Inverse Quadratic Eigenvalue Problems

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Given $A \in \mathcal{S}^n$ (the space of real symmetric matrices of order n), how can we find a positive semidefinite (PSD) matrix $X \in \mathcal{S}^n_+$ such that $\|X - A\|_F$ is minimized?

Mathematically,

$$\begin{array}{rl} X \in \operatorname{argmin} & \frac{1}{2}\langle X - A, X - A\rangle_F \\ \text{s.t.} & X \succeq 0\,. \end{array}$$

The answer is straightforward. Let $\mathcal{O}^n$ denote the set of all orthonormal matrices in $\mathbb{R}^{n\times n}$. Let $P \in \mathcal{O}^n$ be such that

$$A = P\mathrm{diag}(\sigma_1(A), \sigma_2(A), \cdots, \sigma_n(A))P^T.$$

Then the unique solution, A_+, which is actually the metric projection of A onto $\mathcal{S}^n_+$, is given by [Higham'88, Tseng'98]

$$P\mathrm{diag}((\sigma_1(A))_+, (\sigma_2(A))_+, \cdots, (\sigma_n(A))_+)P^T.$$

For given $M, C, K \in \mathbb{R}^{n\times n}$, let

$$Q(\lambda) := \lambda^2 M + \lambda C + K.$$

Then the quadratic eigenvalue problem (QEP) is to find scalars $\lambda \in \mathbb{C}$ and nonzero vectors $\mathbf{x}$ such that

$$Q(\lambda)\mathbf{x} = 0.$$

where λ and $\mathbf{x}$ are called the eigenvalue and the eigenvector, respectively.

The general Inverse QEP (IQEP) can be defined as follows:

- Given a measured partial eigenpair $(\Lambda, X) \in \mathbb{R}^{k\times k} \times \mathbb{R}^{n\times k}$ with $1 \le k \le n, \quad \mathrm{rank}(X) = k$,

$$\Lambda = \mathrm{diag}\{\Lambda_1, \ldots, \Lambda_\mu, \Lambda_{\mu+1}, \ldots, \Lambda_\nu\},$$

$$\Lambda_i = \mathrm{diag}\{\overbrace{\lambda_i^{[2]}, \ldots, \lambda_i^{[2]}}^{s_i}\} \quad \text{for} \quad 1 \le i \le \mu,$$

$$\Lambda_i = \lambda_i I_{s_i} \quad \text{for} \quad \mu + 1 \le i \le \nu,$$

$$\lambda_i^{[2]} = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix} \in \mathbb{R}^{2\times 2}, \quad \beta_i \neq 0,$$

$$\sigma(\Lambda_i) \cap \sigma(\Lambda_j) = \emptyset, \quad \forall\, 1 \leq i \neq j \leq \mu,$$

$$\lambda_i \in \mathbb{R}, \quad \lambda_i \neq \lambda_j, \quad \forall\, \mu + 1 \leq i \neq j \leq \nu,$$

- find $M, C, K \in \mathcal{S}^n$ with $M \succ 0$ and $K \succeq 0$ such that

$$\boxed{MX\Lambda^2 + CX\Lambda + KX = 0.}$$

M. Chu, Kuo, and Lin (2004) showed that the general IQEP admits a nontrivial solution, i.e, there exist

$$M \succ 0, C = C^T, K \succeq 0 \text{ satisfying}$$

$$MX\Lambda^2 + CX\Lambda + KX = 0.$$

For given $M_a, C_a, K_a \in \mathcal{S}^n$, which are called the estimated analytic mass, damping, and stiffness matrix, the IQEP is

$$
\begin{aligned}
\inf \quad & \frac{c_1}{2}\|M - M_a\|^2 + \frac{c_2}{2}\|C - C_a\|^2 + \frac{1}{2}\|K - K_a\|^2 \\
\text{s.t.} \quad & MX\Lambda^2 + CX\Lambda + KX = 0, \\
& M \succ 0 (M \succeq 0), \quad C = C^T, \quad K \succeq 0,
\end{aligned}
$$

where $c_1 > 0$ and $c_2 > 0$.

Let the QR factorization of X be given by

$$X = Q \begin{bmatrix} R \\ 0 \end{bmatrix},$$

where $Q \in \mathbb{R}^{n\times n}$ is orthogonal and $R \in \mathbb{R}^{k\times k}$ is nonsingular and upper triangular.

By doing variables substitution,

$$M := \sqrt{c_1} Q^T M Q, M_a := \sqrt{c_1} Q^T M_a Q, etc.$$

The IQEP becomes

$$
\begin{array}{ll}
\min & \frac{1}{2}\|M - M_a\|^2 + \frac{1}{2}\|C - C_a\|^2 + \frac{1}{2}\|K - K_a\|^2 \\
\text{s.t.} & \frac{1}{\sqrt{c_1}} M \begin{bmatrix} R \\ 0 \end{bmatrix} \Lambda^2 + \frac{1}{\sqrt{c_2}} C \begin{bmatrix} R \\ 0 \end{bmatrix} \Lambda + K \begin{bmatrix} R \\ 0 \end{bmatrix} = 0, \\
& (M, C, K) \in \Omega,
\end{array}
$$

where

$$
\Omega_0 := \mathcal{S}^n \times \mathcal{S}^n \times \mathcal{S}^n
$$

$$
\Omega := \{(M, C, K) \in \Omega_0 \,:\, M \succeq 0, \quad K \succeq 0\}.
$$

Theorem 1. The IQEP has a strictly feasible solutuion iff

$$\mathrm{Det}(\Lambda) \neq 0.$$

Remark: If $\mathrm{Det}(\Lambda) = 0$, we do not lose generality as we can reduce the IQEP to another problem with a strictly feasible solution.

The IQEP is a special case of

$$\begin{aligned} \min \quad & \frac{1}{2}\langle x - x^0, x - x^0 \rangle \\ \text{s.t.} \quad & \mathcal{A}x = b, \\ & x \in \mathcal{Q}, \end{aligned}$$

where $x^0 \in \mathcal{X}$, $A : \mathcal{X} \to \mathcal{Y}$ is a linear operator, $b \in \mathcal{Y}$, $\mathcal{Q}$ is a closed convex cone in $\mathcal{X}$, and $\mathcal{X}$ and $\mathcal{Y}$ are finite dimensional real vector spaces each equipped with a scalar inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. Let $\mathcal{A}^* : \mathcal{Y} \to \mathcal{X}$ be the adjoint of $\mathcal{A}$.

Let $\mathcal{D} \subseteq \mathcal{X}$ be a closed convex set. For any $x \in \mathcal{X}$, let $\Pi_{\mathcal{D}}(x)$ denote the metric projection of x onto $\mathcal{D}$,

$$\begin{array}{ll} \min & \frac{1}{2}\langle z - x, z - x\rangle \\ \text{s.t.} & z \in \mathcal{D}. \end{array}$$

The dual problem is

$$
\begin{array}{ll}
\min & \theta(y) \\
\text{s.t.} & y \in \mathcal{Y},
\end{array}
$$

where

$$
\begin{array}{rcl}
\theta(y) : & = & \frac{1}{2}\|x^0 + \mathcal{A}^* y\|^2 \\
& & -\frac{1}{2}\|x^0 + \mathcal{A}^* y - \Pi_{\mathcal{Q}}(x^0 + \mathcal{A}^* y)\|^2 \\
& & -\langle b, y\rangle - \frac{1}{2}||x^0\|^2
\end{array}
$$

and if $\mathcal{Q}$ is a closed convex cone

$$\theta(y) = \frac{1}{2}\|\Pi_{\mathcal{Q}}(x^0 + \mathcal{A}^* y)\|^2 - \langle b, y\rangle - \frac{1}{2}\|x^0\|^2.$$

Consider the following equation:

$$F(y) := \nabla\theta(y) = \mathcal{A}\,\Pi_{\mathcal{Q}}(x^0 + \mathcal{A}^* y) - b = 0, \quad y \in \mathcal{Y}.$$

Under Slater's condition

$$\begin{cases} \mathcal{A} : \mathcal{X} \to \mathcal{Y} \text{ is onto}, \\ \exists\ \bar{x} \in \mathcal{X} \text{ such that } \ \mathcal{A}\bar{x} = b, \ \bar{x} \in \operatorname{int}(\mathcal{Q}), \end{cases}$$

where "int" denotes the topological interior, the classical duality theorem [Rockafellar'74] says that

$x^* := \Pi_{\mathcal{Q}}(x^0 + \mathcal{A}^* y^*)$ solves the original problem if y^* solves $F(y^*) = 0$.

Let $\mathcal{Z}$ be an arbitrary finite dimensional real vector space.

Let $\mathcal{O}$ be an open set in $\mathcal{Y}$ and $\Xi : \mathcal{O} \subseteq \mathcal{Y} \to \mathcal{Z}$ be a locally Lipschitz continuous function on the open set $\mathcal{O}$.

Rademacher's theorem says that Ξ is almost everywhere Fréchet differentiable in $\mathcal{O}$.

We denote by $\mathcal{O}_\Xi$ the set of points in $\mathcal{O}$ where Ξ is Fréchet differentiable.

Let $\Xi'(y)$ denote the Jacobian of Ξ at $y \in \mathcal{O}_\Xi$.

Then Clarke's generalized Jacobian of Ξ at $y \in \mathcal{O}$ is defined by [Clarke'83]

$$\partial \Xi(y) := \mathrm{conv}\{\partial_B \Xi(y)\},$$

where "conv" denotes the convex hull and

$$\partial_B \Xi(y) := \left\{ V \,:\, V = \lim_{j \to \infty} \Xi'(y^j)\,, y^j \to y\,,\ y^j \in \mathcal{O}_\Xi \right\}.$$

When $F : \mathcal{O} \subseteq \mathcal{Y} \to \mathcal{Y}$ is continuously differentiable (smooth), the most effective approach for solving

$$F(y) = 0$$

is probably Newton's method. For example, in 1987, S. Smale wrote

> If any algorithm has proved itself for the problem of nonlinear systems, it is Newton's method and its many modifications. ... Thus a relation between the simplex method of linear programming and Newton's method, is no surprise. ..."

The extension of Newton's methods to Lipschitz systems:

• Friedland, Nocedal, and Overton [87] for inverse eigenvalue problems.

• Kojima and Shindoh [86] for piecewise smooth equations.

• Kummer [88] proposed a condition

(ii) for any $x \to y$ and $V \in \partial\Xi(x)$,

$$\Xi(x) - \Xi(y) - V(x-y) = o(||x-y||)\,.$$

• Finally, Qi and J. Sun [93] showed what needed is semismoothness.

The function Ξ is (strongly) *semismooth* at a point $y \in \mathcal{O}$ if

(i) Ξ is directionally differentiable at y; and

(ii) for any $x \to y$ and $V \in \partial \Xi(x)$,

$$\Xi(x) - \Xi(y) - V(x-y) = o(||x-y||) \ (O(||x-y||^2)).$$

Condition (ii) can be replaced by

(ii)' for any $x \to y$ and $x \in \mathcal{O}_\Xi$,

$$\Xi(x) - \Xi(y) - \Xi'(x)(x-y) = o(||x-y||) \ (O(||x-y||^2)).$$

Let $A \in \mathcal{S}^n$. Then A admits the following spectral decomposition

$$A = P\Sigma P^T,$$

where Σ is the diagonal matrix of eigenvalues of A and P is a corresponding orthogonal matrix of orthonormal eigenvectors.

Define three index sets of positive, zero, and negative eigenvalues of A, respectively, as

$$\alpha := \{i : \sigma_i > 0\},$$

$$\beta := \{i : \sigma_i = 0\},$$

$$\gamma := \{i : \sigma_i < 0\}.$$

Write

$$\Sigma = \begin{bmatrix} \Sigma_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Sigma_\gamma \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} P_\alpha & P_\beta & P_\gamma \end{bmatrix}$$

with $P_\alpha \in \mathbb{R}^{n\times|\alpha|}$, $P_\beta \in \mathbb{R}^{n\times|\beta|}$, and $P_\gamma \in \mathbb{R}^{n\times|\gamma|}$.

Define the matrix $U \in \mathcal{S}^n$ with entries

$$U_{ij} := \frac{\max\{\sigma_i, 0\} + \max\{\sigma_j, 0\}}{|\sigma_i| + |\sigma_j|}, \quad i,j = 1,\ldots,n,$$

where $0/0$ is defined to be 1.

Sun and J. Sun [02] showed $\Pi_{\mathcal{S}^n_+}(\cdot)$ is strongly semismooth everywhere and the directional derivative $\Pi'_{\mathcal{S}^n_+}(A;H)$ is given by

$$P\begin{bmatrix} P_\alpha^T H P_\alpha & P_\alpha^T H P_\beta & U_{\alpha\gamma}\circ P_\alpha^T H P_\gamma \\ P_\beta^T H P_\alpha & \Pi_{\mathcal{S}^{|\beta|}_+}\left(P_\beta^T H P_\beta\right) & 0 \\ P_\gamma^T H P_\alpha \circ U_{\alpha\gamma}^T & 0 & 0 \end{bmatrix} P^T,$$

where $\circ$ denotes the Hadamard product.

When A is nonsingular, i.e., $|\beta| = 0$, $\Pi_{\mathcal{S}^n_+}(\cdot)$ is continuously differentiable around A and the above formula reduces to the classical result of Löwner [34].

The tangent cone of $\mathcal{S}^n_+$ at $A_+ = \Pi_{\mathcal{S}^n_+}(A)$:

$$\mathcal{T}_{\mathcal{S}^n_+}(A_+) = \{B \in \mathcal{S}^P \ : \ P_{\bar{\alpha}}^T B P_{\bar{\alpha}} \succeq 0\}$$

and the lineality space of $\mathcal{T}_{\mathcal{S}^n_+}(A_+)$, i,e, the largest linear space in $\mathcal{T}_{\mathcal{S}^n_+}(A_+)$,

$$\mathrm{lin}\left(\mathcal{T}_{\mathcal{S}^n_+}(A_+)\right) = \{B \in \mathcal{S}^n \ : \ P_{\bar{\alpha}}^T B P_{\bar{\alpha}} = 0\},$$

where $\bar{\alpha} := \{1, \ldots, n\} \backslash \alpha$ and $P_{\bar{\alpha}} := [P_\beta \ P_\gamma]$.

Let $W(H)$ be defined by

$$P \begin{bmatrix} P_\alpha^T H P_\alpha & P_\alpha^T H P_\beta & U_{\alpha\gamma} \circ P_\alpha^T H P_\gamma \\ P_\beta^T H P_\alpha^T & 0 & 0 \\ P_\gamma^T H P_\alpha \circ U_{\alpha\gamma}^T & 0 & 0 \end{bmatrix} P^T$$

for all $H \in \mathcal{S}^n$. Then W is an element in $\partial_B \Pi_{\mathcal{S}_+^n}(A)$.

Let us come back to the IEQP. Denote

$$M =: \begin{bmatrix} M_1 & M_2 \\ M_2^T & M_4 \end{bmatrix}, \; C := \begin{bmatrix} C_1 & C_2 \\ C_2^T & C_4 \end{bmatrix},$$

$$K := \begin{bmatrix} K_1 & K_2 \\ K_2^T & K_4 \end{bmatrix},$$

where $M_1, C_1, K_1 \in \mathcal{S}^k$, M_2, C_2, $K_2 \in \mathbb{R}^{k\times(n-k)}$, and $M_4, C_4, K_4 \in \mathcal{S}^{(n-k)}$. Let $S := R\Lambda R^{-1}$.

For $(M, C, K) \in \Omega_0$, let $\mathcal{H}(M, C, K)$ be given by

$$\frac{1}{\sqrt{c_1}}(\Lambda^2)^T \left(R^T M_1 R\right) + \frac{1}{\sqrt{c_2}}\Lambda^T \left(R^T C_1 R\right) + \left(R^T K_1 R\right)$$

and $\mathcal{G}(M, C, K)$ be given by

$$\frac{1}{\sqrt{c_1}}(S^2)^T M_2 + \frac{1}{\sqrt{c_2}} S^T C_2 + K_2.$$

While $\mathcal{G} : \Omega_0 \longrightarrow \mathbb{R}^{k\times(n-k)}$ is onto, $\mathcal{H} : \Omega_0 \longrightarrow \mathbb{R}^{k\times k}$ is not. Let

$$\mathrm{Range}(\mathcal{H}) := \{\mathcal{H}(M, C, K) \;:\; (M, C, K) \in \Omega_0\}.$$

$\mathcal{H}:\Omega_0 \to \mathrm{Range}(\mathcal{H})$ is surjective. The dimension of $\mathrm{Range}(\mathcal{H})$ is given by

$$k^2 - \sum_{i=1}^{\mu} s_i(s_i-1) - \frac{1}{2}\sum_{i=\mu+1}^{\nu} s_i(s_i-1).$$

In particular, if $s_1 = \cdots = s_\mu = s_{\mu+1} = \cdots = s_\nu = 1$, it is equal to k^2.

Define the linear operator $\mathcal{A}:\Omega_0 \to \mathrm{Range}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$ by

$$\mathcal{A}(M,C,K) := \left(\mathcal{H}(M,C,K), \mathcal{G}(M,C,K)\right).$$

The IQEP takes the following compact form

$$\begin{aligned} \min \quad & \tfrac{1}{2}\|(M, C, K) - (M_a, C_a, K_a)\|^2 \\ \text{s.t.} \quad & \mathcal{A}(M, C, K) = 0, \\ & (M, C, K) \in \Omega. \end{aligned}$$

Define $\theta : \text{Range}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)} \to \mathbb{R}$ by

$$\begin{aligned} \theta(Y, Z) : \;&=\; \tfrac{1}{2}\|\Pi_\Omega((M_a, C_a, K_a) + \mathcal{A}^*(Y, Z))\|^2 \\ & \quad -\tfrac{1}{2}\|(M_a, C_a, K_a)\|^2, \end{aligned} \tag{1}$$

where $(Y, Z) \in \text{Range}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$.

The dual problem is

$$\begin{array}{ll} \min & \theta(Y,Z) \\ \text{s.t.} & (Y,Z) \in \mathrm{Range}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}. \end{array} \tag{2}$$

Define $F : \mathrm{Range}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)} \to \mathrm{Range}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$ by

$$\begin{array}{rcl} F(Y,Z): & = & \nabla\theta(Y,Z) \\ & = & \mathcal{A}\,\Pi_\Omega\left((M_a, C_a, K_a) + \mathcal{A}^*(Y,Z)\right), \end{array} \tag{3}$$

where $(Y,Z) \in \mathrm{Range}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$.

(Newton's Method)

[**Step 0.**] Given $(Y^0, Z^0) \in \mathrm{Range}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$, $\eta \in (0,1)$, $\rho, \delta \in (0, 1/2)$. $j := 0$.

[**Step 1.**] Select an element

$$W_j \in \partial\Pi_\Omega\left((M_a, C_a, K_a) + \mathcal{A}^*(Y^j, Z^j)\right)$$

and let

$$V_j := \mathcal{A}W_j\mathcal{A}^*.$$

Apply the conjugate gradient method to find an approximate solution

$$(\Delta Y^j, \Delta Z^j) \in \mathrm{Range}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$$

to the linear system

$$F(Y^j, Z^j) + V_j(\Delta Y, \Delta Z) = 0 \tag{4}$$

such that

$$\|F(Y^j, Z^j) + V_j(\Delta Y^j, \Delta Z^j)\| \le \eta_j \|F(Y^j, Z^j)\| \tag{5}$$

and

$$\begin{aligned} &\left\langle F(Y^j, Z^j), (\Delta Y^j, \Delta Z^j) \right\rangle \\ &\le -\eta_j \left\langle (\Delta Y^j, \Delta Z^j), (\Delta Y^j, \Delta Z^j) \right\rangle, \end{aligned} \tag{6}$$

where $\eta_j := \min\{\eta, \|F(Y^j, Z^j)\|\}$.

If (5) and (6) are not achievable, let

$$
\begin{aligned}
(\Delta Y^j, \Delta Z^j) &:= -F(Y^j, Z^j) \\
&= -\mathcal{A}\,\Pi_\Omega\left((M_a, C_a, K_a) + \mathcal{A}^*(Y^j, Z^j)\right).
\end{aligned}
$$

[**Step 2.**] Let m_j be the smallest nonnegative integer m such that

$$
\begin{aligned}
&\theta\left((Y^j, Z^j) + \rho^m(\Delta Y^j, \Delta Z^j)\right) - \theta(Y^j, Z^j) \\
&\le \delta\rho^m \left\langle F(Y^j, Z^j), (\Delta Y^j, \Delta Z^j)\right\rangle.
\end{aligned}
$$

Set

$$
(Y^{j+1}, Z^{j+1}) := (Y^j, Z^j) + \rho^{m_j}(\Delta Y^j, \Delta Z^j).
$$

[**Step 3.**] Replace j by $j+1$ and go to Step 1.

Theorem 2. The algorithm generates an infinite sequence $\{(Y^j, Z^j)\}$ with the properties that for each $j \geq 0$, $(Y^j, Z^j) \in \text{Range}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)}$, $\{(Y^j, Z^j)\}$ is bounded, and any accumulation point of $\{(Y^j, Z^j)\}$ is a solution to the dual problem.

For discussions on the rate of convergence, we need the constraint nondegenerate condition ("LICQ")

$$\mathcal{A}\left(\mathrm{lin}\left(\mathcal{T}_{\mathcal{S}^n_+}(\overline{M})\right), \mathcal{S}^n, \mathrm{lin}\left(\mathcal{T}_{\mathcal{S}^n_+}(\overline{K})\right)\right)$$
$$= \mathrm{Range}(\mathcal{H}) \times \mathbb{R}^{k\times(n-k)},$$

where $(\overline{M}, \overline{C}, \overline{K}) \in \Omega_0$ is a feasible solution to the original problem.

Theorem 3. Let $(\overline{Y}, \overline{Z})$ be an accumulation point of the infinite sequence $\{(Y^j, Z^j)\}$ generated by the algorithm. Let

$$(\overline{M}, \overline{C}, \overline{K}) := \Pi_\Omega\left((M_a, C_a, K_a) + \mathcal{A}^*(\overline{Y}, \overline{Z})\right).$$

Assume that the constraint nondegenerate condition holds at $(\overline{M}, \overline{C}, \overline{K})$. Then the whole sequence $\{(Y^j, Z^j)\}$ converges to $(\bar{Y}, \bar{Z})$ <u>quadratically</u>.

The stopping criterion is

$$\texttt{To}l. := \frac{\|\nabla\theta(Y_k, Z_k)\|}{\max\left\{1, \left\|(\frac{1}{\sqrt{c_1}}M_a, \frac{1}{\sqrt{c_2}}C_a, K_a)\right\|\right\}} \leq 10^{-7}\,.$$

We set other parameters used in our algorithm as $\eta = 10^{-6}$, $\rho = 0.5$, and $\delta = 10^{-4}$.

$k = 30, \quad c_1 = c_2 = 1.0$				
n	cputime	It.	Func.	Tol.
100	01 m 26 s	18	24	3.9×10^{-11}
200	04 m 39 s	14	15	3.9×10^{-11}
500	21 m 16 s	11	12	1.3×10^{-10}
1,000	44 m 13 s	9	10	1.1×10^{-9}
1,500	08 h 49 m 11 s	7	8	1.6×10^{-8}
2,000	05 h 24 m 37 s	9	10	3.3×10^{-8}

$k \approx n/3, \quad c_1 = 10.0, c_2 = 0.10$					
n	k	cputime	It.	Func.	Tol.
100	33	46.1 s	9	11	1.4×10^{-9}
200	66	42 m 42 s	13	15	5.8×10^{-8}
300	100	02 h 24 m 23 s	17	20	6.5×10^{-9}
400	133	04 h 38 m 42 s	10	11	4.0×10^{-8}
450	150	12 h 23 m 44 s	13	14	8.8×10^{-9}

The largest numerical examples that we tested in this paper are: (i) $n = 2,000$ and $k = 30$ and (ii) $n = 450$ and $k = 150$.

For case (i), there are roughly $6,000,000$ unknowns in the primal problem and $60,000$ unknowns in the dual problem while for case (ii), these numbers are roughly $300,000$ and $67,000$, respectively. In consideration of the scales of problems solved, our algorithm is very effective.