Preferences for travel time under risk and ambiguity: Implications in path selection and network equilibrium

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ABSTRACT

In this paper, we study the preferences for uncertain travel times in which probability distributions may not be fully characterized. In evaluating an uncertain travel time, we explicitly distinguish between *risk*, where the probability distribution is precisely known, and *ambiguity*, where it is not. In particular, we propose a new criterion called *ambiguity-aware CARA travel time* (ACT) for evaluating uncertain travel times under various attitudes of risk and ambiguity, which is a preference based on blending the Hurwicz criterion and Constant Absolute Risk Aversion (CARA). More importantly, we show that when the uncertain link travel times are independently distributed, finding the path that minimizes travel time under the ACT criterion is essentially a shortest path problem. We also study the implications on Network Equilibrium (NE) model where travelers on the traffic network are characterized by their knowledge of the network uncertainty as well as their risk and ambiguity attitudes under the ACT. We derive and analyze the existence and uniqueness of solutions under NE. Finally, we obtain the Price of Anarchy that characterizes the inefficiency of this new equilibrium. The computational study suggests that as uncertainty increases, the influence of selfishness on inefficiency diminishes.

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1. Introduction

In real world transportation networks, travel times are almost always uncertain, which is found to be one the most important factors in the path selection decisions (Abdel-Aty et al., 1995). Individuals' preferences greatly depend on their knowledge about the uncertain travel time as well as their attitudes towards uncertainty. In transportation literatures, an uncertain travel time is often associated with a random variable with the known probability distribution. In other words, the traveler knows the exact frequency of travel time outcomes, and his/her preference relies on his/her risk attitude, that is usually characterized by taking an expectation over a disutility function (an increase in the travel time amounts to a loss). Deliberating on reliability, Fan et al. (2005), Mirchandani (1976), and Nie and Wu (2009) consider the probability of punctuality as a preference criterion, which could be treated as a step disutility function. Unfortunately, since in general, computing the probability of a sum of random variables is NP-hard (Khachiyan, 1989), it is a computationally intractable problem to find the path with the minimum expected disutility over a transportation network, which severely limits our

analysis and implementation. Murthy and Sarkar (1998) consider a piece-wise linear concave disutility function, and solve the problem with certain enumeration algorithms. Loui (1983) and Eiger et al. (1985) consider disutility functions in the form of linear, quadratic or exponential, in which the resultant static path selection problems are computationally tractable. In particular, De Palma and Picard (2005) justify empirically the relevance of the exponential disutility function, which appeals to travelers with Constant Absolute Risk Aversion (CARA) and has the best fit on path selection behavior amongst common disutility functions.

Implications of risk in Network Equilibrium (NE) problems, which model a collective behavior of a large population of travelers, have also been studied. One stream suggests using disutility function to capture travel time uncertainty, and travelers' attitudes towards risk (see Chen et al., 2002; Mirchandani and Soroush, 1987; Nagurney and Dong, 2002; Yin and Ieda, 2001; and Yin et al., 2004). The second stream discusses the travel time variability by adding the mean travel time with a safety margin, which can be described by a penalty function (see Noland and Polak, 2002; Watling, 2006), or the standard deviation (see Connors et al., 2007; Lo et al., 2006; Nikolova and Stier-Moses, 2014; Siu and Lo, 2008; Uchida and Iida, 1993). However, adding the safety margin in these ways may violate first-order stochastic dominance, and it generally cannot be separated by links, which makes the model hard to solve. We refer interested readers to the review papers of Noland and Polak (2002) and Connors and Sumalee (2009). We would like to distinguish our work with stochastic NE (see for instance, Sheffi, 1985). Stochastic NE model captures the variations in travelers' perception on the travel time but still assumes that the travel time is deterministic.

The assumption that travelers know the exact frequency of travel time outcomes is unrealistic. In a real world, it is conceivable that a traveler is incapable of knowing the entire probability distributions of the transportation network. Major exceptional events (e.g., natural disasters) and minor regular events (e.g., minor accident, traffic signal) will incur uncertainty to travel time. Hence, complete distribution of travel time is seldom known exactly, and even the estimated one could be considerably affected by the sampling procedure. If the actual travel time probability distribution is not fully known, then it would be impossible to establish the preferences for travel times based on the expected disutility criterion. In fact, the distinction between risk, where outcome frequency is known, and ambiguity, where it is not, can be retrospected to Knight (1921), and has been extensively studied in economics (see for instance Camerer and Weber, 1992; Gilboa et al., 2008; Mukerji and Tallon, 2004; Wakker, 2008), finance (see for instance Bossaerts et al., 2010; Chen and Epstein, 2002; Dow and Werlang, 1992; Epstein and Schneider, 2008; Guidolin and Rinaldi, 2013), and marketing (see for instance Erdem and Swait, 2007; Muthukrishnan et al., 2009). Ellsberg (1961) shows convincingly by means of paradoxes that ambiguity preference cannot be reconciled by classical expected utility theory. Inspired by this seminal work, numerous experimental and theoretical studies spring up to verify and accommodate this behavior issue. Notably, in Hsu et al. (2005) ground-breaking experiments, economists and neuroscientists collaborate to establish significant physiological evidence via functional brain imaging that humans have varying and distinct attitudes towards risk and ambiguity. The results also indicate that people's attitudes towards risk and ambiguity are not fully correlated, i.e., there exists a population of people that are ambiguity averse and risk-seeking, or ambiguity seeking and risk-averse.

From the normative perspective, ambiguity is also an active area of research within the domains of decision theory and operations research. Gilboa and Schmeidler (1989) consider ambiguity as a set of possible probability distributions, and present the Max-min Expected Utility (MEU) model, which appeals to ambiguity averse decision makers. To accommodate the heterogeneity of ambiguity and risk attitudes found in experiments, Ghirardato et al. (2004), based on Hurwicz criterion (Arrow and Hurwicz, 1972; Hurwicz, 1951), axiomatize the α−MEU model, which represents a compromise via a convex combination of the worst and best case expected utility. The parameter α is an index of pessimism or optimism. However, the discussion on travel time ambiguity is relatively new. Yu and Yang (1998) propose a worst-case shortest path problem over a set of discrete scenarios, which results in an *NP*-hard problem. Bertsimas and Sim (2003) introduce the "budget of uncertainty" in characterizing uncertain travel time and show that the worst-case shortest path problem is a tractable optimization problem. Ordóñes and Stier-Moses (2010) extend the work to address an NE problem. They generally consider three cases of equilibrium with uncertain travel times: α-percentile equilibrium, added-variability equilibrium, and robust Wardrop equilibrium. The α-percentile equilibrium assumes travelers minimize the α quantile (or Value-at-Risk) of their experienced travel times, which are generally computationally intractable optimization problems. Added-variability equilibrium provides a safety margin to the expected travel time as a proxy to account for risk-averse behavior, an approach that may not be coherent with decision analysis such as violating first order stochastic dominance. Robust Wardrop equilibrium borrows the idea of Bertsimas and Sim (2003), and assumes that ambiguity averse travelers minimize the worst-case travel time given that the total variation is bounded by a certain parameter. However, the assumptions that the entire population of travelers are only ambiguity averse and not risk sensitive limit the application of this model.

In contrast to the aforementioned works that consider risk and ambiguity separately, our main contribution is to explicitly distinguish between risk and ambiguity in a unified framework in articulating travelers' preferences for travel times. We present a new criterion named *ambiguity-aware CARA travel time* (ACT) for evaluating uncertain travel times for travelers with various attitudes of risk and ambiguity. Apart from the behavioral relevance of the ACT, we also present a computational justification by showing that when the uncertain link travel times are independently distributed, finding the path that minimizes travel time under the ACT criterion is essentially a shortest path problem. We also study the implications on NE problem, in which travelers minimize their own travel times under the ACT criterion, and no traveler can improve his/her travel time under the ACT by unilaterally changing routes. Our new NE model under the ACT criterion shares similar properties with deterministic multi-class NE model, and can be solved by the traditional Frank–Wolfe algorithm. We also

examine the inefficiency of this NE model compared with System Optimum (SO), which minimizes the aggregate travel time under the ACT criterion of all travelers, by deriving its Price of Anarchy. The computational study suggests that as uncertainty increases, the influence of selfishness on inefficiency diminishes. Moreover, when uncertainty is neglected in traffic equilibrium analysis, the social optimum solution may become more inefficient than the solution under selfish routing.

The remainder of this paper is organized as follows. In Section 2, we formally define the ACT criterion and its properties. In Section 3, we investigate a path selection problem under the ACT criterion. In Section 4, we extend to the study of the NE problem under the ACT criterion and discuss its computational solvability when the uncertain link travel time is independent with each other. We also analyze the corresponding NE inefficiency by calculating its Price of Anarchy. Finally, in Section 5, we make our conclusions and some suggestions for future research.

Notations. We use boldface, e.g., $\boldsymbol{x}$ to represent a vector. We use tilde (˜) sign to denote uncertain quantities, for example $\tilde{t}$ denotes uncertain travel time. We model uncertainty $\tilde{t}$ by a state-space Ω and a $\sigma-$algebra of events in Ω. We use $\mathcal{V}$ to represent the set of all real-valued random variables. $\tilde{x} \geq \tilde{y}$ denotes state-wise dominance, i.e., $x(\omega) \geq y(\omega)$ for all $\omega \in \Omega$. To model ambiguity, $\tilde{t}$'s true distribution $\mathbb{P}$ is not necessarily specified but is known to lie in a distributional uncertainty set $\mathbb{F}$, i.e., $\mathbb{P} \in \mathbb{F}$. Correspondingly, known distribution case is only a special case, in which $\mathbb{F} = \{\mathbb{P}\}$. We denote by $E_{\mathbb{P}}(\tilde{t})$ the expectation of $\tilde{t}$ under the probability distribution $\mathbb{P}$.

2. Preferences for travel time under risk and ambiguity

In the empirical study of De Palma and Picard (2005), they conclude that exponential disutility function, which is the unique disutility function that appeals to travelers with Constant Absolute Risk Aversion (CARA), aptly characterizes travelers' preferences for travel times under risk. Besides, Cheu and Kreinovich (2007) also verify that exponential disutility function is the only function that is consistent with common sense and could simplify the model. Hence, we first introduce the exponential disutility function in the following form,

$$u(x) = \begin{cases} \frac{1}{\lambda}\exp(\lambda x), & \text{when } \lambda \neq 0, \\ ax + b, & \text{when } \lambda = 0, \end{cases}$$

in which $a \in \Re_+$ and the parameter $\lambda \in \Re$ is known as the coefficient of absolute risk aversion. The corresponding certainty equivalent of $\tilde{t}$, $CE_\lambda(\tilde{t}) : \mathcal{V} \to \Re$ is defined as

$$u\big(CE_\lambda(\tilde{t})\big) = E_{\mathbb{P}}\big(u(\tilde{t})\big).$$

The concept of certainty equivalent $CE_\lambda(\tilde{t})$ is popularized in economic literature (see for instance, Chapter 6 in Mas-Colell et al., 1995), and represents a fixed interval of travel time that the traveler with risk tolerance parameter λ will view equally acceptable as the uncertain travel time $\tilde{t}$ under disutility function $u(\cdot)$. When $u(\cdot)$ is exponential disutility function, we have

$$CE_\lambda(\tilde{t}) = \begin{cases} \frac{1}{\lambda}\ln E_{\mathbb{P}}\big(\exp\big(\lambda\tilde{t}\big)\big), & \text{when } \lambda \neq 0, \\ E_{\mathbb{P}}(\tilde{t}), & \text{when } \lambda = 0. \end{cases}$$

Parameter λ specifies the traveler's risk attitude. If $\lambda > 0$, he/she is risk-averse and evaluates an uncertain travel time longer than its average. In contrast, a traveler with risk-seeking attitude has $\lambda < 0$ and perceives the uncertain travel time shorter than its average. At neutrality ($\lambda = 0$), the traveler is indifferent between the uncertain travel time and its mean. When travel time is deterministic, we have $CE_\lambda(\text{constant}) = \text{constant}$ for all $\lambda \in \Re$. When travel time follows certain probability distribution, function $CE_\lambda(\tilde{t})$ can be derived through calculating the moment generating function of random variable $\tilde{t}$. For example, if $\tilde{t}$ is normally distributed $N(\mu, \sigma^2)$, we have $E_{\mathbb{P}}\big(\exp\big(\lambda\tilde{t}\big)\big) = \exp\big(\lambda\mu + \frac{1}{2}\sigma^2\lambda^2\big)$, and certainty equivalent $CE_\lambda(\tilde{t})$ is

$$CE_\lambda(\tilde{t}) = \mu + \frac{1}{2}\lambda\sigma^2,$$

which is consistent with mean-variance measure (Markowitz, 1959) of uncertain travel time $\tilde{t}$. Note that $CE_\lambda(\tilde{t})$ is different from the mean-variance measure when $\tilde{t}$ follows other kinds of distributions. Moreover, the nice thing about $CE_\lambda(\tilde{t})$ is it preserves first-order stochastic dominance (see for instance Föllmer and Schied, 2011), which is violated by the mean-variance measure. Take two paths as an example, one with travel time equal to 1 or 2 with 0.5 probabilities and the other with travel time equal to 3 (with certainty). Though the first path stochastically dominates the second, mean-variance measure would favor the second path for an extremely risk-averse traveler, while the CARA model always supports the first path, as the certainty equivalent of the first is always less than that of the second.

If the actual travel time probability distribution is not fully known, then it would be impossible to establish preferences for travel times based on the expected disutility criterion. The CARA model could not reveal travelers' preferences when facing ambiguity. We study the preference for uncertain travel times in which the traveler is oblivious to the true probability distribution $\mathbb{P}$ but knows the distributional uncertainty set $\mathbb{F}$, which can be characterized by certain descriptive statistics. The "size" of the set $\mathbb{F}$ indicates the level of ambiguity perceived by the traveler. For instance, the distributional uncertainty

set perceived by an informed traveler may be a subset of that perceived by a clueless traveler. To evaluate an ambiguity preference, the Hurwicz criterion (Arrow and Hurwicz, 1972; Hurwicz, 1951) represents a compromise between the worst-case and the best-case evaluation of travel time under distributional ambiguity as follows:

$$H_\alpha(\tilde{t}) = \alpha \sup_{\mathbb{P} \in \mathbb{F}} E_\mathbb{P}(\tilde{t}) + (1-\alpha) \inf_{\mathbb{P} \in \mathbb{F}} E_\mathbb{P}(\tilde{t}),$$

where the parameter $\alpha \in [0, 1]$ indicates the level of optimism, with $\alpha = 0$ being the most optimistic and $\alpha = 1$ being the most pessimistic.

2.1. Ambiguity-aware CARA travel time (ACT)

Instead of considering risk and ambiguity separately, we explicitly distinguish between them in a unified framework for articulating travelers' preferences for travel times. We propose the ambiguity-aware CARA travel time (ACT) criterion for evaluating an uncertain travel time under various attitudes of risk and ambiguity, which is based on blending Hurwicz and Constant Absolute Risk Aversion (CARA) criteria.

The traveler has a distributional uncertainty set $\mathbb{F}$ to characterize the uncertain travel time. Similar to the Hurwicz criterion, his/her attitude towards ambiguity is described by parameter $\alpha \in [0, 1]$ and risk attitude under CARA is given by parameter $\lambda \in \Re$. Accordingly, we identify the traveler under the ACT by $V = (\alpha, \lambda, \mathbb{F})$.

Definition 1. The ambiguity-aware CARA travel time $ACT_V(\tilde{t}) : \mathcal{V} \to \Re$ specified by the traveler with parameter $V = (\alpha, \lambda, \mathbb{F})$ is

$$ACT_V(\tilde{t}) = \begin{cases} \alpha \sup_{\mathbb{P} \in \mathbb{F}} \dfrac{1}{\lambda} \ln E_\mathbb{P}(\exp(\lambda\tilde{t})) + (1-\alpha) \inf_{\mathbb{P} \in \mathbb{F}} \dfrac{1}{\lambda} \ln E_\mathbb{P}(\exp(\lambda\tilde{t})), & \text{when } \lambda \neq 0, \\ \alpha \sup_{\mathbb{P} \in \mathbb{F}} E_\mathbb{P}(\tilde{t}) + (1-\alpha) \inf_{\mathbb{P} \in \mathbb{F}} E_\mathbb{P}(\tilde{t}), & \text{when } \lambda = 0. \end{cases}$$

Observing that if the probability distribution is known, i.e., $\mathbb{F} = \{\mathbb{P}\}$, we have

$$\begin{aligned} ACT_V(\tilde{t}) &= ACT_{(\alpha, \lambda, \mathbb{P})}(\tilde{t}) \\ &= \begin{cases} \alpha \dfrac{1}{\lambda} \ln E_\mathbb{P}(\exp(\lambda\tilde{t})) + (1-\alpha) \dfrac{1}{\lambda} \ln E_\mathbb{P}(\exp(\lambda\tilde{t})), & \text{when } \lambda \neq 0, \\ \alpha E_\mathbb{P}(\tilde{t}) + (1-\alpha) E_\mathbb{P}(\tilde{t}), & \text{when } \lambda = 0. \end{cases} \\ &= CE_\lambda(\tilde{t}). \end{aligned}$$

Hence, the ACT criterion is a generalization of certainty equivalent function under CARA. To quantitatively characterize travelers' attitudes towards risk and ambiguity, economists have summarized the procedure to sought these two parameters α, λ in experimental studies (see for instance Wakker, 2010 §11.7; Abdellaoui et al., 2011). We believe this could shed some light on the future empirical studies on travelers' preferences. Next, we provide some useful properties of the ACT criterion. For any given distributional uncertainty set $\mathbb{F}$, we first define the corresponding bound as $\bar{t}_\mathbb{F} = \inf\{t \in \Re | \mathbb{P}(\tilde{t} \leq t) = 1, \forall \mathbb{P} \in \mathbb{F}\}$ and $\underline{t}_\mathbb{F} = \sup\{t \in \Re | \mathbb{P}(\tilde{t} \geq t) = 1, \forall \mathbb{P} \in \mathbb{F}\}$.

Proposition 1.

(a) $ACT_V(\tilde{t})$ is nondecreasing in $\lambda \in \Re$ and $\alpha \in [0, 1]$, and

$$\lim_{\lambda \to \infty} ACT_{(1, \lambda, \mathbb{F})}(\tilde{t}) = \bar{t}_\mathbb{F}, \qquad \lim_{\lambda \to -\infty} ACT_{(0, \lambda, \mathbb{F})}(\tilde{t}) = \underline{t}_\mathbb{F}.$$

(b) For any $\tilde{x}, \tilde{y} \in \mathcal{V}$, if $\tilde{x} \geq \tilde{y}$, we have $ACT_V(\tilde{x}) \geq ACT_V(\tilde{y})$;

(c) Suppose $\tilde{t}_1, \ldots, \tilde{t}_J$ are independent random variables, and $t_0 \in \Re$. Then

$$ACT_V\left(t_0 + \sum_{j=1}^{J} \tilde{t}_j\right) = t_0 + \sum_{j=1}^{J} ACT_V(\tilde{t}_j).$$

Proof. (a) Note that $ACT_V(\tilde{t})$ being nondecreasing in α follows directly from $\sup_{\mathbb{P} \in \mathbb{F}} \frac{1}{\lambda} \ln E_\mathbb{P}(\exp(\lambda\tilde{t})) \geq \inf_{\mathbb{P} \in \mathbb{F}} \frac{1}{\lambda} \ln E_\mathbb{P}(\exp(\lambda\tilde{t}))$ and $\sup_{\mathbb{P} \in \mathbb{F}} E_\mathbb{P}(\tilde{t}) \geq \inf_{\mathbb{P} \in \mathbb{F}} E_\mathbb{P}(\tilde{t})$. Based on Jensen's inequality, for any $\lambda_1 \leq \lambda_2 < 0$ or $0 < \lambda_1 \leq \lambda_2$, we can get

$$\frac{1}{\lambda_2} \ln E_\mathbb{P}(\exp(\lambda_2\tilde{t})) = \frac{1}{\lambda_2} \ln E_\mathbb{P}\left((\exp(\lambda_1\tilde{t}))^{\lambda_2/\lambda_1}\right) \geq \frac{1}{\lambda_2}\frac{\lambda_2}{\lambda_1} \ln E_\mathbb{P}(\exp(\lambda_1\tilde{t})) = \frac{1}{\lambda_1} \ln E_\mathbb{P}(\exp(\lambda_1\tilde{t})).$$

When $\lambda_1 < 0 < \lambda_2$, we have

$$\frac{1}{\lambda_2} \ln E_\mathbb{P}(\exp(\lambda_2\tilde{t})) \geq \frac{1}{\lambda_2} \ln \exp(E_\mathbb{P}(\lambda_2\tilde{t})) = E_\mathbb{P}(\tilde{t}) \geq \frac{1}{\lambda_1} \ln E_\mathbb{P}(\exp(\lambda_1\tilde{t})).$$

Therefore, for any $\lambda_1 \leq \lambda_2$,

$$ACT_{(\alpha,\lambda_2,\mathbb{F})}\left(\tilde{t}\right) = \alpha \sup_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda_2} \ln E_{\mathbb{P}}\left(\exp\left(\lambda_2\tilde{t}\right)\right) + (1-\alpha) \inf_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda_2} \ln E_{\mathbb{P}}\left(\exp\left(\lambda_2\tilde{t}\right)\right)$$

$$\geq \alpha \sup_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda_1} \ln E_{\mathbb{P}}\left(\exp\left(\lambda_1\tilde{t}\right)\right) + (1-\alpha) \inf_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda_1} \ln E_{\mathbb{P}}\left(\exp\left(\lambda_1\tilde{t}\right)\right)$$

$$= ACT_{(\alpha,\lambda_1,\mathbb{F})}\left(\tilde{t}\right).$$

Equivalently, $ACT_V(\tilde{t})$ is nondecreasing in λ.

When $\alpha = 1$, the traveler is most pessimistic towards ambiguity, then

$$ACT_{(1,\lambda,\mathbb{F})}\left(\tilde{t}\right) = \begin{cases} \sup_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda} \ln E_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right), & \text{when } \lambda \neq 0, \\ \sup_{\mathbb{P}\in\mathbb{F}} E_{\mathbb{P}}\left(\tilde{t}\right), & \text{when } \lambda = 0. \end{cases}$$

We have for any $\mathbb{P} \in \mathbb{F}$ and $\lambda \in \mathfrak{R}\backslash\{0\}$,

$$\frac{1}{\lambda} \ln E_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right) \leq \frac{1}{\lambda} \ln\left(\exp\left(\lambda\bar{t}_{\mathbb{F}}\right)\right) = \bar{t}_{\mathbb{F}}.$$

Therefore,

$$\lim_{\lambda\to\infty} ACT_{(1,\lambda,\mathbb{F})}\left(\tilde{t}\right) \leq \bar{t}_{\mathbb{F}}.$$

Moreover, according to the definition of $\bar{t}_{\mathbb{F}}$, for any $\epsilon > 0$, $\exists \mathbb{P} \in \mathbb{F}$ such that $\mathbb{P}\left(\tilde{t} \in \left[\bar{t}_{\mathbb{F}} - \epsilon, \bar{t}_{\mathbb{F}}\right]\right) > 0$, we have

$$\lim_{\lambda\to\infty} \frac{1}{\lambda} \ln E_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right)$$

$$= \lim_{\lambda\to\infty} \frac{1}{\lambda} \ln E_{\mathbb{P}}\left(\exp\left(\lambda\bar{t}_{\mathbb{F}}\right)\exp\left(\lambda\left(\tilde{t} - \bar{t}_{\mathbb{F}}\right)\right)\right)$$

$$= \bar{t}_{\mathbb{F}} + \lim_{\lambda\to\infty} \frac{1}{\lambda} \ln E_{\mathbb{P}}\left(\exp\left(\lambda\left(\tilde{t} - \bar{t}_{\mathbb{F}}\right)\right)\right)$$

$$\geq \bar{t} + \lim_{\lambda\to\infty} \frac{1}{\lambda} \ln\left(\exp\left(\lambda\left(\bar{t}_{\mathbb{F}} - \epsilon - \bar{t}_{\mathbb{F}}\right)\right)\mathbb{P}\left(\tilde{t} \in \left[\bar{t}_{\mathbb{F}} - \epsilon, \bar{t}_{\mathbb{F}}\right]\right)\right)$$

$$= \bar{t}_{\mathbb{F}} + \lim_{\lambda\to\infty} \frac{1}{\lambda} \ln\left(\exp(-\lambda\epsilon)\right) + \lim_{\lambda\to\infty} \frac{1}{\lambda} \ln\left(\mathbb{P}\left(\tilde{t} \in \left[\bar{t}_{\mathbb{F}} - \epsilon, \bar{t}_{\mathbb{F}}\right]\right)\right)$$

$$= \bar{t}_{\mathbb{F}} - \epsilon,$$

which means,

$$\lim_{\lambda\to\infty} \sup_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda} \ln E_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right) \geq \lim_{\lambda\to\infty} \frac{1}{\lambda} \ln E_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right) \geq \bar{t}_{\mathbb{F}} - \epsilon \qquad \forall\, \epsilon > 0.$$

Combining these two inequalities together, we have

$$\lim_{\lambda\to\infty} ACT_{(1,\lambda,\mathbb{F})}\left(\tilde{t}\right) = \bar{t}_{\mathbb{F}}.$$

Similarly, we can modify the above proof to show that

$$\lim_{\lambda\to-\infty} ACT_{(0,\lambda,\mathbb{F})}\left(\tilde{t}\right) = \underline{t}_{\mathbb{F}}.$$

(b) If $\tilde{x} \geq \tilde{y}$ i.e., $x(\omega) \geq y(\omega)$ for all $\omega \in \Omega$, we have when $\lambda = 0$,

$$ACT_V(\tilde{x}) = \alpha \sup_{\mathbb{P}\in\mathbb{F}} E_{\mathbb{P}}(\tilde{x}) + (1-\alpha) \inf_{\mathbb{P}\in\mathbb{F}} E_{\mathbb{P}}(\tilde{x}) \geq \alpha \sup_{\mathbb{P}\in\mathbb{F}} E_{\mathbb{P}}(\tilde{y}) + (1-\alpha) \sup_{\mathbb{P}\in\mathbb{F}} E_{\mathbb{P}}(\tilde{y}) = ACT_V(\tilde{y}).$$

When $\lambda \neq 0$, noting that $\frac{1}{\lambda} \ln E_{\mathbb{P}}(\exp(\lambda\tilde{x})) \geq \frac{1}{\lambda} \ln E_{\mathbb{P}}(\exp(\lambda\tilde{y}))$ for all $\mathbb{P} \in \mathbb{F}$, we have

$$ACT_V(\tilde{x}) = \alpha \sup_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda} \ln E_{\mathbb{P}}(\exp(\lambda\tilde{x})) + (1-\alpha) \inf_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda} \ln E_{\mathbb{P}}(\exp(\lambda\tilde{x}))$$

$$\geq \alpha \sup_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda} \ln E_{\mathbb{P}}(\exp(\lambda\tilde{y})) + (1-\alpha) \inf_{\mathbb{P}\in\mathbb{F}} \frac{1}{\lambda} \ln E_{\mathbb{P}}(\exp(\lambda\tilde{y}))$$

$$= ACT_V(\tilde{y}).$$

(c) Since random variables $\tilde{t}_1, \ldots, \tilde{t}_J$ are independently distributed, we define the true probability distribution of the random variable $\tilde{t}_j$ as $\mathbb{P}^j$ which belongs to the distributional uncertainty set as $\mathbb{F}^j$ for all $j = 1, \ldots, J$. We let $\hat{\mathbb{P}} = \mathbb{P}^1 \times \ldots \times \mathbb{P}^J$ and
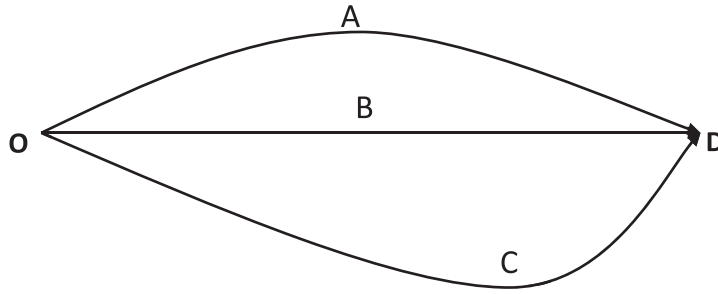
Fig. 1. A simple network with uncertain travel time.

$\hat{\mathbb{F}} = \mathbb{F}^1 \times \ldots \times \mathbb{F}^J$ represent the joint probability distribution of $\tilde{t}_1, \ldots, \tilde{t}_J$ and its corresponding distributional uncertainty set, respectively. Hence, we get

$$ACT_V\left(t_0 + \sum_{j=1}^{J} \tilde{t}_j\right)$$

$$= \alpha \sup_{\hat{\mathbb{P}} \in \hat{\mathbb{F}}} \frac{1}{\lambda} \ln E_{\hat{\mathbb{P}}}\left(\exp\left(\lambda\left(t_0 + \sum_{j=1}^{J} \tilde{t}_j\right)\right)\right) + (1-\alpha) \inf_{\hat{\mathbb{P}} \in \hat{\mathbb{F}}} \frac{1}{\lambda} \ln E_{\hat{\mathbb{P}}}\left(\exp\left(\lambda\left(t_0 + \sum_{j=1}^{J} \tilde{t}_j\right)\right)\right)$$

$$= \alpha t_0 + \alpha \sup_{\mathbb{P}^1 \in \mathbb{F}^1} \ldots \sup_{\mathbb{P}^J \in \mathbb{F}^J} \frac{1}{\lambda} \ln \left(\prod_{j=1}^{J} E_{\mathbb{P}^j}\left(\exp\left(\lambda \tilde{t}_j\right)\right)\right) + (1-\alpha) t_0$$

$$+ (1-\alpha) \inf_{\mathbb{P}^1 \in \mathbb{F}^1} \ldots \inf_{\mathbb{P}^J \in \mathbb{F}^J} \frac{1}{\lambda} \ln \left(\prod_{j=1}^{J} E_{\mathbb{P}^j}\left(\exp\left(\lambda \tilde{t}_j\right)\right)\right)$$

$$= t_0 + \alpha \sup_{\mathbb{P}^1 \in \mathbb{F}^1} \ldots \sup_{\mathbb{P}^J \in \mathbb{F}^J} \frac{1}{\lambda}\left(\sum_{j=1}^{J} \ln E_{\mathbb{P}^j}\left(\exp(\lambda \tilde{t}_j)\right)\right) + (1-\alpha) \inf_{\mathbb{P}^1 \in \mathbb{F}^1} \ldots \inf_{\mathbb{P}^J \in \mathbb{F}^J} \frac{1}{\lambda}\left(\sum_{j=1}^{J} \ln E_{\mathbb{P}^j}\left(\exp(\lambda \tilde{t}_j)\right)\right)$$

$$= t_0 + \alpha \frac{1}{\lambda}\left(\sum_{j=1}^{J} \sup_{\mathbb{P}^j \in \mathbb{F}^j} \ln E_{\mathbb{P}^j}\left(\exp(\lambda \tilde{t}_j)\right)\right) + (1-\alpha) \frac{1}{\lambda}\left(\sum_{j=1}^{J} \inf_{\mathbb{P}^j \in \mathbb{F}^j} \ln E_{\mathbb{P}^j}\left(\exp(\lambda \tilde{t}_j)\right)\right)$$

$$= t_0 + \sum_{j=1}^{J}\left(\alpha \sup_{\mathbb{P}^j \in \mathbb{F}^j} \frac{1}{\lambda} \ln E_{\mathbb{P}^j}\left(\exp(\lambda \tilde{t}_j)\right) + (1-\alpha) \inf_{\mathbb{P}^j \in \mathbb{F}^j} \frac{1}{\lambda} \ln E_{\mathbb{P}^j}\left(\exp(\lambda \tilde{t}_j)\right)\right)$$

$$= t_0 + \sum_{j=1}^{J} ACT_V\left(\tilde{t}_j\right)$$

The second and fourth equalities hold since random variables $\tilde{t}_1, \ldots, \tilde{t}_J$ are independently distributed. $\quad\square$

Remark 1. Property (a) is a trivial statement, it indicates that when a traveler is more risk-averse or ambiguity averse than the others, he/she perceives the uncertain travel time longer than the others' perception. The extreme cases occur when $\lambda = \infty, \alpha = 1$ and $\lambda = -\infty, \alpha = 0$, respectively. When a traveler is extremely risk-averse and ambiguity averse, he/she pessimistically regards the uncertain travel time from the worst-case perspective, and the corresponding $ACT_V(\tilde{t})$ takes the largest possible value. Property (b) captures traveler's essential preference for a shorter travel time. His/her perceived travel time becomes longer when the travel time increases. Property (c) suggests that $ACT_V(\cdot)$ is additive for independent random variables. This property is quite helpful for modeling, since $ACT_V(\cdot)$ along a path could be easily separated by links.

Next, we will provide an example to illustrate travelers' preferences for travel times under the ACT criterion. Fig. 1 shows three paths from the origin O to the destination D. Travel time on path A is deterministic, 1.5 h; travel time on path B is stochastic and the duration is 1 h or 2 h with equal probability; travel time on path C is uncertain, and bounded by 1 h and 2 h. We present in Table 1 the path preferences induced by the ACT criterion under various attitudes and degrees of risk and ambiguity.

When a traveler is extremely risk-averse and pessimistic towards ambiguity ($\lambda \to \infty, \alpha = 1$) as property (a) described, he/she will perceive the uncertain travel time as taking the longest duration. Hence, path A is preferred as it has the smallest *ACT*. On the other hand, when the traveler is radically risk-seeking and optimistic towards ambiguity ($\lambda \to -\infty, \alpha = 0$), then path A would be least preferred. At risk neutrality, both paths A and B are equally preferred and the preference for path C

Table 1
Preferences for travel times under the ACT criterion.

Risk attitude λ	Ambiguity attitude α	$ACT_V(\tilde{t}_A)$	$ACT_V(\tilde{t}_B)$	$ACT_V(\tilde{t}_C)$	Preferences
∞	1	1.5	2	2	$A \succ B \sim C$
0	1	1.5	1.5	2	$A \sim B \succ C$
0	0	1.5	1.5	1	$C \succ A \sim B$
$-\infty$	0	1.5	1	1	$B \sim C \succ A$

depends on the traveler's attitude towards ambiguity. For instance, if he/she is optimistic towards ambiguity, then path C will be preferred over paths A and B.

2.2. Two simple uncertainty models for travel time

If the probability distribution of an uncertain travel time $\tilde{t}$ is completely known, there exists no ambiguity, and $ACT_V(\tilde{t})$ reduces to $CE_\lambda(\tilde{t})$, which can be calculated directly. When the probability distribution is not fully available, the characterization of uncertain travel time can be in various ways depending on the available information. We then propose two simple models for characterizing the uncertain travel time and provide analytical forms of the ACT criterion.

Uncertainty model I

Driven by pragmatism, the traveler may have a simple description of the uncertain travel time by providing the ranges in which travel time and average travel time would fall within. Specifically, the travel time takes values in $[\underline{t}, \overline{t}]$, $0 < \underline{t} \leq \overline{t}$ and the average travel time falls within the range $[\underline{\mu}, \overline{\mu}] \subseteq [\underline{t}, \overline{t}]$. Hence, the distributional uncertainty set $\mathbb{F}$ of the uncertain travel time $\tilde{t}$ is given by

$$\mathbb{F} = \left\{ \mathbb{P} \middle| E_{\mathbb{P}}(\tilde{t}) \in [\underline{\mu}, \overline{\mu}], \mathbb{P}(\tilde{t} \in [\underline{t}, \overline{t}]) = 1 \right\}. \tag{1}$$

Proposition 2. *Given a distributional uncertainty set $\mathbb{F}$ described by (1), the uncertain travel time under the ACT criterion is*

$$ACT_V(\tilde{t}) = \begin{cases} \dfrac{\alpha}{\lambda} \ln \left(\dfrac{(\overline{t} - \overline{\mu}) \exp(\lambda \underline{t}) + (\overline{\mu} - \underline{t}) \exp(\lambda \overline{t})}{\overline{t} - \underline{t}} \right) + (1 - \alpha)\underline{\mu}, & \text{when } \lambda > 0, \\[4mm] \alpha\overline{\mu} + \dfrac{1 - \alpha}{\lambda} \ln \left(\dfrac{(\overline{t} - \underline{\mu}) \exp(\lambda \underline{t}) + (\underline{\mu} - \underline{t}) \exp(\lambda \overline{t})}{\overline{t} - \underline{t}} \right), & \text{when } \lambda < 0, \\[4mm] \alpha\overline{\mu} + (1 - \alpha)\underline{\mu}, & \text{when } \lambda = 0. \end{cases}$$

Moreover,

$$\lim_{\lambda \to \infty} ACT_V(\tilde{t}) = \alpha\overline{t} + (1 - \alpha)\underline{\mu},$$
$$\lim_{\lambda \to -\infty} ACT_V(\tilde{t}) = (1 - \alpha)\underline{t} + \alpha\overline{\mu}.$$

Proof. We first provide the analytical expressions for $\sup_{\mathbb{P} \in \mathbb{F}} E_{\mathbb{P}}(\exp(\lambda\tilde{t}))$ and $\inf_{\mathbb{P} \in \mathbb{F}} E_{\mathbb{P}}(\exp(\lambda\tilde{t}))$. According to Proposition 3 in Brown et al. (2012),

$$\sup_{\mathbb{P} \in \mathbb{F}} E_{\mathbb{P}}(\exp(\lambda\tilde{t})) = \begin{cases} \dfrac{(\overline{t} - \overline{\mu}) \exp(\lambda \underline{t}) + (\overline{\mu} - \underline{t}) \exp(\lambda \overline{t})}{\overline{t} - \underline{t}}, & \text{when } \lambda > 0, \\[4mm] \dfrac{(\overline{t} - \underline{\mu}) \exp(\lambda \underline{t}) + (\underline{\mu} - \underline{t}) \exp(\lambda \overline{t})}{\overline{t} - \underline{t}}, & \text{when } \lambda < 0. \end{cases}$$

To determine $\inf_{\mathbb{P} \in \mathbb{F}} E_{\mathbb{P}}(\exp(\lambda\tilde{t}))$, we note that by Jensen's inequality,

$$E_{\mathbb{P}}(\exp(\lambda\tilde{t})) \geq \exp(E_{\mathbb{P}}(\lambda\tilde{t})) = \exp(\lambda E_{\mathbb{P}}(\tilde{t})),$$

consequently,

$$\inf_{\mathbb{P} \in \mathbb{F}} E_{\mathbb{P}}(\exp(\lambda\tilde{t})) \geq \begin{cases} \exp(\lambda\underline{\mu}), & \text{when } \lambda > 0, \\ \exp(\lambda\overline{\mu}), & \text{when } \lambda < 0. \end{cases}$$

Equality holds when $\tilde{t}$ is deterministic,

$$\begin{cases} \mathbb{P}(\tilde{t} = \underline{\mu}) = 1, & \text{when } \lambda > 0; \\ \mathbb{P}(\tilde{t} = \overline{\mu}) = 1, & \text{when } \lambda < 0. \end{cases}$$

Table 2
Path preferences under ACT.

Risk attitude λ	Ambiguity attitude α	Preferences
$[0, \infty)$	$[f(\lambda), 1]$[a]	$A \succeq B \succeq C$
$[0, \infty)$	$\left[\frac{1}{2}, f(\lambda)\right]$	$A \succeq C \succeq B$
$[0, \infty)$	$\left[0, \frac{1}{2}\right]$	$C \succeq A \succeq B$
$(-\infty, 0]$	$\left[\frac{1}{2}, 1\right]$	$B \succeq A \succeq C$
$(-\infty, 0]$	$\left[g(\lambda), \frac{1}{2}\right]$[b]	$B \succeq C \succeq A$
$(-\infty, 0]$	$[0, g(\lambda)]$	$C \succeq B \succeq A$

[a] $f(\lambda) = \frac{\lambda}{3\lambda + 2\ln 2 - 2\ln(1 + \exp(\lambda))}$;
[b] $g(\lambda) = \frac{2\ln(1 + \exp(\lambda)) - 2\ln 2}{\lambda + 2\ln(1 + \exp(\lambda)) - 2\ln 2}$.

Note that this distribution also belongs to the distributional uncertainty set $\mathbb{F}$, and $ACT_V(\tilde{t})$ can be accordingly calculated. Based on L'Hôpital's rule, when $\lambda > 0$,

$$
\begin{aligned}
\lim_{\lambda \to \infty} ACT_V(\tilde{t}) &= \lim_{\lambda \to \infty} \left(\frac{\alpha}{\lambda} \ln \left(\frac{(\bar{t} - \overline{\mu}) \exp(\lambda \underline{t}) + (\overline{\mu} - \underline{t}) \exp(\lambda \bar{t})}{\bar{t} - \underline{t}} \right) + (1 - \alpha)\underline{\mu} \right) \\
&= \lim_{\lambda \to \infty} \left(\frac{\alpha}{\lambda} \ln \left(\frac{(\bar{t} - \overline{\mu}) \exp(\lambda \underline{t}) + (\overline{\mu} - \underline{t}) \exp(\lambda \bar{t})}{\bar{t} - \underline{t}} \right) \right) + (1 - \alpha)\underline{\mu} \\
&= \alpha \lim_{\lambda \to \infty} \left(\frac{(\bar{t} - \overline{\mu}) \exp(\lambda \underline{t})\underline{t} + (\overline{\mu} - \underline{t}) \exp(\lambda \bar{t})\bar{t}}{(\bar{t} - \overline{\mu}) \exp(\lambda \underline{t}) + (\overline{\mu} - \underline{t}) \exp(\lambda \bar{t})} \right) + (1 - \alpha)\underline{\mu} \\
&= \alpha \lim_{\lambda \to \infty} \left(\frac{(\bar{t} - \overline{\mu}) \exp(\lambda(\underline{t} - \bar{t}))\underline{t} + (\overline{\mu} - \underline{t})\bar{t}}{(\bar{t} - \overline{\mu}) \exp(\lambda(\underline{t} - \bar{t})) + (\overline{\mu} - \underline{t})} \right) + (1 - \alpha)\underline{\mu} \\
&= \alpha \bar{t} + (1 - \alpha)\underline{\mu}.
\end{aligned}
$$

Likewise, the result could extend to

$$
\lim_{\lambda \to -\infty} ACT_V(\tilde{t}) = (1 - \alpha)\underline{t} + \alpha\overline{\mu}.
$$

$\square$

We further analyze paths preferences on the simple network depicted in Fig. 1 as an example.

Example: In Fig. 1, travel times on path A and C remain unchanged. As for path B, we now assume that the travel time is within 1 h to 2 h, and the mean travel time is exactly 1.5 h. Given the above information of three paths, travelers' preferences ranked by the ACT criterion are summarized in Table 2.

To show the results in Table 2, from Proposition 2, we calculate the travel time under the ACT criterion for each of the three paths. The information is specified as follows:

$$
\begin{array}{llll}
\underline{t}_A = 1.5, & \bar{t}_A = 1.5 & \underline{\mu}_A = 1.5, & \overline{\mu}_A = 1.5, \\
\underline{t}_B = 1, & \bar{t}_B = 2, & \underline{\mu}_B = 1.5, & \overline{\mu}_B = 1.5, \\
\underline{t}_C = 1, & \bar{t}_C = 2, & \underline{\mu}_C = 1, & \overline{\mu}_C = 2.
\end{array}
$$

Therefore, the *ACT* can be calculated correspondingly,

$$
ACT_V(t_A) = \frac{3}{2};
$$

$$
ACT_V(\tilde{t}_B) = \begin{cases}
\frac{\alpha}{\lambda} \ln \left(\frac{1}{2} \exp(\lambda) + \frac{1}{2} \exp(2\lambda) \right) + \frac{3}{2}(1 - \alpha), & \text{when } \lambda > 0, \\
\frac{3}{2}\alpha + \frac{1 - \alpha}{\lambda} \ln \left(\frac{1}{2} \exp(\lambda) + \frac{1}{2} \exp(2\lambda) \right), & \text{when } \lambda < 0, \\
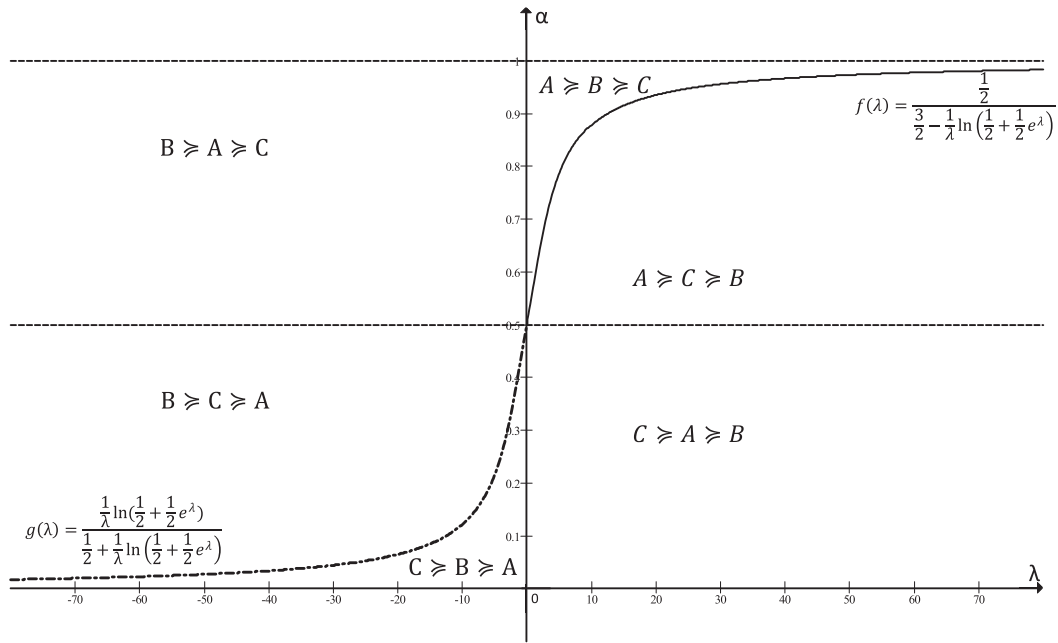\frac{3}{2}\alpha + \frac{3}{2}(1 - \alpha), & \text{when } \lambda = 0;
\end{cases}
$$

Fig. 2. Path preferences under different attitudes towards risk and ambiguity.

$$ACT_V\left(\tilde{t}_C\right) = \begin{cases} \dfrac{\alpha}{\lambda}\ln\left(\exp(2\lambda)\right) + (1-\alpha), & \text{when } \lambda > 0, \\[2mm] 2\alpha + \dfrac{1-\alpha}{\lambda}\ln\left(\exp(\lambda)\right), & \text{when } \lambda < 0, \\[2mm] 2\alpha + (1-\alpha), & \text{when } \lambda = 0 \end{cases}$$
$$= 1 + \alpha.$$

Since the travel time under the ACT criterion is nondecreasing in both λ and α, the preference relationships between paths A and B, and between paths A and C can be readily established. When $\lambda \geq 0$, we have $A \succeq B$. Likewise, when $1 \geq \alpha \geq \frac{1}{2}$, then $A \succeq C$. Hence, we focus on the preferences between paths B and C.

$ACT_V\left(\tilde{t}_B\right) \geq ACT_V\left(\tilde{t}_C\right)$ implies

$$\begin{cases} \dfrac{\alpha}{\lambda}\ln\left(\dfrac{1}{2}\exp(\lambda) + \dfrac{1}{2}\exp(2\lambda)\right) + \dfrac{3}{2}(1-\alpha) \geq 1+\alpha, & \text{when } \lambda > 0, \\[3mm] \dfrac{3}{2}\alpha + \dfrac{1-\alpha}{\lambda}\ln\left(\dfrac{1}{2}\exp(\lambda) + \dfrac{1}{2}\exp(2\lambda)\right) \geq 1+\alpha, & \text{when } \lambda < 0, \\[3mm] \dfrac{3}{2} \geq 1+\alpha, & \text{when } \lambda = 0. \end{cases}$$

Equivalently, path C is preferred to path B when

$$\begin{cases} \alpha \leq f(\lambda) = \dfrac{\frac{1}{2}}{\frac{3}{2} - \frac{1}{\lambda}\ln\left(\frac{1}{2} + \frac{1}{2}\exp(\lambda)\right)} = \dfrac{\lambda}{3\lambda + 2\ln 2 - 2\ln\left(1+\exp(\lambda)\right)}, & \text{when } \lambda > 0, \\[4mm] \alpha \leq g(\lambda) = \dfrac{\frac{1}{\lambda}\ln\left(\frac{1}{2} + \frac{1}{2}\exp(\lambda)\right)}{\frac{1}{2} + \frac{1}{\lambda}\ln\left(\frac{1}{2} + \frac{1}{2}\exp(\lambda)\right)} = \dfrac{2\ln\left(1+\exp(\lambda)\right) - 2\ln 2}{\lambda + 2\ln\left(1+\exp(\lambda)\right) - 2\ln 2}, & \text{when } \lambda < 0. \end{cases}$$

The preferences expressed by travelers with varied λ and α are depicted in Fig. 2. When the traveler is risk-averse ($\lambda > 0$), he/she prefers path A over path B, and the converse is true when the traveler is risk-seeking. With α decreases from 1 to 0, the traveler's attitude towards ambiguity shifts from being pessimistic to optimistic, in which case, path C, which has complete ambiguity, will become more favorable. This example may suggest a way to empirically identify travelers' attitudes towards risk and ambiguity by providing travelers with different choice scenarios.

Uncertainty model II

In practice, the uncertain travel time only takes a set of discrete values. Next, we present a general model for this discrete case where the uncertain travel time has finite realizations, for example, $t_1, \ldots, t_M$, and more statistics on the moment are

available, i.e., the distributional uncertainty set $\mathbb{F}$ is given by

$$\mathbb{F} = \left\{ \mathbb{P} \,\middle|\, E_{\mathbb{P}}\left(\tilde{t}^{l_k}\right) \in \left[\underline{\mu}_k, \overline{\mu}_k\right],\ k = 1, \ldots, K, \quad \mathbb{P}\left(\tilde{t} \in \{t_1, \ldots, t_M\}\right) = 1 \right\}, \tag{2}$$

where $l_k \in \mathcal{Z}_+, k = 1, \ldots, K$.

Proposition 3. *If the distributional uncertainty set $\mathbb{F}$ is described by (2), the uncertain travel time under the ACT criterion can be derived by solving two linear optimization problems,*

$$ACT_V(\tilde{t}) = \begin{cases} \alpha \dfrac{1}{\lambda} \ln\left(\sup\limits_{\mathbb{P} \in \mathbb{F}} E_{\mathbb{P}}\left(\exp(\lambda\tilde{t})\right)\right) + (1-\alpha)\dfrac{1}{\lambda}\ln\left(\inf\limits_{\mathbb{P} \in \mathbb{F}} E_{\mathbb{P}}\left(\exp(\lambda\tilde{t})\right)\right), & \text{when } \lambda > 0, \\[3mm] (1-\alpha)\dfrac{1}{\lambda} \ln\left(\sup\limits_{\mathbb{P} \in \mathbb{F}} E_{\mathbb{P}}\left(\exp(\lambda\tilde{t})\right)\right) + \alpha\dfrac{1}{\lambda}\ln\left(\inf\limits_{\mathbb{P} \in \mathbb{F}} E_{\mathbb{P}}\left(\exp(\lambda\tilde{t})\right)\right), & \text{when } \lambda < 0, \\[3mm] \alpha\overline{\mu}_1 + (1-\alpha)\underline{\mu}_1, & \text{when } \lambda = 0. \end{cases}$$

where

$$\sup\limits_{\mathbb{P} \in \mathbb{F}} E_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right) = \max\limits_{(p_1, \ldots, p_M) \in \mathcal{P}} \sum\limits_{m=1}^{M} p_m \exp(\lambda t_m)$$

$$\inf\limits_{\mathbb{P} \in \mathbb{F}} E_{\mathbb{P}}\left(\exp\left(\lambda\tilde{t}\right)\right) = \min\limits_{(p_1, \ldots, p_M) \in \mathcal{P}} \sum\limits_{m=1}^{M} p_m \exp(\lambda t_m)$$

and

$$\mathcal{P} = \left\{ (p_1, \ldots, p_M) \,\middle|\, \begin{array}{ll} \sum\limits_{m=1}^{M} p_m t_m^{l_k} \leq \overline{\mu}_k, & k = 1, \ldots, K, \\[2mm] \sum\limits_{m=1}^{M} p_m t_m^{l_k} \geq \underline{\mu}_k, & k = 1, \ldots, K, \\[2mm] \sum\limits_{m=1}^{M} p_m = 1, & \\[2mm] p_m \geq 0, & m = 1, \ldots, M. \end{array} \right\}.$$

Proof. The proof for this proposition is rather straight forward. $\square$

3. Path selection under the ACT criterion

In this section, we study the problem of selecting the path that minimizes the ACT criterion when the link travel times on the network are uncertain. We consider a directed network $\mathcal{G} = (\mathcal{N}, \mathcal{A})$ and let $\mathcal{R}$ be the set of all admissible paths, which are sets of links connecting the origin node to the destination node. The uncertain travel time along the link $a \in \mathcal{A}$ is denoted by $\tilde{t}_a$.

The deterministic version of this path selection problem or shortest path problem is well known to be polynomial time solvable. When the travel times are uncertain, the path selection problem that minimizes the travel time under the ACT criterion is given by

$$\min_{r \in \mathcal{R}} ACT_V\left(\sum_{a \in r} \tilde{t}_a\right). \tag{3}$$

In Proposition 4 below, we show that the solvability of Problem (3) depends on whether the uncertain link travel times are correlated.

Proposition 4.

(a) *If the uncertain link travel times are independently distributed, then Problem (3) is a shortest path problem on the same network in which the link travel time on $a \in \mathcal{A}$ is given by $ACT_V(\tilde{t}_a)$.*

(b) *If the uncertain link travel times are correlated, then the recognition version of Problem (3) is NP-complete.*

Proof. (a) According to Proposition 1, if the link travel times are independently distributed, the objective function in Problem (3) can be written additively as

$$ACT_V\left(\sum_{a \in r} \tilde{t}_a\right) = \sum_{a \in r} ACT_V(\tilde{t}_a).$$

In this case, we can regard the travel time under the ACT criterion along each link as the deterministic link travel time, and polynomially solve it by the shortest path algorithm.

(b) We will prove its NP-complete by reduction from the following problem, which is proved to be NP-complete by Yu and Yang (1998):

$$\min_{r \in \mathcal{R}} \max \left\{ \sum_{a \in r} t_a^1, \sum_{a \in r} t_a^2 \right\}, \tag{4}$$

where t_a^1 and t_a^2 are two travel time scenarios on link $a \in \mathcal{A}$.

We construct an instance of Problem (3), in which the uncertain travel time on link a is

$$\tilde{t}_a = \frac{1}{2}\left(t_a^1 + t_a^2\right) + \frac{1}{2}\left(t_a^1 - t_a^2\right)\tilde{z}, \quad \forall\, a \in \mathcal{A},$$

that is, the travel times of all the links are influenced by a common random variable $\tilde{z}$, which we assume is $+1$ or -1 with equal probability. Hence, for an extremely risk-averse and pessimistic towards ambiguity traveler ($\lambda \to \infty$, $\alpha = 1$), finding a path with minimum travel time under the ACT criterion from the origin node to the destination node can be written as

$$\min_{r \in \mathcal{R}} \lim_{\lambda \to \infty} \sup_{\mathbb{P} \in \mathbb{F}} \frac{1}{\lambda} \ln \mathbb{E}_{\mathbb{P}} \left(\exp\left(\lambda \sum_{a \in r} \left(\frac{1}{2}\left(t_a^1 + t_a^2\right) + \frac{1}{2}\left(t_a^1 - t_a^2\right)\tilde{z} \right) \right) \right).$$

According to Proposition 1, it can be simplified further as

$$\min_{r \in \mathcal{R}} \sum_{a \in r} \frac{1}{2}\left(t_a^1 + t_a^2\right) + \max\left\{ \sum_{a \in r} \frac{1}{2}\left(t_a^1 - t_a^2\right), \sum_{a \in r} \frac{1}{2}\left(t_a^2 - t_a^1\right) \right\},$$

which could be equivalently written as Problem (4). Thus, Problem (3) is NP-complete. □

Proposition 4 shows that when the link travel times are independently distributed, we can easily find the optimal path under the ACT criterion, which accounts for both risk and ambiguity. The result, though simple, shows that the ACT criterion not only is descriptive relevant by being able to account for a traveler's different attitudes of risk and ambiguity over uncertain travel times, but also can be used normatively to find the most preferred path using modest computational effort.

4. Analysis of network equilibrium with risk and ambiguity aware travelers

We study the network equilibrium problem when travelers are sensitive to risk and ambiguity and evaluate the travel times along paths using the ACT criterion. In Section 4.1, we characterize the network equilibrium such that no traveler could improve his/her travel time under the ACT criterion by unilaterally changing routes. In Section 4.2, we investigate the inefficiency of the NE by comparing with the System Optimal solution that minimizes the total travel time under the ACT criterion of all travelers. We also provide a simple network equilibrium study in Section 4.3.

4.1. Network equilibrium formulation

Given a network $\mathcal{G} = (\mathcal{N}, \mathcal{A})$, we let $\mathcal{W} \subseteq \mathcal{N} \times \mathcal{N}$ be a set of Origin-Destination (OD) pairs, and $\mathcal{R}_w$ be a set of all simple paths connecting a given OD pair $w \in \mathcal{W}$. To derive a tractable model, we assume that the uncertain link travel times are independently distributed. We define the uncertain travel time along link $a \in \mathcal{A}$ as

$$\tilde{t}_a(v_a) = s_a(v_a)\tilde{z}_a + \tilde{\tau}_a,$$

where $s_a(v_a)$ is a differentiable, monotonically increasing function in its own link traffic flow v_a, and $\tilde{z}_a, \tilde{\tau}_a, a \in \mathcal{A}$ are independently distributed nonnegative random variables. The multiplicative uncertainty $\tilde{z}_a$ can be interpreted as the flow dependent disturbance, while $\tilde{\tau}_a$, the additive uncertainty, is the flow independent disturbance.

For generality, we allow travelers to have different perceptions on uncertainty in link travel times. For example, a local resident, who is very familiar with the area, would be less ambiguous, compared to a tourist, in characterizing the uncertain travel times along the network links. To characterize the heterogeneity, we classify all travelers on the network into n types. The ith type of travelers, $i \in \mathcal{I} = \{1, \ldots, n\}$ are characterized by their risk parameter λ_i, ambiguity parameter α_i, and their distributional uncertainty set $\mathbb{F}_i$ of the travel times on the network. For notational convenience, we denote $V_i = (\lambda_i, \alpha_i, \mathbb{F}_i)$. Under the ACT criterion, the uncertain travel time $\tilde{t}_a(v_a)$ perceived by the ith type of travelers is given by

$$\begin{aligned}
t_{ai}(v_a) &= ACT_{V_i}\left(\tilde{t}_a(v_a)\right) \\
&= ACT_{V_i}(s_a(v_a)\tilde{z}_a + \tilde{\tau}_a) \\
&= ACT_{V_i}(s_a(v_a)\tilde{z}_a) + ACT_{V_i}(\tilde{\tau}_a).
\end{aligned}$$

For a given OD pair $w \in \mathcal{W}$, let d_{wi} be the number of trips made by the ith type of travelers and f_{ri} be the flow on path $r \in \mathcal{R}_w$ contributed by the ith type of travelers, and $\boldsymbol{f} = (f_{ri})_{r \in \mathcal{R}_w, w \in \mathcal{W}, i \in \mathcal{I}}$ is the vector of flows of all travelers along all paths. The aggregate flow on link $a \in \mathcal{A}$ is

$$v_a(\boldsymbol{f}) = \sum_{i \in \mathcal{I}} \sum_{w \in \mathcal{W}} \sum_{r \in \mathcal{R}_w} f_{ri}\delta_{ar},$$

where δ_{ar} equals 1 if the link a is along the path r and 0 otherwise. Moreover, since the travel time along any path r is given by

$$\tilde{c}_r(\boldsymbol{f}) = \sum_{a \in \mathcal{A}} \tilde{t}_a(v_a(\boldsymbol{f}))\delta_{ar},$$

the travel time along path $r \in \mathcal{R}_w$ under the ACT criterion perceived by the ith type of travelers is given by

$$
\begin{aligned}
c_{ri}(\boldsymbol{f}) &= ACT_{V_i}(\tilde{c}_r(\boldsymbol{f})) \\
&= ACT_{V_i}\left(\sum_{a \in \mathcal{A}}(s_a(v_a(\boldsymbol{f}))\tilde{z}_a + \tilde{\tau}_a)\delta_{ar}\right) \\
&= \sum_{a \in \mathcal{A}}\left(ACT_{V_i}(s_a(v_a(\boldsymbol{f}))\tilde{z}_a\delta_{ar}) + ACT_{V_i}(\tilde{\tau}_a\delta_{ar})\right) \\
&= \sum_{a \in \mathcal{A}} t_{ai}(v_a(\boldsymbol{f}))\delta_{ar}.
\end{aligned}
$$

Let $\boldsymbol{c}(\boldsymbol{f}) = (c_{ri}(\boldsymbol{f}))_{r \in \mathcal{R}_w, w \in \mathcal{W}, i \in \mathcal{I}}$ be the vector of the travel time under the ACT criterion of all types of travelers over all paths, and $\mathcal{F}$ be the feasible set of possible flows on all paths denoted by

$$\mathcal{F} = \left\{\boldsymbol{f} \geq \boldsymbol{0} \,\middle|\, \sum_{r \in \mathcal{R}_w} f_{ri} = d_{wi}, \quad w \in \mathcal{W}, i \in \mathcal{I}\right\},$$

in which the constraints are OD demand conservation conditions for all classes of travelers among all OD pairs. Hence, the path flow $\boldsymbol{f}^* \in \mathcal{F}$ is in equilibrium if for all $r \in \mathcal{R}_w, w \in \mathcal{W}, i \in \mathcal{I}$,

$$c_{ri}(\boldsymbol{f}^*)\begin{cases} = \mu_{wi}, & \text{if } f_{ri}^* > 0, \\ \geq \mu_{wi}, & \text{if } f_{ri}^* = 0, \end{cases}$$

where μ_{wi} is the equilibrium travel time under the ACT criterion associated with the OD pair w and traveler type i. The travel time along any path connecting the OD pair w perceived by the ith type of travelers under the ACT criterion is at least μ_{wi}. Moreover, on the paths that have been actually traveled ($f_{ri}^* > 0$), the perceived travel times are exactly at the minimum $c_{ri}(\boldsymbol{f}^*) = \mu_{wi}$. In other words, no traveler can improve his/her travel time under the ACT criterion by unilaterally changing routes.

Clearly, we can formulate the NE by means of Variational Inequalities (VI). We let $\boldsymbol{v} = (v_{ai})_{a \in \mathcal{A}, i \in \mathcal{I}}$ be the vector of flows of all travelers along all links, and we have $v_a = \sum_{i \in \mathcal{I}} v_{ai}, a \in \mathcal{A}$. Let $\boldsymbol{t}(v) = (t_{ai}(v_a))_{a \in \mathcal{A}, i \in \mathcal{I}}$ be the vector of travel time under the ACT criterion of all traveler types and along all links. The set of feasible link flows is represented by

$$\mathcal{V} = \left\{\boldsymbol{v} \,\middle|\, v_{ai} = \sum_{w \in \mathcal{W}} \sum_{r \in \mathcal{R}_w} f_{ri}\delta_{ar}, \quad a \in \mathcal{A}, i \in \mathcal{I}, \boldsymbol{f} \in \mathcal{F}\right\}.$$

Proposition 5. *The path flow of the NE can be equivalently characterized by the following VI problem:*
Find $\boldsymbol{f}^* \in \mathcal{F}$, such that

$$\langle \boldsymbol{f} - \boldsymbol{f}^*, \boldsymbol{c}(\boldsymbol{f}^*) \rangle \geq 0, \qquad \forall \boldsymbol{f} \in \mathcal{F},$$

where $\langle \cdot \rangle$ denotes the Euclidean inner product. Likewise, the link flow of NE is characterized by finding $\boldsymbol{v}^* \in \mathcal{V}$, such that

$$\langle \boldsymbol{v} - \boldsymbol{v}^*, \boldsymbol{t}(v^*) \rangle \geq 0, \qquad \forall \boldsymbol{v} \in \mathcal{V}. \tag{5}$$

Proof. This is an extension of the single class deterministic NE problem and we refer interested readers to Smith (1979) and Dafermos (1980). □

If travelers are homogeneous, i.e., $n = 1$, the NE defined under the ACT criterion reduces to a single class deterministic NE model. For the general case, $n > 1$, we could adopt algorithms for solving the generic VI (see Facchinei and Pang, 2003; Nagurney, 1998).

Corollary 1. *The link flow of NE exists, but may not be unique.*

Proof. Since set $\mathcal{V}$ is a compact set, and function $\boldsymbol{t}(v)$ is continuous, Problem (5) admits at least one solution $\boldsymbol{v}^*$. Furthermore, this link flow of NE may not be unique, as $\boldsymbol{t}(v)$ is not strictly monotone in $\mathcal{V}$. □

For the special case in which uncertainty along links is flow independent, we show that the corresponding NE problem can be solved via a convex optimization problem. Under this case, the uncertain travel time on link $a \in \mathcal{A}$ can be simplified as

$$\tilde{t}_a(v_a) = s_a(v_a) + \tilde{\tau}_a,$$

and travel time perceived by the ith type of travelers under the ACT criterion is

$$t_{ai}(v_a) = s_a(v_a) + ACT_{V_i}(\tilde{\tau}_a).$$

Proposition 6. *When the uncertainty is flow independent, we can compute the NE traffic flow by solving the following convex optimization problem:*

$$\min_{\boldsymbol{v} \in \mathcal{V}} \quad \sum_{a \in \mathcal{A}} \int_0^{v_a} s_a(x)dx + \sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} ACT_{V_i}(\tilde{\tau}_a)v_{ai}. \tag{6}$$

Proof. First, set $\mathcal{V}$ is convex and compact. Let $Z(\boldsymbol{v}) = \sum_{a \in \mathcal{A}} \int_0^{v_a} s_a(x)dx + \sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} ACT_{V_i}(\tilde{\tau}_a)v_{ai}$, we can easily verify that

$$\frac{\partial Z(\boldsymbol{v})}{\partial v_{ai}} = s_a(v_a) + ACT_{V_i}(\tilde{\tau}_a) = t_{ai}(v_a), \quad \forall\, a \in \mathcal{A}, i \in \mathcal{I},$$

and $Z(\boldsymbol{v})$ is convex in $\boldsymbol{v}$. Therefore, from the necessary optimality condition, we know $\boldsymbol{v}^*$ is an optimal solution to the convex optimization problem

$$\min_{\boldsymbol{v} \in \mathcal{V}} Z(\boldsymbol{v}),$$

if and only if it solves VI Problem (5) when the uncertainty is flow independent. $\square$

We next derive the uniqueness of the NE traffic flow under the assumption that uncertainty along links is flow independent.

Corollary 2. *If the travel time function is a strictly monotonically increasing function of its own link flow, then the optimal solution of aggregate flow on each link is unique.*

Proof. Suppose two distinct link flow solutions $\boldsymbol{v}^1$ and $\boldsymbol{v}^2$ are both optimal solutions to Problem (6). That is, $\exists\, a \in \mathcal{A}, v_a^1 \neq v_a^2$, and $Z(\boldsymbol{v}^1) = Z(\boldsymbol{v}^2)$. Then we will show the contradiction.

Since $s_a(x)$ is a strictly monotonic increasing function, $\int_0^{v_a} s_a(x)dx$ is a strictly convex function in v_a. For any $\eta \in (0, 1)$,

$$Z\big(\eta\boldsymbol{v}^1 + (1-\eta)\boldsymbol{v}^2\big) - \big(\eta Z(\boldsymbol{v}^1) + (1-\eta)Z(\boldsymbol{v}^2)\big)$$

$$= \sum_{a \in \mathcal{A}} \int_0^{\eta v_a^1 + (1-\eta)v_a^2} s_a(x)dx + \sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} ACT_{V_i}(\tilde{\tau}_a)\big(\eta v_{ai}^1 + (1-\eta)v_{ai}^2\big)$$

$$\quad - \eta\left(\sum_{a \in \mathcal{A}} \int_0^{v_a^1} s_a(x)dx + \sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} ACT_{V_i}(\tilde{\tau}_a)v_{ai}^1\right)$$

$$\quad - (1-\eta)\left(\sum_{a \in \mathcal{A}} \int_0^{v_a^2} s_a(x)dx + \sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} ACT_{V_i}(\tilde{\tau}_a)v_{ai}^2\right)$$

$$= \sum_{a \in \mathcal{A}} \int_0^{\eta v_a^1 + (1-\eta)v_a^2} s_a(x)dx - \left(\eta \sum_{a \in \mathcal{A}} \int_0^{v_a^1} s_a(x)dx + (1-\eta)\sum_{a \in \mathcal{A}} \int_0^{v_a^2} s_a(x)dx\right)$$

$$< 0,$$

it follows that

$$Z\big(\eta\boldsymbol{v}^1 + (1-\eta)\boldsymbol{v}^2\big) < \eta Z(\boldsymbol{v}^1) + (1-\eta)Z(\boldsymbol{v}^2) = Z(\boldsymbol{v}^1) = Z(\boldsymbol{v}^2).$$

Now we have a contradiction to the assumption that $\boldsymbol{v}^1$ and $\boldsymbol{v}^2$ are both optimal. Therefore, the optimal solution of aggregate flow on each link is unique. $\square$

We can interpret Problem (6) as a deterministic multi-class NE problem, which is easily solved by the traditional Frank–Wolfe algorithm (see for instance Frank and Wolfe, 1956; Yang and Huang, 2004).

4.2. Inefficiency of network equilibrium

Another concept accompanied with NE is to compare with the so-called System Optimum (SO) in which the aggregate travel time of all travelers is minimized (Nash, 1951; Wardrop, 1952). As travelers choose routes without considering about possible negative impacts on the system performance, it is obvious that the NE solution usually deviates from SO and is less efficient in attaining the minimum aggregate travel time. Led by the seminal work of Koutsoupias and Papadimitriou (1999), the loss of efficiency in NE is an active area of research. The authors propose the concept of Price of Anarchy, which is formally defined as the worst-case inefficiency or the ratio between the aggregate cost of NE and that of SO. In particular, Roughgarden and Tardos (2002) and Correa et al. (2004) present a surprising, but welcome result that NE is near optimal in the sense that the aggregate travel time of all travelers under NE is at most that under SO with double traffic in the

same network. In addition, when the travel time function depends linearly on traffic flow, the aggregate travel time of all travelers under NE is at most 4/3 times that under SO. This topic and many of its variants are extensively studied later on. Chau and Sim (2003), Perakis (2007), Roughgarden (2003) and Correa et al. (2008) extend the results to more general link travel time functions. Guo et al. (2010) study the inefficiency for logit-based stochastic NE. Wang et al. (2014) and O'Hare et al. (2016) study the case with the variation of travel demand. In this section, we derive similar results in the NE problem for the case when travelers are sensitive to risk and ambiguity. To obtain analytical results, we again assume that the uncertainty along links is flow independent. Since in the network, each traveler may not have complete information about uncertain travel times, his/her perceived travel time (the travel time under the ACT criterion) is the only foundation to make route choice decisions. Therefore, to be consistent with travelers' choices, we define the system performance as the aggregate perceived travel time, i.e., aggregate travel time under the ACT criterion. Hence, we define SO as the minimum aggregate perceived travel time. This definition is in line with the definition of Nikolova and Stier-Moses (2014), which also accounts for the uncertain travel time.

For a given traffic flow, $v \in \mathcal{V}$, we represent the aggregate travel time under the ACT criterion on the entire network by

$$\mathcal{C}_v(v) = \langle t(v), v \rangle = \sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} t_{ai}(v_a) v_{ai} = \sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} \left(s_a(v_a) + ACT_{V_i}(\tilde{\tau}_a) \right) v_{ai}.$$

By defining $\mathcal{C}_{v^*}(v) = \langle t(v^*), v \rangle$, variational inequalities (5) can be replaced as $\mathcal{C}_{v^*}(v^*) \leq \mathcal{C}_{v^*}(v)$, where $v^* = (v_{ai}^*)_{a \in \mathcal{A}, i \in \mathcal{I}}$ is traffic flow vector at NE for types of travelers along all links, and $v \in \mathcal{V}$ is the vector of any feasible flows. Let $x^* = (x_{ai}^*)_{a \in \mathcal{A}, i \in \mathcal{I}}$ denote the traffic flow vector at SO, which minimizes aggregate travel time under the ACT criterion. We can analyze the inefficiency of NE by comparing $\mathcal{C}_{v^*}(v^*)$ and $\mathcal{C}_{x^*}(x^*)$. In particular, we are interested in the Price of Anarchy, which is the worst-case ratio between the aggregate travel time of NE and that of SO under the ACT criterion.

Proposition 7. *Consider an instance of Problem (6). The vectors $v^* = (v_{ai}^*)_{a \in \mathcal{A}, i \in \mathcal{I}}$ and $x^* = (x_{ai}^*)_{a \in \mathcal{A}, i \in \mathcal{I}}$ represent link flows at NE and SO, respectively.*

(a) *Let vector $u = (u_{ai})_{a \in \mathcal{A}, i \in \mathcal{I}}$ be a feasible flow for the same network but with twice as many travelers of the same type. Then*

$$\mathcal{C}_{v^*}(v^*) \leq \mathcal{C}_u(u).$$

(b) *If travel time function is a monomial function $s_a(v_a) = b_a(v_a)^m \ (m \geq 0)$, then*

$$\frac{\mathcal{C}_{v^*}(v^*)}{\mathcal{C}_{x^*}(x^*)} \leq \left(1 - m(m+1)^{-(m+1)/m} \right)^{-1}.$$

(c) *If travel time function is a general continuous and nondecreasing function, we have*

$$\frac{\mathcal{C}_{v^*}(v^*)}{\mathcal{C}_{x^*}(x^*)} \leq \frac{1}{1 - \beta(\mathcal{A})},$$

where

$$\beta(\mathcal{A}) = \sup_{a \in \mathcal{A}} \sup_{v \geq 0} \frac{\max_{x \geq 0} x(s_a(v) - s_a(x))}{s_a(v)v}, \quad \text{and} \quad 0 \leq \beta(\mathcal{A}) \leq 1.$$

The proof of this result follows from Correa et al. (2004) and for completeness, we present it in the Appendix. The bound can be attained by a network with two parallel links a and b. Link a has constant travel time as $s(v)$ and link b has uncertain travel time as $s(x) + \tilde{\tau}$, where x is the traffic flow on link b and $\tilde{\tau}$ is an uncertain variable with its lower bound as 0. The total number of travelers is v and we assume all these travelers are extremely risk and ambiguity seeking, i.e., $\lambda \to -\infty$ and $\alpha = 0$. Based on the definition of ACT, the travel time under the ACT criterion on link b for the above travelers is $s(x)$. This setting reduces to the example discussed in Correa et al. (2004), in which the bound is attained. As far as we know, classical Price of Anarchy results on traffic equilibriums do not consider the influence of uncertainty on travelers' choice and our result is possibly the first attempt in this direction. It is interesting to observe that after accounting for travelers preferences for risk and ambiguity in the traffic equilibrium problem, the Price of Anarchy results remain similar to the classical ones where travel times are deterministic.

4.3. A network equilibrium example

The following example explicitly illustrates the calculation of NE and SO under the ACT criterion, and demonstrates the inefficiency issues under various mixtures of travelers' profiles. It elucidates the importance of taking travelers' risk and ambiguity attitudes into account in analyzing traffic networks.

We consider a two paths network from origin O to destination D depicted in Fig. 3. The traffic rate is assumed to be 1. The paths have travel times as follows:

$$\tilde{t}_A(v_A) = (v_A)^4 + \tilde{\tau}_A, \quad \tilde{t}_B(v_B) = \frac{6}{5},$$
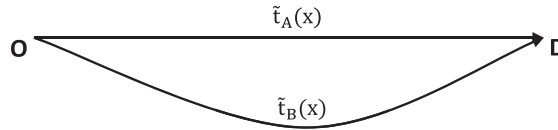
Fig. 3. Two paths network with uncertain travel time.

Table 3
Travelers' profile in Case 3.

Type i	Demand	α_i	λ_i	$ACT_{V_i}(\tilde{\tau}_A)$
1	$\frac{2}{3}$	$\frac{4}{5}$	5	$\frac{1}{25} + \frac{4}{25}\ln\left(1 + \frac{1}{5\Delta}(\exp(5\Delta) - 1)\right)$
2	$\frac{1}{3}$	$\frac{1}{5}$	-5	$\frac{1}{25} - \frac{4}{25}\ln\left(1 + \frac{1}{5\Delta}(\exp(-5\Delta) - 1)\right)$

where $\tilde{\tau}_A$ is uncertain. We assume that all travelers have the same information on the uncertain parameter $\tilde{\tau}_A$. Specifically, $\tilde{\tau}_A$ has a mean value of $\frac{1}{5}$ and support in $[0, \Delta]$, $\Delta > \frac{1}{5}$. Hence, the corresponding distributional uncertainty set is given by

$$\mathbb{F}(\Delta) = \left\{ \mathbb{P} \middle| E_{\mathbb{P}}(\tilde{\tau}_A) = \frac{1}{5}, \mathbb{P}(\tilde{\tau}_A \in [0, \Delta]) = 1 \right\}.$$

Note that the parameter Δ represents the worst-case delay of $\tilde{\tau}_A$ and implies the level of uncertainty along Path *A*. On the other hand, Path *B* has deterministic travel time and is unaffected by Δ. With various compositions of travelers in terms of risk and ambiguity attitudes, the NE and SO under the ACT criterion will yield different flow patterns. To explore the impact of Δ on these flow patterns, we consider the following three cases:

Case 1: All travelers are risk-neutral and ambiguity neutral ($\lambda = 0, \alpha = \frac{1}{2}$);
Case 2: All travelers are extremely risk-averse and pessimistic towards ambiguity ($\lambda \to \infty, \alpha = 1$).
Case 3: Travelers composition with profiles is shown in Table 3.

In Case 1, all travelers are risk and ambiguity neutral and they intuitively perceive the uncertain term as its mean value. Hence, the solutions are consistent with traditional deterministic NE and SO models. In Case 2, travelers who are radically risk-averse and pessimistic towards ambiguity consider the worst-case travel time in deciding between paths. In Case 3, type 1 travelers are risk-averse and pessimistic towards ambiguity, while type 2 travelers are risk-seeking and optimistic towards ambiguity. We derive flow solutions of NE and SO under the ACT criterion in Table 4. For notational simplicity, in this example, we let $t_{A1} = ACT_{V_1}(\tilde{\tau}_A)$ and $t_{A2} = ACT_{V_2}(\tilde{\tau}_A)$. Note that Δ_1 and Δ_2 are the unique solutions satisfying $(\frac{6}{25} - \frac{1}{5}t_{A1})^{1/4} - \frac{1}{3} = 0$ and $(\frac{6}{5} - t_{A1})^{1/4} - \frac{1}{3} = 0$, and $\Delta_1 \approx 1.8136$, and $\Delta_2 \approx 1.8829$, respectively.

We now study the inefficiency of NE under the ACT criterion with respect to the parameter Δ. We represent the aggregate travel times under the ACT criterion in Case i under the NE and SO model by ACT_i^{NE} and ACT_i^{SO} respectively, and quantify the inefficiency of NE via the ratio $\frac{ACT_i^{NE}}{ACT_i^{SO}}$. For these three cases, the ratios are calculated as:

$$\frac{ACT_1^{NE}}{ACT_1^{SO}} = \frac{6}{6 - 4 \times 5^{-1/4}};$$

$$\frac{ACT_2^{NE}}{ACT_2^{SO}} = \begin{cases} \dfrac{6}{6 - 20\left(\frac{6}{25} - \frac{1}{5}\Delta\right)^{5/4}}, & \text{when } \frac{1}{5} < \Delta \leq \frac{6}{5}, \\ 1, & \text{when } \frac{6}{5} \leq \Delta; \end{cases}$$

$$\frac{ACT_3^{NE}}{ACT_3^{SO}} = \begin{cases} \dfrac{\frac{6}{5} - \frac{1}{3}t_{A1} + \frac{1}{3}t_{A2}}{\frac{6}{5} - \frac{1}{3}t_{A1} + \frac{1}{3}t_{A2} - 4\left(\frac{6}{25} - \frac{1}{5}t_{A1}\right)^{5/4}}, & \text{when } 0.2 < \Delta \leq \Delta_1, \\ \dfrac{\frac{6}{5} - \frac{1}{3}t_{A1} + \frac{1}{3}t_{A2}}{\frac{4}{5} + \frac{1}{243} + \frac{1}{3}t_{A2}}, & \text{when } \Delta_1 \leq \Delta \leq \Delta_2, \\ 1, & \text{when } \Delta_2 \leq \Delta. \end{cases}$$

Fig. 4 depicts the ratios of Case 2 and 3. We observe that the ratios decrease with the increase of upper bound Δ. For this specific example, when the travel time becomes more uncertain, the change of traffic flow has less impact on the traveler's path selection decisions, correspondingly, the flow pattern at NE will approach to that at SO. In other words, it suggests that if the travel time along a traffic network is highly uncertain, then there is little benefit from having the system optimal solution in which the aggregate travel time under the ACT criterion is minimized.

Next, we highlight that it is essential for traffic managers to consider travelers' risk and ambiguity attitudes when determining the system optimal flow pattern. Specifically, if we ignore uncertainty and calculate the deterministic system optimal (DSO) flow pattern, its aggregate travel time under the ACT criterion may be worse than that of NE. We represent the DSO

Table 4
Flow patterns of NE and SO under the ACT criterion for three cases.

Case	Condition	Criterion	Type	Traffic flow		Path ACT[a]		Network ACT[b]
				A	B	A	B	
1		NE		1	0	$\frac{6}{5}$	$\frac{6}{5}$	$\frac{6}{5}$
		SO		$5^{-1/4}$	$1-5^{-1/4}$	$\frac{2}{5}$	$\frac{6}{5}$	$\frac{6}{5}-4\times5^{-5/4}$
2	$\frac{1}{5}<\Delta$	NE		$\left(\frac{6}{5}-\Delta\right)^{1/4}$	$1-\left(\frac{6}{5}-\Delta\right)^{1/4}$	$\frac{6}{5}$	$\frac{6}{5}$	$\frac{6}{5}$
	$\le\frac{6}{5}$	SO		$\left(\frac{6}{25}-\frac{1}{5}\Delta\right)^{1/4}$	$1-\left(\frac{6}{25}-\frac{1}{5}\Delta\right)^{1/4}$	$\frac{6}{25}+\frac{4}{5}\Delta$	$\frac{6}{5}$	$\frac{6}{5}-4\left(\frac{6}{25}-\frac{1}{5}\Delta\right)^{5/4}$
	$\frac{6}{5}\le\Delta$	NE		0	1	Δ	$\frac{6}{5}$	$\frac{6}{5}$
		SO		0	1	Δ	$\frac{6}{5}$	$\frac{6}{5}$
3		NE	1	$\left(\frac{6}{5}-t_{A1}\right)^{1/4}-\frac{1}{3}$	$1-\left(\frac{6}{5}-t_{A1}\right)^{1/4}$	$\frac{6}{5}$	$\frac{6}{5}$	$\frac{6}{5}-\frac{1}{3}t_{A1}+\frac{1}{3}t_{A2}$
	$\frac{1}{5}<\Delta$		2	$\frac{1}{3}$	0	$\frac{6}{5}-t_{A1}+t_{A2}$	$\frac{6}{5}$	
	$\le\Delta_1$	SO	1	$\left(\frac{6}{25}-\frac{1}{5}t_{A1}\right)^{1/4}-\frac{1}{3}$	$1-\left(\frac{6}{25}-\frac{1}{5}t_{A1}\right)^{1/4}$	$\frac{6}{25}+\frac{4}{5}t_{A1}$	$\frac{6}{5}$	$\frac{6}{5}-\frac{1}{3}t_{A1}+\frac{1}{3}t_{A2}-4\left(\frac{6}{25}-\frac{1}{5}t_{A1}\right)^{5/4}$
			2	$\frac{1}{3}$	0	$\frac{6}{25}-\frac{1}{5}t_{A1}+t_{A2}$	$\frac{6}{5}$	
		NE	1	$\left(\frac{6}{5}-t_{A1}\right)^{1/4}-\frac{1}{3}$	$1-\left(\frac{6}{5}-t_{A1}\right)^{1/4}$	$\frac{6}{5}$	$\frac{6}{5}$	$\frac{6}{5}-\frac{1}{3}t_{A1}+\frac{1}{3}t_{A2}$
	$\Delta_1\le\Delta$		2	$\frac{1}{3}$	0	$\frac{6}{5}-t_{A1}+t_{A2}$	$\frac{6}{5}$	
	$\le\Delta_2$	SO	1	0	$\frac{2}{3}$	$\frac{1}{81}+t_{A1}$	$\frac{6}{5}$	$\frac{4}{5}+\frac{1}{243}+\frac{1}{3}t_{A2}$
			2	$\frac{1}{3}$	0	$\frac{1}{81}+t_{A2}$	$\frac{6}{5}$	
	$\Delta_2\le\Delta$	NE	1	0	$\frac{2}{3}$	$\frac{1}{81}+t_{A1}$	$\frac{6}{5}$	$\frac{4}{5}+\frac{1}{243}+\frac{1}{3}t_{A2}$
			2	$\frac{1}{3}$	0	$\frac{1}{81}+t_{A2}$	$\frac{6}{5}$	
		SO	1	0	$\frac{2}{3}$	$\frac{1}{81}+t_{A1}$	$\frac{6}{5}$	$\frac{4}{5}+\frac{1}{243}+\frac{1}{3}t_{A2}$
			2	$\frac{1}{3}$	0	$\frac{1}{81}+t_{A2}$	$\frac{6}{5}$	

[a] **Path ACT** refers to the travel time along the path under the ACT criterion;
[b] **Network ACT** refers to the aggregate travel times under the ACT criterion for all travelers on the network.
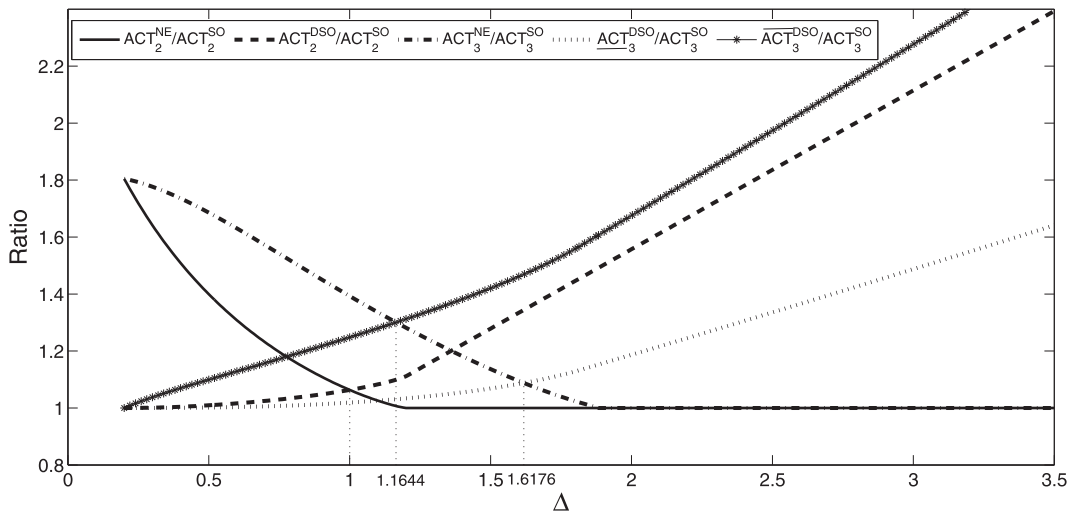

Fig. 4. Inefficiency of NE and DSO under the ACT criterion in Case 2 and 3.

flow pattern by $\boldsymbol{u}^*=(u_a^*)_{a\in\mathcal{A}}$, which is the unique optimal solution of

$$\min \quad \sum_{a\in\mathcal{A}}(s_a(u_a)+\mathrm{E}(\tilde{\tau}_a))u_a$$

$$\text{s.t.} \quad u_a=\sum_{w\in\mathcal{W}}\sum_{r\in\mathcal{R}_w}f_r\delta_{ar}, \qquad \forall\, a\in\mathcal{A},$$

$$\sum_{r\in\mathcal{R}_w}f_r=\sum_{i\in\mathcal{I}}d_{wi}, \qquad \forall\, w\in\mathcal{W},$$

$$f_r\ge 0, \qquad \forall\, r\in\mathcal{R}_w, w\in\mathcal{W}.$$

Note that the flow pattern $\boldsymbol{u}^* = (u_a^*)_{a \in \mathcal{A}}$ only identifies the aggregate traffic flow on each link. Therefore, with the mixture of travelers, the traffic flow for each type of travelers on each link may not be unique. We represent its feasible set as

$$\mathcal{U} = \left\{ \boldsymbol{v} \in \mathcal{V} \,\middle|\, \sum_{i \in \mathcal{I}} v_{ai} = u_a^*, \forall \, a \in \mathcal{A} \right\}.$$

Then, for any $\boldsymbol{v} \in \mathcal{U}$, we define $ACT^{DSO}(\boldsymbol{v})$ as the total travel time under the ACT criterion of all travelers when the traffic flow is $\boldsymbol{v}$, that is,

$$ACT^{DSO}(\boldsymbol{v}) = \sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} \big(s_a(u_a^*) + ACT_{V_i}(\tilde{\tau}_a) \big) v_{ai}.$$

Since $ACT^{DSO}(\boldsymbol{v})$ is a function of $\boldsymbol{v} \in \mathcal{U}$, we define its lower and upper bound by $\underline{ACT}^{DSO}$ and $\overline{ACT}^{DSO}$ respectively, where

$$\underline{ACT}^{DSO} = \min_{\boldsymbol{v} \in \mathcal{U}} \sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} \big(s_a(u_a^*) + ACT_{V_i}(\tilde{\tau}_a) \big) v_{ai};$$

$$\overline{ACT}^{DSO} = \max_{\boldsymbol{v} \in \mathcal{U}} \sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} \big(s_a(u_a^*) + ACT_{V_i}(\tilde{\tau}_a) \big) v_{ai}.$$

Hence, for any $\boldsymbol{v} \in \mathcal{U}$, $ACT^{DSO}(\boldsymbol{v}) \in [\underline{ACT}^{DSO}, \overline{ACT}^{DSO}]$. Similarly, we quantify the inefficiency of DSO under the ACT criterion via the ratios $\frac{\underline{ACT}_i^{DSO}}{ACT_i^{SO}}$ and $\frac{\overline{ACT}_i^{DSO}}{ACT_i^{SO}}$ as follows:

$$\frac{ACT_2^{DSO}}{ACT_2^{SO}} = \frac{\underline{ACT}_2^{DSO}}{ACT_2^{SO}} = \frac{\overline{ACT}_2^{DSO}}{ACT_2^{SO}} = \begin{cases} \dfrac{6 + 5^{3/4}(\Delta - 1)}{6 - 20\big(\frac{6}{25} - \frac{1}{5}\Delta\big)^{5/4}}, & \text{when } \dfrac{1}{5} < \Delta \leq \dfrac{6}{5}, rcl \\[2mm] \dfrac{6 + 5^{3/4}(\Delta - 1)}{6}, & \text{when } \dfrac{6}{5} \leq \Delta; \end{cases}$$

$$\frac{\underline{ACT}_3^{DSO}}{ACT_3^{SO}} = \begin{cases} \dfrac{\frac{6}{5} - 5^{-1/4} + \big(5^{-1/4} - \frac{1}{3}\big)t_{A1} + \frac{1}{3}t_{A2}}{\frac{6}{5} - \frac{1}{3}t_{A1} + \frac{1}{3}t_{A2} - 4\big(\frac{6}{25} - \frac{1}{5}t_{A1}\big)^{5/4}}, & \text{when } \dfrac{1}{5} < \Delta \leq \Delta_1, \\[3mm] \dfrac{\frac{6}{5} - 5^{-1/4} + \big(5^{-1/4} - \frac{1}{3}\big)t_{A1} + \frac{1}{3}t_{A2}}{\frac{4}{5} + \frac{1}{243} + \frac{1}{3}t_{A2}}, & \text{when } \Delta_1 \leq \Delta; \end{cases}$$

$$\frac{\overline{ACT}_3^{DSO}}{ACT_3^{SO}} = \begin{cases} \dfrac{\frac{6}{5} - 5^{-1/4} + \frac{2}{3}t_{A1} + \big(5^{-1/4} - \frac{2}{3}\big)t_{A2}}{\frac{6}{5} - \frac{1}{3}t_{A1} + \frac{1}{3}t_{A2} - 4\big(\frac{6}{25} - \frac{1}{5}t_{A1}\big)^{5/4}}, & \text{when } \dfrac{1}{5} < \Delta \leq \Delta_1, \\[3mm] \dfrac{\frac{6}{5} - 5^{-1/4} + \frac{2}{3}t_{A1} + \big(5^{-1/4} - \frac{2}{3}\big)t_{A2}}{\frac{4}{5} + \frac{1}{243} + \frac{1}{3}t_{A2}}, & \text{when } \Delta_1 \leq \Delta. \end{cases}$$

Fig. 4 demonstrates the inefficiency of NE and DSO under the ACT criterion of Cases 2 and 3. In Case 2, for the network where travelers are extremely risk-averse and pessimistic towards ambiguity, with the increase of Δ, the NE flow pattern under the ACT criterion becomes less inefficient, while the inefficiency of DSO grows increasingly severe. When $\Delta > 1$, $ACT_2^{DSO} > ACT_2^{NE}$ suggests if we instruct the traffic flow following DSO criterion, which does not account for travelers' attitudes towards risk and ambiguity, the performance will turn worse than its original anarchy state. Similarly, in Case 3, with two types of travelers, the ratio $\frac{ACT_3^{DSO}(\boldsymbol{v})}{ACT_3^{SO}}$ lies between the two curves $\frac{\underline{ACT}_3^{DSO}}{ACT_3^{SO}}$ and $\frac{\overline{ACT}_3^{DSO}}{ACT_3^{SO}}$. The increase of upper bound Δ will cut down the inefficiency of NE, but result in the deterioration of DSO in terms of system performance. Moreover, when the level of travel time uncertainty increases to some specific value, the DSO performance will be no better than the NE performance, which suggests this guidance effort would be in vain.

Following the same strategy, we extend our computational study from this two links small network to a five-nodes complete network, which includes 5 nodes, and 20 links. Since calculating $\underline{ACT}^{DSO}$ and $\overline{ACT}^{DSO}$ is generally a hard problem, we only use this simple network for illustrative purpose. The demand on each OD pair for each type of travelers is uniformly generated from the set $\{101, 102, \ldots, 800\}$. Uncertain travel time on each link is written as

$$\tilde{t}_a(v_a) = s_a(0) + 0.15 \left(\frac{v_a}{c_a} \right)^4 + \tilde{\tau}_a, \quad \forall \, a \in \mathcal{A}.$$

Free flow travel time $s_a(0)$ follows uniform distribution $U(2, 6)$, and capacity c_a is generated from uniform distribution $U(200, 1000)$. Instead of deterministic travel time, we assume that uncertainties occur on each link independently. Moreover, the disturbance is flow independent, with the mean equal to 20% of free flow travel time on that link, and lower bound equal to zero. We vary the upper bound of uncertainties by Δ. The uncertainty $\tilde{\tau}_a$ is characterized as

$$\mathbb{F} = \{ \mathbb{P} | E_{\mathbb{P}}(\tilde{\tau}_a) = 0.2 s_a(0), \ \mathbb{P}(\tilde{\tau}_a \in [0, \Delta \times E_{\mathbb{P}}(\tilde{\tau}_a)]) = 1, \ \forall \, a \in \mathcal{A} \}.$$
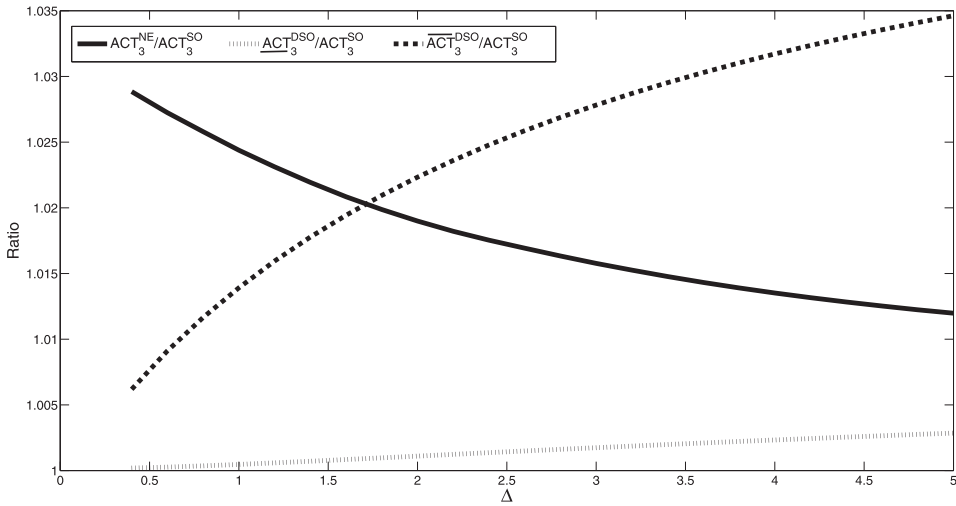
Fig. 5. Inefficiency of NE and DSO under the ACT criterion in Case 3 in five-nodes Network.

Travelers' characteristics are consistent with Case 3. We randomly generate 50 instances, and summarize the average performance. The inefficiency results of NE and DSO under the ACT criterion of five-nodes network are listed in Fig. 5. Similar conclusions could be derived here. When the flow independent disturbance on travel time becomes highly uncertain, the influence of selfishness on inefficiency diminishes.

5. Conclusion

This paper studies the preferences for uncertain travel times in which the probability distributions may not be fully characterized. By explicitly distinguishing risk and ambiguity concepts, we propose a new criterion called ambiguity-aware CARA travel time for ranking the uncertain travel time, which systematically integrates the travelers' inability to capture the exact information of uncertain travel times, and their attitudes towards risk and ambiguity. This setting is based on the Hurwicz criterion and constant absolute risk aversion, which is empirically supported and provides computational benefits.

With this criterion, we explore computational solvability of the path selection problem on a network where travel times are uncertain. We show that finding a path with the minimum travel time under the ACT criterion is polynomially solvable when link travel times are independently distributed. We also prove that the problem becomes intractable when link travel times are correlated. Focusing on independently distributed link travel times, we present the general VI formulation of NE under the ACT criterion. We analyze the case when the uncertainty along links is flow independent and show that it can be addressed as a convex optimization problem. We also determine the inefficiency of NE by deriving the Price of Anarchy, which is similar to the deterministic NE case.

The ACT criterion could potentially enhance the predictive capability of path selection and traffic equilibrium. First, it does not require travelers to know the probability distributions of the network. Second, it has the potential to incorporate risk and ambiguity in travelers' decision making. Third, the path selection problem and network equilibrium established retain the computational tractability of their deterministic counterparts. It will be valuable to establish empirically the risk and ambiguity profiles of a population of travelers residing in different cities and possibly having different cultures. We hope that our work could encourage future research in this direction.

Appendix A. Proof of Proposition 7

Proof. (a) Note that $s_a(v_a)$ is a differentiable, monotonically increasing function in v_a, and $\boldsymbol{u} = (u_{ai})_{a\in\mathcal{A}, i\in\mathcal{I}}$ is a feasible flow for the same network but with double demands. We have

$$s_a(u_a)u_a + s_a(v_a^*)v_a^* - s_a(v_a^*)u_a \geq s_a(u_a)u_a \geq 0, \quad \text{if} \quad u_a \leq v_a^*;$$
$$s_a(u_a)u_a + s_a(v_a^*)v_a^* - s_a(v_a^*)u_a \geq s_a(v_a^*)v_a^* \geq 0, \quad \text{if} \quad u_a \geq v_a^*.$$

Therefore,

$$\mathcal{C}_{\boldsymbol{u}}(\boldsymbol{u}) + \mathcal{C}_{v^*}(v^*) - \mathcal{C}_{v^*}(\boldsymbol{u})$$
$$= \sum_{a\in\mathcal{A}}\sum_{i\in\mathcal{I}}\left(t_{ai}(u_a)u_{ai} + t_{ai}(v_a^*)v_{ai}^* - t_{ai}(v_a^*)u_{ai}\right)$$
$$= \sum_{a\in\mathcal{A}}\sum_{i\in\mathcal{I}}\left(s_a(u_a)u_{ai} + s_a(v_a^*)v_{ai}^* + ACT_{V_i}(\tilde{\tau}_a)v_{ai}^* - s_a(v_a^*)u_{ai}\right)$$

$$\begin{aligned}
&= \sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} ACT_{V_i}(\tilde{\tau}_a) v_{ai}^* + \sum_{a \in \mathcal{A}} \left(s_a(u_a) u_a + s_a(v_a^*) v_a^* - s_a(v_a^*) u_a \right) \\
&\geq \sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} ACT_{V_i}(\tilde{\tau}_a) v_{ai}^* \\
&\geq 0.
\end{aligned}$$

Besides, we note that $\boldsymbol{u}/2 = (\frac{u_{ai}}{2})_{a \in \mathcal{A}, i \in \mathcal{I}}$ is a feasible flow for the original instance. From the NE property,

$$\mathcal{C}_{\boldsymbol{u}}(\boldsymbol{u}) \geq \mathcal{C}_{v^*}(\boldsymbol{u}) - \mathcal{C}_{v^*}(\boldsymbol{v}^*) = 2\mathcal{C}_{v^*}\left(\frac{\boldsymbol{u}}{2}\right) - \mathcal{C}_{v^*}(\boldsymbol{v}^*) \geq 2\mathcal{C}_{v^*}(\boldsymbol{v}^*) - \mathcal{C}_{v^*}(\boldsymbol{v}^*) = \mathcal{C}_{v^*}(\boldsymbol{v}^*).$$

(b) If travel time is a monomial function, defined as $s_a(v_a) = b_a(v_a)^m$ such that

$$t_{ai}(v_a) = b_a(v_a)^m + ACT_{V_i}(\tilde{\tau}_a).$$

Then, we have

$$\begin{aligned}
\mathcal{C}_{v^*}(\boldsymbol{x}) &= \sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} \left(t_a(v_a^*) + ACT_{V_i}(\tilde{\tau}_a) \right) x_{ai} \\
&= \sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} ACT_{V_i}(\tilde{\tau}_a) x_{ai} + \sum_{a \in \mathcal{A}} b_a(v_a^*)^m x_a \\
&\leq \sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} ACT_{V_i}(\tilde{\tau}_a) x_{ai} + \sum_{a \in \mathcal{A}} b_a \left(x_a^{m+1} + m(m+1)^{-(m+1)/m} (v_a^*)^{m+1} \right) \\
&= \sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} \left(b_a(x_a)^m + ACT_{V_i}(\tilde{\tau}_a) \right) x_{ai} + m(m+1)^{-(m+1)/m} \sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} b_a(v_a^*)^m v_{ai}^* \\
&\leq \mathcal{C}_{\boldsymbol{x}}(\boldsymbol{x}) + m(m+1)^{-(m+1)/m} \sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} \left(b_a(v_a^*)^m + ACT_{V_i}(\tilde{\tau}_a) \right) v_{ai}^* \\
&= \mathcal{C}_{\boldsymbol{x}}(\boldsymbol{x}) + m(m+1)^{-(m+1)/m} \mathcal{C}_{v^*}(\boldsymbol{v}^*),
\end{aligned}$$

where the first inequality is tenable because the function $f(x) = v^m x - x^{m+1}$ ($x \geq 0$) will get its maximum $m(m+1)^{-(m+1)/m} v^{m+1}$ at $x = v(m+1)^{-1/m}$; and the second inequality holds because $\sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} ACT_{V_i}(\tilde{\tau}_a) v_{ai}^* \geq 0$. Then, since $\mathcal{C}_{v^*}(\boldsymbol{v}^*) \leq \mathcal{C}_{v^*}(\boldsymbol{x})$, we get

$$\left(1 - m(m+1)^{-(m+1)/m} \right) \mathcal{C}_{v^*}(\boldsymbol{v}^*) \leq \mathcal{C}_{\boldsymbol{x}^*}(\boldsymbol{x}^*).$$

When $\boldsymbol{x}^* = (x_{ai}^*)_{a \in \mathcal{A}, i \in \mathcal{I}}$ is the system optimum, we can find the Price of Anarchy bounded at $\left(1 - m(m+1)^{-(m+1)/m} \right)^{-1}$, which is the same as that in deterministic cases.

(c) Similar to Correa et al. (2004), we could generalize the travel time function to continuous, nondecreasing case.

$$\begin{aligned}
\mathcal{C}_{v^*}(\boldsymbol{x}) &= \sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} t_{ai}(v_a^*) x_{ai} \\
&= \sum_{a \in \mathcal{A}} s_a(v_a^*) x_a + \sum_{a \in \mathcal{A}} \sum_{i \in \mathcal{I}} ACT_{V_i}(\tilde{\tau}_a) x_{ai} \\
&= \sum_{a \in \mathcal{A}} x_a(s_a(v_a^*) - s_a(x_a)) + \mathcal{C}_{\boldsymbol{x}}(\boldsymbol{x}) \\
&= \sum_{a \in \mathcal{A}} \frac{x_a(s_a(v_a^*) - s_a(x_a))}{s_a(v_a^*) v_a^*} s_a(v_a^*) v_a^* + \mathcal{C}_{\boldsymbol{x}}(\boldsymbol{x}) \\
&\leq \sum_{a \in \mathcal{A}} \beta(v_a^*, s_a(v_a^*)) s_a(v_a^*) v_a^* + \mathcal{C}_{\boldsymbol{x}}(\boldsymbol{x}) \\
&\leq \sup_{a \in \mathcal{A}} \beta(v_a^*, s_a(v_a^*)) \sum_{a \in \mathcal{A}} s_a(v_a^*) v_a^* + \mathcal{C}_{\boldsymbol{x}}(\boldsymbol{x}) \\
&\leq \beta(\mathcal{A}) \mathcal{C}_{v^*}(\boldsymbol{v}^*) + \mathcal{C}_{\boldsymbol{x}}(\boldsymbol{x}),
\end{aligned}$$

where $\beta(v, s(v)) = \frac{1}{s(v)v} \max_{x \geq 0} \{x(s(v) - s(x))\}$, and $\beta(\mathcal{A}) = \sup_{a \in \mathcal{A}} \sup_{v \geq 0} \beta(v, s_a(v))$. Since the travel time function $s(v)$ is a continuous nondecreasing function, the following relationship holds:

$$0 = \frac{v(s(v) - s(v))}{s(v)v} \leq \beta(v, s(v)) \leq \frac{\max_{0 \leq x \leq v} x(s(v) - s(x))}{s(v)v} \leq \frac{\max_{0 \leq x \leq v} xs(v)}{s(v)v} \leq \frac{vs(v)}{s(v)v} = 1.$$

Assuming $\boldsymbol{x}^* = (x_{ai}^*)_{a \in \mathcal{A}, i \in \mathcal{I}}$ is SO solution, we have the Price of Anarchy as

$$\frac{\mathcal{C}_{v^*}(\boldsymbol{v}^*)}{\mathcal{C}_{\boldsymbol{x}^*}(\boldsymbol{x}^*)} \leq \frac{1}{1 - \beta(\mathcal{A})}.$$

$\square$

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