\boldsymbol{x}$**-SOL** denotes the solution to the corresponding TCP$(\mathcal{A},\boldsymbol{q})$, **Iter** denotes the number of iterations, which is also equal to the number of Jacobian evaluations for the function H, **NH** denotes the number of function evaluations for the function H, **NormH** denotes the value of $\|H(\boldsymbol{z})\|$ at the final iterate, and **Time** (s) denotes the elapsed CPU time in seconds.

Example 6.1. Consider the TCP$(\mathcal{A},\boldsymbol{q})$, where $\mathcal{A}\in\mathbb{R}^{[4,2]}$ is defined by

$$a_{1111}=1,\quad a_{1222}=-\frac{1}{3},\quad a_{1122}=1,\quad a_{2222}=1,\quad a_{2111}=-\frac{1}{3},\quad a_{2211}=1,$$

and $a_{i_1i_2i_3i_4}=0$ otherwise.

For $\boldsymbol{x}=(x_1,x_2)^{\mathrm{T}}\in\mathbb{R}^2$, define $P(\boldsymbol{x})=\mathcal{A}\boldsymbol{x}^3$ and let $JP(\boldsymbol{x})$ denote its Jacobian matrix. Then we have

$$P(\boldsymbol{x})=\begin{pmatrix}x_1^3-\frac{1}{3}x_2^3+x_1x_2^2\\ x_2^3-\frac{1}{3}x_1^3+x_1^2x_2\end{pmatrix},\quad JP(\boldsymbol{x})=\begin{pmatrix}3x_1^2+x_2^2 & -x_2^2+2x_1x_2\\ -x_1^2+2x_1x_2 & 3x_2^2+x_1^2\end{pmatrix}.$$

For any $\boldsymbol{d}=(d_1,d_2)^{\mathrm{T}}\in\mathbb{R}^2$ and $\boldsymbol{d}\neq\boldsymbol{0}$, by straightforward computation, we obtain

$$\boldsymbol{d}^{\mathrm{T}}JP(\boldsymbol{x})\boldsymbol{d}=2(x_1d_1+x_2d_2)^2+(x_1^2+x_2^2)(d_1^2+d_2^2-d_1d_2)>0$$

holds for any $\boldsymbol{x}\in\mathbb{R}^2\backslash\{\boldsymbol{0}\}$. Hence, by Proposition 2.2 and Definition 2.4(c), $\mathcal{A}$ is strictly positive definite on $\mathbb{R}^2$. So by Theorem 2.5, the TCP$(\mathcal{A},\boldsymbol{q})$ in Example 6.1 has a unique solution for any $\boldsymbol{q}\in\mathbb{R}^2$.

Example 6.2. Consider the TCP$(\mathcal{A},\boldsymbol{q})$, where $\mathcal{A}\in\mathbb{R}^{[3,2]}$ is given in Example 3.2 as $a_{112}=2$, $a_{211}=-1$ and zero otherwise.

Such an $\mathcal{A}$ is strictly positive semidefinite on $\mathbb{R}^2_+$ and the solvability of the corresponding TCP$(\mathcal{A},\boldsymbol{q})$ was given in (3.2). Using Algorithm 1 to solve it, we can obtain the same results.

Example 6.3. Consider the TCP$(\mathcal{A},\boldsymbol{q})$, where $\mathcal{A}\in\mathbb{R}^{[4,4]}$ is defined by

$$a_{1111}=a_{1122}=a_{2222}=a_{2211}=a_{3333}=a_{4444}=1,$$

and $a_{i_1i_2i_3i_4}=0$ otherwise.

For $\boldsymbol{x}\in\mathbb{R}^4_+$, define $P(\boldsymbol{x})=\mathcal{A}\boldsymbol{x}^3$ and let $JP(\boldsymbol{x})$ denote its Jacobian matrix. Then we have

$$P(\boldsymbol{x})=\begin{pmatrix}x_1^3+x_1x_2^2\\ x_2^3+x_1^2x_2\\ x_3^3\\ x_4^3\end{pmatrix},\quad JP(\boldsymbol{x})=\begin{pmatrix}3x_1^2+x_2^2 & 2x_1x_2 & 0 & 0\\ 2x_1x_2 & 3x_2^2+x_1^2 & 0 & 0\\ 0 & 0 & 3x_3^2 & 0\\ 0 & 0 & 0 & 3x_4^2\end{pmatrix}.$$

Obviously, $JP(\boldsymbol{x})$ is symmetric and every of its principal minors is nonnegative. Hence, $JP(\boldsymbol{x})$ is positive semidefinite for all $\boldsymbol{x}\in\mathbb{R}^4_+$. So by Proposition 2.2 and Definition 3.1, $\mathcal{A}$ is strictly positive semidefinite on $\boldsymbol{x}\in\mathbb{R}^4_+$. Clearly, $\mathcal{A}\boldsymbol{e}^3=P(\boldsymbol{e})>0$. By Theorem 3.8, the TCP$(\mathcal{A},\boldsymbol{q})$ in Example 6.3 is solvable for any $\boldsymbol{q}\in\mathbb{R}^4$.

Example 6.4. Consider the TCP$(\mathcal{A},\boldsymbol{q})$, where $\mathcal{A}\in\mathbb{R}^{[6,6]}$ is defined by

$$\begin{aligned}&a_{111111}=1,\quad a_{222222}=4,\quad a_{112222}=3,\quad a_{221111}=1,\quad a_{111122}=2,\quad a_{222211}=6,\\ &a_{333333}=3,\quad a_{444444}=1,\quad a_{333344}=4,\quad a_{444433}=2,\quad a_{334444}=1,\quad a_{443333}=2,\\ &a_{555555}=a_{666666}=1,\end{aligned}$$

and $a_{i_1i_2i_3i_4i_5i_6}=0$ otherwise.

For $\boldsymbol{x}\in\mathbb{R}^6_+$, define $P(\boldsymbol{x})=\mathcal{A}\boldsymbol{x}^5$ and let $JP(\boldsymbol{x})$ denote its Jacobian matrix. Then we have

$$P(\boldsymbol{x})=\begin{pmatrix}x_1^5+3x_1x_2^4+2x_1^3x_2^2\\ 4x_2^5+x_1^4x_2+6x_1^2x_2^3\\ 3x_3^5+4x_3^3x_4^2+x_3x_4^4\\ x_4^5+2x_3^2x_4^3+2x_3^4x_4\\ x_5^5\\ x_6^5\end{pmatrix}.$$

By straightforward computation, we obtain that $JP(\boldsymbol{x})$ is symmetric and every of its principal minors is nonnegative. Hence, $JP(\boldsymbol{x})$ is positive semidefinite for all $\boldsymbol{x} \in \mathbb{R}^4_+$. So by Proposition 2.2 and Definition 3.1, $\mathcal{A}$ is strictly positive semidefinite on $\boldsymbol{x} \in \mathbb{R}^4_+$. Clearly,

$$\mathcal{A}\boldsymbol{e}^5 > 0.$$

By Theorem 3.8, the TCP$(\mathcal{A}, \boldsymbol{q})$ in Example 6.4 is solvable for any $\boldsymbol{q} \in \mathbb{R}^6$.

Example 6.5. Consider the TCP$(\mathcal{A}, \boldsymbol{q})$, where $\mathcal{A} \in \mathbb{R}^{[m,n]}$ is randomly generated with the *rand* function in MATLAB ($m = 3$, n is a given integer), and entries of $\boldsymbol{q} \in \mathbb{R}^n$ are randomly selected from the interval $[-1, 1]$. In order to make sure $\mathcal{A}$ is a strictly positive semidefinite tensor, we revise the i-th diagonal entries of $\mathcal{A}$ to the summation of all the non-diagonal entries in the i-th row ($i = 1, 2, \ldots, 20$). Then we use the proposed algorithm to solve the TCP$(\mathcal{A}, \boldsymbol{q})$ and the numerical results are reported in Table 5.

The numerical results reported in Tables 1–5 show that our algorithm works well for all the tested problems. Our algorithm can stop in a relatively small number of iterations with a very small residual and the cost of CPU time is very minimal. Surprisingly, the algorithm proposed in this paper can find a nonzero solution, if it exists, for $\boldsymbol{q} \geqslant \boldsymbol{0}$; it can also report the case of no solution (see Example 6.2). We also use Algorithm 1 to solve Examples 6.1–6.4 with the randomly generated $\boldsymbol{q}$, and the performance is very similar to the given $\boldsymbol{q}$. We omit the report. Furthermore, we run it on the randomly generated Example 6.5 and the numerical results in Table 5 show that Algorithm 1 has good performance on large-scale problems.

Table 1 Numerical results of the TCP in Example 6.1

$\boldsymbol{q}$	$\boldsymbol{x}$-SOL	Iter	NH	NormH	Time (s)
$[-10, 0]$	$[2.0976, 0.6397]$	8	9	1.7764E−15	2.8366E−02
$[0, -5]$	$[0.5077, 1.6648]$	8	9	8.8818E−16	2.9254E−02
$[-7, -1]$	$[1.8240, 0.7709]$	8	9	9.1551E−16	2.7534E−02
$[2, -9]$	$[0.2226, 2.0724]$	8	9	1.8310E−15	2.6477E−02
$[-8, 3]$	$[2, 0]$	7	10	2.7756E−16	2.9343E−02

Table 2 Numerical results of the TCP in Example 6.2

$\boldsymbol{q}$	$\boldsymbol{x}$-SOL	Iter	NH	NormH	Time (s)
$[0, 9]$	$[0, 0]$	11	17	2.7893E−17	2.9882E−02
$[5, 3]$	$[0, 0]$	10	12	5.2995E−31	2.4598E−02
$[-12, 9]$	$[3, 2]$	9	11	1.0596E−23	2.3291E−02
$[2, -3]$		—It has no solution—			0.3424
$[-8, -5]$		—It has no solution—			0.3513

Table 3 Numerical results of the TCP in Example 6.3

$\boldsymbol{q}$	$\boldsymbol{x}$-SOL	Iter	NH	NormH	Time (s)
$[0, 5, 8, 9]$	$[3.773\text{E}{-04}, 0.0000, 0.0000, 0.0000]$	20	20	5.3727E−11	5.9401E−02
$[0, -5, 8, -0.5]$	$[0, 1.7100, 0, 0.7937]$	9	10	8.9509E−16	3.0340E−02
$[-7, -1, 0.01, 0]$	$[1.9001, 0.2714, 0, 0.0001]$	24	25	4.7314E−14	6.8719E−02
$[12, -1, -28, 0]$	$[0, 1.0000, 3.0366, 0]$	10	14	7.1055E−15	3.4389E−02
$[-8, 15, 0, -23]$	$[2.0000, 0, 0, 2.8439]$	9	11	3.9464E−17	2.9046E−02

Table 4 Numerical results of the TCP in Example 6.4

$\boldsymbol{q}$	$\boldsymbol{x}$-**SOL**	**Iter**	**NH**	**NormH**	**Time** (s)
$[-10, -100, -1000, -2, 5, -9]$	$[0.26, 1.89, 3.20, 0.01, 0, 1.55]$	10	19	1.1721E−13	9.0945E−02
$[-8, 15, 0, -23, 0, 0]$	$[1.52, 0, 0, 1.87, 0, 0]$	10	14	1.0806E−14	8.5456E−02
$[-7, -1, 0.01, 0, -8, -10]$	$[1.47, 0.20, 0, 0, 1.52, 1.58]$	29	38	9.8348E−14	2.7442E−01
$[12, -1, -28, 0, 0, 0]$	$[0, 0.76, 1.56, 0, 0, 0]$	9	11	2.3001E−14	8.3025E−02

Table 5 Numerical results of the TCP in Example 6.5

n	**Iter**	**NH**	**NormH**	**Time** (s)	$\boldsymbol{x}^{\mathrm{T}}(\mathcal{A}\boldsymbol{x}^{m-1}+\boldsymbol{q})$
8	10	13	1.2518E−13	2.4997E−02	−4.6640E−14
12	13	17	3.9740E−16	3.2583E−02	−5.8251E−17
16	14	26	2.1744E−14	3.7189E−02	−5.4285E−15
20	14	25	2.8631E−14	5.5825E−02	−6.4891E−15

Example 6.6 (Multi-person noncooperative game)**.** Consider the 3-person noncooperative game, where there are three players, player 1 has 2 pure strategies, player 2 has 3 pure strategies and player 3 has 2 pure strategies. Three payoff tensors $\mathcal{A}^{(1)}, \mathcal{A}^{(2)}, \mathcal{A}^{(3)} \in \mathbb{R}^{2\times 3\times 2}$ are given by

$$\mathcal{A}^{(1)}(:,:,1) = \begin{pmatrix} 0.9395 & 0.4731 & 0.3431 \\ 0.6007 & 0.5832 & 0.3720 \end{pmatrix}, \quad \mathcal{A}^{(1)}(:,:,2) = \begin{pmatrix} 0.7080 & 0.9845 & 0.8328 \\ 0.5683 & 0.0159 & 0.8938 \end{pmatrix},$$

$$\mathcal{A}^{(2)}(:,:,1) = \begin{pmatrix} 0.6276 & 0.5103 & 0.0484 \\ 0.8019 & 0.6605 & 0.0797 \end{pmatrix}, \quad \mathcal{A}^{(2)}(:,:,2) = \begin{pmatrix} 0.9473 & 0.7309 & 0.4521 \\ 0.5772 & 0.2621 & 0.0573 \end{pmatrix},$$

$$\mathcal{A}^{(3)}(:,:,1) = \begin{pmatrix} 0.5823 & 0.6985 & 0.3337 \\ 0.0169 & 0.2989 & 0.4609 \end{pmatrix}, \quad \mathcal{A}^{(3)}(:,:,2) = \begin{pmatrix} 0.3019 & 0.8219 & 0.0009 \\ 0.3335 & 0.8720 & 0.8289 \end{pmatrix}.$$

We give an illustration to explain the elements of the above tensors. For example, 0.3720 in $\mathcal{A}^{(1)}(:,:,1)$ denotes the payoff of player 1 when player 1 plays his 2nd pure strategy, player 2 plays his 3rd strategy, and player 3 plays his 1st strategy.

Using $\mathcal{A}^{(1)}$, $\mathcal{A}^{(2)}$ and $\mathcal{A}^{(3)}$, we generate $\mathcal{A} \in \mathbb{R}^{[3,7]}$ and $\boldsymbol{q} = -\boldsymbol{e} \in \mathbb{R}^7$ via [7, Equation (5)]. Thus, the corresponding TCP model of Example 6.6 is obtained. We use Algorithm 1 to solve this TCP and obtain a solution

$$\boldsymbol{y}_* = (0.6236, 0.0000, 3.8388, 0.0000, 0.0000, 4.3058, 0.0000)^{\mathrm{T}}$$

and the corresponding vector

$$\mathcal{A}\boldsymbol{y}_*^2 + \boldsymbol{q} = (0.0000, 5.6000, 0.0000, 0.3150, 1.5553, 0.0000, 0.6713)^{\mathrm{T}}$$

with 13 iterative steps in 0.1320 seconds. Via [7, Theorem 1], we obtain that a Nash equilibrium point of the concerned game is $\boldsymbol{x}_* = (\boldsymbol{x}_*^{(1)}, \boldsymbol{x}_*^{(2)}, \boldsymbol{x}_*^{(3)})^{\mathrm{T}}$ with

$$\boldsymbol{x}_*^{(1)} = (1.0000, 0.0000)^{\mathrm{T}}, \quad \boldsymbol{x}_*^{(2)} = (1.0000, 0.0000, 0.0000)^{\mathrm{T}}, \quad \boldsymbol{x}_*^{(3)} = (1.0000, 0.0000)^{\mathrm{T}}.$$

Obviously, this is a Nash equilibrium of the pure strategy. In the special game given by Example 6.6, player 2 has a dominant strategy. From his payoff tensor $\mathcal{A}^{(2)}$, we can see that no matter what strategy player 1 and player 3 play, player 2 is dominant in this game if he plays his 1st strategy. Thus, it follows from the theory of the Nash equilibrium that $\boldsymbol{x}_*^{(2)} = (1, 0, 0)^{\mathrm{T}}$. It is dominant for player 1 to play his 1st strategy no matter what strategy player 3 plays. Furthermore, player 3 will play his 1st strategy to maximize his profits. In fact, we can guess the Nash equilibrium for this game with the dominant strategy. Therefore, the numerical results are true.

7 Conclusions

In this paper, we first introduce the concept of a strictly positive semidefinite tensor on $\mathbb{R}^n_+$, and we consider the solvability of the TCP$(\mathcal{A}, \boldsymbol{q})$ when the involved tensor $\mathcal{A}$ is strictly positive semidefinite on $\mathbb{R}^n_+$. We prove that the solution set of such a TCP$(\mathcal{A}, \boldsymbol{q})$ is nonempty and compact for any $\boldsymbol{q} \in \mathbb{R}^n$ if it is feasible. This result is novel for the special instance of monotone NCPs.

We then construct a Newton-type algorithm based on the Huber function for solving such a TCP$(\mathcal{A}, \boldsymbol{q})$ by using a smoothing technique. The convergence results discussed in this paper are very favorable. The numerical results show that our algorithm works well for the tested problems. With regard to our nice theoretical results of our algorithm, the computational results reported are very encouraging. Since the involved tensor $\mathcal{A}$ in the TCP$(\mathcal{A}, \boldsymbol{q})$ is strictly positive semidefinite on $\mathbb{R}^n_+$ and it is not so on the whole space $\mathbb{R}^n$, the existing smoothing Newton algorithms in the literature for TCPs cannot be directly used to solve such a TCP$(\mathcal{A}, \boldsymbol{q})$. The novelty of our algorithm is to adopt the Huber function and the normal equation.

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