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The sign and the co-monotonicity of Z for a class of decoupled FBSDEs: Theory and applications ☆

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ABSTRACT

In this paper, we investigate the properties of the control process (z) in a class of backward stochastic differential equation (BSDE) with Markovian terminal data driven by a forward stochastic differential equation (SDE). We focus on determining the sign of (z), and find that it is determined via certain information about the BSDE driver and the terminal data. Notably, we explore the co-monotonicity property of (z) when the terminal value of the BSDE within the forward–backward SDE (FBSDE) system is non-monotonic. Three applications are also showed in the paper. First, we provide a sufficient condition ensuring that the nonlinear g-expectation is additive. Second, we obtain the explicit solution for a class of nonlinear BSDEs. Third, we offer a closed form representation for the standard aggregator utility under ambiguity, as discussed in Chen and Epstein (2002).

1. Introduction

Pardoux and Peng (1990) [1] have shown that under some proper conditions on the non-linear driver g and the terminal data ξ, there exists a unique pair of adapted and square integrable solutions (y, z) to the following backward stochastic differential equations (BSDEs for short)

$$y_t = \xi + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s dW_s, \quad t \in [0, T]. \tag{1.1}$$

Since their seminal work, the theory of BSDEs has been studied by many researchers and many remarkable results about the solution pair (y, z) have been obtained. Over the past two decades, BSDEs have found applications in various fields, including mathematical finance, stochastic control, and partial differential equations (PDEs), see for instance El Karoui, Peng and Quenez [2], El Karoui and Quenez [3], Chen and Epstein [4], Pardoux and Zhang [5], Darling [6], Ma, Protter and Yong [7] and etc. However, the precise information about the solution pair (y, z) remains to be fully explored. Indeed, there are few results on the solution (z), which plays an important role in applications to quantitative finance. In fact, in BSDE models of financial derivatives, (z) is interpreted as the volatility. Therefore, accurately determining the properties of (z), such as whether it is positive, is of significant importance.

In the paper by Chen, Kulperger and Wei [8], the authors established a co-monotonicity theorem for the control process (z). Specifically, they examined BSDEs with terminal data $\xi = \Phi(X_T)$, where Φ is a monotone function on $\mathbb{R}$, and $(X_s)_{s\in[t,T]}$ is the solution to the following forward stochastic differential equation (SDE):

$$\begin{cases} dX_s = b(s, X_s)ds + \sigma(s, X_s)dW_s, \\ X_t = x, \quad x \in \mathbb{R}, \quad s \in [t, T]. \end{cases} \tag{1.2}$$

They showed that if (y^1, z^1), (y^2, z^2) are solutions of the BSDE (1.1) corresponding to the terminal values $\xi_1 = \Phi_1(X_T^1)$ and $\xi_2 = \Phi_2(X_T^2)$ respectively, then $z^1 \cdot z^2 \geq 0$ if Φ_1 and Φ_2 are both increasing or both decreasing on $\mathbb{R}$. Subsequently, G. Dos Reis and R.J.N. Dos Reis [9] studied solutions of Forward–Backward SDEs (FBSDE) with drivers that grow quadratically in the control component (z). They found that when the terminal values of BSDEs are co-monotonic, the control process (z) are also co-monotonic, provided the terminal Φ is monotone. Recently, R. Likibi Pellat and O. Menoukeu Pamen [10] explored solutions to coupled quadratic FBSDEs under weaker conditions on the drift coefficient of the forward component. They derived a co-monotonicity theorem for the control variable, extending the works [8,9].

However, all the previous works mentioned above have focused on monotone terminal values. By applying comparison theorem for FBSDEs, they obtained the desired results. When the terminal is not monotone, comparison theorem does not work any more. More recently, Chen, Liu, Qian and Xu [11] determined the sign of (z) and obtained the explicit solutions for certain nonlinear BSDEs with non-monotone terminal values ξ. Specifically, they considered the case

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where $\xi = \Phi(W_T)$, with $\Phi(x)$ being symmetric about a point c and monotonic on $\{x > c\}$, such as $\Phi(x) = x^2$ with $c = 0$. By analyzing the corresponding PDEs, they demonstrated that $\operatorname{sgn}(z_t) = \operatorname{sgn}(W_t - c)$ under certain regularity conditions on Φ.

The objective of this paper is to investigate the sign of (z) for a more general terminal $\xi = \Phi(X_T)$, where $(X_t)_{t\in[0,T]}$ is given by (1.2). The SDE (1.2) and the BSDE (1.1) together form a decoupled FBSDE system. For general results on the well-posedness of FBSDEs, see [12–15] et al. In this paper, we aim to extend the co-monotonic theorem for BSDEs in [8] to the case where Φ_1 and Φ_2 may not be monotone. Since the comparison theorem for BSDEs is no longer applicable in these scenarios, we address this challenge by carefully analyzing the corresponding PDEs derived via the Feynman–Kac formula. By constructing a martingale and employing stopping techniques, we determine the sign of the solutions to the corresponding PDEs. Specifically, we deal with the case where the terminal value $\xi = \Phi(X_T)$, with Φ being a 'piecewise symmetric' function having more than one symmetric points, such as $\Phi(x) = \sin x$. We show that the sign of z in BSDE (1.1) depends on Φ' but is independent of g under some suitable conditions. This allows us to analyze the co-monotonicity of (z) within a more general framework. Suppose (y^1, z^1), (y^2, z^2) are solutions of BSDE (1.1) corresponding to terminal values $\xi_1 = \Phi_1(X_T^1)$ and $\xi_2 = \Phi_2(X_T^2)$ respectively, where Φ_1 and Φ_2 are general co-monotonic functions as defined in Definition 4.1. Assume X_t^i $(i = 1, 2)$ are the unique solution of the following SDEs,

$$\begin{cases} dX_s^i = b_i(s, X_s^i)ds + \sigma_i(s, X_s^i)dW_s, \\ X_0^i = x_i, \quad x_i \in \mathbb{R}. \end{cases}$$

Then, under some technical conditions on Φ_i, b_i and σ_i $(i = 1, 2)$, we show that

$$z_t^1 \cdot z_t^2 \geq 0, \text{ a.e. } t \in [0, T].$$

We also discuss three applications of our main results in the last part of the paper. First, we discuss the so-called g-expectation of ξ (denoted by $\mathbb{E}_g[\xi]$ in literature), defined as y_0, where y_t is the solution of BSDE (1.1) with driver g. Unlike ordinary probability expectation, the g-expectation is typically non-additive due to the nonlinearity of g. This concept has significant applications in quantitative finance and has been studied by various researchers, including El Karoui, Peng and Quenez [2], Briand, Coquet, Hu, Memin and Peng [16], Chen and Epstein [4], Coquet, Hu, Memin and Peng [17] and Rosazza Gianin [18] and the literature therein. In this paper, we show that g-expectation $\mathbb{E}_g$ is *additive* for some nonlinear g. Specifically, if the terminal values ξ_1 and ξ_2 are co-monotonic in our sense (see Definition 4.1), and g satisfies a technical condition, then $\mathbb{E}_g[\xi]$ becomes additive. Second, by using the result on the sign of z, we give the explicit solution of the nonlinear BSDE when the terminal value $\xi = \Phi(X_T)$ and the generator $g(s, y, z) = k|z|$. The third application pertains to the standard additive utility, as discussed in works such as Duffie and Epstein [19]. Chen and Epstein [4] considered this utility under ambiguity and derived a closed-form expression. We use our results on the sign of (z) to obtain a more explicit representation of the standard aggregator utility for specific multiple priors.

The paper is organized as follows. In Section 2, we collect some results of BSDEs. In Section 3, we show the relationship between the sign of z and the terminal value ξ, In Section 4, we give the definition of general co-monotonic considered in this paper, and show the co-monotonic theorem. The applications of the main results, sufficient conditions for the additivity of g-expectation, the explicit solution of a nonlinear BSDE with a Markovian terminal condition and the explicit form of the standard aggregator utility under ambiguity, are given in Section 5.

2. Preliminary notions

In this section, we briefly recall some basic notions and results about BSDEs and establish notations as well, the reader may refer to Pardoux and Peng [1] for more information. Let $L^2(0, T; \mathbb{R})$ be the set of Lebesgue function $\varphi : [0, T] \to \mathbb{R}$ such that $\int_0^T |\varphi(t)|^2 dt < \infty$, and $L^2(\Omega, \mathcal{F}_t, P)$ denote the space of $\mathcal{F}_t$-measurable and square integrable random variables on $(\Omega, \mathcal{F}, P)$ for each $t \geq 0$. Denote

$$\mathcal{S}^2(0, T; \mathbb{R}) := \left\{ (y_t)_{t\in[0,T]} : \text{continuous } (\mathcal{F}_t)\text{-adapted process with } E\left[\sup_{t\in[0,T]} |y_t|^2\right] < \infty \right\}.$$

and

$$\mathcal{M}^2(0, T; \mathbb{R}) := \left\{ (z_t)_{t\in[0,T]} : \text{real valued } (\mathcal{F}_t)\text{-adapted process with } E\left[\int_0^T |z_t|^2 dt\right] < \infty \right\}.$$

The following assumptions will be enforced throughout the paper.

(A1) There exists a constant C such that

$|b(t, x) - b(t, x')|, |\sigma(t, x) - \sigma(t, x')| \leq C|x - x'|$

for $x,\ x' \in \mathbb{R}$, and moreover

$|b(\cdot, 0)| + |\sigma(\cdot, 0)| \in L^2(0, T; \mathbb{R}), \quad \forall T > 0.$

(A2) There exists a constant C such that

$|g(t, y, z) - g(t, y, z')| \leq C(|y - y'| + |z - z'|)$

for $y,\ y' \in \mathbb{R}$ and $z,\ z' \in \mathbb{R}$.

(A3) $\int_0^T |g(s, y, z)|^2 ds < \infty$ for any $(y, z) \in \mathbb{R} \times \mathbb{R}$.

Under the conditions *(A2), (A3)*, if moreover $\xi \in L^2(\Omega, \mathcal{F}_T, P)$, Pardoux and Peng [1] showed that BSDEs (2.1) admits a unique solution, i.e., there is a pair of adapted processes $(y, z) \in \mathcal{S}^2(0, T; \mathbb{R}) \times \mathcal{M}^2(0, T; \mathbb{R})$ such that

$$y_t = \xi + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s dW_s. \tag{2.1}$$

In this paper, we mainly consider the case when $\xi = \Phi(X_T)$, where $(X_s)_{s\in[t,T]}$ is the solution of the following SDE

$$\begin{cases} dX_s = b(s, X_s)ds + \sigma(s, X_s)dW_s, \\ X_t = x, \quad s \in [t, T]. \end{cases} \tag{2.2}$$

and $\Phi(x)$ is a continuous function defined on $\mathbb{R}$ such that $\Phi(X_T) \in L^2(\Omega, \mathcal{F}_T, P)$. That is, we consider the following BSDE:

$$y_t = \Phi(X_T) + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s dW_s. \tag{2.3}$$

We know that under condition (A1), SDE (2.2) admits a unique strong solution in $L^p(\Omega, \mathcal{F}_T, P)$ $(p \geq 1)$; and BSDE (2.3) admits a unique solution under condition (A2) and (A3).

Let $C^{m,n}(\mathbb{R} \times \mathbb{R})$ denote the set of functions $\phi(x, y) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that the mth order partial derivative with respect to x and the nth order partial derivative with respect y exist and are continuous, and $C_b^{m,n}(\mathbb{R} \times \mathbb{R})$ denotes the space of those functions with bounded partial derivatives. Suppose $b(t, \cdot)$, $\sigma(t, \cdot)$ and $\Phi(\cdot) \in C^3(\mathbb{R})$, $g(t, \cdot, \cdot) \in C^{1,3,3}([0, T] \times \mathbb{R} \times \mathbb{R})$. Let $u(t, x)$ be the unique solution of the following quasi-linear partial differential equations (PDEs) for a fixed $T > 0$:

$$\begin{cases} \partial_t u(t, x) + \mathcal{L}u(t, x) + g\left(t, u(t, x), \sigma(t, x)\partial_x u(t, x)\right) = 0, \\ u(T, x) = \Phi(x), \quad (t, x) \in (0, T] \times \mathbb{R}, \end{cases} \tag{2.4}$$

where

$$\mathcal{L}u(t, x) := \frac{1}{2}\sigma^2(t, x)\partial_{xx}u(t, x) + b(t, x)\partial_x u(t, x).$$

Define $y_t = u(t, X_t)$ and $z_t = \sigma(t, X_t) \cdot \partial_x u(t, X_t)$. Then by Feynman–Kac's formula, we know that $(y_t, z_t)_{t\in[t,T]}$ solves BSDE (2.3), see [2] for more details.

3. The sign of Z

In this section, we study the sign of z of the following decoupled FBSDE system:

$$y_t = \Phi(X_T) + \int_t^T g(s, y_s, z_s)ds - \int_t^T z_s dW_s, \tag{3.1}$$

where $(X_s)_{s\in[t,T]}$ is the solution of SDE (2.2). We will show that the sign of z has a connection with the terminal value $\xi = \Phi(X_T)$ and is independent of g under some conditions.

Denote $w(t,x) := \partial_x u(t,x)$. Now we give the first main result.

Theorem 3.1. *Suppose that $(A1)$–$(A3)$ hold, and*

(i) $\Phi^{(m)}$ $(m = 0,1,2,3)$ possess at most polynomial growth, $g(t,y,z) \in C_b^{1,3,3}(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R})$;

(ii) $\mathbb{D} := \{d_j, j = 0, \pm1, \pm2, \ldots\}$ is a nodal set of $w(t,x)$. This is, $w(t,x)|_{x\in\mathbb{D}} = 0$ for every $t \in [0,T]$.

Then the following conclusions hold:

(i) *If $\Phi(x)$ is monotone on $[d_j, d_{j+1}]$, then $z_t \cdot \sigma(t,X_t)\Phi'(X_t) \ge 0$.*

(ii) *If $\Phi(x)$ is strictly monotone on $[d_j, d_{j+1}]$, then $sgn(z_t) = sgn(\sigma(t,X_t)\Phi'(X_t))$.*

Remark 3.2. It should be noted that when z is high-dimensional, by defining $z \odot x := (z_1x_1, z_2x_2, \ldots, z_dx_d)$, where $z \odot x \ge 0$ means $z_ix_i \ge 0$ and $z \odot x > 0$ means $z_ix_i > 0$, the results in this paper still hold. So without loss of generality, we only consider the case where z is one-dimensional.

We now introduce some assumptions regarding the functions Φ, b and σ under which assumption (ii) in Theorem 3.1 is satisfied. Assume that Φ is a 'piecewise symmetric' function. Specifically, there exists an increasing sequence

$$\cdots d_{-n} < \cdots < d_{-2} < d_{-1} < d_0 < d_1 < d_2 < \cdots < d_n < \cdots$$

such that $\Phi(x)$ in the BSDE (3.1), and $b(t,x)$ and $\sigma(t,x)$ in the SDE (2.2), satisfy the following condition for $t \in [0,T]$:

(H) $\Phi(d_j - x) = \Phi(d_j + x)$, $b(t, d_j - x) = -b(t, d_j + x)$, $\sigma(t, d_j - x) = \sigma(t, d_j + x)$ for $x \in \mathbb{R}$, $j = 0, \pm1, \pm2, \ldots$

Then we have the following result:

Theorem 3.3. *Under assumptions (H) and $(A1)$–$(A3)$, if moreover, $\Phi(x) \in C^3(\mathbb{R})$ and $\Phi^{(m)}$ $(m = 0,1,2,3)$ possess at most polynomial growth; $g(t,y,z) = g(t,y,-z)$ and $g(t,y,z) \in C_b^{1,3,3}(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R})$. Then*

(i) *$z_t \cdot \sigma(t,X_t)\Phi'(X_t) \ge 0$ if $\Phi(x)$ is monotone on $[d_j, d_{j+1}]$.*

(ii) *$sgn(z_t) = sgn(\sigma(t,X_t)\Phi'(X_t))$ if $\Phi(x)$ is strictly monotone on $[d_j, d_{j+1}]$.*

Proof. Under assumption (H) and $g(t,y,z) = g(t,y,-z)$, we can verify that $u(t, d_j - x)$ and $u(t, x + d_j)$ both are the solutions of PDE (2.4). By the theory of PDEs, Eq. (2.4) has a unique solution under the regularity conditions on Φ, b, σ and g. Then we have $u(t, d_j - x) = u(t, d_j + x)$ for $x \in \mathbb{R}$, which means $u(t,x)$ is an even function about $x = d_j$ $(j = 0, \pm1, \pm2, \ldots)$. Since $w(t,x) = \partial_x u(t,x)$, we have $w(t, d_j) = \partial_x u(t,x)|_{x=d_j} = 0$, for all $t \in [0,T]$. Then the results follow from Theorem 3.1. □

Example 3.4. When the terminal value ξ of BSDEs (3.1) is $\xi = \Phi(W_T) = \cos(W_T)$. We know $\Phi(x) = \cos x$ is symmetric with respect to $x = d_k = k\pi$, $k \in \mathbb{N}$. Then if g of BSDE (3.1) satisfies the conditions of Theorem 3.3, the solution z of BSDE (3.1) satisfies $\mathrm{sgn}(z_t) = \mathrm{sgn}(-\sin(W_t))$, *a.e.* $t \in [0,T]$.

The following lemma is important when proving Theorem 3.1.

Lemma 3.5. *Under the assumptions of Theorem 3.1, the solution of PDE (2.4) can be represented as*

$$\partial_x u(t,x) = E\left[N_T \Phi'(X_T) e^{\int_t^T D(r,X_r)dr} 1_{\{T<\tau_j\}}\right],$$

where j is chosen such that $X_t = x \in [d_j, d_{j+1}]$, $D(s,X_s)$ and $G(s,X_s)$ are given by (3.2) and (3.3) respectively,

$$\tau_j = \inf\left\{s \ge t,\ X_s = d_j \text{ or } d_{j+1}\right\} \quad (j = 0, \pm1, \pm2, \ldots), \text{ and}$$

$$N_s = \exp\left\{\int_t^s G(r,X_r)dW_r - \int_t^s \frac{1}{2}G(r,X_r)^2 dr\right\}.$$

Proof. Since $g(t,y,z) \in C_b^{1,3,3}(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R})$, $\Phi(\cdot) \in C^3(\mathbb{R})$, $b(\cdot), \sigma(\cdot) \in C^{1,3}(\mathbb{R})$ and $\Phi^{(i)}$ (where $i = 0,1,2,3$) possess at most polynomial growth, the unique solution $u(t,x)$ to PDE (2.4) belongs to $C^{1,3}([0,T]\times\mathbb{R})$, see for example Friedman [20]. In particular we have $\partial_x u(t,x) \in C^{1,2}([0,T]\times\mathbb{R})$.

Let us first consider the case when $\Phi^{(i)}$ (where $i = 0,1,2,3$) are bounded. For this case, the second order derivative of $u(t,x)$, that is, $\partial_x w(t,x)$ is bounded in $[0,T]\times\mathbb{R}$. For $0 \le s \le t$, set

$$a_s = \partial_y g\left(s, u(s,X_s), \sigma(s,X_s)w(s,X_s)\right),$$
$$b_s = \partial_z g\left(s, u(s,X_s), \sigma(s,X_s)w(s,X_s)\right).$$

Then, taking derivative of PDE (2.4) with respect to x and using Itô's formula,

$$\begin{aligned} dw(s,X_s) &= \left(\partial_s w(s,X_s) + \frac{1}{2}\sigma^2(s,X_s)\cdot\partial_{xx}^2 w(s,X_s) + \partial_x w(s,X_s)\cdot b(s,X_s)\right)ds \\ &\qquad + \partial_x w(s,X_s)\cdot\sigma(s,X_s)dW_s \\ &= \Big\{-\left[\partial_x b(s,X_s) + a_s + b_s\cdot\partial_x\sigma(s,X_s)\right]\cdot w(s,X_s) \\ &\qquad - \left[\sigma(s,X_s)\partial_x\sigma(s,X_s) + b(s,X_s) + b_s\cdot\partial_x\sigma(s,X_s)\right]\cdot\partial_x w(s,X_s)\Big\} \\ &\quad \times ds \\ &\quad + \partial_x w(s,X_s)\cdot\sigma(s,X_s)dW_s. \end{aligned}$$

Let $M_s = q_s N_s w(s,X_s)$, where

$$\begin{cases} dq_s = D(s,X_s)q_s ds,\ q_t = 1, \\ dN_s = N_s G(s,X_s)dW_s,\ N_t = 1, \end{cases} \qquad s \in [t,T].$$

with

$$\begin{aligned} D(s,x) :=& \partial_x b(s,x) + \partial_y g(t,u(t,x),\sigma(s,x)\partial_x u(t,x)) \\ &+ \partial_z g(t,u(t,x),\sigma(s,x)\partial_x u(t,x))\partial_x\sigma(s,x), \end{aligned} \tag{3.2}$$

$$G(t,x) := \partial_x\sigma(s,x) + \partial_z g(t,u(t,x),\sigma(t,x)\partial_x u(t,x)). \tag{3.3}$$

Therefore,

$$\begin{aligned} dM_s &= q(s)d\left[N_s w(s,X_s)\right] + N_s w(s,X_s)D(s,X_s)q(s)ds \\ &= q(s)N_s dw(s,X_s) + q(s)w(s,X_s)N_s G(s,X_s)dW_s \\ &\qquad + q(s)N_s G(s,X_s)\sigma(s,X_s)\partial_x w(s,X_s)ds \\ &\qquad + q(s)N_s w(s,X_s)D(s,X_s)ds. \end{aligned}$$

Substituting $dw(s,X_s)$ into dM_s, by the definition of $D(s,x)$ and $G(s,x)$, we obtain that

$$dM_s = [q(s)N_s\partial_x w(s,X_s)\cdot\sigma(s,X_s) + q(s)N_s w(s,X_s)G(s,X_s)]dW_s.$$

We now claim that M is a square integral martingale. In the following illustration, C will denote a constant whose values vary from line to line, which dependent on t and independent of $s \le t$. Since $\sigma(t,\cdot), b(t,\cdot) \in C_b^1(\mathbb{R})$ and $g(t,y,z) \in C_b^{1,3,3}(\mathbb{R}_+ \times \mathbb{R} \times \mathbb{R})$, it holds that

$$|q(s)N_s\partial_x w(s,X_s)\cdot\sigma(s,X_s) + q(s)N_s w(s,X_s)G(s,X_s)| \le C|N_s|,$$

and

$$E[|N_s|^2] = E\left[\exp\left\{2\int_t^s G(r,X_r)dW_r - \int_t^s |G(r,X_r)|^2 d_r\right\}\right]$$

$$\leq CE\left[\exp\left\{2\int_t^s G(r,X_r)dW_r - 2\int_t^s |G(r,X_r)|^2 d_r\right\}\right]$$
$$= C < +\infty.$$

Then,

$$E|M_s|^2 = E\left\{M_t + \int_t^s [q(r)N_r\partial_x w(r,X_r)\cdot\sigma(r,X_r) + q(r)N_r w(r,X_r) \times G(r,X_r)]dW_r\right\}^2$$
$$\leq 2E\left\{\int_t^s [q(r)N_r\partial_x w(r,X_r)\cdot\sigma(r,X_r) + q(r)N_r w(r,X_r) \times G(r,X_r)]^2 dr\right\} + 2E[M_t^2]$$
$$\leq C$$

Thus, M_s is a square integrable martingale up to time t. Since

$\tau_j = \inf\{s \geq t,\ X_s = d_j \text{ or } d_{j+1}\}$

is a stopping time, finite almost surely (see (1.2) in [21]), by stopping theorem for martingales, we have $E(M_t) = E\left(M_{T\wedge\tau_j}\right)$. Due to $w(s,d_j) = 0$ for all $s \in [t,T]$ $(j = 0, \pm 1, \pm 2, \ldots)$, then

$$w(t,x) = E\left[q(T\wedge\tau_j)N_{T\wedge\tau_j} w(T\wedge\tau_j, X_{T\wedge\tau_j})\right]$$
$$= E\left[N_T\Phi'(X_T)e^{\int_t^T D(r,X_r)dr}1_{\{T<\tau_j\}}\right] + E\left[q(\tau_j)N_{\tau_j} w(\tau_j, X_{\tau_j})1_{\{\tau_j\leq T\}}\right]$$
$$= E\left[N_T\Phi'(X_T)e^{\int_t^T D(r,X_r)dr}1_{\{T<\tau_j\}}\right].$$

For the case that $\Phi^{(i)}$ $(i = 0,1,2,3)$ possess polynomial growth, we can obtain the representation by the simple approximation procedure. The proof is complete. □

Now we give the proof of Theorem 3.1.

Proof of Theorem 3.1. By the illustration in Section 2, let $y_t = u(t,X_t)$, $z_t = \sigma(X_t)\cdot\partial_x u(t,X_t)$, where $u(t,x)$ is the unique solution of PDEs (2.4). Then (y_t,z_t) solve BSDEs (2.3), see [2] for more details.

Set $w(t,x) := \partial_x u(t,x)$, by Lemma 3.5,

$$w(t,x) = E\left[N_T\Phi'(X_T)e^{\int_t^T D(r,X_r)dr}1_{\{t<\tau_j\}}\right].$$

Due to the definition of τ_j, we know $X_T \in [d_j, d_{j+1}]$ on $\{T < \tau_j\}$ when $x \in [d_j, d_{j+1}]$. Since $\Phi(x)$ is monotone on $[d_j, d_{j+1}]$, we have $\Phi'(X_T) \geq 0$ (or ≤ 0) on $\{T < \tau_j\}$ if $\Phi(x)$ is increasing (or decreasing) on $[d_j, d_{j+1}]$. Thus, $w(t,x) = E\left[N_T\Phi'(X_T)e^{\int_t^T D(r,X_r)dr}1_{\{T<\tau_j\}}\right] \geq 0$ (or ≤ 0). From this we have $w(t,x)\Phi'(x) \geq 0$ for all $(t,x) \in (0,T]\times\mathbb{R}$, which means

$w(t,X_t)\cdot\Phi'(X_t) \geq 0,\quad a.e.\ t\in[0,T].$

Therefore, $z_t\cdot\sigma(t,X_t)\Phi'(X_t) = \sigma^2(t,X_t)w(t,X_t)\cdot\Phi'(X_t) \geq 0$.

Furthermore, if $\Phi(x)$ is strictly monotone on $[d_j, d_{j+1}]$, then $\text{sgn}(\Phi'(x)) = \text{sgn}\left(\Phi'(X_T^x)\right)$ on $\{T < \tau_j\}$. Since $w(t,x) = E\left[N_T\Phi'(X_T^x)e^{\int_t^T D(r,X_r)dr}1_{\{T<\tau_j\}}\right]$, it is not difficult to verify that sgn $(w(t,x)) = \text{sgn}(\Phi'(x))$ for all $(t,x) \in (0,T]\times\mathbb{R}$, which means, sgn $(w(t,X_t)) = \text{sgn}(\Phi'(X_t))$, $a.e.$ $t \in [0,T]$. Therefore, $\text{sgn}(z_t) = \text{sgn}$ $(\sigma(t,X_t)w(t,X_t)) = \text{sgn}(\sigma(t,X_t))\text{sgn}(\Phi'(X_t))$. The proof is complete. □

4. Co-monotonicity of z

In this section, we shall establish a more general co-monotonicity theorem for z based on the study in the last section. Let us first generalize the definition of the co-monotonicity, which was initially introduced in [8].

Definition 4.1 (*Generally Co-monotonic*)**.** Two functions $\Phi,\Psi : \mathbb{R}\to\mathbb{R}$ are said to be co-monotonic on set A if Φ is increasing (resp. decreasing) on A, then Ψ is also increasing (resp. decreasing) on A. If there exist subsets $A_1, A_2, \ldots$ such that Φ and Ψ are co-monotonic on each subset A_i for $i \geq 1$, then Φ and Ψ are said to be generally co-monotonic on the set $A = A_1 \cup A_2 \cup \cdots$.

Furthermore, $\Phi,\Psi : \mathbb{R}\to\mathbb{R}$ are said to be strictly co-monotonic on a set A if Φ is strictly increasing (resp. strictly decreasing) on A, then Ψ is strictly increasing (resp. strictly decreasing) on A. They are said to be strictly generally co-monotonic on the set $A = A_1\cup A_2\cup\cdots$, if Φ,Ψ are strictly co-monotonic on each subset A_i for $i\geq 1$.

Now we are in a position to show the co-monotonic theorem. For $i = 1,2$, let $\sigma_i(s,x)$ and $b_i(s,x)$ satisfy assumption $(A1)$ and $(X_s^i)_{s\in[t,T]}$ $(i = 1,2)$ be the solutions of the following SDEs, respectively,

$$\begin{cases} dX_s^i = b_i(s,X_s^i)ds + \sigma_i(s,X_s^i)dW_s \\ X_t^i = x_i,\quad x_i\in\mathbb{R},\quad i = 1,2 \end{cases} \tag{4.1}$$

For any $\Phi_i(X_T^i) \in L^2(\Omega,\mathcal{F}_T,P)$ $(i = 1,2)$, let (y^i,z^i) be solutions of the following BSDEs,

$$y_t^i = \Phi_i(X_T^i) + \int_t^T g_i(s,y_s^i,z_s^i)ds - \int_t^T z_s^i dW_s. \tag{4.2}$$

Let $u^i(t,x)(i = 1,2)$ denote the solutions to the PDE (2.4) with respect to $(\Phi_i,b_i,\sigma_i,g_i),(i = 1,2)$. Define $w^i(t,x) := \partial_x u^i(t,x)$ for $i = 1,2$. We then present the co-monotonicity theorem as follows:

Theorem 4.2. *Suppose that Φ_i, b_i, σ_i and g_i $(i = 1,2)$ satisfy the assumptions in Theorem 3.1. Let (y^1,z^1) and (y^2,z^2) be the solutions of BSDE (4.2) corresponding to the terminal values $\xi_1 = \Phi_1(X_T^1)$ and $\xi_2 = \Phi_2(X_T^2)$, respectively. Furthermore, assume that $w^1(s,x)$ and $w^2(s,x)$ have the same nodal set, meaning $w^1(s,d_j) = w^2(s,d_j) \equiv 0$ for $d_j\in\mathbb{D}$.*

(i) *If $\Phi_1'(X_s^1)\cdot\Phi_2'(X_s^2) > 0$ and $\sigma_1(s,X_s^1)\cdot\sigma_2(s,X_s^2)\geq 0$ a.e. $t\in[0,T]$, then*

$$z_s^1\cdot z_s^2 \geq 0,\quad a.e.\quad s\in[t,T].$$

(ii) *Furthermore, if $\Phi_1'(X_s^1)\cdot\Phi_2'(X_s^2) > 0$ and $\sigma_1(s,X_s^1)\cdot\sigma_2(s,X_s^2) > 0$ a.e. $s\in[t,T]$, then*

$$sgn(z_t^1) = sgn(z_t^2)\quad a.e.\quad t\in[0,T].$$

Proof. Let $y_s^i = u^i(s,X_s^i)$, $z_s^i = \sigma(s,X_s^i)\cdot\partial_x u^i(s,X_s^i)$, where $u^i(t,x)$ are the solutions of PDE (2.4) with respect to $(\Phi_i,b_i,\sigma_i,g_i)$, $(i = 1,2)$. By Feynman–Kac formula, see [2,22], we know (y_s^i,z_s^i) are the solutions of BSDE (4.2). Set $w^i(s,X_s^i) := \partial_x u^i(s,X_s^i)$.

(i) Since $z_s^i = \sigma_i(s,X_s^i)\cdot\partial_x u^i(s,X_s^i)$ $(i = 1,2)$, and $\Phi_1'(X_s^1)\cdot\Phi_2'(X_s^2) > 0$,

$$z_s^1\cdot z_s^2 = \sigma_1(X_s^1)w^1(s,X_s^1)\cdot\sigma_2(s,X_s^2)w^2(s,X_s^2) = \frac{\sigma_1(X_s^1)w^1(s,X_s^1)\Phi_1'(X_s^1)\cdot\sigma_2(s,X_s^2)w^2(s,X_s^2)\Phi_2'(X_s^2)}{\Phi_1'(X_s^1)\Phi_2'(X_s^2)}$$

Moreover, by the proof of Theorem 3.1, we have $w^i(s,X_s^i)\cdot\Phi_i'(X_s^i)\geq 0$. Then, $z_s^1\cdot z_s^2\geq 0$, $a.e.$ $s\in[t,T]$.

(ii) Since $\Phi_1'(X_s^1)\cdot\Phi_2'(X_s^2) > 0$, a.e. $s\in[t,T]$, then $\Phi_i(x)$ satisfy the assumption of Theorem 3.1. By the proof of Theorem 3.1, we have $\text{sgn}(w^i(s,X_s^i)) = \text{sgn}(\Phi_i'(X_s^i))$, then

$$\text{sgn}(z_s^i) = \text{sgn}(w^i(s,X_s^i))\text{sgn}(\sigma_i(s,X_s^i)) = \text{sgn}(\Phi_i'(X_s^i))\text{sgn}(\sigma_i(s,X_s^i)).$$

Moreover, by the assumptions $\Phi_1'(X_s^1)\cdot\Phi_2'(X_s^2) > 0$ and $\sigma_1(X_s^1)\cdot\sigma_2(X_s^2) > 0$, then $\text{sgn}(\Phi_1'(X_s^1)) = \text{sgn}(\Phi_2'(X_s^2))$ and $\text{sgn}(\sigma_1(X_s^1)) = \text{sgn}(\sigma_2(X_s^2))$. Thus, $\text{sgn}(z_s^1) = \text{sgn}(z_s^2)$ $a.e.$ $s\in[t,T]$. The proof is complete. □

When considering the same underlying process X_t, we have the following result.

Theorem 4.3. *Under the assumptions of Theorem 4.2, suppose that* (y^1, z^1) *and* (y^2, z^2) *are the solutions of BSDE (4.2) corresponding to terminal values* $\xi_1 = \Phi_1(X_T)$ *and* $\xi_2 = \Phi_2(X_T)$*, respectively.*

(i) *If* Φ_1 *and* Φ_2 *are strictly generally co-monotonic on* $\mathbb{R}$*, then* $z_s^1 \cdot z_s^2 \geq 0$*, a.e.* $s \in [t, T]$.

(ii) *Furthermore, if* $\sigma(X_s) \neq 0$ *a.e., then*

$$sgn(z_s^1) = sgn(z_s^2), \ a.e. \ s \in [t, T].$$

Proof. Since Φ_1 and Φ_2 are strictly general co-monotonic, from the proof of Theorem 4.2, we have

$$z_s^1 \cdot z_s^2 = \sigma(s, X_s)w^1(s, X_s) \cdot \sigma(X_s)w^2(s, X_s) = \frac{\sigma^2(X_s)w^1(s, X_s)\Phi_1'(X_s) \cdot w^2(s, X_s)\Phi_2'(X_s)}{\Phi_1'(X_s)\Phi_2'(X_s)}$$

By Theorem 4.2, $w^i(s, X_s)\Phi_i'(X_s) \geq 0$. Moreover, $\sigma^2(X_s) \geq 0$. Therefore, $z_s^1 \cdot z_s^2 \geq 0$, *a.e.* $s \in [t, T]$. Furthermore, $\sigma^2(s, X_s) > 0$ a.e. if $\sigma(s, X_s) \neq 0$ almost surely. Then $z_s^1 \cdot z_s^2 > 0$, *a.e.* $s \in [t, T]$, which means $\text{sgn}(z_s^1) = \text{sgn}(z_s^2)$, *a.e.* $s \in [t, T]$. This completes the proof. □

5. Applications

5.1. Additivity of g-expectations

Under the assumptions $(A2)$–$(A3)$ and the following condition $(A4)$, Peng [23] introduced the notion of g-expectation.

(A4) $g(t, y, 0) = 0$ for each $(t, y) \in [0, T] \times \mathbb{R}$.

Definition 5.1 (*g-expectation*). Suppose g satisfies $(A2)$–$(A4)$. For any $\xi \in L^2(\Omega, \mathcal{F}, P)$, let (y^ξ, z^ξ) be the solution of BSDE (2.1) with terminal value ξ. Considering the mapping $E_g[\cdot] : L^2(\Omega, \mathcal{F}, P) \to \mathbb{R}$ denoted by

$$\mathbb{E}_g[\xi] = y_0^\xi.$$

We call $\mathbb{E}_g[\xi]$ the g-expectation of ξ.

Usually, $\mathbb{E}_g[\xi]$ with respect to BSDE (2.1) is nonlinear on $L^2(\Omega, \mathcal{F}, P)$. However, for some special cases, $\mathbb{E}_g[\cdot]$ is still additive even though g is nonlinear. In this subsection, we will give a sufficient condition for the additivity of $\mathbb{E}_g[\xi]$. In order to do this, first we introduce a definition on the generator g, which will be used in the main results of this subsection later.

Definition 5.2. A function $g(t, y, z) : [0, T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called positively additive, if it holds that,

$$g(t, y_1 + y_2, z_1 + z_2) = g(t, y_1, z_1) + g(t, y_2, z_2)$$

for any (y_1, z_1) and (y_2, z_2) whenever $y_1 y_2 \geq 0$, $z_1 z_2 \geq 0$, $\forall t \in [0, T]$.

Proposition 5.3. *Suppose that* $\Phi_1(X_T^1)$ *and* $\Phi_2(X_T^2)$ *are the random variables defined in Section 3 and* g *is a positively additive function. Moreover,* $\Phi_i(X_T^i)$ $(i = 1, 2)$ *and* g *satisfy the assumptions of Theorem 3.1 and* $(A4)$.

(i) *Suppose that* $\Phi_1(X_T^1) \geq 0$ *and* $\Phi_2(X_T^2) \geq 0$ *(or* $\Phi_1(X_T^1) \leq 0$ *and* $\Phi_2(X_T^2) \leq 0$*). If* $\sigma_1(X_t^1) \cdot \sigma_2(X_t^2) \geq 0$ *and* $\Phi_1'(X_t^1) \cdot \Phi_2'(X_t^2) > 0$*, a.e.* $t \in [0, T]$*, then*

$$\mathbb{E}_g[\Phi_1(X_T^1) + \Phi_2(X_T^2)] = \mathbb{E}_g[\Phi_1(X_T^1)] + \mathbb{E}_g[\Phi_2(X_T^2)].$$

(ii) *If* g *does not depend on* y*, the assumptions* $\Phi_1(X_T^1) \geq 0$ *and* $\Phi_2(X_T^2) \geq 0$ *(or* $\Phi_1(X_T^1) \leq 0$ *and* $\Phi_2(X_T^2) \leq 0$*) in (i) can be dropped.*

Proof. Due to Theorem 4.2, we know if $\sigma_1(X_t^1) \cdot \sigma_2(X_t^2) \geq 0$ and $\Phi_1'(X_t^1) \cdot \Phi_2'(X_t^2) > 0$, then $z_t^1 \cdot z_t^2 \geq 0$ a.e. $t \in [0, T]$. Moreover, by the comparison theorem of BSDEs, see [23] for details, we have that $y_t^1 \geq 0$ and $y_t^2 \geq 0$ if $\Phi_1(X_T^1) \geq 0$ and $\Phi_2(X_T^2) \geq 0$. Hence $y_t^1 \cdot y_t^2 \geq 0$, $t \in [0, T]$. Since g is a positively additive function, then

$$g(t, y_t^1 + y_t^2, z_t^1 + z_t^2) = g(t, y_t^1, z_t^1) + g(t, y_t^2, z_t^2).$$

Thus, $(y_t^1 + y_t^2, z_t^1 + z_t^2)_{0 \leq t \leq T}$ is the solution of BSDE

$$y_t = \Phi_1(X_T^1) + \Phi_2(X_T^2) + \int_t^T g(s, y_s, z_s) ds - \int_t^T z_s dW_s.$$

Therefore, $\mathbb{E}_g[\Phi_1(X_T^1) + \Phi_2(X_T^2)] = y_0^1 + y_0^2 = \mathbb{E}_g[\Phi_1(X_T^1)] + E_g[\Phi_2(X_T^2)]$.

If g does not depend on y, then it holds that

$$g(t, z_t^1 + z_t^2) = g(t, z_t^1) + g(t, z_t^2)$$

when $z_t^1 \cdot z_t^2 \geq 0$ a.e. $t \in [0, T]$. Thus, $\mathbb{E}_g[\Phi_1(X_T^1) + \Phi_2(X_T^2)] = \mathbb{E}_g[\Phi_1(X_T^1)] + \mathbb{E}_g[\Phi_2(X_T^2)]$.

The proof is complete. □

When we consider the same process $X_t = X_t^1 = X_t^2$, it is obviously that $\sigma_1(X_t^1) \cdot \sigma_2(X_t^2) = \sigma^2(X_t) \geq 0$. Thus, we have the following corollary.

Corollary 5.4. *Suppose that* $\Phi_1(X_T)$ *and* $\Phi_2(X_T)$ *are the random variables defines in Section 3 and* g *is a positively additive function. Moreover,* $\Phi_i(X_T)$ $(i = 1, 2)$ *and* g *satisfy the assumptions of Theorem 3.1 and* $(A4)$.

(i) *Suppose that* Φ_1 *and* Φ_2 *are strictly generally co-monotonic on* $\mathbb{R}$ *with* $\Phi_1(X_T) \geq 0$ *and* $\Phi_2(X_T) \geq 0$ *(or* $\Phi_1(X_T) \leq 0$ *and* $\Phi_2(X_T) \leq 0$*), then*

$$\mathbb{E}_g[\Phi_1(X_T) + \Phi_2(X_T)] = \mathbb{E}_g[\Phi_1(X_T)] + \mathbb{E}_g[\Phi_2(X_T)].$$

(ii) *If* g *does not depend on* y*, then the assumptions* $\Phi_1(X_T) \geq 0$ *and* $\Phi_2(X_T) \geq 0$ *(or* $\Phi_1(X_T) \leq 0$ *and* $\Phi_2(X_T) \leq 0$*) in (i) can be dropped.*

5.2. Explicit solutions of nonlinear BSDEs

In this subsection, we will use the sign of z in Theorem 3.1 to obtain the explicit solutions of one kind of nonlinear BSDEs. Consider the following nonlinear BSDE,

$$y_t = \Phi(X_T) + \int_t^T k|Z_s| ds - \int_t^T z_s dW_s, \tag{5.1}$$

where $\Phi(X_T)$ is defined in Section 3. We should mention that Chen et al. [11] obtained the explicit solution of the above BSDE when the terminal is $\Phi(W_T)$. For Eq. (5.1), we have the following result.

Proposition 5.5. *Assume that* $\Phi(\cdot)$, $b(\cdot)$, $\sigma(\cdot)$ *satisfy the assumptions in Theorem 3.1. Then the explicit solution of BSDE (5.1) is given by*

$$y_t = E_Q[\Phi(X_T)|\mathcal{F}_t] =: u(t, X_t),$$

$$z_t = \sigma(t, X_t)\partial_x u(t, X_t),$$

where $\frac{dQ}{dP}|_{\mathcal{F}_T} = \exp\left\{\int_0^T k \cdot sgn(\sigma(s, X_s)) \cdot sgn(\Phi'(X_s)) dW_s - \frac{1}{2}k^2 T\right\}$.

Proof. Let $y_t^\epsilon = u^\epsilon(t, X_t)$, $z_t^\epsilon = \sigma(X_t) \cdot \partial_x u^\epsilon(t, X_t)$, where $u^\epsilon(t, x)$, $t \in [0, T]$ is the solution of PDE (2.4) with $g^\epsilon(s, y, z) = k\sqrt{z^2 + \epsilon}$, $u^\epsilon(T, x) = \Phi(x)$. It is easy to check that g satisfy the assumptions of Theorem 3.1. By Itô's formula, we know $(y_t^\epsilon, z_t^\epsilon)$ is the solution of BSDE (3.1) with $g^\epsilon(s, y, z) = k\sqrt{z^2 + \epsilon}$, $\xi = \Phi(X_T)$. Then due to Theorem 3.1, we have

$$\text{sgn}(z_t^\epsilon) = \text{sgn}(\sigma(t, X_t) \cdot \partial_x u^\epsilon(t, X_t)) = \text{sgn}(\sigma(t, X_t)) \cdot \text{sgn}(\partial_x u^\epsilon(t, X_t)),$$
$$a.e. \ t \in [0, T].$$

By the proof of Theorem 3.1, $\text{sgn}(\partial_x u^\epsilon(t, X_t)) = \text{sgn}(\Phi'(X_t))$ *a.e.* $t \in [0, T]$. Thus, $\text{sgn}(z_t^\epsilon) = \text{sgn}(\sigma(t, X_t)) \cdot \text{sgn}(\Phi'(X_t))$, which means the sign of z_t^ϵ is independent of ϵ. Therefore, let $\epsilon \to 0$, by [11, Theorem 3.3] or [24, Theorem 4.2.1], we have

$$\text{sgn}(z_t) = \text{sgn}(\sigma(t, X_t)) \cdot \text{sgn}(\Phi'(X_t)) \; a.e. \; t \in [0, T].$$

Now using Girsanov's theorem, we can get

$$y_t = E_P\left[\Phi(X_T) \cdot \exp\left\{\int_0^T k \cdot \text{sgn}(\sigma(s, X_s)) \cdot \text{sgn}(\Phi'(X_s)) dW_s - \frac{1}{2}k^2 T\right\} \Big| \mathcal{F}_t\right].$$

Then by the relationship between y_t and z_t, see [11, Theorem 4] for details, we get the representation of z_t. □

5.3. Standard aggregator utility under ambiguity

In this subsection, we will use Theorem 3.1 to obtain a closed form expression of standard aggregator utility for specific recursive multiple priors. For any given consumption process c, Duffie and Epstein [19] define the stochastic differential utility (SDU) as follows:

$$Y_t^P = E\left[\int_t^T f(c_s, Y_s^P) ds | \mathcal{F}_t\right],$$

where the function f is called an aggregator. The special case $f(c, y) = U(c) - \beta y$ delivers the standard aggregator utility

$$Y_t^P = E\left[\int_t^T e^{-\beta(s-t)} U(c_s) ds | \mathcal{F}_t\right].$$

Chen and Epstein [4] show that the above utility takes the following form for recursive multiple-priors $\mathcal{P}^\Theta$,

$$Y_t = \min_{Q \in \mathcal{P}^\Theta} E_Q\left[\int_t^\tau e^{-\beta(s-t)} U(c_s) ds + e^{-\beta(\tau-t)} Y_\tau | \mathcal{F}_t\right], \qquad 0 \le t < \tau < T.$$

We shall show that the above utility under ambiguity has more explicit representation when $\mathcal{P}^\Theta$ is given by

$$\mathcal{P}^\Theta = \left\{Q : \frac{dQ}{dP} = \exp\left\{\frac{1}{2}\int_0^T |\theta_s|^2 ds - \int_0^T \theta_s dW_s\right\}, \sup_{t \in [0,T]} |\theta_t| \le k\right\}. \tag{5.2}$$

By [4, Theorem 2.2], Y_t satisfies the following nonlinear BSDE when $Y_T = \Phi(X_T)$ and $\mathcal{P}^\Theta$ is given by (5.2):

$$dY_t = \left(-U(c_t) + \beta Y_t + k|Z_t|\right) dt + Z_t dW_t, \qquad Y_T = \Phi(X_T). \tag{5.3}$$

In the following, we assume the consumption process c_t is a real valued continuous function on $[0, T]$. We have the following result.

Proposition 5.6. *Suppose that $\Phi(\cdot)$, $b(\cdot)$, $\sigma(\cdot)$ satisfy the assumptions in Theorem 3.1. Then the standard aggregator utility Y_t with terminal $Y_T = \Phi(X_T)$ has the following closed form representation:*

$$Y_t = E_Q\left[e^{-\beta(T-t)}\Phi(X_T) + \int_t^T U(c_s) e^{-\beta(s-t)} ds \Big| \mathcal{F}_t\right], \qquad t \in [0, T], \tag{5.4}$$

where $\frac{dQ}{dP}|_{\mathcal{F}_T} = \exp\left\{-\int_0^T k \cdot sgn(\sigma(X_s)) \cdot sgn(\Phi'(X_s)) dW_s - \frac{1}{2}k^2 T\right\}$.

Proof. By the illustration in this subsection, Y_t is the solution of BSDE (5.3). Let $u^\epsilon(t, x)$, $t \in [0, T]$ be the solution of PDE (2.4) with $g^\epsilon(s, y, z) = U(c_s) - \beta y - k\sqrt{z^2 + \epsilon}$, $u^\epsilon(T, x) = \Phi(x)$. It is obvious g^ϵ satisfies the assumptions of Theorem 3.1. By Itô's formula, we know $Y_t^\epsilon = u^\epsilon(t, X_t)$ and $Z_t^\epsilon = \sigma(t, X_t) \cdot \partial_x u^\epsilon(t, X_t)$ are solutions of BSDE (3.1) with $g^\epsilon(s, y, z) = U(c_s) - \beta y - k\sqrt{z^2 + \epsilon}$ and $\xi = \Phi(X_T)$. Using the similar method as Proposition 5.5, it holds

$$\text{sgn}(Z_t) = \text{sgn}(\sigma(t, X_t)) \cdot \text{sgn}(\Phi'(X_t)) \; a.e. \; t \in [0, T]. \tag{5.5}$$

Let $\tilde{Y}_t = e^{-\beta t} Y_t$ and $\tilde{Z}_t = e^{-\beta t} Z_t$. Applying the Itô formula to $\tilde{Y}_t$, we know that $\tilde{Y}_t$ solves the following BSDE:

$$d\tilde{Y}_t = \left(-e^{-\beta t} U(c_t) + k|\tilde{Z}_t|\right) dt + \tilde{Z}_t dW_t, \qquad \tilde{Y}_T = e^{-\beta T}\Phi(X_T). \tag{5.6}$$

By (5.5), we have $\text{sgn}(\tilde{Z}_t) = \text{sgn}(\sigma(t, X_t)) \cdot \text{sgn}(\Phi'(X_t))$. Thus, BSDE (5.6) is actually a linear equation. Applying Girsanov's theorem, $\tilde{Y}_t$ is given by

$$\tilde{Y}_t = E_Q\left[e^{-\beta T}\Phi(X_T) + \int_t^T U(c_s) e^{-\beta s} ds \Big| \mathcal{F}_t\right].$$

Therefore, we obtain Y_t as (5.4). □

CRediT authorship contribution statement

Shuhui Liu: Writing – review & editing, Writing – original draft, Investigation, Formal analysis, Conceptualization. **Defeng Sun:** Writing – review & editing, Supervision, Funding acquisition.

Declaration of competing interest

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Data availability

No data was used for the research described in the article.

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