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A DUAL OPTIMIZATION APPROACH TO INVERSE QUADRATIC EIGENVALUE PROBLEMS WITH PARTIAL EIGENSTRUCTURE*

ZHENG-JIAN BAI†, DELIN CHU‡, AND DEFENG SUN‡

Abstract. The inverse quadratic eigenvalue problem (IQEP) arises in the field of structural dynamics. It aims to find three symmetric matrices, known as the mass, the damping, and the stiffness matrices, such that they are closest to the given analytical matrices and satisfy the measured data. The difficulty of this problem lies in the fact that in applications the mass matrix should be positive definite and the stiffness matrix positive semidefinite. Based on an equivalent dual optimization version of the IQEP, we present a quadratically convergent Newton-type method. Our numerical experiments confirm the high efficiency of the proposed method.

Key words. nonlinear optimization, quadratic eigenvalue problem, inverse eigenvalue problem, partial eigenstructure

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1. Introduction.

Throughout this paper the following notation will be used:

- $A \succeq 0$ ($A \succ 0$, respectively) means that the $n \times n$ real matrix A is symmetric and positive semidefinite (symmetric and positive definite, respectively).
- A^T, $\sigma(A)$, and $\|A\|$ denote the transpose, the spectrum, and the Frobenius norm of $A \in \mathbb{R}^{n\times n}$, respectively.
- I denotes the identity matrix of an appropriate dimension.
- $\mathcal{S}^n$ and $\mathcal{S}^n_+$ denote the set of all real $n \times n$ symmetric matrices and the cone of positive semidefinite matrices in this set, respectively. Let

$$\Omega_0 := \mathcal{S}^n \times \mathcal{S}^n \times \mathcal{S}^n \quad \text{and} \quad \Omega := \mathcal{S}^n_+ \times \mathcal{S}^n \times \mathcal{S}^n_+.$$

For given matrices $M, C, K \in \mathbb{R}^{n\times n}$, let

$$Q(\lambda) := \lambda^2 M + \lambda C + K.$$

Then the quadratic eigenvalue problem (QEP) is to find scalars $\lambda \in \mathbb{C}$ and nonzero vectors x such that

$$Q(\lambda)x = 0, \tag{1.1}$$

where λ and x are called the eigenvalue and the eigenvector, respectively. For various applications, mathematical properties, and numerical solution techniques of the QEP,

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†Department of Information and Computational Mathematics, Xiamen University, Xiamen 361005, People's Republic of China (zjbai@xmu.edu.cn). This author's research was partially supported by the National Natural Science Foundation of China grant 10601043 and by Xiamen University grant 0000-X07152.

‡Department of Mathematics, National University of Singapore, 2 Science Drive 2, Republic of Singapore (matchudl@nus.edu.sg, matsundf@nus.edu.sg). The work of the second author was partially supported by grant R146-000-047-112 of the National University of Singapore. The work of the third author was partially supported by grant R146-00-061-112 of the National University of Singapore.

we refer the reader to the survey paper [48]. In many practical applications, the matrices M, C, and K are required to be symmetric. In particular, in one of the most popular numerical models, the finite element model (FEM) [19, 48]

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = 0, \tag{1.2}$$

the mass matrix M, the damping matrix C, and the stiffness matrix K are required to be symmetric with M and K being positive definite and positive semidefinite, respectively. It is shown in [48] that the general solution to the homogeneous equation (1.2) is given in terms of the solution of the QEP (1.1). Unfortunately, the natural frequencies and model shapes (eigenvalues and eigenvectors) predicted by the FEM often disagree with the physical experimental results. The FEM updating aims to improve the numerical model (1.2) by the measured data. Various model updating methods have been presented in the literature [19, 27, 29, 33]. Mathematically, the model updating problem is an inverse quadratic eigenvalue problem (IQEP) which aims to find matrices M, C, and K such that they are closest to the given estimated analytical matrices and satisfy the partially measured eigendata. The general IQEP can be defined as follows: Given a measured partial eigenpair $(\Lambda, X) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k}$ with

$$1 \le k \le n, \quad \mathrm{rank}(X) = k, \quad \text{and} \quad \Lambda = \mathrm{diag}\{\Lambda_1, \ldots, \Lambda_\mu, \Lambda_{\mu+1}, \ldots, \Lambda_\nu\}, \tag{1.3}$$

where

$$\Lambda_i = \mathrm{diag}\{\overbrace{\lambda_i^{[2]}, \ldots, \lambda_i^{[2]}}^{s_i}\} \quad \text{for} \quad 1 \le i \le \mu \quad \text{and} \quad \Lambda_i = \lambda_i I_{s_i} \quad \text{for} \quad \mu+1 \le i \le \nu,$$

$$\lambda_i^{[2]} = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix} \in \mathbb{R}^{2\times 2}, \quad \beta_i \ne 0, \quad \sigma(\Lambda_i) \cap \sigma(\Lambda_j) = \emptyset, \quad \forall\, 1 \le i \ne j \le \mu,$$

$$\lambda_i \in \mathbb{R}, \quad \lambda_i \ne \lambda_j, \quad \forall\, \mu+1 \le i \ne j \le \nu,$$

$$k = \sum_{i=1}^{\mu} 2s_i + \sum_{i=\mu+1}^{\nu} s_i,$$

find $M, C, K \in \mathcal{S}^n$ with $M \succ 0$ and $K \succeq 0$ such that

$$MX\Lambda^2 + CX\Lambda + KX = 0. \tag{1.4}$$

The model updating problem for given $M_a, C_a, K_a \in \mathcal{S}^n$, which are called the estimated analytic mass, damping, and stiffness matrices, respectively, in the FEM updating [19], is then to find $M, C, K \in \mathcal{S}^n$ such that (M, C, K) is a solution to the following optimization problem:

$$\begin{array}{ll} \inf & \dfrac{c_1}{2}\|M - M_a\|^2 + \dfrac{c_2}{2}\|C - C_a\|^2 + \dfrac{1}{2}\|K - K_a\|^2 \\ \text{subject to (s.t.)} & MX\Lambda^2 + CX\Lambda + KX = 0, \\ & M \succ 0, \quad C = C^T, \quad K \succeq 0, \end{array}$$

where c_1 and c_2 are two positive weighting parameters. It is noted that in numerical computation it is better to replace the condition $M \succ 0$ by $M \succeq 0$. This consideration leads us to focus on the following optimization problem:

$$\begin{array}{ll} \min & \dfrac{c_1}{2}\|M - M_a\|^2 + \dfrac{c_2}{2}\|C - C_a\|^2 + \dfrac{1}{2}\|K - K_a\|^2 \\ \text{s.t.} & MX\Lambda^2 + CX\Lambda + KX = 0, \\ & M \succeq 0, \quad C = C^T, \quad K \succeq 0. \end{array} \tag{1.5}$$

To avoid confusion, we refer to the optimization problem (1.5) as the IQEP.

As in [25], let the QR factorization of X be given by

$$X = Q \begin{bmatrix} R \\ 0 \end{bmatrix},$$

where $Q \in \mathbb{R}^{n\times n}$ is orthogonal and $R \in \mathbb{R}^{k\times k}$ is nonsingular and upper triangular. By renaming $M := \sqrt{c_1}Q^TMQ$, $C := \sqrt{c_2}Q^TCQ$, $K := Q^TKQ$, $M_a := \sqrt{c_1}Q^TM_aQ$, $C_a := \sqrt{c_2}Q^TC_aQ$, and $K_a := Q^TK_aQ$, without loss of generality, we may assume from now on that problem (1.5) takes the following form:

$$\begin{aligned} \min \quad & \frac{1}{2}\|M - M_a\|^2 + \frac{1}{2}\|C - C_a\|^2 + \frac{1}{2}\|K - K_a\|^2 \\ \text{s.t.} \quad & \frac{1}{\sqrt{c_1}}M \begin{bmatrix} R \\ 0 \end{bmatrix} \Lambda^2 + \frac{1}{\sqrt{c_2}}C \begin{bmatrix} R \\ 0 \end{bmatrix} \Lambda + K \begin{bmatrix} R \\ 0 \end{bmatrix} = 0, \\ & M \succeq 0, \quad C = C^T, \quad K \succeq 0. \end{aligned} \tag{1.6}$$

THEOREM 1.1. *Problem* (1.6) *admits a strictly feasible solution if and only if* Λ *is nonsingular.*

Proof. First we show the *necessity* by contradiction. Assume that Λ is singular. Then there exists a nonzero vector y such that

$$\Lambda y = 0.$$

As a result, for any feasible solution (M, C, K) to problem (1.6), we have

$$\left(\frac{1}{\sqrt{c_1}}M \begin{bmatrix} R \\ 0 \end{bmatrix} \Lambda^2 + \frac{1}{\sqrt{c_2}}C \begin{bmatrix} R \\ 0 \end{bmatrix} \Lambda + K \begin{bmatrix} R \\ 0 \end{bmatrix}\right) y = 0; \quad \text{i.e.,} \quad K \begin{bmatrix} Ry \\ 0 \end{bmatrix} = 0.$$

Note that R is nonsingular, which yields that $Ry \neq 0$. Hence K must be singular. This is a contradiction to the existence of a strictly feasible solution to problem (1.6). Therefore, problem (1.6) can admit a strictly feasible solution only if Λ is nonsingular.

Next we show the *sufficiency*. Following the idea of [12], we can obtain a special feasible solution to (1.6) as follows:

$$\begin{cases} \widehat{M} = \sqrt{c_1} \begin{bmatrix} (RR^T)^{-1} & 0 \\ 0 & I \end{bmatrix}, \\ \widehat{C} = \sqrt{c_2} \begin{bmatrix} -R^{-T}(\Lambda + \Lambda^T)R^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \\ \widehat{K} = \begin{bmatrix} R^{-T}\Lambda^T\Lambda R^{-1} & 0 \\ 0 & I \end{bmatrix}. \end{cases} \tag{1.7}$$

Since both R and Λ are nonsingular, $\widehat{M}$ and $\widehat{K}$ are symmetric and positive definite. This implies that $(\widehat{M}, \widehat{C}, \widehat{K})$ above is a strictly feasible solution to problem (1.6). ∎

Because of Theorem 1.1 and the generalized Slater condition (2.4) (see section 2) used in our dual approach, in the following development we make a blanket assumption, unless stated otherwise, that the following condition holds.

Assumption 1.2. Λ is nonsingular.

Remark 1.3. Problem (1.6) fails to possess a strictly feasible solution if Λ is singular. As a remedy, we can reduce problem (1.6) equivalently to a similar problem which admits a strictly feasible solution. See Appendix B for discussions on this.

Due to complicated structures of the IQEP and a lack of tools for dealing with it, many authors considered various simplified versions of the IQEP by dropping or relaxing the requirement on the positive semidefiniteness of the matrices M and K [3, 4, 5, 6, 14, 18, 19, 26, 51, 52, 53]. As observed in the literature on structural dynamics [18, 19], however, these approaches may fail to guarantee the positive semidefiniteness of M and K, which is vital in applications.

In this paper, we deal directly with the IQEP with the required positive semidefiniteness condition on the matrices M and K. We shall introduce a generalized Newton method to the dual of problem (1.6), whose objective function is continuously differentiable but not twice differentiable. This approach is motivated by two recent papers due to Chu, Kuo, and Lin [12] and Qi and Sun [37]. In [12], Chu, Kuo, and Lin made an important step by showing for the first time that the general IQEP admits a nontrivial solution; i.e., there exist symmetric matrices M, C, and K with $M \succ 0$ and $K \succeq 0$ satisfying (1.4). In [37], based on recent developments in strongly semismooth matrix-valued functions [47], Qi and Sun presented a highly efficient quadratically convergent Newton method for solving the nearest correlation matrix problem proposed by Higham in [22] (see Boyd and Xiao [10], Malick [30], and Toh, Tütüncü, and Todd [50] for more discussions on this problem and its extensions). Theoretically, the Newton method introduced in [37] can be used to solve a much broader class of problems that includes the IQEP. However, there exist several practical difficulties in applying the approach by Qi and Sun [37] to the IQEP. In particular, for the IQEP we must first circumvent the incomputability of an element in Clarke's generalized Jacobian of the gradient mapping of the dual function needed in the approach by Qi and Sun [37]. We shall address this important issue under a more general setting in this paper by introducing a modified version of Clarke's generalized Jacobian. Consequently, the convergence analysis must be modified, too. Note that the IQEP is a special constrained least-squares problem, which can be solved by applying conventional approaches such as first order methods or a quasi-Newton method (e.g., Malick [30] suggested using the BFGS update) to its dual problem. Here, we show that a generalized Newton method can be applied to the dual of the IQEP with guaranteed quadratic convergence and high efficiency.

The organization of this paper is as follows. In section 2, we discuss some basic convex optimization theory used in this paper and present some results regarding Clarke's generalized Jacobian of certain locally Lipschitz functions. In section 3, we present the dual forms of the IQEP. In section 4, we give the details of our generalized Newton method and its convergence analysis. We report our numerical results in section 5 and make some final conclusions in section 6.

2. Preliminaries. Let $\mathcal{X}$ and $\mathcal{Y}$ be two finite dimensional real vector spaces, each equipped with a scalar inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. Let $\mathcal{A} : \mathcal{X} \to \mathcal{Y}$ be a linear operator and $\mathcal{A}^* : \mathcal{Y} \to \mathcal{X}$ be the adjoint of $\mathcal{A}$. Let x^0 be a given vector in $\mathcal{X}$. Consider the following constrained least-squares problem:

$$\begin{array}{ll} \min & \dfrac{1}{2}\langle x - x^0, x - x^0 \rangle \\ \text{s.t.} & \mathcal{A}x = b, \\ & x \in \mathcal{Q}, \end{array} \tag{2.1}$$

where $b \in \mathcal{Y}$ and $\mathcal{Q}$ is a closed convex cone in $\mathcal{X}$. In the literature, problem (2.1) is also known as the best approximation problem in Hilbert space. For a survey on the latter, see [36]. Let $\mathcal{D} \subseteq \mathcal{X}$ be a closed convex set. For any $x \in \mathcal{X}$, let $\Pi_{\mathcal{D}}(x)$ denote

the metric projection of x onto $\mathcal{D}$; i.e., $\Pi_{\mathcal{D}}(x)$ is the unique optimal solution to the following convex programming problem:

$$\begin{array}{ll} \min & \dfrac{1}{2}\langle z-x, z-x\rangle \\ \text{s.t.} & z \in \mathcal{D}. \end{array}$$

Then, the dual of problem (2.1) (cf. Rockafellar and Wets [44]) has the following form:

$$\begin{array}{ll} \max & -\theta(y) \\ \text{s.t.} & y \in \mathcal{Y}, \end{array} \tag{2.2}$$

where $\theta : \mathcal{Y} \to \mathbb{R}$ is given by

$$\theta(y) := \frac{1}{2}\|x^0 + \mathcal{A}^* y\|^2 - \frac{1}{2}\|x^0 + \mathcal{A}^* y - \Pi_{\mathcal{Q}}(x^0 + \mathcal{A}^* y)\|^2 - \langle b, y\rangle - \frac{1}{2}\|x^0\|^2. \tag{2.3}$$

Recall that the generalized Slater condition is said to hold for the convex optimization problem (2.1) if

$$\begin{cases} \mathcal{A} : \mathcal{X} \to \mathcal{Y} \text{ is onto,} \\ \exists\ \bar{x} \in \mathcal{X} \text{ such that } \mathcal{A}\bar{x} = b,\ \bar{x} \in \operatorname{int}(\mathcal{Q}), \end{cases} \tag{2.4}$$

where "int" denotes the topological interior of a given set. The following result is well known in the classical duality theory for convex programming [43, Theorems 17 and 18].

PROPOSITION 2.1. *Under the generalized Slater condition* (2.4)*, the following hold:*

(i) *There exists at least one $\bar{y} \in \mathcal{Y}$ that solves the dual problem* (2.2)*. The unique solution to the original problem* (2.1) *is given by*

$$\bar{x} = \Pi_{\mathcal{Q}}(x^0 + \mathcal{A}^* \bar{y}). \tag{2.5}$$

(ii) *For every real number τ, the level set $\{y \in \mathcal{Y} : \theta(y) \le \tau\}$ is closed, bounded, and convex.*

For more recent discussions about the above duality theory for the constrained least-squares problem (2.1) and its extensions, see [10, 30, 37].

Since $\mathcal{Q}$ is assumed to be a closed convex cone, by [54], we know that the function θ defined by (2.3) takes the following form:

$$\theta(y) = \frac{1}{2}\|\Pi_{\mathcal{Q}}(x^0 + \mathcal{A}^* y)\|^2 - \langle b, y\rangle - \frac{1}{2}\|x^0\|^2, \quad y \in \mathcal{Y}. \tag{2.6}$$

From [54] we also know that θ is a continuously differentiable convex function and that its gradient at $y \in \mathcal{Y}$ is given by

$$\nabla\theta(y) = \mathcal{A}\,\Pi_{\mathcal{Q}}(x^0 + \mathcal{A}^* y) - b.$$

Furthermore, since $\Pi_{\mathcal{Q}}(\cdot)$ is globally Lipschitz continuous with modulus 1, $\nabla\theta(\cdot)$ is globally Lipschitz continuous. Therefore, if the Slater condition (2.4) holds, then one may use any gradient-based method, such as the steepest descent method or quasi-Newton methods (e.g., the BFGS method suggested by Malick [30]), to find an optimal solution y^* to the dual problem (2.2) first and then use (2.5) to get an optimal

solution to problem (2.1). This is exactly the dual approach outlined by Rockafellar in [43, page 4] for general convex optimization problems and tailored to the constrained least-squares problem (2.1). Since in general $\Pi_{\mathcal{Q}}(\cdot)$ is not continuously differentiable, $\theta(\cdot)$ is not twice differentiable. So we cannot directly use the Newton method to solve (2.2); i.e.,

$$\begin{array}{ll} \min & \theta(y) \\ \text{s.t.} & y \in \mathcal{Y}. \end{array} \tag{2.7}$$

Fortunately, we may apply Clarke's Jacobian-based Newton method developed in the last two decades for solving locally Lipschitz equations [24, 39]. The success of using Clarke's Jacobian-based Newton method to solve the nearest correlation matrix problem in Qi and Sun [37] inspires us to explore this idea further.

Let $\mathcal{Z}$ be an arbitrary finite dimensional real vector space. Let $\mathcal{O}$ be an open set in $\mathcal{Y}$ and let $\Xi : \mathcal{O} \subseteq \mathcal{Y} \to \mathcal{Z}$ be a locally Lipschitz continuous function on the open set $\mathcal{O}$. Rademacher's theorem [44, Chapter 9.J] says that Ξ is almost everywhere Fréchet differentiable in $\mathcal{O}$. We denote by $\mathcal{O}_\Xi$ the set of points in $\mathcal{O}$ where Ξ is Fréchet differentiable. Let $\Xi'(y)$ denote the Jacobian of Ξ at $y \in \mathcal{O}_\Xi$. Then Clarke's generalized Jacobian of Ξ at $y \in \mathcal{O}$ is defined by [13]

$$\partial \Xi(y) := \mathrm{conv}\{\partial_B \Xi(y)\},$$

where "conv" denotes the convex hull and the Bouligand subdifferential $\partial_B \Xi(y)$ is defined by Qi in [38] as

$$\partial_B \Xi(y) := \left\{ V \,:\, V = \lim_{j \to \infty} \Xi'(y^j),\, y^j \to y,\, y^j \in \mathcal{O}_\Xi \right\}.$$

Suppose that $F : \mathcal{O} \subseteq \mathcal{Y} \to \mathcal{Y}$ is a locally Lipschitz continuous function on the open set $\mathcal{O}$. Let y^0 be an arbitrary point in $\mathcal{O}$. Then Clarke's Jacobian-based Newton method for solving the locally Lipschitz equation $F(y) = 0$ can be described by

$$y^{j+1} = y^j - V_j^{-1} F(y^j), \qquad V_j \in \partial F(y^j), \quad j = 0, 1, 2, \ldots. \tag{2.8}$$

In general, the above iterative method does not converge even if $\{\|V_j^{-1}\|\}$ is uniformly bounded and y^0 is arbitrarily close to a given solution of $F(y) = 0$ (see [24]). Kummer in [24] proposed a general condition guaranteeing the superlinear convergence of (2.8), which generalized Kojima and Shindo's condition for superlinear (quadratic) convergence of the Newton method for piecewise smooth equations [23]. Kummer's result was largely unnoticed until Qi and Sun in [39] published their now well-known work by showing that the iteration sequence generated by (2.8) can converge superlinearly if F is a *semismooth function*.

The concept of semismoothness was introduced by Mifflin [32] for functionals. In order to study the convergence of the iterative method (2.8), Qi and Sun [39] extended the definition of semismoothness to vector-valued functions. The following definition of semismoothness is convenient to use.

DEFINITION 2.2. *Let $\Xi : \mathcal{O} \subseteq \mathcal{Y} \to \mathcal{Z}$ be a locally Lipschitz continuous function on the open set $\mathcal{O}$. We say that Ξ is semismooth at a point $y \in \mathcal{O}$ if*

(i) *Ξ is directionally differentiable at y, and*

(ii) *for any $x \to y$ and $V \in \partial \Xi(x)$,*

$$\Xi(x) - \Xi(y) - V(x - y) = o(\|x - y\|). \tag{2.9}$$

The function $\Xi : \mathcal{O} \subseteq \mathcal{Y} \to \mathcal{Z}$ is said to be strongly semismooth at a point $y \in \mathcal{O}$ if F is semismooth at y, and for any $x \to y$ and $V \in \partial\Xi(x)$,

$$\Xi(x) - \Xi(y) - V(x-y) = O(\|x-y\|^2). \tag{2.10}$$

The following convergence result is established in [39, Theorem 3.2], while the result on superlinear convergence can also be found in [24, Proposition 3].

PROPOSITION 2.3. *Let $F : \mathcal{O} \subseteq \mathcal{Y} \to \mathcal{Y}$ be a locally Lipschitz continuous function on the open set $\mathcal{O}$. Let $\bar{y} \in \mathcal{O}$ be such that $F(\bar{y}) = 0$. Assume that F is semismooth at $\bar{y}$ and that every element in $\partial F(\bar{y})$ is nonsingular. Then every sequence generated by* (2.8) *converges to $\bar{y}$ superlinearly provided that the starting point y^0 is sufficiently close to $\bar{y}$. Moreover, if F is strongly semismooth at $\bar{y}$, the rate of convergence is quadratic.*

Now, we consider the following equation:

$$F(y) := \nabla\theta(y) = \mathcal{A}\,\Pi_{\mathcal{Q}}(x^0 + \mathcal{A}^* y) - b = 0, \quad y \in \mathcal{Y}. \tag{2.11}$$

We have already shown that F is a globally Lipschitz continuous mapping, and thus $\partial F(y)$ is well defined everywhere in $\mathcal{Y}$. In order to apply Clarke's Jacobian-based Newton method (2.8) to find a solution of (2.11), we need to assume that the two conditions imposed in Proposition 2.3 are valid. Certainly, the (strong) semismoothness of F at $\bar{y}$ follows easily if the metric projector $\Pi_{\mathcal{Q}}(\cdot)$ is (strongly) semismooth at $x^0 + \mathcal{A}^*\bar{y}$. This means that if one knows in advance that $\Pi_{\mathcal{Q}}(\cdot)$ is (strongly) semismooth everywhere in $\mathcal{Y}$ (as is the case for the IQEP discussed in this paper), then there is no need to be concerned about the semismoothness assumption on F. However, the second condition, i.e., the nonsingularity assumption on the generalized Jacobian $\partial F(\bar{y})$, needs more elaboration. We shall discuss this issue in the remaining part of this section.

Let $F : \mathcal{Y} \to \mathcal{Y}$ be defined by (2.11). The next proposition on Clarke's Jacobian of F follows easily from [13, page 75].

PROPOSITION 2.4. *Let $\bar{y} \in \mathcal{Y}$ and $\bar{x} := x^0 + \mathcal{A}^*\bar{y}$. Then, for any $d \in \mathcal{Y}$, it holds that*

$$\partial F(\bar{y})d \subseteq \mathcal{A}\,\partial\Pi_{\mathcal{Q}}(\bar{x})\mathcal{A}^* d. \tag{2.12}$$

Proposition 2.4 relates the Jacobian of F to the Jacobian of the metric projector $\Pi_{\mathcal{Q}}(\cdot)$. Note, however, that (2.12) does not mean that $\partial F(\bar{y}) \subseteq \mathcal{A}\,\partial\Pi_{\mathcal{Q}}(\bar{x})\mathcal{A}^*$. For the metric projector, we have the following result from [31, Proposition 1].

LEMMA 2.5. *Let $\mathcal{D} \subseteq \mathcal{X}$ be a closed convex set. Then, for any $x \in \mathcal{X}$ and $V \in \partial\Pi_{\mathcal{D}}(x)$,*

(i) *V is self-adjoint,*
(ii) *$\langle d, Vd\rangle \geq 0$ for all $d \in \mathcal{X}$,*
(iii) *$\langle Vd, d - Vd\rangle \geq 0$ for all $d \in \mathcal{X}$.*

Let $\mathcal{D} \subseteq \mathcal{X}$ be a closed convex set. For any $x \in \mathcal{X}$, denote

$$\mathrm{dist}(x, \mathcal{D}) := \inf\{\|x - d\| \,:\, d \in \mathcal{D}\}.$$

The *tangent cone* of $\mathcal{D}$ at a point $x \in \mathcal{D}$, which we denote by $\mathcal{T}_{\mathcal{D}}(x)$, consisting of all tangent vectors of $\mathcal{D}$ at x, is defined by (cf. [9, section 2.2.4])

$$\mathcal{T}_{\mathcal{D}}(x) := \{d \in \mathcal{X} \,:\, \mathrm{dist}(x + td, \mathcal{D}) = o(t),\ t \geq 0\}. \tag{2.13}$$

For any $x \in \mathcal{D}$, we denote by $\lin(\mathcal{T}_{\mathcal{D}}(x))$ the lineality space of $\mathcal{T}_{\mathcal{D}}(x)$, i.e., the largest linear space in $\mathcal{T}_{\mathcal{D}}(x)$.

The following definition of constraint nondegeneracy is adapted from the works of Robinson, who defined it for general optimization problems and variational inequalities [40, 41, 42].

DEFINITION 2.6. *We say that a feasible point x to problem* (2.1) *is constraint nondegenerate if*

$$\begin{pmatrix} \mathcal{A} \\ I \end{pmatrix} \mathcal{X} + \begin{pmatrix} 0 \\ \lin(\mathcal{T}_{\mathcal{Q}}(x)) \end{pmatrix} = \begin{pmatrix} \mathcal{Y} \\ \mathcal{X} \end{pmatrix} \tag{2.14}$$

or, equivalently, if

$$\mathcal{A} \lin(\mathcal{T}_{\mathcal{Q}}(x)) = \mathcal{Y}. \tag{2.15}$$

The constraint nondegenerate condition (2.14) in its general form is extensively used by Bonnans and Shapiro in [9] for perturbation analysis of optimization problems. In the context of semidefinite programming, condition (2.14) reduces to the primal nondegenerate condition proposed by Alizadeh, Haeberly, and Overton [1]. For the conventional nonlinear programming problem with finitely many equality and inequality constraints, the constraint nondegenerate condition is equivalent to the well-known linear independence constraint qualification [40, 46].

DEFINITION 2.7. *Let $\mathcal{D} \subseteq \mathcal{X}$ be a closed convex set. We say that the metric projector $\Pi_{\mathcal{D}}(\cdot)$ is Jacobian amicable at $x \in \mathcal{X}$ if, for any $V \in \partial\Pi_{\mathcal{D}}(x)$ and $d \in \mathcal{X}$ such that $Vd = 0$, it holds that*

$$d \in \left(\lin\left(\mathcal{T}_{\mathcal{D}}(x_+)\right)\right)^{\perp},$$

where $x_+ := \Pi_{\mathcal{D}}(x)$ and $\left(\lin\left(\mathcal{T}_{\mathcal{D}}(x_+)\right)\right)^{\perp}$ is defined by

$$\left(\lin\left(\mathcal{T}_{\mathcal{D}}(x_+)\right)\right)^{\perp} := \{d \in \mathcal{X} \,:\, \langle d, h\rangle = 0 \quad \forall\, h \in \lin\left(\mathcal{T}_{\mathcal{D}}(x_+)\right)\}.$$

We say that $\Pi_{\mathcal{D}}(\cdot)$ is Jacobian amicable if it is Jacobian amicable at every point in $\mathcal{X}$.

Now, we are ready to state our result on the nonsingularity of Clarke's Jacobian of the mapping F defined by (2.11).

PROPOSITION 2.8. *Let $\bar{y} \in \mathcal{Y}$ be such that $F(\bar{y}) = 0$. Let $\bar{x} := x^0 + \mathcal{A}^*\bar{y}$ and $\bar{x}_+ := \Pi_{\mathcal{Q}}(\bar{x})$. Assume that the constraint nondegenerate condition* (2.15) *holds at $\bar{x}_+$ and that $\Pi_{\mathcal{Q}}(\cdot)$ is Jacobian amicable at $\bar{x}$. Then every element in $\mathcal{A}\partial\Pi_{\mathcal{Q}}(\bar{x})\mathcal{A}^*$ is self-adjoint and positive definite.*

Proof. Let V be an arbitrary element in $\mathcal{A}\partial\Pi_{\mathcal{Q}}(\bar{x})\mathcal{A}^*$. Then, there exists an element $W \in \partial\Pi_{\mathcal{Q}}(\bar{x})$ such that

$$V = \mathcal{A}W\mathcal{A}^*.$$

Since, by Lemma 2.5, W is self-adjoint and positive semidefinite, we know that V is also self-adjoint and positive semidefinite.

Next, we show the positive definiteness of V. Let $d \in \mathcal{Y}$ be such that $Vd = 0$. Then, by (iii) of Lemma 2.5, we obtain that

$$0 = \langle d, Vd\rangle = \langle d, \mathcal{A}W\mathcal{A}^*d\rangle = \langle \mathcal{A}^*d, W\mathcal{A}^*d\rangle \geq \langle W\mathcal{A}^*d, W\mathcal{A}^*d\rangle,$$

which implies

$$W\mathcal{A}^*d = 0.$$

Therefore, by the assumption that $\Pi_{\mathcal{Q}}(\cdot)$ is Jacobian amicable at $\bar{x}$, we know that

$$\mathcal{A}^* d \in (\mathrm{lin}\,(\mathcal{T}_{\mathcal{Q}}(\bar{x}_+)))^{\perp}.$$

Since the constraint nondegenerate condition (2.15) holds at $\bar{x}_+$, there exists a vector $h \in \mathrm{lin}\,(\mathcal{T}_{\mathcal{Q}}(\bar{x}_+))$ such that $\mathcal{A}h = d$. Hence, since $\mathcal{A}^* d \in (\mathrm{lin}\,(\mathcal{T}_{\mathcal{Q}}(\bar{x}_+)))^{\perp}$ and $h \in \mathrm{lin}\,(\mathcal{T}_{\mathcal{Q}}(\bar{x}_+))$, it holds that

$$\langle d, d\rangle = \langle d, \mathcal{A}h\rangle = \langle \mathcal{A}^* d, h\rangle = 0.$$

Thus $d = 0$. This, together with the fact that V is self-adjoint and positive semidefinite, shows that V is self-adjoint and positive definite. □

Proposition 2.8 motivates us to define the following modified version of the Jacobian-based Newton method (2.8) to solve (2.11):

$$y^{j+1} = y^j - V_j^{-1} F(y^j), \quad V_j := \mathcal{A} W_j \mathcal{A}^*, \quad W_j \in \partial \Pi_{\mathcal{Q}}(x^0 + \mathcal{A}^* y^j), \quad j = 0, 1, 2, \ldots, \tag{2.16}$$

where $y^0 \in \mathcal{Y}$ is an initial point.

By Proposition 2.8, we can obtain the following convergence results for the iterative method (2.16) in a way similar to that of Proposition 2.3. We include a brief proof for the sake of completeness.

PROPOSITION 2.9. *Let $\bar{y} \in \mathcal{Y}$ be such that $F(\bar{y}) = 0$. Let $\bar{x} := x^0 + \mathcal{A}^* \bar{y}$ and $\bar{x}_+ := \Pi_{\mathcal{Q}}(\bar{x})$. Assume that the constraint nondegenerate condition* (2.15) *holds at $\bar{x}_+$, $\Pi_{\mathcal{Q}}(\cdot)$ is semismooth at $\bar{x}$, and $\Pi_{\mathcal{Q}}(\cdot)$ is Jacobian amicable at $\bar{x}$. Then every sequence generated by* (2.16) *converges to $\bar{y}$ superlinearly provided that the starting point y^0 is sufficiently close to $\bar{y}$. Moreover, if $\Pi_{\mathcal{Q}}(\cdot)$ is strongly semismooth at $\bar{x}$, the rate of convergence is quadratic.*

Proof. Since, by Proposition 2.8, every element in $\mathcal{A}\partial\Pi_{\mathcal{Q}}(\bar{x})\mathcal{A}^*$ is positive definite, we know from the basic properties of $\partial\Pi_{\mathcal{Q}}(\cdot)$ (cf. [13, Proposition 2.6.2]) that there exist a constant $c > 0$ and an open neighborhood $\mathcal{N}$ of $\bar{y}$ such that for any $y \in \mathcal{N}$ and $W \in \partial\Pi_{\mathcal{Q}}(x^0 + \mathcal{A}^* y)$, $\mathcal{A}W\mathcal{A}^*$ is positive definite and satisfies

$$\| (\mathcal{A} W \mathcal{A}^*)^{-1} \| \le c. \tag{2.17}$$

By using (2.9) in Definition 2.2, we obtain for any $y \in \mathcal{N}$ and $W \in \partial\Pi_{\mathcal{Q}}(x^0 + \mathcal{A}^* y)$ with $y \to \bar{y}$ that

$$\|\Pi_{\mathcal{Q}}(x^0 + \mathcal{A}^* y) - \Pi_{\mathcal{Q}}(\bar{x}) - W(\mathcal{A}^* y - \mathcal{A}^* \bar{y})\| = o(\|\mathcal{A}^*(y - \bar{y})\|) = o(\|(y - \bar{y})\|), \tag{2.18}$$

which implies that for any $y \in \mathcal{N}$ (shrinking $\mathcal{N}$ if necessary) and $W \in \partial\Pi_{\mathcal{Q}}(x^0 + \mathcal{A}^* y)$ we have

$$\|\Pi_{\mathcal{Q}}(x^0 + \mathcal{A}^* y) - \Pi_{\mathcal{Q}}(\bar{x}) - W(\mathcal{A}^* y - \mathcal{A}^* \bar{y})\| \le (2c\|\mathcal{A}\|)^{-1} \|y - \bar{y}\|. \tag{2.19}$$

Then from (2.16), (2.17), and (2.19), we know that for any $y^j \in \mathcal{N}$, $W_j \in \partial\Pi_{\mathcal{Q}}(x^0 + \mathcal{A}^* y^j)$, and $V_j = \mathcal{A} W_j \mathcal{A}^*$,

$$\begin{aligned}
\|y^{j+1} - \bar{y}\| &= \|y^j - \bar{y} - V_j^{-1} F(y^j)\| \\
&= \|V_j^{-1}[V_j(y^j - \bar{y}) - F(y^j) + F(\bar{y})]\| \\
&\le c\|F(y^j) - F(\bar{y}) - V_j(y^j - \bar{y})\| \\
&\le c\|\mathcal{A}\|\|\Pi_{\mathcal{Q}}(x^0 + \mathcal{A}^* y^j) - \Pi_{\mathcal{Q}}(\bar{x}) - W_j(\mathcal{A}^* y^j - \mathcal{A}^* \bar{y})\| \\
&\le \frac{1}{2}\|y^j - \bar{y}\|.
\end{aligned} \tag{2.20}$$

This shows that once $y^0 \in \mathcal{N}$, the whole iteration sequence $\{y^j\}$ converges to $\bar{y}$ linearly. The superlinear convergence of $\{y^j\}$ follows from (2.18) and the second inequality in (2.20).

If $\Pi_{\mathcal{Q}}(\cdot)$ is strongly semismooth at $\bar{x}$, then by repeating the above process and using (2.10) in Definition 2.2 we can easily prove the quadratic convergence of $\{y^j\}$. □

In Proposition 2.9, besides the well-studied constraint nondegenerate condition (2.15) at $\bar{x}_+$ in the literature, we also assume that $\Pi_{\mathcal{Q}}(\cdot)$ is Jacobian amicable at $\bar{x}$, which may hold for many interesting closed convex cones. Here we discuss the Jacobian amicability of $\Pi_{\mathcal{Q}}(\cdot)$ only for the case that $\mathcal{Q} = \mathcal{S}^n_+$, the cone of symmetric and positive semidefinite matrices in $\mathcal{S}^n$.

Let $A \in \mathcal{S}^n$. Then A admits the following spectral decomposition:

$$A = P\Sigma P^T, \tag{2.21}$$

where

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \sigma_{k_1} & & & & & & \\ & & & 0 & & & & & \\ & & & & \ddots & & & & \\ & & & & & 0 & & & \\ & & & & & & \sigma_{k_1+k_2+1} & & \\ & & & & & & & \ddots & \\ & & & & & & & & \sigma_n \end{bmatrix}$$

with $\sigma_i > 0$ $(i = 1, \ldots, k_1)$ and $\sigma_i < 0$ $(i = k_1 + k_2 + 1, \ldots, n)$ is the diagonal matrix of eigenvalues of A, and

$$P = \begin{bmatrix} P_1 & P_2 & P_3 \end{bmatrix}$$

with $P_1 \in \mathbb{R}^{n\times k_1}$, $P_2 \in \mathbb{R}^{n\times k_2}$, and $P_3 \in \mathbb{R}^{n\times(n-k_1-k_2)}$ is a corresponding orthogonal matrix of orthonormal eigenvectors. Under the Frobenius inner product, the projection $A_+ := \Pi_{\mathcal{S}^n_+}(A)$ of a matrix $A \in \mathcal{S}^n$ onto the cone $\mathcal{S}^n_+$ has an analytic formula [21, 49]

$$A_+ = P \begin{bmatrix} \sigma_1 & & & & & \\ & \ddots & & & & \\ & & \sigma_{k_1} & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{bmatrix} P^T.$$

Define the matrix $U \in \mathbb{R}^{k_1\times(n-k_1-k_2)}$ with entries

$$U_{ij} := \frac{\max\{\sigma_i, 0\} + \max\{\sigma_{k_1+k_2+j}, 0\}}{|\sigma_i| + |\sigma_{k_1+k_2+j}|}, \quad i = 1, \ldots, k_1, \quad j = 1, \ldots, n - k_1 - k_2.$$

Bonnans, Cominetti, and Shapiro [8] showed that $\Pi_{\mathcal{S}^n_+}(\cdot)$ is directionally differentiable everywhere in $\mathcal{S}^n$. For any $H \in \mathcal{S}^n$, D. Sun and J. Sun [47] gave the following explicit

formula for $\Pi'_{\mathcal{S}^n_+}(A;H)$:

$$\Pi'_{\mathcal{S}^n_+}(A;H) = P \begin{bmatrix} P_1^T H P_1 & P_1^T H P_2 & U \circ P_1^T H P_3 \\ P_2^T H P_1 & \Pi_{\mathcal{S}^{k_2}_+}\left(P_2^T H P_2\right) & 0 \\ P_3^T H P_1 \circ U^T & 0 & 0 \end{bmatrix} P^T, \tag{2.22}$$

where $\circ$ denotes the Hadamard product. More importantly, Sun and Sun proved that $\Pi_{\mathcal{S}^n_+}$ is a strongly semismooth matrix-valued function [47]. When A is nonsingular, i.e., $k_2 = 0$, $\Pi_{\mathcal{S}^n_+}(\cdot)$ is continuously differentiable around A, and the formula (2.22) reduces to the classical result of Löwner [28]. See Donoghue [16] and Bhatia [7] for detailed discussions on the latter and Chen, Qi, and Tseng [11] for a generalization.

From (2.13) and (2.22), one can obtain Arnold's well-known characterization of the tangent cone of $\mathcal{S}^n_+$ at A_+ [2],

$$\begin{aligned} \mathcal{T}_{\mathcal{S}^n_+}(A_+) &= \{B \in \mathcal{S}^n \,:\, B = \Pi'_{\mathcal{S}^n_+}(A_+;B)\} \\ &= \left\{B \in \mathcal{S}^n \,:\, \begin{bmatrix} P_2 & P_3 \end{bmatrix}^T B \begin{bmatrix} P_2 & P_3 \end{bmatrix} \succeq 0\right\}, \end{aligned}$$

and its lineality space,

$$\operatorname{lin}\left(\mathcal{T}_{\mathcal{S}^n_+}(A_+)\right) = \left\{B \in \mathcal{S}^n \,:\, \begin{bmatrix} P_2 & P_3 \end{bmatrix}^T B \begin{bmatrix} P_2 & P_3 \end{bmatrix} = 0\right\}. \tag{2.23}$$

Let $\Phi(\cdot) := \Pi'_{\mathcal{S}^n_+}(A;\cdot)$. It is proved in [35, Lemma 11] that

$$\partial_B \Pi_{\mathcal{S}^n_+}(A) = \partial_B \Phi(0) \tag{2.24}$$

and that the operator $W : \mathcal{S}^n \to \mathcal{S}^n$ defined by

$$W(H) = P \begin{bmatrix} P_1^T H P_1 & P_1^T H P_2 & U \circ P_1^T H P_3 \\ P_2^T H P_1 & 0 & 0 \\ P_3^T H P_1 \circ U^T & 0 & 0 \end{bmatrix} P^T \quad \forall\, H \in \mathcal{S}^n \tag{2.25}$$

is an element in $\partial_B \Pi_{\mathcal{S}^n_+}(A)$.

The next proposition shows that $\Pi_{\mathcal{S}^n_+}(\cdot)$ is Jacobian amicable at A.

PROPOSITION 2.10. *The projection operator $\Pi_{\mathcal{S}^n_+}(\cdot)$ is Jacobian amicable everywhere in $\mathcal{S}^n$.*

Proof. Let $A \in \mathcal{S}^n$ admit the spectral decomposition (2.21). Let $W \in \partial\Pi_{\mathcal{S}^n_+}(A)$ and $H \in \mathcal{S}^n$ be such that $W(H) = 0$. Then from (2.22), (2.24), and the definition of the matrix U, we obtain

$$P_1^T H P_1 = 0, \quad P_1^T H P_2 = 0, \quad \text{and} \quad P_1^T H P_3 = 0,$$

which, together with (2.23), implies

$$H \in \left(\operatorname{lin}\left(\mathcal{T}_{\mathcal{S}^n_+}(A_+)\right)\right)^{\perp}.$$

Since A is chosen arbitrarily, we know from the definition that $\Pi_{\mathcal{S}^n_+}(\cdot)$ is Jacobian amicable everywhere in $\mathcal{S}^n$. ∎

3. Dual reformulations. In this section, we will consider the dual problem of the IQEP, i.e., problem (1.6). As mentioned in section 1, we need only focus on the IQEP under Assumption 1.2, i.e., Λ is nonsingular. Denote

$$M := \begin{bmatrix} M_1 & M_2 \\ M_2^T & M_4 \end{bmatrix}, \quad C := \begin{bmatrix} C_1 & C_2 \\ C_2^T & C_4 \end{bmatrix}, \quad K := \begin{bmatrix} K_1 & K_2 \\ K_2^T & K_4 \end{bmatrix},$$

where $M_1, C_1, K_1 \in \mathcal{S}^k$, $M_2, C_2, K_2 \in \mathbb{R}^{k\times(n-k)}$, and $M_4, C_4, K_4 \in \mathcal{S}^{(n-k)}$. Let $S := R\Lambda R^{-1}$. Then problem (1.6) can be written equivalently as

$$\begin{array}{ll} \min & \frac{1}{2}\|M - M_a\|^2 + \frac{1}{2}\|C - C_a\|^2 + \frac{1}{2}\|K - K_a\|^2 \\ \text{s.t.} & \frac{1}{\sqrt{c_1}}(\Lambda^2)^T(R^T M_1 R) + \frac{1}{\sqrt{c_2}}\Lambda^T(R^T C_1 R) + (R^T K_1 R) = 0, \\ & \frac{1}{\sqrt{c_1}}(S^2)^T M_2 + \frac{1}{\sqrt{c_2}}S^T C_2 + K_2 = 0, \\ & (M, C, K) \in \Omega. \end{array} \tag{3.1}$$

Let Ω_0 be equipped with the natural inner product

$$\left\langle (M, C, K), (\widetilde{M}, \widetilde{C}, \widetilde{K}) \right\rangle := \langle M, \widetilde{M}\rangle + \langle C, \widetilde{C}\rangle + \langle K, \widetilde{K}\rangle, \quad (M, C, K), (\widetilde{M}, \widetilde{C}, \widetilde{K}) \in \Omega_0,$$

and its induced norm $\|\cdot\|$. Define

$$\begin{aligned} &\mathcal{H}(M, C, K) \\ &:= \frac{1}{\sqrt{c_1}}(\Lambda^2)^T\left(R^T M_1 R\right) + \frac{1}{\sqrt{c_2}}\Lambda^T\left(R^T C_1 R\right) + \left(R^T K_1 R\right), \quad (M, C, K) \in \Omega_0, \end{aligned} \tag{3.2}$$

$$\mathcal{G}(M, C, K) := \frac{1}{\sqrt{c_1}}(S^2)^T M_2 + \frac{1}{\sqrt{c_2}}S^T C_2 + K_2, \quad (M, C, K) \in \Omega_0, \tag{3.3}$$

and

$$\mathcal{A}(M, C, K) := \left(\mathcal{H}(M, C, K), \mathcal{G}(M, C, K)\right), \quad (M, C, K) \in \Omega_0. \tag{3.4}$$

Then, problem (3.1), which is a special case of problem (2.1), becomes

$$\begin{array}{ll} \min & \frac{1}{2}\|(M, C, K) - (M_a, C_a, K_a)\|^2 \\ \text{s.t.} & \mathcal{A}(M, C, K) = 0, \\ & (M, C, K) \in \Omega. \end{array} \tag{3.5}$$

Let

$$\mathcal{R}(\mathcal{H}) := \{\mathcal{H}(M, C, K) \,:\, (M, C, K) \in \Omega_0\}. \tag{3.6}$$

Obviously, the linear operator $\mathcal{H} : \Omega_0 \to \mathcal{R}(\mathcal{H})$ defined in (3.2) is surjective. The adjoint $\mathcal{H}^* : \mathcal{R}(\mathcal{H}) \to \Omega_0$ of $\mathcal{H}$ is given by

$$\mathcal{H}^*(Y) := \left(\mathcal{H}_1^*(Y), \mathcal{H}_2^*(Y), \mathcal{H}_3^*(Y)\right), \quad Y \in \mathcal{R}(\mathcal{H}), \tag{3.7}$$

where for each $Y \in \mathcal{R}(\mathcal{H})$,

$$\mathcal{H}_1^*(Y) = \frac{1}{2\sqrt{c_1}} \begin{bmatrix} R\Lambda^2 Y R^T + (R\Lambda^2 Y R^T)^T & 0 \\ 0 & 0 \end{bmatrix},$$
$$\mathcal{H}_2^*(Y) = \frac{1}{2\sqrt{c_2}} \begin{bmatrix} R\Lambda Y R^T + (R\Lambda Y R^T)^T & 0 \\ 0 & 0 \end{bmatrix},$$
$$\mathcal{H}_3^*(Y) = \frac{1}{2} \begin{bmatrix} RYR^T + (RYR^T)^T & 0 \\ 0 & 0 \end{bmatrix}.$$

Similarly, $\mathcal{G} : \Omega_0 \to \mathbb{R}^{k\times(n-k)}$ defined in (3.3) is also surjective, and its adjoint $\mathcal{G}^* : \mathbb{R}^{k\times(n-k)} \to \Omega_0$ is given by

$$\mathcal{G}^*(Z) := (\mathcal{G}_1^*(Z), \mathcal{G}_2^*(Z), \mathcal{G}_3^*(Z)), \tag{3.8}$$

where for each $Z \in \mathbb{R}^{k\times(n-k)}$,

$$\mathcal{G}_1^*(Z) = \frac{1}{2\sqrt{c_1}} \begin{bmatrix} 0 & S^2 Z \\ (S^2 Z)^T & 0 \end{bmatrix},$$
$$\mathcal{G}_2^*(Z) = \frac{1}{2\sqrt{c_2}} \begin{bmatrix} 0 & SZ \\ (SZ)^T & 0 \end{bmatrix},$$
$$\mathcal{G}_3^*(Z) = \frac{1}{2} \begin{bmatrix} 0 & Z \\ Z^T & 0 \end{bmatrix}.$$

Since both $\mathcal{H} : \Omega_0 \to \mathcal{R}(\mathcal{H})$ and $\mathcal{G} : \Omega_0 \to \mathbb{R}^{k\times(n-k)}$ are surjective, the linear operator $\mathcal{A} : \Omega_0 \to \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$ defined by (3.4) is also surjective. The adjoint $\mathcal{A}^* : \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)} \to \Omega_0$ of $\mathcal{A}$ takes the following form:

$$\begin{aligned} \mathcal{A}^*(Y,Z) &:= (\mathcal{A}_1^*(Y,Z), \mathcal{A}_2^*(Y,Z), \mathcal{A}_3^*(Y,Z)) \\ &= \mathcal{H}^*(Y) + \mathcal{G}^*(Z), \quad (Y,Z) \in \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}, \end{aligned} \tag{3.9}$$

where for each $(Y,Z) \in \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$, $\mathcal{H}^*(Y)$ and $\mathcal{G}^*(Z)$ are defined by (3.7) and (3.8), respectively. Define $\theta : \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)} \to \mathbb{R}$ by

$$\theta(Y,Z) := \frac{1}{2}\|\Pi_\Omega((M_a, C_a, K_a) + \mathcal{A}^*(Y,Z))\|^2 - \frac{1}{2}\|(M_a, C_a, K_a)\|^2. \tag{3.10}$$

By (2.6) and (2.7), the dual of problem (3.5) is

$$\begin{aligned} \min \quad & \theta(Y,Z) \\ \text{s.t.} \quad & (Y,Z) \in \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}. \end{aligned} \tag{3.11}$$

Since Λ is nonsingular, by Theorem 1.1, problem (1.6), or, equivalently, problem (3.5), admits a strictly feasible solution; i.e., there exists a triplet $(\widehat{M}, \widehat{C}, \widehat{K}) \in \Omega_0$ such that $\mathcal{A}(\widehat{M}, \widehat{C}, \widehat{K}) = 0$ and $(\widehat{M}, \widehat{C}, \widehat{K}) \in \operatorname{int}(\Omega)$. Moreover, the linear operator $\mathcal{A} : \Omega_0 \to \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$ defined by problem (3.5) is surjective. Thus, for the convex optimization problem (3.5), the generalized Slater condition (2.4) is satisfied. Consequently, the following result follows directly from Proposition 2.1.

PROPOSITION 3.1. *There exists at least one pair* $(\overline{Y}, \overline{Z}) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{k\times(n-k)}$ *that solves the dual problem* (3.11). *The unique solution to the original problem* (1.6) *is given by*

$$(\overline{M}, \overline{C}, \overline{K}) = \Pi_\Omega\left((M_a, C_a, K_a) + \mathcal{A}^*(\overline{Y}, \overline{Z})\right).$$

Furthermore, for every real number τ, the level set

$$\{(Y,Z)\in\mathbb{R}^{k\times k}\times\mathbb{R}^{k\times(n-k)}:\ \theta(Y,Z)\le\tau\}$$

is closed, bounded, and convex.

It is clear from Proposition 3.1 that we need only find a solution of the dual problem (3.11) in order to obtain the unique solution of the original problem (1.6).

Remark 3.2. The operator $\mathcal{H}$ defined above is not necessarily surjective from Ω_0 to $\mathbb{R}^{k\times k}$. To see this, we refer the reader to Proposition A.1 in Appendix A.

Next, we discuss the following simplified version of the IQEP:

$$\begin{array}{ll}\min & \dfrac{1}{2}\|(M,C,K)-(M_a,C_a,K_a)\|^2\\ \text{s.t.} & \mathcal{A}(M,C,K)=0,\\ & (M,C,K)\in\Omega_0,\end{array}\tag{3.12}$$

which admits an explicit optimal solution and has been considered by many authors; for example, see [14, 19]. The dual of problem (3.12) is

$$\begin{array}{ll}\min & \theta_0(Y,Z)\\ \text{s.t.} & (Y,Z)\in\mathcal{R}(\mathcal{H})\times\mathbb{R}^{k\times(n-k)},\end{array}\tag{3.13}$$

where $\theta_0:\mathcal{R}(\mathcal{H})\times\mathbb{R}^{k\times(n-k)}\to\mathbb{R}$ is defined by

$$\theta_0(Y,Z):=\frac{1}{2}\|\Pi_{\Omega_0}((M_a,C_a,K_a)+\mathcal{A}^*(Y,Z))\|^2-\frac{1}{2}\|(M_a,C_a,K_a)\|^2.$$

Let $(Y^0,Z^0)\in\mathcal{R}(\mathcal{H})\times\mathbb{R}^{k\times(n-k)}$ be a solution to the dual problem (3.13). By section 2, the solution (M^0,C^0,K^0) to problem (3.12) is given by

$$(M^0,C^0,K^0)=\Pi_{\Omega_0}((M_a,C_a,K_a)+\mathcal{A}^*(Y^0,Z^0))=(M_a,C_a,K_a)+\mathcal{A}^*(Y^0,Z^0).\tag{3.14}$$

Although (Y^0,Z^0) may not be a solution to the dual problem (3.11) as M^0 and K^0 above may fail to be positive semidefinite, it may be a good guess at the solution of problem (3.11). Hence, we will use it as the starting point to our Newton method to be proposed in the next section.

Now we discuss the computation of (Y^0,Z^0) based on (3.13). Since $(Y^0,Z^0)\in\mathcal{R}(\mathcal{H})\times\mathbb{R}^{k\times(n-k)}$ satisfies

$$\nabla\theta_0(Y^0,Z^0)=\mathcal{A}\left((M_a,C_a,K_a)+\mathcal{A}^*(Y^0,Z^0)\right)=0,$$

thus, by (3.4), $(Y^0,Z^0)\in\mathcal{R}(\mathcal{H})\times\mathbb{R}^{k\times(n-k)}$ is a solution of

$$\left(\mathcal{H}\left((M_a,C_a,K_a)+\mathcal{A}^*(Y,Z)\right),\mathcal{G}\left((M_a,C_a,K_a)+\mathcal{A}^*(Y,Z)\right)\right)=0.\tag{3.15}$$

Therefore, we can obtain (Y^0,Z^0) by solving (3.15). Since the linear operator $\mathcal{A}:\Omega_0\to\mathcal{R}(\mathcal{H})\times\mathbb{R}^{k\times(n-k)}$ defined by (3.4) is surjective, $\mathcal{A}\mathcal{A}^*$ is self-adjoint and positive definite. This implies that (Y^0,Z^0) is the unique solution to (3.15).

By (3.7), (3.8), and (3.9), we obtain that for any $(Y,Z)\in\mathcal{R}(\mathcal{H})\times\mathbb{R}^{k\times(n-k)}$,

$$\begin{aligned}&\mathcal{H}((M_a,C_a,K_a)+\mathcal{A}^*(Y,Z))=\mathcal{H}(M_a,C_a,K_a)+\mathcal{H}(\mathcal{H}^*(Y))\\&\quad=\mathcal{H}(M_a,C_a,K_a)+U_1YR^TR\\&\qquad+\frac{1}{2}\left(\frac{1}{c_1}(\Lambda^2)^TR^TRY^T(\Lambda^2)^TR^TR+\frac{1}{c_2}\Lambda^TR^TRY^T\Lambda^TR^TR+R^TRY^TR^TR\right)\end{aligned}$$

and

$$\mathcal{G}((M_a, C_a, K_a) + \mathcal{A}^*(Y, Z)) = \mathcal{G}(M_a, C_a, K_a) + \mathcal{G}(\mathcal{G}^*(Z)) = \mathcal{G}(M_a, C_a, K_a) + U_2 Z,$$

where

$$U_1 := \frac{1}{2}\left(\frac{1}{c_1}(\Lambda^2)^T R^T R \Lambda^2 + \frac{1}{c_2}\Lambda^T R^T R \Lambda + R^T R\right) \tag{3.16}$$

and

$$U_2 := \frac{1}{2}\left(\frac{1}{c_1}(S^2)^T S^2 + \frac{1}{c_2} S^T S + I\right). \tag{3.17}$$

Since the linear operator $\mathcal{H} : \Omega_0 \to \mathcal{R}(\mathcal{H})$ defined by (3.2) is surjective, $\mathcal{H}\mathcal{H}^*$ is self-adjoint and positive definite. So we may use the conjugate gradient (CG) method[1] [20, Algorithm 10.2.1] to solve the first equation of (3.15), i.e.,

(3.18)
$$\begin{aligned} &-\mathcal{H}(M_a, C_a, K_a) = \mathcal{H}(\mathcal{H}^*(Y)) \\ &= U_1 Y R^T R + \frac{1}{2}\left(\frac{1}{c_1}(\Lambda^2)^T R^T R Y^T (\Lambda^2)^T R^T R + \frac{1}{c_2}\Lambda^T R^T R Y^T \Lambda^T R^T R + R^T R Y^T R^T R\right), \end{aligned}$$

to get $Y^0 \in \mathcal{R}(\mathcal{H})$. The required total computational cost of computing Y^0 is $O(k^5)$ flops (the CG method needs at most k^2 steps with each step costing $O(k^3)$ flops). The solution to the second equation of (3.15), i.e,

$$-\mathcal{G}(M_a, C_a, K_a) = \mathcal{G}(\mathcal{G}^*(Z)) = U_2 Z,$$

is given explicitly by

$$Z^0 = -U_2^{-1}\mathcal{G}(M_a, C_a, K_a).$$

We need $O(nk^2)$ flops to compute Z^0. The optimal solution to problem (3.12) can then be obtained via (3.14).

Remark 3.3. The above arguments imply that if we drop the positive semidefiniteness requirement on M and K, we can obtain the solution to problem (3.12) with a total cost of $O(nk^2 + k^5)$. This cost amount is small as in practice $k \ll n$. It is noted that in our approach the presence of the weighting parameters c_1 and c_2 presents no difficulty at all.

4. Algorithm and convergence analysis. In this section, we shall first present a globalized version of Clarke's Jacobian-based Newton method (2.16) for solving the dual problem (3.11) with $\theta : \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)} \to \mathbb{R}$ being defined by (3.10).

Let $F : \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)} \to \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$ be defined by

$$F(Y, Z) = \nabla\theta(Y, Z) = \mathcal{A}\,\Pi_\Omega\left((M_a, C_a, K_a) + \mathcal{A}^*(Y, Z)\right), \quad (Y, Z) \in \mathcal{R}(\mathcal{H})\times\mathbb{R}^{k\times(n-k)},$$

where $\mathcal{A} : \Omega_0 \to \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$ is defined by (3.4) and $\mathcal{A}^*$ is the adjoint of $\mathcal{A}$.

[1]Strictly speaking, we should apply the CG method to the equivalent equation form $Ax = b$ of (3.18), where $A \in \mathcal{S}^m$ is positive definite, $b \in \mathbb{R}^m$, and m is the dimension of $\mathcal{R}(\mathcal{H})$. In the implementation of CG methods there is no need to form A and b explicitly. One may also consider the preconditioned CG method to solve (3.18). However, caution should be taken as the preconditioned CG method may not guarantee that the generated matrices Y stay in $\mathcal{R}(\mathcal{H})$.

ALGORITHM 4.1 (Newton method).

Step 0. Given $(Y^0, Z^0) \in \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$, $\eta \in (0,1)$, $\rho, \delta \in (0, 1/2)$. $j := 0$.

Step 1. Select an element $W_j \in \partial\Pi_\Omega\left((M_a, C_a, K_a) + \mathcal{A}^*(Y^j, Z^j)\right)$ and let $V_j := \mathcal{A}W_j\mathcal{A}^*$. Apply the conjugate gradient method [20, Algorithm 10.2.1] to find an approximate solution $(\Delta Y^j, \Delta Z^j) \in \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$ to

$$F(Y^j, Z^j) + V_j(\Delta Y, \Delta Z) = 0 \tag{4.1}$$

such that

$$\|F(Y^j, Z^j) + V_j(\Delta Y^j, \Delta Z^j)\| \le \eta_j \|F(Y^j, Z^j)\| \tag{4.2}$$

and

$$\left\langle F(Y^j, Z^j), (\Delta Y^j, \Delta Z^j)\right\rangle \le -\eta_j \left\langle (\Delta Y^j, \Delta Z^j), (\Delta Y^j, \Delta Z^j)\right\rangle, \tag{4.3}$$

where $\eta_j := \min\{\eta, \|F(Y^j, Z^j)\|\}$. If (4.2) and (4.3) are not achievable, let

$$(\Delta Y^j, \Delta Z^j) := -F(Y^j, Z^j) = -\mathcal{A}\,\Pi_\Omega\left((M_a, C_a, K_a) + \mathcal{A}^*(Y^j, Z^j)\right).$$

Step 2. Let m_j be the smallest nonnegative integer m such that

$$\theta\left((Y^j, Z^j) + \rho^m(\Delta Y^j, \Delta Z^j)\right) - \theta(Y^j, Z^j) \le \delta\rho^m \left\langle F(Y^j, Z^j), (\Delta Y^j, \Delta Z^j)\right\rangle.$$

Set

$$(Y^{j+1}, Z^{j+1}) := (Y^j, Z^j) + \rho^{m_j}(\Delta Y^j, \Delta Z^j).$$

Step 3. Replace j by $j+1$ and go to Step 1.

Next we make several comments on the above algorithm before we present our convergence analysis.

- *The starting point.* One may choose an arbitrary starting point (Y^0, Z^0) as long as $(Y^0, Z^0) \in \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$. In numerical computation, we recommend choosing the solution to (3.15) as (Y^0, Z^0).
- *The generalized Jacobian.* In Step 1 of Algorithm 4.1, we need to select an element

$$W_j \in \partial\Pi_\Omega\left((M_a, C_a, K_a) + \mathcal{A}^*(Y^j, Z^j)\right)$$

at the jth iteration. This can be computed according to (2.25) once the spectral decompositions of $M_a + \mathcal{A}_1^*(Y^j, Z^j)$ and $K_a + \mathcal{A}_3^*(Y^j, Z^j)$ are available.
- *The conjugate gradient method.* In Step 1 of Algorithm 4.1, if we apply the CG method [20, Algorithm 10.2.1] to (4.1), we can always keep $(\Delta Y^j, \Delta Z^j) \in \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$ when the initial guess is in $\mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$. One may also consider the preconditioned CG method to solve (3.18) as long as the preconditioned CG method can guarantee that $(\Delta Y^j, \Delta Z^j) \in \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$.

The next theorem is our global convergence result.

THEOREM 4.2. *For any $(Y^0, Z^0) \in \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$, Algorithm 4.1 generates an infinite sequence $\{(Y^j, Z^j)\}$ with the properties that for each $j \ge 0$, $(Y^j, Z^j) \in \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$, $\{(Y^j, Z^j)\}$ is bounded, and any accumulation point of $\{(Y^j, Z^j)\}$ is a solution to problem (3.11).*

Proof. Since, by Step 1 of Algorithm 4.1, for any $j \geq 0$, $(\Delta Y^j, \Delta Z^j)$ is always a descent direction of $\theta(\cdot)$ at (Y^j, Z^j) and $(\Delta Y^j, \Delta Z^j) \in \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$, Algorithm 4.1 is well defined. Then an infinite sequence $\{(Y^j, Z^j)\}$ is generated and for each $j \geq 0$, $(Y^j, Z^j) \in \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$.

From Theorem 1.1 we know that problem (3.1) has a strictly feasible solution. This, together with the surjectivity of $\mathcal{A} : \Omega_0 \to \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$, implies that the generalized Slater condition (2.4) holds for problem (3.1). Thus, by Proposition 3.1, the level set

$$\{(Y, Z) \in \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)} \,:\, \theta(Y, Z) \leq \theta(Y^0, Z^0)\}$$

is bounded. Therefore, the sequence $\{(Y^j, Z^j)\}$ is bounded. Furthermore, by employing standard convergence analysis of optimization methods (cf. [15, Theorem 6.3.3]), we can conclude that

$$\lim_{j\to\infty} \nabla\theta(Y^j, Z^j) = 0,$$

which, together with the convexity of $\theta(\cdot)$ and the boundedness of $\{(Y^j, Z^j)\}$, implies that any accumulation point of $\{(Y^j, Z^j)\}$ is a solution to (3.11). ∎

Theorem 4.2 shows that our algorithm converges globally. For discussions on the rate of convergence, we need the constraint nondegenerate condition (2.15) at the solution $(\overline{M}, \overline{C}, \overline{K})$ of problem (1.6). In our case, this condition can be written as

$$\mathcal{A}\left(\operatorname{lin}\left(\mathcal{T}_{\mathcal{S}^n_+}(\overline{M})\right), \mathcal{S}^n, \operatorname{lin}\left(\mathcal{T}_{\mathcal{S}^n_+}(\overline{K})\right)\right) = \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}. \tag{4.4}$$

PROPOSITION 4.3. *Condition* (4.4) *holds if the solution* $(\overline{M}, \overline{C}, \overline{K}) \in \Omega_0$ *to problem* (1.6) *satisfies* $\overline{K} \succ 0$ *or* $\overline{M} \succ 0$.

Proof. Since

$$\operatorname{lin}\left(\mathcal{T}_{\mathcal{S}^n_+}(\overline{K})\right) = \mathcal{S}^n \qquad \text{if } \overline{K} \succ 0,$$
$$\operatorname{lin}\left(\mathcal{T}_{\mathcal{S}^n_+}(\overline{M})\right) = \mathcal{S}^n \qquad \text{if } \overline{M} \succ 0,$$

the proof of Proposition A.1 in Appendix A gives that

$$\mathcal{H}\left(\operatorname{lin}\left(\mathcal{T}_{\mathcal{S}^n_+}(\overline{M})\right), \mathcal{S}^n, \mathcal{S}^n\right) = \mathcal{R}(\mathcal{H})$$

and

$$\mathcal{G}\left(\operatorname{lin}\left(\mathcal{T}_{\mathcal{S}^n_+}(\overline{M})\right), \mathcal{S}^n, \mathcal{S}^n\right) = \mathbb{R}^{k\times(n-k)},$$

and a similar discussion also yields that

$$\mathcal{H}\left(\mathcal{S}^n, \mathcal{S}^n, \operatorname{lin}\left(\mathcal{T}_{\mathcal{S}^n_+}(\overline{K})\right)\right) = \mathcal{R}(\mathcal{H})$$

and

$$\mathcal{G}\left(\mathcal{S}^n, \mathcal{S}^n, \operatorname{lin}\left(\mathcal{T}_{\mathcal{S}^n_+}(\overline{M})\right)\right) = \mathbb{R}^{k\times(n-k)},$$

Proposition 4.3 follows readily. ∎

Another interesting special case that guarantees condition (4.4) is $\Lambda = \tau I$ for some $\tau \neq 0$. This can be seen easily because in this case, for the solution $(\overline{M}, \overline{C}, \overline{K})$ to problem (1.6), we have

$$\mathcal{H}\left(\lin\left(\mathcal{T}_{\mathcal{S}^n_+}(\overline{M})\right), \mathcal{S}^n, \lin\left(\mathcal{T}_{\mathcal{S}^n_+}(\overline{K})\right)\right) = \mathcal{S}^k = \mathcal{R}(\mathcal{H})$$

and

$$\mathcal{G}\left(\lin\left(\mathcal{T}_{\mathcal{S}^n_+}(\overline{M})\right), \mathcal{S}^n, \lin\left(\mathcal{T}_{\mathcal{S}^n_+}(\overline{K})\right)\right) = \mathbb{R}^{k\times(n-k)} = \mathcal{R}(\mathcal{G}),$$

where $\mathcal{R}(\mathcal{G})$ denotes the range of the linear operator $\mathcal{G} : \Omega_0 \to \mathbb{R}^{k\times(n-k)}$ defined by (3.3).

We are now ready to show the quadratic convergence of Algorithm 4.1.

THEOREM 4.4. *Let $(\overline{Y}, \overline{Z})$ be an accumulation point of the infinite sequence $\{(Y^j, Z^j)\}$ generated by Algorithm* 4.1 *for solving the dual problem* (3.11). *Then let $(\overline{M}, \overline{C}, \overline{K})$ be the solution to problem* (1.6). *Assume that the constraint nondegenerate condition* (4.4) *holds at $(\overline{M}, \overline{C}, \overline{K})$. Then the whole sequence $\{(Y^j, Z^j)\}$ converges to $(\overline{Y}, \overline{Z})$ quadratically.*

Proof. By Proposition 3.1, we know

$$(\overline{M}, \overline{C}, \overline{K}) := \Pi_\Omega\left((M_a, C_a, K_a) + \mathcal{A}^*(\overline{Y}, \overline{Z})\right).$$

By Theorem 4.2, the whole sequence $\{(Y^j, Z^j)\}$ is bounded and $(\overline{Y}, \overline{Z})$ is a solution of the dual problem (3.11) such that

$$F(\overline{Y}, \overline{Z}) = \nabla\theta(\overline{Y}, \overline{Z}) = 0.$$

From the constraint nondegenerate condition (4.4) and Propositions 2.8 and 2.10, we know that any element $V := \mathcal{A}W\mathcal{A}^*$ with $W \in \partial\Pi_\Omega(\overline{M}, \overline{C}, \overline{K}))$ is self-adjoint and positive definite. Then, by Proposition 2.4, any element in $\partial F(\overline{Y}, \overline{Z})$ is also positive definite. This, together with the convexity of θ, implies that $(\overline{Y}, \overline{Z})$ is the unique solution of (3.11). Consequently, by Theorem 4.2, the whole sequence $\{(Y^j, Z^j)\}$ converges to $(\overline{Y}, \overline{Z})$. Thus, for all j sufficiently large, V_j is positive definite and $\{\|V_j^{-1}\|\}$ is uniformly bounded. Hence, for all j sufficiently large, the CG method can find $(\Delta Y^j, \Delta Z^j) \in \mathcal{R}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$ such that both (4.2) and (4.3) are satisfied. Then, by Propositions 2.9 and 2.10, for all j sufficiently large,

(4.5)
$$\begin{aligned}\|(Y^j, Z^j) + (\Delta Y^j, \Delta Z^j) - (\overline{Y}, \overline{Z})\| &\le O(\|(Y^j, Z^j) - (\overline{Y}, \overline{Z})\|^2) + \eta_j\|V_j^{-1}\|\|F(Y^j, Z^j)\| \\ &= O(\|(Y^j, Z^j) - (\overline{Y}, \overline{Z})\|^2) + O(\|F(Y^j, Z^j)\|^2) \\ &= O(\|(Y^j, Z^j) - (\overline{Y}, \overline{Z})\|^2),\end{aligned}$$

where in the last inequality the global Lipschitz continuity of F was used. By using (4.5) and the fact that $\{(Y^j, Z^j)\}$ converges to $(\overline{Y}, \overline{Z})$, we have for all j sufficiently large that

$$(Y^j, Z^j) - (\overline{Y}, \overline{Z}) = -(\Delta Y^j, \Delta Z^j) + O(\|(\Delta Y^j, \Delta Z^j)\|^2) \text{ and } (\Delta Y^j, \Delta Z^j) \to 0. \tag{4.6}$$

For each $j \ge 0$, let

$$r_j := F(Y^j, Z^j) + V_j(\Delta Y^j, \Delta Z^j).$$

Then for all j sufficiently large,

$$
\begin{aligned}
&-\left\langle F(Y^j,Z^j),(\Delta Y^j,\Delta Z^j)\right\rangle\\
&=\left\langle(\Delta Y^j,\Delta Z^j),V_j(\Delta Y^j,\Delta Z^j)\right\rangle-\left\langle(\Delta Y^j,\Delta Z^j),r_j\right\rangle\\
&\geq\left\langle(\Delta Y^j,\Delta Z^j),V_j(\Delta Y^j,\Delta Z^j)\right\rangle-\|(\Delta Y^j,\Delta Z^j)\|\|r_j\|\\
&\geq\left\langle(\Delta Y^j,\Delta Z^j),V_j(\Delta Y^j,\Delta Z^j)\right\rangle-\eta_j\|(\Delta Y^j,\Delta Z^j)\|\|F(Y^j,Z^j)\|\\
&\geq\left\langle(\Delta Y^j,\Delta Z^j),V_j(\Delta Y^j,\Delta Z^j)\right\rangle-\|(\Delta Y^j,\Delta Z^j)\|\|F(Y^j,Z^j)\|^2\\
&=\left\langle(\Delta Y^j,\Delta Z^j),V_j(\Delta Y^j,\Delta Z^j)\right\rangle-\|(\Delta Y^j,\Delta Z^j)\|\|F(Y^j,Z^j)-F(\overline{Y},\overline{Z})\|^2\\
&\geq\left\langle(\Delta Y^j,\Delta Z^j),V_j(\Delta Y^j,\Delta Z^j)\right\rangle-\|(\Delta Y^j,\Delta Z^j)\|\|\mathcal{A}\|^2\|\mathcal{A}^*\|^2\|(Y^j,Z^j)-(\overline{Y},\overline{Z})\|^2,
\end{aligned}
$$

which, together with (4.6) and the uniform positive definiteness of V_j, implies that there exists a positive constant $\hat{\rho}>0$ such that for all j sufficiently large we have

$$-\left\langle F(Y^j,Z^j),(\Delta Y^j,\Delta Z^j)\right\rangle\geq\hat{\rho}\|(\Delta Y^j,\Delta Z^j)\|^2.$$

Then, since F is strongly semismooth at $(\overline{Y},\overline{Z})$ as $\Pi_{\mathcal{S}^n_+}(\cdot)$ is strongly semismooth [47], we know from [17, Theorem 3.3 and Remark 3.4] that for all j sufficiently large,

$$\theta\left((Y^j,Z^j)+(\Delta Y^j,\Delta Z^j)\right)-\theta(Y^j,Z^j)\leq\delta\left\langle F(Y^j,Z^j),(\Delta Y^j,\Delta Z^j)\right\rangle,$$

which implies that for all j sufficiently large,

$$(Y^{j+1},Z^{j+1})=(Y^j,Z^j)+(\Delta Y^j,\Delta Z^j).$$

This, together with (4.5), completes the proof. □

Theorems 4.2 and 4.4 present global and local convergence analysis of Algorithm 4.1 for solving the IQEP under Assumption 1.2. In fact, our algorithm can also be used to solve the IQEP of general matrix weights even if Assumption 1.2 fails to hold. Without giving details, we briefly outline the way of doing so in Appendix B.

5. Numerical experiments. In this section, we report our numerical experiments of Algorithm 4.1 for solving the IQEP (1.6) carried out in `MATLAB 7.0.1` running on a PC Intel Pentium IV of 2.40 GHz CPU. In both of the following two numerical examples, we randomly generate $\Lambda\in\mathbb{R}^{k\times k}$ by using the built-in function `randn` in `MATLAB 7.0.1`. The total number of complex-valued eigenvalues is chosen to be around $k/2$. The matrix $R\in\mathbb{R}^{k\times k}$ is obtained by applying the QR factorization to a random generated $n\times k$ matrix X by using `randn`.

Example 5.1. Let $\widehat{M}$, $\widehat{C}$, and $\widehat{K}$ be given by (1.7). Set

$$M_a:=\widehat{M}+\tau R_M,\quad C_a:=\widehat{C}+\tau R_C,\quad K_a:=\widehat{K}+\tau R_K,$$

where R_M, R_C, and R_K are $n\times n$ symmetric matrices with random entries uniformly distributed between -1.0 and 1.0, and $\tau\in\mathbb{R}$ is a perturbed parameter. We report our numerical results for (a) $k=30$, $n=100,200,500,1000,1500,2000$, and $\tau=0.1,1.0$ and (b) $k\approx n/3$, $n=100,200,300,400,450$, and $\tau=0.1,1.0$.

Example 5.2. The matrices G_M and G_K are random $n\times n$ correlation matrices generated by `MATLAB 7.0.1`'s `gallery ('randcorr',n)`, and the matrix G_C is a random $n\times n$ symmetric matrix with entries $(G_C)_{ij}\in[-1,1]$ and $(G_C)_{ii}=1.0$ for $i,j=1,2,\ldots,n$. Then M_a, C_a, and K_a are, respectively, obtained via perturbing $\sqrt{c_1}G_M$, $\sqrt{c_2}G_C$, and G_K by a random $n\times n$ symmetric matrix with entries in $[-\tau,\tau]$, where $\tau=0.1,1.0$. We report our numerical results for (a) $k=30$,

TABLE 1
Numerical results of Example 5.1(a).

$k = 30$, $c_1 = c_2 = 1$							
τ	n	cputime	`It.`	`Func.`	`Res0.`	`Res*.`	`Tol.`
0.1	100	01 m 26 s	18	24	1.2×10^2	5.4×10^{-10}	3.9×10^{-11}
	200	04 m 39 s	14	15	3.2×10^2	9.0×10^{-10}	3.9×10^{-11}
	500	21 m 16 s	11	12	1.8×10^3	6.1×10^{-9}	1.3×10^{-10}
	1,000	44 m 13 s	9	10	4.2×10^3	9.3×10^{-8}	1.1×10^{-9}
	1,500	08 h 49 m 11 s	7	8	9.0×10^3	1.9×10^{-6}	1.6×10^{-8}
	2,000	05 h 24 m 37 s	9	10	1.4×10^4	5.0×10^{-6}	3.3×10^{-8}
1.0	100	40.6 s	10	11	1.2×10^3	1.9×10^{-9}	2.7×10^{-11}
	200	01 m 37 s	9	11	3.7×10^3	1.8×10^{-6}	1.3×10^{-8}
	500	09 m 03 s	10	12	1.4×10^4	7.0×10^{-7}	2.0×10^{-9}
	1,000	01 h 09 m 50 s	10	11	4.5×10^4	1.7×10^{-5}	2.4×10^{-8}
	1,500	08 h 12 m 53 s	14	18	6.7×10^4	3.9×10^{-5}	3.6×10^{-8}
	2,000	08 h 54 m 57 s	10	12	1.0×10^5	8.0×10^{-5}	5.7×10^{-8}
$k = 30$, $c_1 = 10$, and $c_2 = 0.1$							
τ	n	cputime	`It.`	`Func.`	`Res0.`	`Res*.`	`Tol.`
0.1	100	35.5 s	9	10	7.7×10^1	5.1×10^{-9}	2.8×10^{-10}
	200	04 m 18 s	9	10	2.8×10^2	1.7×10^{-6}	5.1×10^{-8}
	500	20 m 14 s	7	8	1.4×10^3	5.4×10^{-7}	7.3×10^{-9}
	1,000	53 m 30 s	8	9	3.7×10^3	1.8×10^{-6}	1.3×10^{-8}
	1,500	05 h 50 m 22 s	10	11	6.0×10^3	7.9×10^{-6}	3.7×10^{-8}
	2,000	05 h 22 m 25 s	9	10	8.1×10^3	1.3×10^{-6}	4.8×10^{-9}
1.0	100	36.4 s	6	7	1.3×10^3	1.3×10^{-6}	9.4×10^{-9}
	200	03 m 10 s	7	8	3.7×10^3	5.1×10^{-8}	1.9×10^{-10}
	500	19 m 33 s	8	9	1.2×10^4	5.3×10^{-6}	7.8×10^{-9}
	1,000	02 h 02 m 35 s	10	11	2.8×10^4	8.9×10^{-7}	6.6×10^{-10}
	1,500	04 h 52 m 15 s	11	14	6.9×10^4	6.7×10^{-7}	3.3×10^{-10}
	2,000	12 h 30 m 15 s	11	12	8.9×10^4	3.2×10^{-5}	1.2×10^{-8}

$n = 100, 200, 500, 1000, 1500, 2000$, and $\tau = 0.1, 1.0$ and (b) $k \approx n/3$, $n = 100, 200, 300, 400, 450$, and $\tau = 0.1, 1.0$.

In our numerical experiments, the initial point (Y^0, Z^0) is chosen to be the solution to (3.15), and the stopping criterion is

$$\texttt{Tol.} := \frac{\|\nabla\theta(Y_k, Z_k)\|}{\max\left\{1, \left\|\left(\frac{1}{\sqrt{c_1}}M_a, \frac{1}{\sqrt{c_2}}C_a, K_a\right)\right\|\right\}} \le 10^{-7}.$$

We set other parameters used in our algorithm as $\eta = 10^{-6}$, $\rho = 0.5$, and $\delta = 10^{-4}$. Our numerical results are given in Tables 1–4, where `It.`, `Func.`, `Res0.`, and `Res*.` stand for the number of iterations, the number of function evaluations, and the residuals $\|\nabla\theta(\cdot)\|$ at the starting point (Y^0, Z^0) and at the final iterate of our algorithm (the largest number of iterations in CG is set to be $\max(5000, nk)$), respectively. In Algorithm 4.1, the major cost at the jth iteration is solving the linear system (4.1), i.e.,

$$\begin{pmatrix} \mathcal{H}\left(\Pi_\Omega((M_a, C_a, K_a) + \mathcal{A}^*(Y^j, Z^j)) + W_j\mathcal{A}^*(\Delta Y^j, \Delta Z^j)\right) \\ \mathcal{G}\left(\Pi_\Omega((M_a, C_a, K_a) + \mathcal{A}^*(Y^j, Z^j)) + W_j\mathcal{A}^*(\Delta Y^j, \Delta Z^j)\right) \end{pmatrix} = 0,$$

where $W_j \in \partial\Pi_\Omega\left((M_a, C_a, K_a) + \mathcal{A}^*(Y^j, Z^j)\right)$. For better numerical performance, instead of solving the above linear system directly, we apply the CG method to the following preconditioned linear system [45, Chapter 9]:

TABLE 2
Numerical results of Example 5.1(b).

$k \approx n/3$, $c_1 = c_2 = 1$								
τ	n	k	cputime	It.	Func.	Res0.	Res*.	Tol.
0.1	100	33	02 m 31 s	18	23	1.0×10^{2}	9.8×10^{-7}	7.2×10^{-8}
	200	66	01 h 05 m 26 s	40	69	5.3×10^{2}	1.4×10^{-8}	6.3×10^{-10}
	300	100	01 h 37 m 32 s	20	21	1.0×10^{3}	4.8×10^{-7}	1.6×10^{-8}
	400	133	10 h 23 m 23 s	15	16	2.0×10^{3}	1.6×10^{-8}	4.5×10^{-10}
	450	150	10 h 24 m 04 s	14	15	2.7×10^{3}	3.0×10^{-8}	7.4×10^{-10}
1.0	100	33	01 m 43 s	10	12	1.4×10^{3}	1.7×10^{-7}	2.3×10^{-9}
	200	66	22 m 02 s	19	35	5.5×10^{3}	1.4×10^{-5}	9.5×10^{-8}
	300	100	01 h 32 m 58 s	12	14	1.2×10^{4}	1.3×10^{-5}	5.9×10^{-8}
	400	133	02 h 46 m 54 s	14	16	2.4×10^{4}	2.4×10^{-6}	8.3×10^{-9}
	450	150	09 h 38 m 15 s	18	21	2.9×10^{4}	4.9×10^{-7}	1.5×10^{-9}
$k \approx n/3$, $c_1 = 10$, and $c_2 = 0.1$								
τ	n	k	cputime	It.	Func.	Res0.	Res*.	Tol.
0.1	100	33	46.1 s	9	11	7.5×10^{1}	2.6×10^{-8}	1.4×10^{-9}
	200	66	42 m 42 s	13	15	3.6×10^{2}	1.8×10^{-6}	5.8×10^{-8}
	300	100	02 h 24 m 23 s	17	20	1.1×10^{3}	2.9×10^{-7}	6.5×10^{-9}
	400	133	04 h 38 m 42 s	10	11	1.8×10^{3}	2.4×10^{-6}	4.0×10^{-8}
	450	150	12 h 23 m 44 s	13	14	2.2×10^{3}	5.8×10^{-7}	8.8×10^{-9}
1.0	100	33	02 m 29 s	10	11	1.5×10^{3}	2.5×10^{-8}	1.8×10^{-10}
	200	66	23 m 26 s	10	11	5.0×10^{3}	5.5×10^{-8}	2.0×10^{-10}
	300	100	01 h 17 m 47 s	10	11	1.0×10^{4}	2.9×10^{-7}	7.1×10^{-10}
	400	133	06 h 24 m 49 s	17	22	1.9×10^{4}	8.4×10^{-6}	1.5×10^{-8}
	450	150	06 h 39 m 33 s	11	12	2.6×10^{4}	6.7×10^{-7}	1.1×10^{-9}

TABLE 3
Numerical results of Example 5.2(a).

$k = 30$, $c_1 = c_2 = 1$							
τ	n	cputime	It.	Func.	Res0.	Res*.	Tol.
0.1	100	25.1 s	9	10	6.2×10^{2}	1.2×10^{-8}	2.6×10^{-10}
	200	02 m 17 s	8	9	2.4×10^{3}	8.9×10^{-9}	1.0×10^{-10}
	500	16 m 55 s	10	53	9.9×10^{3}	6.5×10^{-8}	3.1×10^{-10}
	1,000	01 h 39 m 23 s	11	12	2.8×10^{4}	1.9×10^{-5}	4.6×10^{-8}
	1,500	04 h 48 m 57 s	11	27	5.5×10^{4}	3.9×10^{-5}	6.2×10^{-8}
	2,000	05 h 01 m 57 s	7	8	8.7×10^{4}	1.9×10^{-5}	2.2×10^{-8}
1.0	100	40.3 s	13	15	1.7×10^{3}	1.3×10^{-7}	1.5×10^{-9}
	200	01 m 30 s	11	13	4.2×10^{3}	1.4×10^{-6}	8.4×10^{-9}
	500	31 m 46 s	15	45	1.3×10^{4}	2.5×10^{-8}	6.1×10^{-11}
	1,000	03 h 04 m 28 s	12	15	4.7×10^{4}	1.8×10^{-7}	2.2×10^{-10}
	1,500	05 h 09 m 11 s	10	11	8.7×10^{4}	1.6×10^{-5}	1.3×10^{-8}
	2,000	12 h 47 m 19 s	9	10	1.2×10^{5}	2.3×10^{-5}	1.4×10^{-8}
$k = 30$, $c_1 = 10$, and $c_2 = 0.1$							
τ	n	cputime	It.	Func.	Res0.	Res*.	Tol.
0.1	100	31.4 s	8	9	1.7×10^{2}	9.5×10^{-9}	2.0×10^{-10}
	200	04 m 30 s	11	12	3.0×10^{2}	5.3×10^{-6}	5.9×10^{-8}
	500	17 m 31 s	8	9	1.8×10^{3}	1.8×10^{-6}	8.2×10^{-9}
	1,000	02 h 17 m 39 s	9	10	2.8×10^{3}	1.9×10^{-6}	4.4×10^{-9}
	1,500	04 h 32 m 10 s	7	8	6.4×10^{3}	3.5×10^{-7}	5.4×10^{-10}
	2,000	Failed	7	8	1.0×10^{4}	5.6×10^{-4}	6.5×10^{-7}
1.0	100	02 m 36 s	13	14	1.2×10^{3}	7.0×10^{-7}	4.8×10^{-9}
	200	03 m 30 s	7	8	2.8×10^{3}	3.0×10^{-8}	1.0×10^{-10}
	500	45 m 44 s	11	14	1.2×10^{4}	7.0×10^{-6}	9.8×10^{-9}
	1,000	54 m 36 s	9	11	2.9×10^{4}	1.4×10^{-5}	9.6×10^{-9}
	1,500	04 h 53 m 19 s	8	9	6.1×10^{4}	9.8×10^{-5}	4.6×10^{-8}
	2,000	10 h 53 m 54 s	6	7	9.3×10^{4}	2.2×10^{-4}	7.7×10^{-8}

TABLE 4
Numerical results of Example 5.2(b).

$k \approx n/3$, $c_1 = c_2 = 1$								
τ	n	k	cputime	It.	Func.	Res0.	Res*.	Tol.
0.1	100	33	47.8 s	12	13	8.3×10^2	1.5×10^{-6}	3.3×10^{-8}
	200	66	12 m 01 s	16	22	3.3×10^3	1.6×10^{-8}	1.8×10^{-10}
	300	100	01 h 58 m 22 s	14	15	7.7×10^3	2.2×10^{-7}	1.7×10^{-9}
	400	133	05 h 39 m 37 s	17	18	1.6×10^4	6.2×10^{-8}	3.7×10^{-10}
	450	150	05 h 28 m 26 s	20	21	1.7×10^4	5.1×10^{-8}	2.7×10^{-10}
1.0	100	33	37.4 s	9	10	1.6×10^3	2.1×10^{-6}	2.5×10^{-8}
	200	66	01 h 24 m 43 s	45	75	5.2×10^3	1.8×10^{-7}	1.1×10^{-9}
	300	100	01 h 38 m 58 s	29	33	1.5×10^4	6.3×10^{-7}	2.6×10^{-9}
	400	133	04 h 36 m 30 s	16	18	2.5×10^4	2.6×10^{-6}	8.0×10^{-9}
	450	150	04 h 55 m 35 s	18	20	3.1×10^4	1.0×10^{-5}	2.7×10^{-8}
$k \approx n/3$, $c_1 = 10$, and $c_2 = 0.1$								
τ	n	k	cputime	It.	Func.	Res0.	Res*.	Tol.
0.1	100	33	52.0 s	8	9	1.8×10^2	1.0×10^{-7}	2.2×10^{-9}
	200	66	11 m 01 s	9	10	4.5×10^2	5.6×10^{-7}	6.3×10^{-9}
	300	100	02 h 20 m 24 s	13	14	1.1×10^3	1.4×10^{-7}	1.1×10^{-9}
	400	133	03 h 06 m 49 s	10	11	3.0×10^3	1.3×10^{-7}	7.5×10^{-10}
	450	150	06 h 01 m 45 s	15	16	2.4×10^3	1.3×10^{-6}	6.6×10^{-9}
1.0	100	33	01 m 54 s	9	10	9.8×10^2	1.7×10^{-8}	1.2×10^{-10}
	200	66	22 m 27 s	8	9	4.7×10^3	2.2×10^{-5}	7.6×10^{-8}
	300	100	01 h 21 m 14 s	11	13	9.8×10^3	4.3×10^{-6}	1.0×10^{-8}
	400	133	03 h 22 m 32 s	12	14	1.9×10^4	2.5×10^{-5}	4.4×10^{-8}
	450	150	06 h 59 m 29 s	16	20	2.3×10^4	1.7×10^{-5}	2.5×10^{-8}

$$\begin{pmatrix} U_1^{-1}\mathcal{H}\left(\Pi_\Omega((M_a, C_a, K_a) + \mathcal{A}^*(Y^j, Z^j)) + W_j\mathcal{A}^*(\Delta Y^j, \Delta Z^j)\right)(R^T R)^{-1} \\ U_2^{-1}\mathcal{G}\left(\Pi_\Omega((M_a, C_a, K_a) + \mathcal{A}^*(Y^j, Z^j)) + W_j\mathcal{A}^*(\Delta Y^j, \Delta Z^j)\right) \end{pmatrix} = 0,$$

where U_1 and U_2 are given by (3.16) and (3.17), respectively. That is, we apply a preconditioned CG method to the linear system (4.1).

Based on our numerical experiments, we make the following observations.

- Our algorithm converges to the required accuracy at a relatively small number of iterations for all but one case ($n = 2,000$, $k = 30$, $c_1 = 10$, $c_2 = 0.1$, $\tau = 1$) in Table 3. Here, Failed means that the CG method fails to converge within the required maximum number of iterations. Therefore, it is not contradictive with the global convergence result as in Theorem 4.2. Quadratic convergence was observed for all convergent cases. This confirms our theoretical result on quadratic convergence.
- The largest numerical examples that we tested in this paper are (i) $n = 2{,}000$ and $k = 30$ and (ii) $n = 450$ and $k = 150$. For case (i), there are roughly 6,000,000 unknowns in the primal problem and 60,000 unknowns in the dual problem, while for case (ii), these numbers are roughly 300,000 and 67,000, respectively. In consideration of the scales of problems solved, our algorithm is very effective.
- We also tested the cases where $k \approx n/4$, which have better numerical performance than the cases where $k \approx n/3$. Generally speaking, the smaller the ratio $\frac{k}{n}$ is, the better our algorithm performs. This is no surprise as a smaller ratio $\frac{k}{n}$ implies that the constraint nondegenerate condition (4.4) is more likely to hold.
- Finally, note that the numbers of iterations for some small cases (see, for instance, $n = 100, 200$, $\tau = 0.1$, $c_1 = c_2 = 1$ in Table 1 and $n = 300$, $\tau = 0.1$,

TABLE 5
Numerical results of Example 5.1(I).

$k = 10$, $c_1 = c_2 = 1$, and $\tau = 0.1$					
Alg.	n	cputime	`It.`	`Func.`	`Res*.`
Newton	50	1.0 s	6	7	3.1×10^{-5}
	100	2.0 s	8	9	1.0×10^{-5}
	150	3.0 s	8	9	2.9×10^{-5}
	200	3.9 s	7	8	1.0×10^{-6}
BFGS	50	14.3 s	163	328	4.9×10^{-5}
	100	51.4 s	197	395	4.8×10^{-5}
	150	01 m 59 s	202	406	4.7×10^{-5}
	200	LS failed	214	447	5.1×10^{-5}
$k = 10$, $c_1 = 10$, $c_2 = 0.1$, and $\tau = 0.1$					
Alg.	n	cputime	`It.`	`Func.`	`Res*.`
Newton	50	0.9 s	5	6	4.7×10^{-5}
	100	1.6 s	6	7	9.0×10^{-6}
	150	2.2 s	6	7	4.5×10^{-5}
	200	4.1 s	7	8	3.2×10^{-6}
BFGS	50	10.9 s	128	257	3.1×10^{-5}
	100	34.6 s	135	271	3.7×10^{-5}
	150	01 m 20 s	134	269	4.1×10^{-5}
	200	02 m 29 s	135	270	4.4×10^{-5}

$c_1 = c_2 = 1$ in Table 2) are larger than those of some large-scale cases. This is normal, as these problems are randomly generated so that the resulting linear system (4.1) may have different condition numbers and the total number of iterations may be different.

To further illustrate the effectiveness of our method, we compared the performance of our method with that of the well-known BFGS method with the Wolfe line search, which was described in detail in [34, Chapter 8] and was suggested by Malick [30] to solve some semidefinite least-squares problems.

In our experiments for the comparison, the starting point is the solution of (3.15), and the stopping tolerance is

$$\|\nabla\theta(Y^j, Z^j)\| \le 5.0 \times 10^{-5}.$$

The reason that we did not choose a higher accuracy is because the BFGS method may run into difficulties for an accuracy greater than 5.0×10^{-5}. In Tables 5–8, we report the numerical results of Examples 5.1 and 5.2 for the following two cases: (I) $k = 10$, $n = 50, 100, 150, 200$, and $\tau = 0.1$ and (II) $k \approx n/3$, $n = 50, 75, 100$, and $\tau = 0.1$. Table 9 includes the numerical results of Example 5.2 for the fixed $n = 100$ and $k = 10$ with the varying τ: $\tau = 0.01, 0.1, 1.0$. Here, `Res*.` stands for the residuals $\|\nabla\theta(\cdot)\|$ at the final iterate of an algorithm (the maximal number of iterations is set to be 5,000), while `It.` and `Func.` still denote the number of iterations and the number of function evaluations, respectively. LS failed means that the line search failed (i.e., the step length is too small to proceed) during the computation.

We see from the numerical performances that our method needs fewer than 15 iterations for all the problems to obtain the required accuracy, while the BFGS method needs more iterations. In terms of the cputime, our method also performs much better than the BFGS method.

6. Conclusions. In this paper, we considered the inverse quadratic eigenvalue problem (IQEP) with the positive semidefiniteness condition on matrices M and K.

TABLE 6
Numerical results of Example 5.1(II).

$k \approx n/3$, $c_1 = c_2 = 1$, and $\tau = 0.1$						
Alg.	n	k	cputime	`It.`	`Func.`	`Res*.`
Newton	50	16	2.3 s	10	12	2.6×10^{-6}
	75	25	4.8 s	10	11	2.3×10^{-6}
	100	33	18.1 s	13	15	1.1×10^{-5}
BFGS	50	16	48.2 s	326	653	3.3×10^{-5}
	75	25	07 m 40 s	714	1429	4.3×10^{-5}
	100	33	43 m 51 s	1416	2833	4.8×10^{-5}
$k \approx n/3$, $c_1 = 10$, $c_2 = 0.1$, and $\tau = 0.1$						
Alg.	n	k	cputime	`It.`	`Func.`	`Res*.`
Newton	50	16	2.3 s	8	9	1.1×10^{-5}
	75	25	4.9 s	8	9	2.1×10^{-5}
	100	33	13.3 s	10	11	2.0×10^{-5}
BFGS	50	16	40.6 s	284	569	4.3×10^{-5}
	75	25	06 m 39 s	624	1249	4.7×10^{-5}
	100	33	38 m 09 s	1223	2448	4.9×10^{-5}

TABLE 7
Numerical results of Example 5.2(I).

$k = 10$, $c_1 = c_2 = 1$, and $\tau = 0.1$					
Alg.	n	cputime	`It.`	`Func.`	`Res*.`
Newton	50	2.6 s	13	15	5.1×10^{-7}
	100	5.3 s	14	16	1.0×10^{-6}
	150	8.7 s	13	14	2.1×10^{-5}
	200	13.4 s	12	13	2.1×10^{-5}
BFGS	50	22.3 s	294	471	4.2×10^{-5}
	100	01 m 14 s	316	420	3.2×10^{-5}
	150	02 m 57 s	344	449	3.7×10^{-5}
	200	05 m 50 s	372	474	4.3×10^{-5}
$k = 10$, $c_1 = 10$, $c_2 = 0.1$, and $\tau = 0.1$					
Alg.	n	cputime	`It.`	`Func.`	`Res*.`
Newton	50	1.4 s	8	9	6.1×10^{-7}
	100	2.3 s	7	8	2.0×10^{-5}
	150	5.7 s	8	9	3.4×10^{-5}
	200	9.6 s	10	11	1.3×10^{-6}
BFGS	50	16.3 s	203	387	4.6×10^{-5}
	100	56.9 s	229	419	3.8×10^{-5}
	150	03 m 00 s	274	507	2.4×10^{-5}
	200	05 m 15 s	300	551	3.9×10^{-5}

We expressed the IQEP as a semidefinite constraint nonlinear optimization problem and introduced a quadratically convergent Newton method for solving the problem. Our numerical experiments show that our method is very efficient. We also observed that the BFGS method converges much more slowly than the Newton method and, moreover, that the Newton method is capable of solving problems of large scales. Since the vast majority of our computer cputime is spent on the preconditioned conjugate gradient method for solving the linear equation (4.1), we would save much computing time by finding a better preconditioner for (4.1). We leave this as our future research topic. Another interesting topic would be to study the IQEP with M, C, and K being of various special structures.

Appendix A. In this appendix, we study the linear space $\mathcal{R}(\mathcal{H})$ defined by (3.6), which varies according to the distribution of the prescribed eigenvalues. For the sake

TABLE 8
Numerical results of Example 5.2(II).

$k \approx n/3$, $c_1 = c_2 = 1$, and $\tau = 0.1$						
Alg.	n	k	cputime	`It.`	`Func.`	`Res*.`
Newton	50	16	2.2 s	7	8	5.6×10^{-6}
	75	25	4.0 s	7	8	1.7×10^{-5}
	100	33	13.1 s	8	9	2.1×10^{-6}
BFGS	50	16	47.0 s	331	664	4.0×10^{-5}
	75	25	08 m 23 s	789	1580	4.5×10^{-5}
	100	33	39 m 10 s	1260	2523	4.9×10^{-5}
$k \approx n/3$, $c_1 = 10$, $c_2 = 0.1$, and $\tau = 0.1$						
Alg.	n	k	cputime	`It.`	`Func.`	`Res*.`
Newton	50	16	1.9 s	6	7	3.1×10^{-6}
	75	25	5.3 s	6	7	4.3×10^{-5}
	100	33	10.5 s	7	8	6.0×10^{-6}
BFGS	50	16	42.8 s	298	597	4.9×10^{-5}
	75	25	07 m 39 s	717	1435	4.7×10^{-5}
	100	33	37 m 34 s	1206	2413	3.3×10^{-5}

TABLE 9
Numerical results of Example 5.2.

$n = 100$, $k = 10$, and $c_1 = c_2 = 1$					
Alg.	τ	cputime	`It.`	`Func.`	`Res*.`
Newton	0.01	2.8 s	10	11	8.0×10^{-7}
	0.1	2.8 s	10	11	7.6×10^{-7}
	1.0	2.7 s	9	10	8.1×10^{-6}
BFGS	0.01	01 m 10 s	273	538	2.9×10^{-5}
	0.1	01 m 18 s	279	559	4.0×10^{-5}
	1.0	LS failed	472	997	8.7×10^{-5}
$n = 100$, $k = 10$, $c_1 = 10$, and $c_2 = 0.1$					
Alg.	τ	cputime	`It.`	`Func.`	`Res*.`
Newton	0.01	1.4 s	6	7	1.3×10^{-5}
	0.1	1.6 s	6	7	7.2×10^{-6}
	1.0	1.9 s	7	8	2.4×10^{-5}
BFGS	0.01	48.3 s	185	369	4.9×10^{-5}
	0.1	53.9 s	209	417	2.0×10^{-5}
	1.0	01 m 29 s	348	694	4.7×10^{-5}

of convenience, we define a new linear operator $\mathcal{J} : \mathcal{S}^k \times \mathcal{S}^k \times \mathcal{S}^k \to \mathbb{R}^{k \times k}$ by

$$\mathcal{J}(B, D, E) := \frac{1}{\sqrt{c_1}}(\Lambda^2)^T B + \frac{1}{\sqrt{c_2}}\Lambda^T D + E, \quad (B, D, E) \in \mathcal{S}^k \times \mathcal{S}^k \times \mathcal{S}^k,$$

where Λ is defined by (1.3).

Denote

$$\mathcal{R}(\mathcal{J}) := \{\mathcal{J}(B, D, E) \,:\, (B, D, E) \in \mathcal{S}^k \times \mathcal{S}^k \times \mathcal{S}^k\}.$$

By the definitions of $\mathcal{R}(\mathcal{H})$ and $\mathcal{R}(\mathcal{J})$, we can see that $\mathcal{R}(\mathcal{H}) = \mathcal{R}(\mathcal{J})$. The following result gives the exact number of $\mathrm{Dim}\,(\mathcal{R}(\mathcal{H}))$, the dimension of $\mathcal{R}(\mathcal{H})$.

PROPOSITION A.1. *Suppose that the prescribed eigenmatrix* Λ *is defined by* (1.3). *Then we have*

$$\mathrm{Dim}\,(\mathcal{R}(\mathcal{H})) = \mathrm{Dim}\,(\mathcal{R}(\mathcal{J})) = k^2 - \sum_{i=1}^{\mu} s_i(s_i - 1) - \frac{1}{2}\sum_{i=\mu+1}^{\nu} s_i(s_i - 1).$$

In particular, if $s_1 = \cdots = s_\mu = s_{\mu+1} = \cdots = s_\nu = 1$, $\mathrm{Dim}\,(\mathcal{R}(\mathcal{H})) = \mathrm{Dim}\,(\mathcal{R}(\mathcal{J})) = k^2$.

Proof. For any $\widetilde{D} \in \mathcal{S}^k$ and $Y \in \mathbb{R}^{k\times k}$, denote

$$\widetilde{D} = \begin{bmatrix} \widetilde{D}_{1,1} & \cdots & \widetilde{D}_{1,\mu} & \widetilde{D}_{1,\mu+1} & \cdots & \widetilde{D}_{1,\nu} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{D}_{1,\mu}^T & \cdots & \widetilde{D}_{\mu,\mu} & \widetilde{D}_{\mu,\mu+1} & \cdots & \widetilde{D}_{\mu,\nu} \\ \widetilde{D}_{1,\mu+1}^T & \cdots & \widetilde{D}_{\mu,\mu+1}^T & \widetilde{D}_{\mu+1,\mu+1} & \cdots & \widetilde{D}_{\mu+1,\nu} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{D}_{1,\nu}^T & \cdots & \widetilde{D}_{\mu,\nu}^T & \widetilde{D}_{\mu+1,\nu}^T & \cdots & \widetilde{D}_{\nu,\nu} \end{bmatrix}$$

and

$$Y = \begin{bmatrix} Y_{1,1} & \cdots & Y_{1,\mu} & Y_{1,\mu+1} & \cdots & Y_{1,\nu} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ Y_{\mu,1} & \cdots & Y_{\mu,\mu} & Y_{\mu,\mu+1} & \cdots & Y_{\mu,\nu} \\ Y_{\mu+1,1} & \cdots & Y_{\mu+1,\mu} & Y_{\mu+1,\mu+1} & \cdots & Y_{\mu+1,\nu} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ Y_{\nu,1} & \cdots & Y_{\nu,\mu} & Y_{\nu,\mu+1} & \cdots & Y_{\nu,\nu} \end{bmatrix}, \tag{A.1}$$

where $\widetilde{D}_{i,j}, Y_{i,j} \in \mathbb{R}^{s_i\times s_j}$ and $Y_{j,i} \in \mathbb{R}^{s_j\times s_i}$ with $1 \le i \le j \le \nu$.

First, we note that for any $Y \in \mathbb{R}^{k\times k}$,

(A.2)

$$\begin{aligned} & Y \in \mathcal{R}(\mathcal{J}) \\ \Leftrightarrow\ & \tfrac{1}{\sqrt{c_1}}(\Lambda^2)^T B + \tfrac{1}{\sqrt{c_2}}\Lambda^T D + E = Y && \text{for some } (B,D,E) \in \mathcal{S}^k\times\mathcal{S}^k\times\mathcal{S}^k \\ \Leftrightarrow\ & E = Y - (\Lambda^2)^T\left(\tfrac{1}{\sqrt{c_1}}B\right) - \Lambda^T\left(\tfrac{1}{\sqrt{c_2}}D\right) && \text{for some } (B,D,E) \in \mathcal{S}^k\times\mathcal{S}^k\times\mathcal{S}^k \\ \Leftrightarrow\ & Y - (\Lambda^2)^T\left(\tfrac{1}{\sqrt{c_1}}B\right) - \Lambda^T\left(\tfrac{1}{\sqrt{c_2}}D\right) = E = E^T \\ & \quad = Y^T - \left(\tfrac{1}{\sqrt{c_1}}B\right)\Lambda^2 - \left(\tfrac{1}{\sqrt{c_2}}D\right)\Lambda && \text{for some } (B,D,E) \in \mathcal{S}^k\times\mathcal{S}^k\times\mathcal{S}^k \\ \Leftrightarrow\ & \left(\tfrac{1}{\sqrt{c_2}}D\right)\Lambda - \Lambda^T\left(\tfrac{1}{\sqrt{c_2}}D\right) \\ & \quad = Y^T - Y + (\Lambda^2)^T\left(\tfrac{1}{\sqrt{c_1}}B\right) - \left(\tfrac{1}{\sqrt{c_1}}B\right)\Lambda^2 && \text{for some } (B,D) \in \mathcal{S}^k\times\mathcal{S}^k \\ \Leftrightarrow\ & \widetilde{D}\Lambda - \Lambda^T\widetilde{D} = Y^T - Y && \text{for some } \widetilde{D} \in \mathcal{S}^k, \end{aligned}$$

where $\widetilde{D} = \frac{1}{\sqrt{c_2}}D + (\frac{1}{\sqrt{c_1}}B)\Lambda + \Lambda^T(\frac{1}{\sqrt{c_1}}B)$. Then it is easy to verify that the last equation in (A.2) holds for some $\widetilde{D} \in \mathcal{S}^k$ if and only if

$$\widetilde{D}_{i,i}\Lambda_i - \Lambda_i^T\widetilde{D}_{i,i} = Y_{i,i}^T - Y_{i,i}, \quad i = 1,\ldots,\nu, \tag{A.3}$$

$$\widetilde{D}_{i,j}\Lambda_j - \Lambda_i^T\widetilde{D}_{i,j} = Y_{j,i}^T - Y_{i,j}, \quad 1 \le i < j \le \nu. \tag{A.4}$$

- Since $\sigma(\Lambda_i) \cap \sigma(\Lambda_j) = \emptyset$, for any $Y_{i,j} \in \mathbb{R}^{s_i\times s_j}$, there always exists a unique $\widetilde{D}_{i,j} \in \mathbb{R}^{s_i\times s_j}$ such that (A.4) holds.
- For $\mu+1 \le i \le \nu$, since $\Lambda_i = \lambda_i I_{s_i}$, (A.3) holds if and only if

$$Y_{i,i}^T = Y_{i,i}. \tag{A.5}$$

- For $1 \le i \le \mu$,

$$\Lambda_i = \alpha_i I + \beta_i J, \quad J = \mathrm{diag}(\overbrace{J_2, \ldots, J_2}^{s_i}), \quad J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Thus, (A.3) holds if and only if

(A.6)
$$(Y_{i,i}(2q+1:2q+2, 2p+1:2p+2))^T - Y_{i,i}(2p+1:2p+2, 2q+1:2q+2) \in \left\{ \begin{bmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{bmatrix} : y_1, y_2 \in \mathbb{R} \right\},$$

where $0 \le p < q \le s_i - 1$ and $Y_{i,i}(u_1 : u_2,\, v_1 : v_2)$ denotes the submatrix obtained by extracting rows u_1 through u_2 and columns v_1 through v_2 from a matrix $Y_{i,i}$.

Hence, we have obtained that $Y \in \mathcal{R}(\mathcal{J})$ if and only if Y is of form (A.1) satisfying (A.5) and (A.6). Therefore,

$$\mathrm{Dim}\,(\mathcal{R}(\mathcal{H}))\,\mathrm{Dim}\,(\mathcal{R}(\mathcal{J})) = k^2 - \sum_{i=1}^{\mu} s_i(s_i-1) - \frac{1}{2}\sum_{i=\mu+1}^{\nu} s_i(s_i-1),$$

which implies that if $s_1 = \cdots = s_\mu = s_{\mu+1} = \cdots = s_\nu = 1$, $\mathrm{Dim}\,(\mathcal{R}(\mathcal{H})) = \mathrm{Dim}\,(\mathcal{R}(\mathcal{J})) = k^2$. □

Remark A.2. From the proof of Proposition A.1 on the dimension of $\mathcal{R}(\mathcal{J})$ (and also $\mathcal{R}(\mathcal{H})$), we actually get the general solution of $\mathcal{J}(B, D, E) = 0$ for any fixed $B \in \mathcal{S}^k$, and thus the general solution of $\mathcal{A}(M, C, K) = 0$ for any fixed $M \in \mathcal{S}^n$. This generalizes a corresponding result of [25] for Λ with simple eigenvalues only.

Appendix B. Let $\Phi_M, \Phi_C, \Phi_K \in \mathcal{S}^n$ be three positive definite matrices. The IQEP of matrix weights can be defined as

(B.1)
$$\begin{aligned} \min \quad & \tfrac{1}{2}\|\Phi_M^{-1}(M - M_a)\Phi_M^{-1}\|^2 + \tfrac{1}{2}\|\Phi_C^{-1}(C - C_a)\Phi_C^{-1}\|^2 + \tfrac{1}{2}\|\Phi_K^{-1}(K - K_a)\Phi_K^{-1}\|^2 \\ \text{s.t.} \quad & M \begin{bmatrix} R \\ 0 \end{bmatrix} \Lambda^2 + C \begin{bmatrix} R \\ 0 \end{bmatrix} \Lambda + K \begin{bmatrix} R \\ 0 \end{bmatrix} = 0, \\ & (M, C, K) \in \Omega. \end{aligned}$$

By renaming $M := \Phi_M^{-1} M \Phi_M^{-1}$, $C := \Phi_C^{-1} C \Phi_C^{-1}$, $K := \Phi_K^{-1} K \Phi_K^{-1}$, $M_a := \phi_M^{-1} M_a \Phi_M^{-1}$, $C_a := \Phi_C^{-1} C_a \Phi_C^{-1}$, and $K_a := \Phi_K^{-1} K_a \Phi_K^{-1}$, we can write problem (B.1) equivalently as

(B.2)
$$\begin{aligned} \min \quad & \tfrac{1}{2}\|M - M_a\|^2 + \tfrac{1}{2}\|C - C_a\|^2 + \tfrac{1}{2}\|K - K_a\|^2 \\ \text{s.t.} \quad & (\Phi_M M \Phi_M) \begin{bmatrix} R \\ 0 \end{bmatrix} \Lambda^2 + (\Phi_C C \Phi_C) \begin{bmatrix} R \\ 0 \end{bmatrix} \Lambda + (\Phi_K K \Phi_K) \begin{bmatrix} R \\ 0 \end{bmatrix} = 0, \\ & (M, C, K) \in \Omega. \end{aligned}$$

Then, from the proof of Theorem 1.1, we can see that problem (B.2) admits a strictly feasible solution if and only if Λ is nonsingular.

Next, we shall show how problem (B.1) can be reduced to a similar problem which has a strictly feasible solution if Λ is singular. Without loss of generality, we assume that

$$\Lambda = \mathrm{diag}\{O_t, \Gamma\},$$

where $O_t = \mathrm{diag}\{\overbrace{0, \ldots, 0}^{t}\}$ and Γ is nonsingular and has the same structure as Λ defined in (1.3). Partition M, C, K, and $\left[\begin{smallmatrix} R \\ 0 \end{smallmatrix}\right]$ by

$$M = \begin{bmatrix} M_1 & M_2 \\ M_2^T & M_4 \end{bmatrix}, \quad C = \begin{bmatrix} C_1 & C_2 \\ C_2^T & C_4 \end{bmatrix}, \quad K = \begin{bmatrix} K_1 & K_2 \\ K_2^T & K_4 \end{bmatrix},$$

$$\text{and} \quad \begin{bmatrix} R \\ 0 \end{bmatrix} = \begin{bmatrix} R_1 & R_2 \\ 0 & R_3 \end{bmatrix},$$

where $M_1, C_1, K_1 \in \mathcal{S}^t$, $M_2, K_2, C_2, \in \mathbb{R}^{t\times(n-t)}$, $M_4, C_4, K_4 \in \mathcal{S}^{(n-t)}$, and $R_1 \in \mathbb{R}^{t\times t}$, $R_2 \in \mathbb{R}^{t\times(k-t)}$, $R_3 \in \mathbb{R}^{(n-t)\times(k-t)}$. The matrix R_3 can be further written as $R_3 = \left[\begin{smallmatrix} R_4 \\ 0 \end{smallmatrix}\right]$, where $R_4 \in \mathbb{R}^{(k-t)\times(k-t)}$ is an invertible upper triangular matrix. Then the first equation in problem (B.1) takes the following form:

$$\text{(B.3)} \qquad \begin{cases} K_1 R_1 = 0, \quad K_2^T R_1 = 0, \\ (M_1 R_2 \Gamma^2 + C_1 R_2 \Gamma) + (M_2 R_3 \Gamma^2 + C_2 R_3 \Gamma) + (K_1 R_2 + K_2 R_3) = 0, \\ (M_2^T R_2 \Gamma^2 + C_2^T R_2 \Gamma) + (M_4 R_3 \Gamma^2 + C_4 R_3 \Gamma + K_4 R_3) + K_2^T R_2 = 0. \end{cases}$$

Since R_1 is invertible, we have

$$K_1 = 0 \quad \text{and} \quad K_2 = 0.$$

Let

$$\widehat{\mathcal{A}}(M, C, K_4) := \begin{pmatrix} (M_1 R_2 \Gamma^2 + C_1 R_2 \Gamma) + (M_2 R_3 \Gamma^2 + C_2 R_3 \Gamma) \\ (M_2^T R_2 \Gamma^2 + C_2^T R_2 \Gamma) + (M_4 R_3 \Gamma^2 + C_4 R_3 \Gamma + K_4 R_3) \end{pmatrix}.$$

Then the last two equations in (B.3) reduce to $\widehat{\mathcal{A}}(M, C, K_4) = 0$. Partition K_a by

$$K_a = \begin{bmatrix} (K_a)_1 & (K_a)_2 \\ (K_a)_2^T & (K_a)_4 \end{bmatrix}$$

with $(K_a)_1 \in \mathcal{S}^t$, $(K_a)_2 \in \mathbb{R}^{t\times(n-t)}$, and $(K_a)_4 \in \mathcal{S}^{(n-t)}$. Let

$$\Psi := \Phi_K^{-1} = \begin{bmatrix} L_1 & L_2 \\ L_2^T & L_4 \end{bmatrix} \quad \text{and} \quad \Psi K_a \Psi := \begin{bmatrix} (H_a)_1 & (H_a)_2 \\ (H_a)_2^T & (H_a)_4 \end{bmatrix},$$

where $L_1, (H_a)_1 \in \mathcal{S}^t$, $L_2, (H_a)_2 \in \mathbb{R}^{t\times(n-t)}$, and $L_4, (H_a)_4 \in \mathcal{S}^{(n-t)}$. By using the fact that

$$\begin{aligned} \|\Psi(K - K_a)\Psi\|^2 &= \left\| \begin{bmatrix} L_2 K_4 L_2^T - (H_a)_1 & L_2 K_4 L_4 - (H_a)_2 \\ L_4 K_4 L_2^T - (H_a)_2^T & L_4 K_4 L_4 - (H_a)_4 \end{bmatrix} \right\|^2 \\ &= \left\| \Psi_K^{-1} (K_4 - \Upsilon_a) \Psi_K^{-1} \right\|^2 + \kappa, \end{aligned}$$

where

$$\begin{aligned}\Psi_K &= (L_2^T L_2 + L_4^2)^{-\frac{1}{2}},\\ \Upsilon_a &= \Psi_K^2\left[(L_2^T(H_a)_1 L_2 + 2L_2^T(H_a)_2 L_4 + L_4(H_a)_4 L_4\right]\Psi_K^2,\\ \kappa &= -\|\Psi_K^{-1}\Upsilon_a\Psi_K^{-1}\|^2 + \|\Psi K_a \Psi\|^2,\end{aligned}$$

we can see that problem (B.1) is equivalent to

(B.4)
$$\begin{aligned}\min\quad & \tfrac{1}{2}\|\Phi_M^{-1}(M-M_a)\Phi_M^{-1}\|^2 + \tfrac{1}{2}\|\Phi_C^{-1}(C-C_a)\Phi_C^{-1}\|^2 + \tfrac{1}{2}\|\Psi_K^{-1}(K_4-\Upsilon_a)\Psi_K^{-1}\|^2\\ \text{s.t.}\quad & \widehat{\mathcal{A}}(M,C,K_4)=0,\\ & M\succeq 0,\quad C^T=C,\quad K_4\succeq 0.\end{aligned}$$

By renaming $M := \Phi_M^{-1}M\Phi_M^{-1}$, $C := \Phi_C^{-1}C\Phi_C^{-1}$, $K_4 := \Psi_K^{-1}K_4\Psi_K^{-1}$, $M_a := \Phi_M^{-1}M_a\Phi_M^{-1}$, $C_a := \Phi_C^{-1}C_a\Phi_C^{-1}$, and $\Upsilon_a := \Psi_K^{-1}\Upsilon_a\Psi_K^{-1}$, we see that problem (B.4) takes the following form:

(B.5)
$$\begin{aligned}\min\quad & \tfrac{1}{2}\|M-M_a\|^2 + \tfrac{1}{2}\|C-C_a\|^2 + \tfrac{1}{2}\|K_4-\Upsilon_a\|^2\\ \text{s.t.}\quad & \widehat{\mathcal{A}}\left(\Phi_M M\Phi_M, \Phi_C C\Phi_C, \Psi_K K_4\Psi_K\right)=0,\\ & M\succeq 0,\quad C^T=C,\quad K_4\succeq 0.\end{aligned}$$

Let $\widehat{M}$, $\widehat{C}$, and $\widehat{K}_4$ be defined by

$$\widehat{M} := \begin{bmatrix}\widehat{M}_1 & \widehat{M}_2\\ \widehat{M}_2^T & \widehat{M}_4\end{bmatrix},\quad \widehat{C} := \begin{bmatrix}\widehat{C}_1 & \widehat{C}_2\\ \widehat{C}_2^T & \widehat{C}_4\end{bmatrix},\quad \text{and}\quad \widehat{K}_4 := \begin{bmatrix}R_4^{-T}\Gamma^T\Gamma R_4^{-1} & 0\\ 0 & I\end{bmatrix},$$

where

$$\widehat{M}_1 = R_1^{-T}R_1^{-1},\quad \widehat{M}_2 = -\begin{bmatrix}R_1^{-T}R_1^{-1}R_2R_4^{-1} & 0\end{bmatrix},\quad \widehat{C}_1=0,\quad \widehat{C}_2=0,$$

and

$$\widehat{M}_4 = \begin{bmatrix}R_4^{-T}R_2^TR_1^{-T}R_1^{-1}R_2R_4^{-1} + R_4^{-T}R_4^{-1} & 0\\ 0 & I\end{bmatrix},$$

$$\widehat{C}_4 = -\begin{bmatrix}R_4^{-T}(\Gamma+\Gamma^T)R_4^{-1} & 0\\ 0 & 0\end{bmatrix}.$$

Clearly,

$$\left(\Phi_M^{-1}\widehat{M}\Phi_M^{-1}, \Phi_C^{-1}\widehat{C}\Phi_C^{-1}, \Psi_K^{-1}\widehat{K}_4\Psi_K^{-1}\right)$$

is a strictly feasible solution to problem (B.5). Therefore, we can apply Algorithm 4.1 to the dual of problem (B.5).

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