AN INTRODUCTION TO A CLASS OF MATRIX OPTIMIZATION PROBLEMS

DING CHAO

(M.Sc., NJU)

A THESIS SUBMITTED

FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS

NATIONAL UNIVERSITY OF SINGAPORE

2012

This thesis is dedicated to

my parents and my wife

Acknowledgements

First and foremost, I would like to state my deepest gratitude to my Ph.D. supervisor Professor Sun Defeng. Without his excellent mathematical knowledge and professional guidance, this work would not have been possible. I am grateful to him for introducing me to the many areas of research treated in this thesis. I am extremely thankful to him for his professionalism and patience. His wisdom and attitude will always be a guide to me. I feel very fortunate to have him as an adviser and a teacher.

My deepest thanks go to Professor Toh Kim-Chuan and Professor Sun Jie, for their collaborations on this research and co-authorship of several papers, and for their helpful advice. I would like to especially acknowledge Professor Jane Ye, for joint work on the conic MPEC problem, and for her friendship and constant support. My grateful thanks also go to Professor Zhao Gongyun for his courses on numerical optimization, which enrich my knowledge in optimization algorithms and software.

I would like to thank all group members of optimization in mathematics department. It has been a pleasure to be a part of the group. I specially like to thank Wu Bin for his collaborations on the study of Moreau-Yosida regularization of k-norm related functions. I should also mention the support and helpful advice given by my friends Miao Weimin, Jiang Kaifeng, Chen Caihua and Gao Yan.

On the personal side, I would like to thank my parents, for their unconditional love and support all though my life. Last but not least, I am also greatly indebted to my wife for her understanding and patience throughout the years of my research. I love you.

Ding Chao

January 2012

Contents

Summary

This thesis focuses on a class of optimization problems, which involve minimizing the sum of a linear function and a proper closed simple convex function subject to an affine constraint in the matrix space. Such optimization problems are said to be matrix optimization problems (MOPs). Many important optimization problems in diverse applications arising from a wide range of fields such as engineering, finance, and so on, can be cast in the form of MOPs.

In order to apply the proximal point algorithms (PPAs) to the MOP problems, as an initial step, we shall study the properties of the corresponding Moreau-Yosida regularizations and proximal point mappings of MOPs. Therefore, we study one kind of matrix-valued functions, so-called spectral operators, which include the gradients of the Moreau-Yosida regularizations and the proximal point mappings. Specifically, the following fundamental properties of spectral operators, including the well-definiteness, the directional differentiability, the Fréchet-differentiability, the locally Lipschitz continuity, the ρ-order B(ouligand)-differentiability ($0 < \rho \leq 1$), the ρ-order G-semismooth ($0 < \rho \leq 1$) and the characterization of Clarke's generalized Jacobian, are studied systemically.

In the second part of this thesis, we discuss the sensitivity analysis of MOP problems. We mainly focus on the linear MCP problems involving Ky Fan k-norm epigraph cone $\mathcal{K}$. Firstly, we study some important geometrical properties of the Ky Fan k-norm epigraph cone $\mathcal{K}$, including the characterizations of tangent cone and the (inner and outer) second order tangent sets of $\mathcal{K}$, the explicit expression of the support function of the second order tangent set, the $\mathcal{C}^2$-cone reducibility of $\mathcal{K}$, the characterization of the critical cone of $\mathcal{K}$. By using these properties, we state the constraint nondegeneracy, the second order necessary condition and the (strong) second order sufficient condition of the linear matrix cone programming (MCP) problem involving the epigraph cone of the Ky Fan k-norm. Variational analysis on the metric projector over the Ky Fan k-norm epigraph cone $\mathcal{K}$ is important for these studies. More specifically, the study of properties of spectral operators in the first part of this thesis plays an essential role. For such linear MCP problem, we establish the equivalent links among the strong regularity of the KKT point, the strong second order sufficient condition and constraint nondegeneracy, and the nonsingularity of both the B-subdifferenitial and Clarke's generalized Jacobian of the nonsmooth system at a KKT point. Finally, the extensions of the corresponding sensitivity results to other MOP problems are also considered.

Summary of Notation

- For any $Z \in \Re^{m\times n}$, we denote by Z_{ij} the (i,j)-th entry of Z.
- For any $Z \in \Re^{m\times n}$, we use z_j to represent the jth column of Z, $j = 1,\ldots,n$. Let $\mathcal{J} \subseteq \{1,\ldots,n\}$ be an index set. We use $Z_{\mathcal{J}}$ to denote the sub-matrix of Z obtained by removing all the columns of Z not in $\mathcal{J}$. So for each j, we have $Z_{\{j\}} = z_j$.
- Let $\mathcal{I} \subseteq \{1,\ldots,m\}$ and $\mathcal{J} \subseteq \{1,\ldots,n\}$ be two index sets. For any $Z \in \Re^{m\times n}$, we use $Z_{\mathcal{I}\mathcal{J}}$ to denote the $|\mathcal{I}| \times |\mathcal{J}|$ sub-matrix of Z obtained by removing all the rows of Z not in $\mathcal{I}$ and all the columns of Z not in $\mathcal{J}$.
- For any $y \in \Re^n$, $\mathrm{diag}(y)$ denotes the diagonal matrix whose i-th diagonal entry is y_i, $i = 1,\ldots,n$.
- $e \in \Re^n$ denotes the vector with all components one. $E \in \Re^{m\times n}$ denotes the m by n matrix with all components one.
- Let $\mathcal{S}^n$ be the space of all real $n \times n$ symmetric matrices and $\mathcal{O}^n$ be the set of all $n \times n$ orthogonal matrices.
- We use "$\circ$" to denote the Hadamard product between matrices, i.e., for any two

matrices X and Y in $\Re^{m\times n}$ the (i,j)-th entry of $Z := X\circ Y \in \Re^{m\times n}$ is $Z_{ij} = X_{ij}Y_{ij}$.

- For any given $Z \in \Re^{m\times n}$, let $Z^\dagger \in \Re^{m\times n}$ be the Moore-Penrose pseudoinverse of Z.

- For each $X \in \Re^{m\times n}$, $\|X\|_2$ denotes the spectral or the operator norm, i.e., the largest singular value of X.

- For each $X \in \Re^{m\times n}$, $\|X\|_*$ denotes the nuclear norm, i.e., the sum of the singular values of X.

- For each $X \in \Re^{m\times n}$, $\|X\|_{(k)}$ denotes the Ky Fan k-norm, i.e., the sum of the k-largest singular values of X, where $0 < k \leq \min\{m,n\}$ is a positive integer.

- For each $X \in \mathcal{S}^n$, $s_{(k)}(X)$ denotes the sum of the k-largest eigenvalues of X, where $0 < k \leq n$ is a positive integer.

- Let $\mathcal{Z}$ and $\mathcal{Z}'$ be two finite dimensional Euclidean spaces. and $\mathcal{A} : \mathcal{Z} \to \mathcal{Z}'$ be a given linear operator. Denote the adjoint of $\mathcal{A}$ by $\mathcal{A}^*$, i.e., $\mathcal{A}^* : \mathcal{Z}' \to \mathcal{Z}$ is the linear operator such that

$$\langle \mathcal{A}z, y\rangle = \langle z, \mathcal{A}^* y\rangle \quad \forall\, z \in \mathcal{Z},\ y \in \mathcal{Z}'\,.$$

- For any subset C of a finite dimensional Euclidean space $\mathcal{Z}$, let

$$\mathrm{dist}(z, C) := \inf\{\|z - y\| \,|\, y \in C\}\,, \quad z \in \mathcal{Z}\,.$$

- For any subset C of a finite dimensional Euclidean space $\mathcal{Z}$, let $\delta^*_C : \mathcal{Z} \to (-\infty, \infty]$ be the support function of the set C, i.e.,

$$\delta^*_C(z) := \sup\{\langle x, z\rangle \,|\, x \in C\}\,, \quad z \in \mathcal{Z}\,.$$

- Given a set C, $\mathrm{int}\, C$ denotes its interior, $\mathrm{ri}\, C$ denotes its relative interior, $\mathrm{cl}\, C$ denotes its closure, and $\mathrm{bd}\, C$ denotes its boundary.

- A backslash denotes the set difference operation, that is $A \setminus B = \{x \in A \,|\, x \notin B\}$.

- Given a nonempty convex cone K of a finite dimensional Euclidean space $\mathcal{Z}$. Let K° be the polar of K, i.e.,

$$K^\circ = \{z \in \mathcal{Z} \,|\, \langle z, x \rangle \leq 0 \;\forall\, x \in K\} \; .$$

All further notations are either standard, or defined in the text.

Chapter 1

Introduction

1.1 Matrix optimization problems

Let $\mathcal{X}$ be the Cartesian product of several finite dimensional real (symmetric or non-symmetric) matrix spaces. More specifically, let s be a positive integer and $0 \le s_0 \le s$ be a nonnegative integer. For the given positive integers $m_1, \ldots, m_{s_0}$ and $n_{s_0+1}, \ldots, n_s$, denote

$$\mathcal{X} := \mathcal{S}^{m_1} \times \ldots \times \mathcal{S}^{m_{s_0}} \times \Re^{m_{s_0+1} \times n_{s_0+1}} \times \ldots \times \Re^{m_s \times n_s} \,. \tag{1.1}$$

Without loss of generality, assume that $m_k \le n_k$, $k = s_0 + 1, \ldots, s$. Let $\langle \cdot, \cdot \rangle$ be the natural inner product of $\mathcal{X}$ and $\| \cdot \|$ be the induced norm. Let $f : \mathcal{X} \to (-\infty, \infty]$ be a closed proper convex function. The primal *matrix optimization problem* (MOP) takes the form:

$$\begin{array}{lll} \text{(P)} & \min & \langle \boldsymbol{C}, \boldsymbol{X} \rangle + f(\boldsymbol{X}) \\ & \text{s.t.} & \mathcal{A}\boldsymbol{X} = b, \quad \boldsymbol{X} \in \mathcal{X} \,, \end{array} \tag{1.2}$$

where $\mathcal{A} : \mathcal{X} \to \Re^p$ is a linear operator; $\boldsymbol{C} \in \mathcal{X}$ and $b \in \Re^p$ are given. Let $f^* : \mathcal{X} \to (-\infty, \infty]$ be the *conjugate function* of f (see, e.g., [83]), i.e.,

$$f^*(\boldsymbol{X}^*) := \sup \{ \langle \boldsymbol{X}^*, \boldsymbol{X} \rangle - f(\boldsymbol{X}) \,|\, \boldsymbol{X} \in \mathcal{X} \} \,, \quad \boldsymbol{X}^* \in \mathcal{X} \,.$$

Then, the dual MOP can be written as

$$\begin{array}{lll} \text{(D)} & \max & \langle b, y\rangle - f^*(\boldsymbol{X}^*) \\ & \text{s.t.} & \mathcal{A}^* y - \boldsymbol{C} = \boldsymbol{X}^*\,, \end{array} \tag{1.3}$$

where $y \in \Re^p$ and $\boldsymbol{X}^* \in \mathcal{X}$ are the dual variables; $\mathcal{A}^* : \Re^p \to \mathcal{X}$ is the adjoint of the linear operator $\mathcal{A}$.

If the closed proper convex function f is the indicator function of some closed convex cone $\mathcal{K}$ of $\mathcal{X}$, i.e., $f \equiv \delta_{\mathcal{K}}(\cdot) : \mathcal{X} \to (-\infty, +\infty]$, then the corresponding MOP is said to be the *matrix cone programming* (MCP) problem. In this case, we have

$$f^*(\boldsymbol{X}^*) = \delta^*_{\mathcal{K}}(\boldsymbol{X}^*) = \delta_{\mathcal{K}^\circ}(\boldsymbol{X}^*), \quad \boldsymbol{X}^* \in \mathcal{X}\,,$$

where $\mathcal{K}^\circ \subseteq \mathcal{X}$ is the polar of the closed convex cone $\mathcal{K}$, i.e.,

$$\mathcal{K}^\circ := \{\boldsymbol{X}^* \in \mathcal{X} \,|\, \langle \boldsymbol{X}, \boldsymbol{X}^*\rangle \leq \delta_{\mathcal{K}}(\boldsymbol{X}) \ \forall\, \boldsymbol{X} \in \mathcal{X}\}\ .$$

Thus, the primal and dual MCPs take the following form

$$\begin{array}{llll} \text{(P)} & \min & \langle \boldsymbol{C}, \boldsymbol{X}\rangle \\ & \text{s.t.} & \mathcal{A}\boldsymbol{X} = b\,, \\ & & \boldsymbol{X} \in \mathcal{K}\,, \end{array} \qquad \begin{array}{lll} \text{(D)} & \max & \langle b, y\rangle \\ & \text{s.t.} & \mathcal{A}^* y - \boldsymbol{C} = \boldsymbol{X}^*\,, \\ & & \boldsymbol{X}^* \in \mathcal{K}^\circ\,. \end{array} \tag{1.4}$$

The MOP is a broad framework, which includes many important optimization problems involving matrices arising from different areas such as engineering, finance, scientific computing, applied mathematics. In such applications, the convex function f usually is simple. For example, let $\mathcal{X} = \mathcal{S}^n$ be real symmetric matrices space and $\mathcal{K} = \mathcal{S}^n_+$ be the cone of real positive semidefinite matrices in $\mathcal{S}^n$. $f \equiv \delta_{\mathcal{S}^n_+}(\cdot)$ and $f^* \equiv \delta_{\mathcal{S}^n_-}(\cdot)$. Then, the corresponding MCP is said to be the *semidefinite programming* (SDP), which has many interesting applications. For an excellent survey on this, see [105]. Below we list some other examples of MOPs.

Matrix norm approximation. Given matrices $B_0, B_1, \ldots, B_p \in \Re^{m\times n}$, the *matrix norm approximation* (MNA) problem is to find an affine combination of the matrices which has the minimal spectral norm (the largest singular value of matrix), i.e.,

$$\min \left\{ \|B_0 + \sum_{k=1}^{p} y_k B_k\|_2 \mid y \in \Re^p \right\}. \tag{1.5}$$

Such problems have been studied in the iterative linear algebra literature, e.g., [38, 99, 100], where the affine combination is a degree-p polynomial function of a given matrix. More specifically, it is easy to see that the problem (1.5) can be written as the dual MOP form (1.3), i.e.,

$$\begin{array}{lll} \text{(D)} & \max & \langle 0, y\rangle - f^*(X^*) \\ & \text{s.t.} & \mathcal{A}^* y - B_0 = X^*, \end{array} \tag{1.6}$$

where $\mathcal{X} \equiv \Re^{m\times n}$, $f^* \equiv \|\cdot\|_2$ is the spectral norm, and $\mathcal{A}^* : \Re^p \to \Re^{m\times n}$ is the linear operator defined by

$$\mathcal{A}^* y = -\sum_{k=1}^{p} y_k B_k, \quad y \in \Re^p. \tag{1.7}$$

Note that for (1.6), the closed proper convex function f^* is positively homogeneous. For positively homogeneous convex functions, we have the following useful result (see, e.g., [83, Theorem 13.5 & 13.2]).

Proposition 1.1. *Suppose $\mathcal{E}$ be a finite dimensional Euclidean space. Let $g : \mathcal{E} \to (-\infty, \infty]$ be a closed proper convex function. Then, g is positively homogeneous if and only if g^* is the indicator function of*

$$C = \{x^* \in \mathcal{E} \mid \langle x, x^*\rangle \le g(x) \ \forall\, x \in \mathcal{E}\}. \tag{1.8}$$

If g is a given norm function in $\mathcal{E}$ and g^D is the corresponding dual norm in $\mathcal{E}$, then by the definition of the dual norm g^D, we know that $C = \partial g(0)$ coincides with the unit ball under the dual norm , i.e.,

$$\partial g(0) = \left\{ x \in \mathcal{E} \mid g^D(x) \le 1 \right\}.$$

In particular, for the case that $g = f^* \equiv \|\cdot\|_2$, by Proposition 1.1, we have

$$f(X) = (f^*)^*(X) = \delta_{\partial f^*(0)}(X)\,.$$

Note that the dual norm of the spectral norm $\|\cdot\|_2$ is the nuclear norm $\|\cdot\|_*$, i.e., the sum of all singular values of matrix. Thus, $\partial f^*(0)$ coincides with the unit ball B_*^1 under the dual norm $\|\cdot\|_*$, i.e.,

$$\partial f^*(0) = B_*^1 := \left\{ X \in \Re^{m\times n} \mid \|X\|_* \le 1 \right\}\,.$$

Therefore, the corresponding primal problem of (1.5) can be written as

$$\begin{array}{lll} \text{(P)} & \min & \langle B_0, X\rangle + \delta_{B_*^1}(X) \\ & \text{s.t.} & \mathcal{A}X = 0\,, \end{array} \tag{1.9}$$

where $\mathcal{A} : \Re^{m\times n} \to \Re^p$ is the adjoint of $\mathcal{A}^*$. Note that in some applications, a sparse affine combination is desired, one can add a penalty term $\rho\|y\|_1$ with some $\rho > 0$ to the objective function in (1.5) meanwhile to use $\frac{1}{2}\|\cdot\|_2^2$ to replace $\|\cdot\|_2$ to get the following model

$$\min\left\{ \frac{1}{2}\|B_0 + \sum_{k=1}^{p} y_k B_k\|_2^2 + \rho\|y\|_1 \mid y \in \Re^p \right\}. \tag{1.10}$$

Correspondingly, we can reformulate (1.10) in terms of the dual MOP form:

$$\begin{array}{lll} \text{(D}'\text{)} & \max & \langle 0, y\rangle - \frac{1}{2}\|X^*\|_2^2 - \rho\|z\|_1 \\ & \text{s.t.} & \mathcal{A}^*y - B_0 = X^*\,, \\ & & y = z\,, \end{array}$$

where $\mathcal{A}^* : \Re^p \to \Re^{m\times n}$ is the linear operator defined by (1.7). Note that for any norm function g in $\mathcal{E}$, we always have

$$(\frac{1}{2}g^2)^* = \frac{1}{2}(g^D)^2\,, \tag{1.11}$$

where g^D is the corresponding dual norm of g. Let B_∞^ρ be the closed ball in $\Re^p$ under the l_∞ norm with radius $\rho > 0$, i.e., $B_\infty^\rho := \{z \in \Re^p \mid \|z\|_\infty \le \rho\}$. Then, the primal form of (1.10) can be written as

$$\begin{array}{lll}(\text{P}) & \min & \langle B_0, X\rangle + \langle 0, x\rangle + \frac{1}{2}\|X\|_*^2 + \delta_{B_\infty^\rho}(x) \\ & \text{s.t.} & \mathcal{A}X + x = 0\,.\end{array}$$

Matrix completion. Given a matrix $M \in \Re^{m\times n}$ with entries in the index set Ω given, the matrix completion problem seeks to find a low-rank matrix X such that $X_{ij} \approx M_{ij}$ for all $(i,j) \in \Omega$. The problem of efficient recovery of a given low-rank matrix has been intensively studied recently. In [15], [16], [39], [47], [77], [78], etc, the authors established the remarkable fact that under suitable incoherence assumptions, an $m \times n$ matrix of rank r can be recovered with high probability from a random uniform sample of $O((m+n)r\text{polylog}(m,n))$ entries by solving the following nuclear norm minimization problem:

$$\min\Big\{\|X\|_* \mid X_{ij} = M_{ij}\ \forall\,(i,j) \in \Omega\Big\}\,.$$

The theoretical breakthrough achieved by Candès et al. has led to the rapid expansion of the nuclear norm minimization approach to model application problems for which the theoretical assumptions may not hold, for example, for problems with noisy data or that the observed samples may not be completely random. Nevertheless, for those application problems, the following model may be considered to accommodate problems with noisy data:

$$\min\Big\{\frac{1}{2}\|P_\Omega(X) - P_\Omega(M)\|_2^2 + \rho\|X\|_* \mid X \in \Re^{m\times n}\Big\}\,, \tag{1.12}$$

where $P_\Omega(X)$ denotes the vector obtained by extracting the elements of X corresponding to the index set Ω in lexicographical order, and ρ is a positive parameter. In the above model, the error term is measured in l_2 norm of vector. One can of course use the l_1-norm or l_∞-norm of vectors if those norms are more appropriate for the applications under consideration. As for the case of the matrix norm approximation, one can easily write (1.12) in the following primal MOP form

$$\begin{array}{lll} \text{(P)} & \min & \langle 0, X\rangle + \langle 0, z\rangle + \dfrac{1}{2}\|z\|_2^2 + \rho\|X\|_* \\ & \text{s.t.} & \mathcal{A}X - z = b\,, \end{array}$$

where $(z, X) \in \mathcal{X} \equiv \Re^{|\Omega|} \times \Re^{m\times n}$, $b = P_\Omega(M) \in \Re^{|\Omega|}$, and the linear operator $\mathcal{A}$: $\Re^{m\times n} \to \Re^{|\Omega|}$ is given by $\mathcal{A}(X) = P_\Omega(X)$. Moreover, by Proposition 1.1 and (1.11), we know that the corresponding dual MOP of (1.12) can be written as

$$\begin{array}{lll} \text{(D)} & \max & \langle b, y\rangle - \dfrac{1}{2}\|z^*\|_2^2 - \delta_{B_2^\rho}(X^*) \\ & \text{s.t.} & \mathcal{A}^* y - X^* = 0, \quad y + z^* = 0\,, \end{array}$$

where $\mathcal{A}^* : \Re^{|\Omega|} \to \Re^{m\times n}$ is the adjoint of $\mathcal{A}$, and $B_2^\rho \subseteq \Re^{m\times n}$ is the closed ball under the spectral norm $\|\cdot\|_2$ with radius $\rho > 0$, i.e., $B_2^\rho := \{Z \in \Re^{m\times n} \mid \|Z\|_2 \le \rho\}$.

Robust matrix completion/Robust PCA. Suppose that $M \in \Re^{m\times n}$ is a partially given matrix for which the entries in the index set Ω are observed, but an unknown sparse subset of the observed entries may be grossly corrupted. The problem here seeks to find a low-rank matrix X and a sparse matrix Y such that $M_{ij} \approx X_{ij} + Y_{ij}$ for all $(i, j) \in \Omega$, where the sparse matrix Y attempts to identify the grossly corrupted entries in M, and X attempts to complete the "cleaned" copy of M. This problem has been considered in [14], and it is motivated by earlier results established in [18], [112]. In [14] the following convex optimization problem is solved to recover M:

$$\min \Big\{ \|X\|_* + \rho\|Y\|_1 \mid P_\Omega(X) + P_\Omega(Y) = P_\Omega(M) \Big\}\,, \tag{1.13}$$

where $\|Y\|_1$ is the l_1-norm of $Y \in \Re^{m\times n}$ defined component-wised, i.e., $\|Y\|_1 = \sum_{i=1}^{m}\sum_{j=1}^{n}|y_{ij}|$, and ρ is a positive parameter. In the event that the "cleaned" copy of M itself in (1.13) is also contaminated with random noise, the following problem could be considered to recover M:

$$\min \Big\{ \frac{1}{2}\|P_\Omega(X) + P_\Omega(Y) - P_\Omega(M)\|_2^2 + \eta\Big(\|X\|_* + \rho\|Y\|_1\Big) \mid X, Y \in \Re^{m\times n} \Big\}\,, \tag{1.14}$$

where η is a positive parameter. Again, the l_2-norm that is used in the first term can be replaced by other norms such as the l_1-norm or l_∞-norm of vectors if they are more appropriate. In any case, both (1.13) and (1.14) can be written in the form of MOP. We omit the details.

Structured low rank matrix approximation. In many applications, one is often faced with the problem of finding a low-rank matrix $X \in \Re^{m\times n}$ which approximates a given target matrix M but at the same time it is required to have certain structures (such as being a Hankel matrix) so as to conform to the physical design of the application problem [21]. Suppose that the required structure is encoded in the constraints $\mathcal{A}(X) \in b + \mathcal{Q}$. Then a simple generic formulation of such an approximation problem can take the following form:

$$\min\left\{\|X - M\|_F \mid \mathcal{A}(X) \in b + \mathcal{Q},\ \text{rank}(X) \le r\right\}. \tag{1.15}$$

Obviously it is generally NP hard to find the global optimal solution for the above problem. However, given a good starting point, it is quite possible that a local optimization method such as variants of the alternating minimization method may be able to find a local minimizer that is close to being globally optimal. One possible strategy to generate a good starting point for a local optimization method to solve (1.15) would be to solve the following penalized version of (1.15):

$$\min\Big\{\|X - M\|_F + \rho \sum_{k=r+1}^{\min\{m,n\}} \sigma_k(X) \mid \mathcal{A}(X) \in b + \mathcal{Q}\Big\}, \tag{1.16}$$

where $\sigma_k(X)$ is the k-th largest singular value of X and $\rho > 0$ is a penalty parameter. The above problem is not convex but we can attempt to solve it via a sequence of convex relaxation problems as proposed in [37] as follows. Start with $X^0 = 0$ or any feasible matrix X^0 such that $\mathcal{A}(X^0) \in b + \mathcal{Q}$. At the k-th iteration, solve

$$\min\Big\{\lambda\|X - X^k\|_F^2 + \|X - M\|_F + \rho(\|X\|_* - \langle H_k, X\rangle) \mid \mathcal{A}(X) \in b + \mathcal{Q}\Big\} \tag{1.17}$$

to get X^{k+1}, where λ is a positive parameter and H_k is a sub-gradient of the convex function $\sum_{k=1}^{r} \sigma_k(\cdot)$ at the point X^k. Once again, one may easily write (1.17) in the form of MOP. Also, we omit the details.

System identification. For system identification problem, the objective is to fit a discrete-time linear time-invariant dynamical system from observations of its inputs and outputs. Let $u(t) \in \Re^m$ and $y_{\rm meas}(t) \in \Re^p$, $t = 0, \ldots, N$ be the sequences of inputs and measured (noise) outputs, respectively. For each time $t \in \{0, \ldots, N\}$, denote the state of the dynamical system at time t by the vectors $x(t) \in \Re^n$, where n is the order of the system. The dynamical system which we need to determine is assumed as following

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t) + Du(t)\,,$$

where the system order n, the matrices A, B, C, D, and the initial state $x(0)$ are the parameters to be estimated. In system identification literatures [52, 106, 104, 107], the SVD low-rank approximation based subspace algorithms are used to estimate the system order, and other model parameters. As mentioned in [59], the disadvantage of this approach is that the matrix structure (e.g., the block Hankel structure) is not taken into account before the model order is chosen. Therefore, it was suggested by [59] (see also [60]) that instead of using the SVD low-rank approximation, one can use nuclear norm minimization to estimate the system order, which preserves the linear (Hankel) structure. The method proposed in [59] is based on computing $y(t) \in \Re^p$, $t = 0, \ldots, N$ by solving the following convex optimization problem with a given positive weighting parameter ρ

$$\min \left\{ \rho \|HU^{\perp}\|_* + \frac{1}{2}\|Y - Y_{\rm meas}\|^2 \right\}\,, \tag{1.18}$$

where $Y = [y(0), \ldots, y(N)] \in \Re^{p\times(N+1)}$, $Y_{\rm meas} = [y_{\rm meas}(0), \ldots, y_{\rm meas}(N)] \in \Re^{p\times(N+1)}$, H

is the block Hankel matrix defined as

$$H = \begin{bmatrix} y(0) & y(1) & y(2) & \cdots & y(N-r) \\ y(1) & y(2) & y(3) & \cdots & y(N-r+1) \\ \vdots & \vdots & \vdots & & \vdots \\ y(r) & y(r+1) & y(r+2) & \cdots & y(N) \end{bmatrix},$$

and $U^{\perp}$ is a matrix whose columns form an orthogonal basis of the null space of the following block Hankel matrix

$$U = \begin{bmatrix} u(0) & u(1) & u(2) & \cdots & u(N-r) \\ u(1) & u(2) & u(3) & \cdots & u(N-r+1) \\ \vdots & \vdots & \vdots & & \vdots \\ u(r) & u(r+1) & u(r+2) & \cdots & u(N) \end{bmatrix}.$$

Note that the optimization variable in (1.18) is the matrix $Y \in \Re^{p\times(N+1)}$. Also, one can easily write (1.18) in the form of MOP. As we mentioned in matrix norm approximation problems, by using (1.11), one can find out the corresponding dual problem of (1.18) directly. Again, we omit the details.

Fastest mixing Markov chain problem. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a connected graph with vertex set $\mathcal{V} = \{1, \ldots, n\}$ and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. We assume that each vertex has a self-loop, i.e., an edge from itself to itself. The corresponding Markov chain can be describe via the transition probability matrix $P \in \Re^{n\times n}$, which satisfies $P \geq 0$, $Pe = e$ and $P = P^T$, where the inequality $P \geq 0$ means elementwise and $e \in \Re^n$ denotes the vector of all ones. The fastest mixing Markov chain problem [10] (FMMC) is finding the edge transition probabilities that give the fastest mixing Markov chain, i.e., that minimize the second largest eigenvalue modulus (SLEM) $\mu(P)$ of P. The eigenvalues of P are real (since it is symmetric), and by Perron-Frobenius theory, no more than 1 in magnitude. Therefore,we have

$$\mu(P) = \max_{i=2,\ldots,n} |\lambda_i(P)| = \sigma_2(P)\,,$$

where $\sigma_2(P)$ is the second largest singular value. Then, the FMMC problem is equivalent to the following optimization problem:

$$\begin{array}{ll} \min & \sigma_1(\mathcal{P}(p)) + \sigma_2(\mathcal{P}(p)) = \|\mathcal{P}(p)\|_{(2)} \\ \text{s.t.} & p \geq 0,\ Bp \leq e\,, \end{array} \tag{1.19}$$

where $\|\cdot\|_{(k)}$ is *Ky Fan k-norm* of matrices, i.e., the sum of the k largest singular values of a matrix; $p \in \Re^m$ denotes the vector of transition probabilities on the non-self-loop edges; $P = I + \mathcal{P}(p) = I + \sum_{l=1}^m p_l E^{(l)}$ with $E^{(l)}_{ij} = E^{(l)}_{ji} = +1$, $E^{(l)}_{ii} = E^{(l)}_{jj} = -1$ and all other entries of $E^{(l)}$ are zero; $B \in \Re^{m\times p}$ is the vertex-edge incidence matrix. Then, the FMMC problem can be reformulated as the following dual MOP form

$$\begin{array}{lll} \text{(D)} & \max & -\|Z\|_{(2)} \\ & \text{s.t.} & \mathcal{P}p - Z = I, \quad p \geq 0,\ Bp - e \leq 0\,. \end{array}$$

Note that for any given positive integer k, the dual norm of Ky Fan k-norm $\|\cdot\|_{(k)}$ (cf. [3, Exercise IV.1.18]) is given by

$$\|X\|_{(k)^*} = \max\{\|X\|_2, \frac{1}{k}\|X\|_*\}\,.$$

Thus, the primal form of (1.19) can be written as

$$\begin{array}{lll} \text{(P)} & \min & \langle \mathbf{1}, v\rangle - \langle I, Y\rangle + \delta_{B^1_{(2)^*}}(Y) \\ & \text{s.t.} & \mathcal{P}^*Y - u + B^T v = 0\,, \\ & & u \geq 0, \quad v \geq 0\,, \end{array}$$

where $\mathcal{P}^* : \Re^{n\times n} \to \Re^m$ is the adjoint of the linear mapping $\mathcal{P}$, and $B^1_{(2)^*} \subseteq \Re^{n\times n}$ is the closed unit ball of the dual norm $\|\cdot\|^*_{(2)}$, i.e.,

$$B^1_{(2)^*} := \{X \in \Re^{n\times n} \mid \|X\|^*_{(2)} \leq 1\} = \{X \in \Re^{n\times n} \mid \|X\|_2 \leq 1,\ \|X\|_* \leq 2\}\,.$$

Fastest distributed linear averaging problem. A matrix optimization problem, which is closely related to the fastest mixing Markov chain (FMMC) problem, is the fastest distributed linear averaging (FDLA) problem. Again, let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a connected graph (network) consisting of the vertex set $\mathcal{V} = \{1, \ldots, n\}$ and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. Suppose that each node i holds an initial scalar value $x_i(0) \in \Re$. Let $x(0) = (x_1(0), \ldots .x_n(0))^T \in \Re^n$ be the vector of the initial node values on the network. Distributed linear averaging is done by considering the following linear iteration

$$x(t+1) = Wx(t), \quad t = 0, 1, \ldots , \tag{1.20}$$

where $W \in \Re^{n\times n}$ is the weight matrix, i.e., W_{ij} is the weight on x_j at node i. Set $W_{ij} = 0$ if the edge $(i, j) \notin \mathcal{E}$ and $i \neq j$. The distributed averaging problem arises in the autonomous agents coordination problem. It has been extensively studied in literature (e.g., [62]). Recently, the distributed averaging problem has found applications in different areas such as formation fight of unmanned airplanes and clustered satellites, and coordination of mobile robots. In such applications, one important problem is how to choose the weight matrix $W \in \Re^{n\times n}$ such that the iteration (1.20) converges and it converges as fast as possible, which is so-called fastest distributed linear averaging problem [58]. It was shown [58, Theorem 1] that the iteration (1.20) converges to the average for any given initial vector $x(0) \in \Re^n$ if and only if $W \in \Re^{n\times n}$ satisfies

$$\begin{cases} e^T W = e^T \, , \\ We = e \, , \\ \rho\left(W - \dfrac{1}{n} ee^T\right) < 1 \, , \end{cases}$$

where $\rho : \Re^{n\times n} \to \Re$ denotes the spectral radius of a matrix. Moreover, the speed of convergence can be measured by the so-called *per-step convergence factor*, which is defined by

$$r_{\text{step}}(W) = \|W - \frac{1}{n} ee^T\|_2 \, .$$

Therefore, the fastest distributed linear averaging problem can be formulated as the following MOP problem:

$$\begin{array}{ll} \min & \|W - \frac{1}{n}ee^T\|_2 \\ \text{s.t.} & e^TW = e^T,\ We = e\,, \\ & W_{ij} = 0,\ (i,j) \notin \mathcal{E},\ i \neq j\,. \end{array} \tag{1.21}$$

The FDLA problem is similar with the FMMC problem. The corresponding dual problem also can be derived easily. We omit the details.

More examples of MOPs such as the reduced rank approximations of transition matrices, the low rank approximations of doubly stochastic matrices, and the low rank nonnegative approximation which preserves the left and right principal eigenvectors of a square positive matrix, can be found in [46].

Finally, by considering the epigraph of the norm function, the MOP problem involving the norm function can be written as the MCP form. In fact, these two concepts can be connected by the following proposition.

Proposition 1.2. *Suppose $\mathcal{E}$ be a finite dimensional Euclidean space. Assume that the proper convex function $g : \mathcal{E} \to (-\infty, \infty]$ is positively homogeneous, then the polar of the epigraph of g is given by*

$$(\operatorname{epi} g)^\circ = \bigcup_{\rho \geq 0} \rho(-1, C)\,,$$

where C is given by (1.8).

For example, consider the MOP problem (1.2) with $f \equiv \|\cdot\|_\sharp$, a given norm function defined in $\mathcal{X}$ (e.g., $\mathcal{X} \equiv \Re^{m\times n}$ and $f \equiv \|\cdot\|_{(k)}$). We know from Proposition 1.2 and Proposition 1.1 that the polar of the epigraph cone $\mathcal{K} \equiv \operatorname{epi}\|\cdot\|_\sharp$ can be written as

$$\mathcal{K}^\circ = \bigcup_{\lambda \geq 0} \lambda(-1, \partial f(0)) = \left\{(-t, -\boldsymbol{Y}) \in \Re \times \mathcal{X} \mid \|\boldsymbol{Y}\|_\sharp^* \leq t\right\} = -\operatorname{epi}\|\cdot\|_\sharp^*\,,$$

where $\|\cdot\|_\sharp^*$ is the dual norm of $\|\cdot\|_\sharp$. Then, the primal and dual MOPs can be rewritten as the following MCP forms

$$\begin{array}{llll} \text{(P)} & \min & \langle \boldsymbol{C}, \boldsymbol{X} \rangle + t \\ & \text{s.t.} & \mathcal{A}\boldsymbol{X} = b\,, \\ & & (t, \boldsymbol{X}) \in \mathcal{K}\,, \end{array} \qquad \begin{array}{llll} \text{(D)} & \max & \langle b, y \rangle \\ & \text{s.t.} & \mathcal{A}^* y - \boldsymbol{C} = \boldsymbol{X}^*\,, \\ & & (-1, \boldsymbol{X}^*) \in \mathcal{K}^\circ\,, \end{array}$$

where $\mathcal{K} = \text{epi}\|\cdot\|_\sharp$ and $\mathcal{K}^\circ = -\text{epi}\|\cdot\|_\sharp^*$.

For many applications in eigenvalue optimization [69, 70, 71, 55], the convex function f in the MOP problem (1.2) is positively homogeneous in $\mathcal{X}$. For example, let $\mathcal{X} \equiv \mathcal{S}^n$ and $f \equiv s_{(k)}(\cdot)$, the sum of k largest eigenvalues of the symmetric matrix. It is clear that $s_k(\cdot)$ is a positively homogeneous closed convex function in $\mathcal{S}^n$. Then, by Proposition 1.2 and Proposition 1.1, we know that the corresponding primal and dual MOPs can be rewritten as the following MCP forms

$$\begin{array}{llll} \text{(P)} & \min & \langle C, X \rangle + t \\ & \text{s.t.} & \mathcal{A}X = b\,, \\ & & (t, X) \in \mathcal{M}\,, \end{array} \qquad \begin{array}{llll} \text{(D)} & \max & \langle b, y \rangle \\ & \text{s.t.} & \mathcal{A}^* y - C = X^*\,, \\ & & (-1, X^*) \in \mathcal{M}^\circ\,, \end{array}$$

where the closed convex cone $\mathcal{M} := \big\{(t, X) \in \Re \times \mathcal{S}^n \,|\, s_{(k)}(X) \le t\big\}$ is the epigraph of $s_{(k)}(\cdot)$, and $\mathcal{M}^\circ$ is the polar of $\mathcal{M}$ given by $\mathcal{M}^\circ = \bigcup_{\rho \ge 0} \rho(-1, C)$ with

$$C = \partial s_{(k)}(0) := \{W \in \mathcal{S}^n \,|\, \text{tr}(W) = k,\ 0 \le \lambda_i(W) \le 1,\ i = 1, \ldots, n\}\ .$$

Since MOPs include many important applications, the first question one must answer is how to solve them. One possible approach is considering the SDP reformulation of the MOP problems. Most of the MOP problems considering in this thesis are semidefinite representable [2, Section 4.2]. For example, if $f \equiv \|\cdot\|_{(k)}$, the Ky Fan k-norm of matrix, then the convex function f is semidefinite representable (SDr) i.e., there exists a linear matrix inequality (LMI) such that

$$(t, X) \in \text{epi} f \quad \Longleftrightarrow \quad \exists\, u \in \Re^q : \quad \mathcal{A}_{SDr}(t, X, u) - C \succeq 0\,,$$

where $\mathcal{A}_{SDr} : \Re \times \Re^{m\times n} \times \Re^q \to \mathcal{S}^r$ is a linear operator and $C \in \mathcal{S}^r$. It is well-known that for any $(t, X) \in \Re \times \Re^{m\times n}$,

$$\|X\|_{(k)} \leq t \quad \Longleftrightarrow \quad \left\{ \begin{aligned} t - kz - \langle Z, I_{m+n}\rangle &\geq 0\,, \\ Z &\succeq 0\,, \\ Z - \begin{bmatrix} 0 & X \\ X^T & 0 \end{bmatrix} + zI_{m+n} &\succeq 0\,, \end{aligned} \right.$$

where $Z \in \mathcal{S}^{m+n}$ and $z \in \Re$. In particular, when $k = 1$, i.e., $f \equiv \|\cdot\|_2$, the spectral norm of matrix, we have

$$\|X\|_2 \leq t \quad \Longleftrightarrow \quad \mathcal{S}^{m+n} \ni \begin{bmatrix} tI_m & X \\ X^T & tI_n \end{bmatrix} \succeq 0\,.$$

See [2, Example 18(c) & 19] for more details on these. By employing the corresponding semidefinite representation of f, most MOPs considering in this thesis can be reformulated as SDP problems with extended dimensions. For instance, consider the matrix norm approximation problem (1.5), which can be reformulated as the following SDP problem:

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & \mathcal{A}^*y - B_0 = Z\,, \\ & \begin{bmatrix} tI_m & Z \\ Z^T & tI_n \end{bmatrix} \succeq 0\,, \end{aligned} \tag{1.22}$$

where $\mathcal{A}^* : \Re^p \to \Re^{m\times n}$ is the linear operator defined by (1.7). Also, it is well-known [10] that the FMMC problem (1.19) has the following SDP reformulation

$$\begin{aligned} \min \quad & s \\ \text{s.t.} \quad & -sI \preceq P - (1/n)ee^T \preceq sI\,, \\ & P \geq 0, \quad Pe = e, \quad P = P^T\,, \\ & P_{ij} = 0, \quad (i,j) \notin \mathcal{E}\,, \end{aligned} \tag{1.23}$$

where $\mathcal{E}$ is the edge set of the given connected graph $\mathcal{G}$. For the semidefinite representations of the other MOPs we mentioned before, one can refer to [71, 1] for more details.

By considering the corresponding SDP reformulations, most MOPs can be solved by the well developed interior point methods (IPMs) based SDP solvers, such as SeDuMi [92] and SDPT3 [103]. This SDP approach is fine as long as the sizes of the reformulated problems are not large. However, for large scale problems, this approach becomes impractical, if possible at all, due to the fact that the computational cost of each iteration of an IPM becomes prohibitively expensive. This is particular the case when $n \gg m$ (if assuming $m \leq n$). For example, for the matrix norm approximation problem (1.5), the matrix variable of the equivalent SDP problem (1.22) has the order $\frac{1}{2}(m+n)^2$. For the extreme case that $m = 1$, instead of solving the SDP problem (1.22), one always want to reformulate (1.5) as the following second order cone programming (SOC) problem:

$$\begin{array}{ll} \min & t \\ \text{s.t.} & \mathcal{A}^* y - B_0 = z \,, \\ & \sqrt{zz^T} \leq t \,, \end{array} \tag{1.24}$$

where $B_0 \in \Re^{1\times n}$, $\mathcal{A}^* : \Re^p \to \Re^{1\times n}$ is the linear operator defined by (1.7), and $z \in \Re^{1\times n}$. Even if $m \approx n$ (e.g., the symmetric case), the expansion of variable dimensions will inevitably lead to extra computational cost. Thus, the SDP approach do not seem to be viable for large scale MOPs. It is highly desirable for us to design algorithms that can solve MOPs in the original matrix spaces.

Our idea for solving MOPs is built on the classical proximal point algorithms (PPAs) [85, 84]. The reason for doing so is because we have witnessed a lot of interests in applying augmented Lagrangian methods, or in general PPAs, to large scale SDP problems during the last several years, e.g., [74, 63, 116, 117, 111]. Depending on how the inner subproblems are solved, these methods can be classified into two categories: first order alternating direction based methods [63, 74, 111] and second order semismooth Newton based methods [116, 117]. The efficiency of all these methods depends on the fact that the metric projector over the SDP cone admits a closed form solution [88, 40, 102]. Furthermore, the semismooth Newton based method [116, 117] also exploits a crucial property – the strong semismoothness of this metric projector established in [95]. It will be shown later that the similar properties of the MOP analogues play a crucial role in the proximal point algorithm (PPA) for solving MOP problems.

Next, we briefly introduce the general framework of the PPA for solving the MOP problem (1.2). The classical PPA is designed to solve the inclusion problems with maximal monotone operators [85, 84]. Let $\mathcal{H}$ be a finite dimensional real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ be a multivalued, maximal monotone operator (see [85] for the definition). Given $x^0 \in \mathcal{H}$, in order to solve the inclusion problem $0 \in \mathcal{T}(x)$ by the PPA, we need to solve iteratively a sequence of regularized inclusion problems:

$$x^{k+1} \text{ approximately solves } 0 \in \mathcal{T}(x) + \eta_k^{-1}(x - x^k)\,. \tag{1.25}$$

Denote $\mathcal{P}_{\eta_k}(\cdot) := (I + \eta_k \mathcal{T})^{-1}(\cdot)$. Then, equivalently, we have

$$x^{k+1} \approx \mathcal{P}_{\eta_k}(x^k)\,,$$

where the given sequence $\{\eta_k\}$ satisfies

$$0 < \eta_k \uparrow \eta_\infty \le \infty\,. \tag{1.26}$$

Two convergence criteria for (1.26) introduced by Rockafellar [85] as follows

$$\|x^{k+1} - \mathcal{P}_{\eta_k}(x^k)\| \;\le\; \varepsilon_k, \quad \varepsilon_k > 0, \quad \sum_{k=0}^{\infty} \varepsilon_k \le \infty\,, \tag{1.27}$$

$$\|x^{k+1} - \mathcal{P}_{\eta_k}(x^k)\| \;\le\; \delta_k \|x^{k+1} - x^k\|, \quad \delta_k > 0, \quad \sum_{k=0}^{\infty} \delta_k < \infty\,. \tag{1.28}$$

For the convergence analysis of the general proximal point method, one may refer to [85, Theorem 1 & 2]. Roughly speaking, under mild assumptions, condition (1.27) guarantees the global convergence of $\{x^k\}$, in the sense that the sequence $\{x^k\}$ converges to one solution of the inclusion problem $0 \in \mathcal{T}(x)$. Moreover, if condition (1.28) holds and $\mathcal{T}^{-1}$ is Lipschitz continuous at the origin, then the sequence $\{x^k\}$ converges locally at a linear rate and in particular, if $\eta_\infty = \infty$, the convergence is superlinear.

Consider the primal and dual MOP problems (1.2) and (1.3). Let $L : \mathcal{X} \times \Re^p \to \Re$ be the ordinary Lagrangian function for (1.2), i.e.,

$$L(\boldsymbol{X}, y) := \langle \boldsymbol{C}, \boldsymbol{X} \rangle + f(\boldsymbol{X}) + \langle b - \mathcal{A}\boldsymbol{X}, y \rangle, \quad \boldsymbol{X} \in \mathcal{X}, \quad y \in \Re^p .$$

The *essential objective function* of the primal and dual MOPs (1.2) and (1.3) are defined by

$$F(\boldsymbol{X}) := \sup_{y \in \Re^p} L(\boldsymbol{X}, y) = \begin{cases} \langle \boldsymbol{C}, \boldsymbol{X} \rangle + f(\boldsymbol{X}) & \text{if } \mathcal{A}\boldsymbol{X} - b = 0 \,, \\ \infty & \text{otherwise} \,, \end{cases} \qquad \boldsymbol{X} \in \mathcal{X} \tag{1.29}$$

and

$$G(y) := \inf_{\boldsymbol{X} \in \mathcal{X}} L(\boldsymbol{X}, y) = \langle b, y \rangle - f^*(\mathcal{A}^* y - C), \quad y \in \Re^p . \tag{1.30}$$

Therefore, the primal and dual MOP problems can be written as the following inclusion problems respectively

$$0 \in \mathcal{T}_F(\boldsymbol{X}) := \partial F(\boldsymbol{X}) \quad \text{and} \quad 0 \in \mathcal{T}_G(y) := \partial G(y) . \tag{1.31}$$

Since F and $-G$ are closed proper convex functions, from [83, Corollary 31.5.2], we know that ∂F and $-\partial G$ are maximal monotone operators. Thus, the proximal point algorithm can be used to solve the inclusion problems (1.31). In order to apply the PPA to MOPs, we need to solve the inner problem (1.25) in each step approximately. For example, consider the primal MOP problem. Let $\eta_k > 0$ be given. Then, we have

$$\boldsymbol{X}^{k+1} \approx (I + \eta_k \mathcal{T}_F)^{-1}(\boldsymbol{X}^k) ,$$

which is equivalent to

$$\boldsymbol{X}^{k+1} \approx \arg \min_{\boldsymbol{X} \in \mathcal{X}} \left\{ F(\boldsymbol{X}) + \frac{1}{2\eta_k} \|\boldsymbol{X} - \boldsymbol{X}^k\|^2 \right\} . \tag{1.32}$$

Let $\psi_{F,\eta_k}(\boldsymbol{X}^k)$ be the optimal function value of (1.32), i.e.,

$$\psi_{F,\eta_k}(\boldsymbol{X}^k) := \min_{\boldsymbol{X}\in\mathcal{X}} \left\{ F(\boldsymbol{X}) + \frac{1}{2\eta_k}\|\boldsymbol{X}-\boldsymbol{X}^k\|^2 \right\} .$$

By the definition of the essential primal objective function (1.29), we have

$$\begin{aligned}
\psi_{F,\eta_k}(\boldsymbol{X}^k) &= \min_{\boldsymbol{X}\in\mathcal{X}} \left\{ F(\boldsymbol{X}) + \frac{1}{2\eta_k}\|\boldsymbol{X}-\boldsymbol{X}^k\|^2 \right\} \\
&= \min_{\boldsymbol{X}\in\mathcal{X}} \left\{ \sup_{y\in\Re^p} L(\boldsymbol{X},y) + \frac{1}{2\eta_k}\|\boldsymbol{X}-\boldsymbol{X}^k\|^2 \right\} \\
&= \sup_{y\in\Re^p} \min_{\boldsymbol{X}\in\mathcal{X}} \left\{ \langle \boldsymbol{C},\boldsymbol{X}\rangle + f(\boldsymbol{X}) + \langle b-\mathcal{A}\boldsymbol{X},y\rangle + \frac{1}{2\eta_k}\|\boldsymbol{X}-\boldsymbol{X}^k\|^2 \right\} \\
&= \sup_{y\in\Re^p} \Theta_{\eta_k}(y;\boldsymbol{X}^k) , \qquad (1.33)
\end{aligned}$$

where $\Theta_{\eta_k}(y;\boldsymbol{X}^k) : \Re^p \to \Re$ is given by

$$\Theta_{\eta_k}(y;\boldsymbol{X}^k) := \psi_{f,\eta_k}(\boldsymbol{X}^k+\eta_k(\mathcal{A}^*y-C)) + \frac{1}{2\eta_k}\left(\|\boldsymbol{X}^k\|^2 - \|\boldsymbol{X}^k+\eta_k(\mathcal{A}^*y-C)\|^2 \right) + \langle b,y\rangle$$

with

$$\psi_{f,\eta_k}(\boldsymbol{X}^k+\eta_k(\mathcal{A}^*y-C)) := \min_{\boldsymbol{X}\in\mathcal{X}} \left\{ f(\boldsymbol{X}) + \frac{1}{2\eta_k}\|\boldsymbol{X}-(\boldsymbol{X}^k+\eta_k(\mathcal{A}^*y-C))\|^2 \right\} . \qquad (1.34)$$

Therefore, from the definition of $\Theta_{\eta_k}(y;\boldsymbol{X}^k)$, we know that in order to solve the inner sub-problem (1.33) efficiently, the properties of the function ψ_{f,η_k} should be studied first. In particular, as we mentioned before, similar as the SDP problems, the success of the PPAs for MOPs depends crucially on the first and second order differential properties of ψ_{f,η_k}. Actually, the function $\psi_{f,\eta_k} : \mathcal{X} \to \Re$ defined in (1.34) is called the Moreau-Yosida regularization of f with respect to η_k. The Moreau-Yosida regularization for the general convex function has many important applications in different optimization problems. There have been great efforts on studying the properties of the Moreau-Yosida regularization (see, e.g., [41, 53]). Several fundamental properties of the Moreau-Yosida regularization will be introduced in Section 1.2.

1.2 The Moreau-Yosida regularization and spectral operators

In this section, we first briefly introduce the Moreau-Yosida regularization and proximal point mapping for general convex functions.

Definition 1.1. *Let $\mathcal{E}$ be a finite dimensional Euclidean space. Suppose that $g : \mathcal{E} \to (-\infty, \infty]$ is a closed proper convex function. Let $\eta > 0$ be given. The Moreau-Yosida regularization $\psi_{g,\eta} : \mathcal{E} \to \Re$ of g with respect to η is defined as*

$$\psi_{g,\eta}(x) := \min_{z \in \mathcal{E}} \left\{ g(z) + \frac{1}{2\eta} \|z - x\|^2 \right\}, \quad x \in \mathcal{E}\,. \tag{1.35}$$

It is well-known that for any given $x \in \mathcal{E}$, the minimization problem (1.35) has unique optimal solution. We denote such unique optimal solution as $P_{g,\eta}(x)$, the *proximal point* of x associated with g. In particular, if $g \equiv \delta_C(\cdot)$ is the indicator function of the nonempty closed convex set C in $\mathcal{E}$ and $\eta = 1$, then the corresponding proximal point of $x \in \mathcal{E}$ is the *metric projection* $\Pi_C(x)$ of x onto C, which is the unique optimal solution to following convex optimization problem:

$$\begin{array}{ll} \min & \dfrac{1}{2}\|y - z\|^2 \\ \text{s.t.} & y \in C\,. \end{array}$$

Next, we list some important properties of the Moreau-Yosida regularization as follows.

Proposition 1.3. *Let $g : \mathcal{E} \to (-\infty, +\infty]$ be a closed proper convex function. Let $\eta > 0$ be given, $\psi_{g,\eta}$ be the Moreau-Yosida regularization of g, and $P_{g,\eta}$ be the associated proximal point mapping. Then, the following properties hold.*

(i) *Both $P_{g,\eta}$ and $Q_{g,\eta} := I - P_{g,\eta}$ are firmly non-expansive, i.e., for any $x, y \in \mathcal{E}$,*

$$\|P_{g,\eta}(x) - P_{g,\eta}(y)\|^2 \leq \langle P_{g,\eta}(x) - P_{g,\eta}(y), x - y\rangle, \tag{1.36}$$

$$\|Q_{g,\eta}(x) - Q_{g,\eta}(y)\|^2 \leq \langle Q_{g,\eta}(x) - Q_{g,\eta}(y), x - y\rangle. \tag{1.37}$$

Consequently, both $P_{g,\eta}$ and $Q_{g,\eta}$ are globally Lipschitz continuous with modulus 1.

(ii) *$\psi_{g,\eta}$ is continuously differentiable, and furthermore, it holds that*

$$\nabla\psi_{g,\eta}(x) = \frac{1}{\eta}Q_{g,\eta}(x) = \frac{1}{\eta}(x - P_{g,\eta}(x)), \quad x \in \mathcal{E}.$$

The following useful property is derived by Moreau [66] and so-called Moreau decomposition.

Theorem 1.4. *Let $g : \mathcal{E} \to (-\infty, \infty]$ be a closed proper convex function and g^* be its conjugate. Then, any $x \in \mathcal{E}$ has the decomposition*

$$P_{g,1}(x) + P_{g^*,1}(x) = x\,. \tag{1.38}$$

Moreover, for any $x \in \mathcal{E}$, we have

$$\psi_{g,1}(x) + \psi_{g^*,1}(x) = \frac{1}{2}\|x\|^2\,. \tag{1.39}$$

Suppose that the closed proper convex function g is positively homogenous. Then, from Proposition 1.1, we can obtain the following result directly.

Corollary 1.5. *Suppose that the closed proper convex function $g : \mathcal{E} \to (-\infty, \infty]$ is positively homogenous. Let g^* be the conjugate of g and $\eta > 0$ be given. For any $x \in \mathcal{E}$, we have*

$$Q_{g,\eta}(x) = x - P_{g,\eta}(x) = \eta P_{g^*,\eta^{-1}}(\eta^{-1}x) = \arg\min_{z}\left\{\frac{1}{2}\|z - x\|^2 \mid z \in \eta C\right\},$$

where the closed convex set C in $\mathcal{E}$ is defined by (1.8). Furthermore, for any $x \in \mathcal{E}$, we have

$$\psi_{g,\eta}(x) + \psi_{g^*,\eta^{-1}}(\eta^{-1}x) = \frac{1}{2\eta}\|x\|^2\,.$$

In applications, the closed proper convex functions $f : \mathcal{X} \to (-\infty, \infty]$ in the MOP problems are *unitarily invariant*, i.e., for any $\boldsymbol{X} = (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{s_0}, \boldsymbol{X}_{s_0+1}, \ldots, \boldsymbol{X}_s) \in \mathcal{X}$, any orthogonal matrices $\boldsymbol{U}_k \in \Re^{m_k \times m_k}$, $k = 1, \ldots, s$ and $\boldsymbol{V}_k \in \Re^{n_k \times n_k}$, $k = s_0 + 1, \ldots, s$,

$$f(\boldsymbol{X}) = f(\boldsymbol{U}_1^T\boldsymbol{X}_1\boldsymbol{U}_1, \ldots, \boldsymbol{U}_{s_0}^T\boldsymbol{X}_{s_0}\boldsymbol{U}_{s_0}, \boldsymbol{U}_{s_0+1}^T\boldsymbol{X}_{s_0+1}\boldsymbol{V}_{s_0+1}, \ldots, \boldsymbol{U}_s^T\boldsymbol{X}_s\boldsymbol{V}_s)\,. \tag{1.40}$$

If the closed proper convex function $f : \mathcal{X} \to (-\infty, \infty]$ is unitarily invariant, then it can be shown (Proposition 3.2 in Chapter 3) that the corresponding Moreau-Yosida regularization $\psi_{f,\eta}$ is also unitarily invariant in $\mathcal{X}$. Moreover, we will show that the proximal mapping $P_{f,\eta} : \mathcal{X} \to \mathcal{X}$ can be written as

$$P_{f,\eta}(\boldsymbol{X}) = (\boldsymbol{G}_1(\boldsymbol{X}), \ldots, \boldsymbol{G}_s(\boldsymbol{X})), \quad \boldsymbol{X} \in \mathcal{X},$$

with

$$\boldsymbol{G}_k(\boldsymbol{X}) := \begin{cases} P_k \mathrm{diag}\big(\boldsymbol{g}_k(\kappa(\boldsymbol{X}))\big) P_k^T & k = 1, \ldots, s_0, \\ U_k \left[\mathrm{diag}\big(\boldsymbol{g}_k(\kappa(\boldsymbol{X}))\big) \ \ 0\right] V_k^T & k = s_0 + 1, \ldots, s, \end{cases}$$

and $P_k \in \mathcal{O}^{m_k}$, $1 \le k \le s_0$, $U_k \in \mathcal{O}^{m_k}$, $V_k \in \mathcal{O}^{n_k}$, $s_0 + 1 \le k \le s$ such that

$$\boldsymbol{X}_k = \begin{cases} P_k \Lambda(\boldsymbol{X}_k) P_k^T & k = 1, \ldots, s_0, \\ U_k [\Sigma(\boldsymbol{X}_k) \ \ 0] V_k^T & k = s_0 + 1, \ldots, s, \end{cases}$$

where $\boldsymbol{g} : \Re^{m_0+m} \to \Re^{m_0+m}$ is a vector valued function satisfying the so-called (mixed) symmetric condition (Definition 3.1). It will be shown in Proposition 3.2 Chapter 3 that the proximal mapping $P_{f,\eta}$ is a *spectral operator* (Definition 3.2).

Spectral operators of matrices have many important applications in different fields, such as matrix analysis [3], eigenvalue optimization [55], semidefinite programming [117], semidefinite complementarity problems [20, 19] and low rank optimization [13]. In such applications, the properties of some special spectral operators have been extensively studied by many researchers. Next, we will briefly review the related work. Usually, the symmetric vector valued function $\boldsymbol{g}$ is either simple or easy to study. Therefore, a natural question one may ask is that how can we study the properties of spectral operators from the vector valued analogues?

For symmetric matrices, *Löwner's (symmetric) operator* [61] is the first spectral operator considered by the mathematical optimization community. Suppose that $X \in \mathcal{S}^n$

has the eigenvalue decomposition

$$X = \overline{P} \begin{bmatrix} \lambda_1(X) & 0 & \cdots & 0 \\ 0 & \lambda_2(X) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n(X) \end{bmatrix} \overline{P}^T , \tag{1.41}$$

where $\lambda_1(X) \geq \lambda_2(X) \geq \ldots \geq \lambda_n(X)$ are the real eigenvalues of X (counting multiplicity) being arranged in non-increasing order. Let $g : \Re \to \Re$ be a scalar function. The corresponding Löwner operator is defined by

$$G(X) := \sum_{i=1}^{n} g(\lambda_i(X))\bar{p}_i\bar{p}_i^T, \quad X \in \mathcal{S}^n , \tag{1.42}$$

where for each $i \in \{1, \ldots, n\}$, $\bar{p}_i$ is the i-th column of $\overline{P}$. Löwner's operator is used in many important applications, such as matrix analysis [3], conic optimization [97] and complementary problems [19]. The properties of Löwner's operator are well-studied in the literature. For example, the well-definiteness can be found, e.g., [3, Chapter V] and [43, Section 6.2]. Chen, Qi and Tseng [19, Proposition 4.6] showed that Löwner's operator G is locally Lipschitz continuous if and only if g is locally Lipschitz continuous. The differentiability result of Löwner's operator G can be largely derived from [31] or [49]. In particular, Chen, Qi and Tseng [19, Proposition 4.3] showed that G is differentiable at X if and only if g is differentiable at every eigenvalue of X. This result is also implied in [56, Theorem 3.3] for the case that $g \equiv \nabla h$ for some differentiable function $h : \Re \to \Re$. Chen, Qi and Tseng [20, Lemma 4 and Proposition 4.4] showed that G is continuously differentiable if and only if g is continuously differentiable near every eigenvalue of X. For the related directional differentiability of G, one may refer to [89] for a nice derivation. Sun and Sun [95, Theorem 4.7] first provided the directional derivative formula for Löwner's operator G with respect to the absolute value function, i.e., $g \equiv |\cdot|$. Also, they proved [95, Theorem 4.13] the strong semismoothness of the corresponding Löwner's operator G. It is an open question whether such a (tractable) characterization can be found for Löwner's operator G with respect to any locally Lipschitz function g. To our knowledge, such characterization can be found only for some special cases. For example, the characterization of Clarke's generalized Jacobian of Löwner's operator G with respect to the absolute value function was provided by [72, Lemma 11]; Chen, Qi and Tseng [20, Proposition 4.8] provided Clarke's generalized Jacobian of G, where the directional derivative of g has the one-side continuity property [20, the condition (24)].

Recently, in order to solve some fundamental optimization problems involving the eigenvalues [55], one needs to consider a kind of (symmetric) spectral operators which are more general than Löwner's operators, in the sense that the functions g in the definition (2.18) are vector-valued. In particular, Lewis [54] defined such kind of (symmetric) spectral operators by considering the gradient of the *symmetric function* ϕ, i.e., $\phi : \Re^n \to \Re$ satisfies that

$$\phi(x) = \phi(Px) \quad \text{for any permutation matrix } P \text{ and any } x \in \Re^n .$$

Let $g := \nabla\phi(\cdot) : \Re^n \to \Re^n$. For any $X \in \mathcal{S}^n$ with the eigenvalue decomposition (2.4), the corresponding (symmetric) spectral operator $G : \mathcal{S}^n \to \mathcal{S}^n$ [54] at X can be defined by

$$G(X) := \sum_{i=1}^{n} g_i(\lambda(X))\bar{p}_i\bar{p}_i^T . \tag{1.43}$$

Lewis [54] proved that such kind of function G is well-defined, by using the "block-refineness" property of g. Also, it is easy to see that Löwner's operator is indeed a special symmetric spectral operator G defined by (1.43), where the vector valued function g is separable. It is well known that the eigenvalue function $\lambda(\cdot)$ is not everywhere differentiable. It is natural to expect that the composite function G could be not everywhere differentiable no matter how smooth g is. It was therefore surprising when Lewis and Sendov claimed in [56] that G is (continuously) differentiable at X if and only if g is (continuously) differentiable at $\lambda(X)$. For the directional differentiability of G, it is well known that the directional differentiability of g is not sufficient. In fact, Lewis provided a count-example in [54] that g is directionally differentiable at $\lambda(X)$ but G is not directionally differentiable at X. Therefore, Qi and Yang [75] proved that G is directionally differentiable at X if g is Hadamard directionally differentiable at $\lambda(X)$, which can be regarded as a sufficient condition. However, they didn't provide the directional derivative formula for G, which is important in nonsmooth analysis. In the same paper, Qi and Yang [75] also proved that G is locally Lipschitz continuous at X if and only if g is locally Lipschitz continuous at $\lambda(X)$, and G is (strongly) semismooth if and only if g is (strongly) semismooth. However, the characterization of Clarke's generalized Jacobian of the general symmetric matrix valued function G is still an open question.

For nonsymmetric matrices, some special Löwner's nonsymmetric operators were considered in applications. One well-known example is the soft thresholding (ST) operator, which is widely used in many applications, such as the low rank optimization [13]. The general Löwner's nonsymmetric operators were first studied by Yang [114]. For the given matrix $Z \in \Re^{m\times n}$ (assume that $m \leq n$), consider the singular value decomposition

$$Z = \overline{U}\,[\Sigma(Z)\ \ 0]\,\overline{V}^T = \overline{U}\,[\Sigma(Z)\ \ 0]\,\left[\overline{V}_1\ \ \overline{V}_2\right]^T = \overline{U}\Sigma(Z)\overline{V}_1^T\,, \tag{1.44}$$

where

$$\Sigma(Z) = \begin{bmatrix} \sigma_1(Z) & 0 & \cdots & 0 \\ 0 & \sigma_2(Z) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_m(Z) \end{bmatrix},$$

and $\sigma_1(Z) \geq \sigma_2(Z) \geq \ldots \geq \sigma_m(Z)$ are the singular values of Z (counting multiplicity) being arranged in non-increasing order. Let $g : \Re_+ \to \Re$ be a scalar function. The corresponding Löwner's nonsymmetric operators [114] is defined by

$$G(Z) := \overline{U}\,[g(\Sigma(Z))\ \ 0]\,\overline{V}^T = \sum_{i=1}^{m} g(\sigma_i(Z))\bar{u}_i\bar{v}_i^T, \quad Z \in \Re^{m\times n}\,, \tag{1.45}$$

where $g(\Sigma(Z)) := \mathrm{diag}(g(\sigma_1(Z)), \ldots, g(\sigma_m(Z)))$. Yang [114] proved that $g(0) = 0$ is the sufficient and necessary condition for the well-definiteness of Löwner's nonsymmetric operators G. By using the connection between the singular value decomposition of Z

and the eigenvalue decomposition of the symmetric transformation [42, Theorem 7.3.7] (see (2.28)-(2.30) in Section 2.2 for more details), Yang [114] studied the corresponding properties of Löwner's nonsymmetric operators. In particular, it was shown that Löwner's nonsymmetric operators G inherit the (continuous) differentiability and the Lipschitz continuity of g. For the (strong) semismoothness of G, Jiang, Sun and Toh [45] first showed that the soft thresholding operator is strongly semismooth. By using similar techniques, Yang [114] showed that the general Löwner's nonsymmetric operators G is (strongly) semismooth at $Z \in \Re^{m\times n}$ if and only if g is (strongly) semismooth at $\sigma(Z)$.

Recently, the metric projection operators over five different matrix cones have been studied in [30]. In particular, they provided the closed form solutions of the metric projection operators over the epigraphs of the spectral and nuclear matrix norm. Such metric projection operators can not be covered by Löwner's nonsymmetric operators. In fact, those metric projection operators are spectral operators defined on $\mathcal{X} \equiv \Re \times \Re^{m\times n}$, which is considered in this thesis. Several important properties, including its closed form solution, ρ-order B(ouligand)-differentiability ($0 < \rho \leq 1$) and strong semismoothness, of the metric projection operators were studied in [30].

Motivated by [30], in this thesis, we study spectral operators under the more general setting, i.e., the spectral operators considered in this thesis are defined on the Cartesian product of several symmetric and nonsymmetric matrix spaces. On one hand, from [30], we know that the directional derivatives of the metric projection operators over the epigraphs of the spectral and nuclear matrix norm are the spectral operators defined on the Cartesian product of several symmetric and nonsymmetric matrix spaces (see Section 3.2 for details). However, most properties of such kind of matrix functions (even the well-definiteness of such functions), which are important to MOPs, are unknown. Therefore, it is desired to start a systemic study of the general spectral operator. On the other hand, in some applications, the convex function f in (1.2) can be defined on the Cartesian product of the symmetric and nonsymmetric matrix space. For example, in applications, one may want to minimize both the largest eigenvalue of a symmetric matrix and the spectral norm of a nonsymmetric matrix under the certain linear constraint, i.e.,

$$\begin{aligned} \min \quad & \langle \boldsymbol{C}, (X,Y)\rangle + \max\{\lambda_1(X), \|Y\|_2\} \\ \text{s.t.} \quad & \mathcal{A}(X,Y) = b\,, \end{aligned} \tag{1.46}$$

where $\boldsymbol{C} \in \mathcal{X} \equiv \mathcal{S}^n \times \Re^{m\times n}$, $(X,Y) \in \mathcal{X}$, $b \in \Re^p$, and $\mathcal{A} : \mathcal{X} \to \Re^p$ is the given linear operator. Therefore, the proximal point mapping $P_{f,\eta}$ and the gradient $\nabla\psi_{f,\eta}$ of the convex function $f \equiv \max\{\lambda_1(X), \|Y\|_2\} : \mathcal{X} \to (-\infty, \infty]$ is the spectral operator defined in $\mathcal{X} = \mathcal{S}^n \times \Re^{m\times n}$, which is not covered by pervious work. Thus, it is necessary to study the properties of spectral operators under such general setting. Specifically, the following fundamental properties of spectral operators, including the well-definiteness, the directional differentiability, the Fréchet-differentiability, the locally Lipschitz continuity, the ρ-order B-differentiability $(0 < \rho \leq 1)$, the ρ-order G-semismooth $(0 < \rho \leq 1)$ and the characterization of Clarke's generalized Jacobian, will be studied in the first part of this thesis. The study of spectral operators is not only interesting in itself, but it is also crucial for the study on the solutions of the Moreau-Yosida regularization of matrix related functions. As mentioned before, in order to make MOPs tractable, we must study the properties of the proximal point mapping $P_{f,\eta}$ and the gradient $\nabla\psi_{f,\eta}$ of the Moreau-Yosida regularization.

It is worth to note that the semismoothness of the proximal point mapping $P_{f,\eta}$ for the MOP problems considered in this thesis, also can be studied by using the corresponding results on tame functions. Firstly, we introduce the concept of the *o(rder)-minimal structure* (cf. [24, Definition 1.4]).

Definition 1.2. *An o-minimal structure of R is a sequence $\mathcal{M} = \{\mathcal{M}_t\}$ with $\mathcal{M}_t$ a collection of subsets of $\Re^n$ satisfying the following axioms.*

(i) For every t, $\mathcal{M}_t$ is closed under Boolean operators (finite unions, intersections and complement).

(ii) *If* $A \in \mathcal{M}_t$ *and* $B \in \mathcal{M}_{t'}$*, then* $A \times B$ *belongs to* $\mathcal{M}_{t+t'}$*.*

(iii) $\mathcal{M}_t$ *contains all the subsets of the form* $\{x \in \Re^n \,|\, p(x) = 0\}$*, where* $p : \Re^n \to \Re$ *is a polynomial function.*

(iv) *Let* $P : \Re^n \to \Re^{n-1}$ *be the projection on the first* n *coordinates. If* $A \in \mathcal{M}_t$*, then* $P(A) \in \mathcal{M}_t$*.*

(v) *The elements of* $\mathcal{M}_1$ *are exactly the finite union of points and intervals.*

The elements of o-minimal structure are called definable sets. A map $F : A \subseteq \Re^n \to \Re^m$ *is called definable if its graph is a definable subset of* $\Re^{n+m}$*.*

A set of $\Re^n$ is called *tame* with respect to an o-minimal structure, if its intersection with the interval $[-r, r]^n$ for every $r > 0$ is definable in this structure, i.e., the element of this structure. A mapping is tame if its graph is tame. One most often used o-minimal structure is the class of semialgebraic subsets of $\Re^n$. A set in $\Re^n$ is *semialgebraic* if it is a finite union of sets of the form

$$\{x \in \Re^n \,|\, p_i(x) > 0,\ q_j(x) = 0, \quad i = 1, \ldots, a,\ j = 1, \ldots, b\}\ ,$$

where $p_i : \Re^n \to \Re$, $i = 1, \ldots, a$ and $q_j : \Re^n \to \Re$, $j = 1, \ldots, b$ are polynomials. A mapping is semialgebraic if its graph is semialgebraic.

For tame functions, the following proposition was firstly established by Bolte et.al in [4]. Also see [44] for another proof of the semismoothness.

Proposition 1.6. *Let* $g : \Re^n \to \Re^m$ *be a locally Lipschitz continuous mapping.*

(i) *If* g *is tame, then* g *is semismooth.*

(ii) *If* g *is semialgebraic, then* g *is* γ*-order semismooth with some* $\gamma > 0$*.*

Let $\mathcal{E}$ be a finite dimensional Euclidean space. If the closed proper convex function $g : \mathcal{E} \to (-\infty, \infty]$ is semialgebraic, then the Moreau-Yosida regularization $\psi_{g,\eta}$ of g with respect to $\eta > 0$ at x is semialgebraic. Moreover, since the graph of the corresponding proximal point mapping $P_{g,\eta}$ is of the form

$$\mathrm{gph}P_{g,\eta} = \left\{ (x, y) \in \mathcal{E} \times \mathcal{E} \mid g(y) + \frac{1}{2\eta}\|y - x\|^2 = \psi_{g,\eta}(x) \right\} ,$$

we know that $P_{g,\eta}$ is also semialgebraic (cf. [44]). Since $P_{g,\eta}$ is globally Lipschitz continuous, according to Proposition 1.6 (ii), it yields that $P_{g,\eta}$ is γ-order semismooth with some $\gamma > 0$. Furthermore, most closed proper convex functions f in the MOP problem (1.2) are semialgebraic. For example, it is easy to verify that the indicator function $\delta_{\mathcal{S}^n_+}(\cdot)$ of the SDP cone and the Ky Fan k-norm $\|\cdot\|_{(k)}$ are semialgebraic. Therefore, we know that the corresponding proximal point mapping $P_{f,\eta}(\cdot)$ for MOPs are γ-order semismooth with some $\gamma > 0$. However, we only know the existence of γ, which means that we may not able to obtain the strong semismoothness of $P_{g,\eta}$ by this approach.

1.3 Sensitivity analysis of MOPs

The second topic of this thesis is the sensitivity analysis of solutions to matrix optimization problems (MOPs) subject to data perturbation. During the last three decades, considerable progress has been made in this area (Bonnans and Shapiro [8], Facchinei and Pang [33], Klatte and Kummer [48], Rockafellar and Wets [86]). Consider the optimization problem

$$\begin{array}{ll} \min & f(x) \\ \text{s.t.} & G(x) \in \mathcal{C} \, , \end{array} \tag{1.47}$$

where $f : \mathcal{E} \to \Re$ and $G : \mathcal{E} \to \mathcal{Z}$ are twice continuously differentiable functions, $\mathcal{E}$ and $\mathcal{Z}$ are two finite dimensional real vector spaces, and $\mathcal{C}$ is a closed convex set in $\mathcal{Z}$. If $\mathcal{C}$ is a polyhedral set (for the conventional nonlinear programming), the corresponding perturbation analysis results are quite complete.

For the general non-polyhedral $\mathcal{C}$, much less has been discovered. However, for the non-polyhedral $\mathcal{C}$ which is $\mathcal{C}^2$-cone reducible, the sensitivity analysis of solutions for (1.47) have been systematically studied in literature [5, 7, 8]. Meanwhile, the theory of second order optimality conditions of the optimization problem (1.47), which are closely related with sensitivity analysis, has also been studied in [6, 8]. Recently, for a local solution of the nonlinear SDP problem, Sun [94] established various characterizations for the strong regularity, which is one of the important concepts in sensitivity and perturbation analysis introduced by Robinson [80]. More specifically, in [94], for a local solution of the nonlinear SDP problem, the author proved that under the Robinson's constraint qualification, the strong second-order sufficient condition and constraint nondegeneracy, the non-singularity of Clarke's Jacobian of the Karush-Kuhn-Tucker (KKT) system and the strong regularity of the KKT point are equivalent. Motived by this, Chan and Sun [17] gained more insightful characterizations about the strong regularity of linear SDP problems. They showed that the primal and dual constraint nondegeneracies, the strong regularity, the non-singularity of the B(ouligand)-subdifferential of the KKT system, and the non-singularity of the corresponding Clarke's generalized Jacobian, at a KKT point are all equivalent. For the (nonlinear and linear) SDP problems, variational analysis on the metric projection operator over the cone of positive semidefinite matrices plays a fundamental role in achieving these goals. One interesting question is that how to extend these stability results on SDP problems to MOPs.

In stead of considering the general MOP problems, as a starting point, we mainly focus on the sensitivity analysis of the MOP problems with some special structures. For example, the proper closed convex function $f : \mathcal{X} \to (-\infty, \infty]$ in (1.2) is assumed to be a unitarily invariant matrix norm (e.g., the Ky Fan k-norm) or a positively homogenous function (e.g., the sum of k largest eigenvalues of the symmetric matrix). Also, we mainly focus on the simple linear model as the MCP problems (1.48). For example, we can study the following linear MCP problem involving Ky Fan k-norm cone

$$\begin{array}{ll} \min & \langle (s,C),(t,X)\rangle \\ \text{s.t.} & \mathcal{A}(t,X) = b\,, \\ & (t,X) \in \mathcal{K}\,, \end{array} \tag{1.48}$$

where $\mathcal{K} \equiv \text{epi}\|\cdot\|_{(k)} = \big\{(t,X) \in \Re \times \Re^{m\times n} \mid \|X\|_{(k)} \le t\big\}$, $(s,C) \in \Re \times \Re^{m\times n}$, $b \in \Re^p$ are given, and $\mathcal{A} : \Re \times \Re^{m\times n} \to \Re^p$ is the given linear operator. Note that the matrix cone $\mathcal{K} = \text{epi}\|\cdot\|_{(k)}$ includes the epigraphs of the spectral norm $\|\cdot\|_2$ $(k = 1)$ and nuclear norm $\|\cdot\|_*$ $(k = m)$ as two special cases. In this thesis, we first study some important geometrical properties of the Ky Fan k-norm epigraph cone $\mathcal{K}$, such as the characterizations of tangent cone and the (inner and outer) second order tangent sets of $\mathcal{K}$, the explicit expression of the support function of the second order tangent set, the $\mathcal{C}^2$-cone reducibility of $\mathcal{K}$, the characterization of the critical cone of $\mathcal{K}$. By using these properties, we state the constraint nondegeneracy, the second order necessary condition and the (strong) second order sufficient condition of the linear MCP problem (1.48). Finally, for the linear MCP problem (1.48), we establish the equivalent links among the strong regularity of the KKT point, the strong second order sufficient condition and constraint nondegeneracy, and the non-singularity of both the B-subdifferential and Clarke's generalized Jacobian of the nonsmooth system at a KKT point. Variational analysis on the metric projector over the Ky Fan k-norm epigraph cone $\mathcal{K}$ is very important for these studies. More specifically, the study of properties of spectral operators, such as the directional differentiability, the F-differentiability, the ρ-order G-semismooth and the characterization of Clarke's generalized Jacobian in the first part of this thesis, plays an essential role.

Since the model is simplified, we may lose some kind of generality, which means that some MOP problems may not be covered by this work. However, it is worth taking into consideration that the study on the basic models as the linear MCP involving the Ky Fan k-norm cone can serve as a basic tools to study the sensitivity analysis of the more complicated MOP problems. For some MOP problems, the corresponding sensitivity results can be obtained similarly by following the derivation of our basic model. For example, we can extend the sensitivity results to the following linear MCP problem involving the epigraph cone of the sum of k largest eigenvalues of the symmetric matrix

$$\begin{array}{ll} \min & \langle (s,C),(t,X)\rangle \\ \text{s.t.} & \mathcal{A}(t,X)=b\,, \\ & (t,X)\in\mathcal{M}\,, \end{array} \tag{1.49}$$

where $\mathcal{M}\equiv \operatorname{epi} s_{(k)}(\cdot)=\left\{(t,X)\in\Re\times\mathcal{S}^n \mid s_{(k)}(X)\le t\right\}$, $(s,C)\in\Re\times\mathcal{S}^n$, $b\in\Re^p$ are given, and $\mathcal{A}:\Re\times\mathcal{S}^n\to\Re^p$ is the given linear operator. In fact, by using the properties of the eigenvalue function $\lambda(\cdot)$ of the symmetric matrix, the corresponding variational properties of $\mathcal{M}$ can be obtained in the similar but simple way to those of the Ky Fan k-norm cone $\mathcal{K}$. Moreover, by using the properties of the spectral operator (the metric projection operator over the epigraph cone $\mathcal{M}$), the corresponding sensitivity results on the linear MCP problem (1.49) can be derived directly. The extensions to other MOP problems are also be discussed in this thesis.

1.4 Outline of the thesis

The thesis is organized as follows: to facilitate later discussions, we give some preliminaries on the eigenvalue decomposition of symmetric matrices and the singular value decomposition of general matrices in Chapter 2. In Chapter 3, we study some fundamental properties of spectral operators. As an example, the corresponding properties of the metric projection operator over the Ky Fan k-norm epigraph cone $\mathcal{K}$ and other matrix cones are studied at the end of this chapter. Chapter 4 focus on the perturbation analysis of the MOP problems. We mainly study some important geometrical properties of the Ky Fan k-norm epigraph cone $\mathcal{K}$ and various characterizations for the strong regularity of the linear matrix cone programming involving Ky Fan k-norm. The extensions to other MOP problems are discussed at the end of the chapter. Chapter 5 presents conclusions and some possible topic for future research.

Chapter 2

Preliminaries

Let $\mathcal{E}$ and $\mathcal{E}'$ be two finite dimensional real Euclidean spaces and $\mathcal{O}$ be an open set in $\mathcal{E}$. Suppose that $\Phi : \mathcal{O} \subseteq \mathcal{E} \to \mathcal{E}'$ is a locally Lipschitz continuous function on the open set $\mathcal{O}$. According to Rademacher's theorem, Φ is almost everywhere differentiable (in the sense of Fréchet) in $\mathcal{O}$. Let $\mathcal{D}_\Phi$ be the set of points in $\mathcal{O}$ where Φ is differentiable. Let $\Phi'(x)$ be the derivative of Φ at $x \in D_\Phi$. Then the *B(ouligand)-subdifferential* of Φ at $x \in \mathcal{O}$ is denoted by [76]:

$$\partial_B \Phi(x) := \left\{ \lim_{\mathcal{D}_\Phi \ni x^k \to x} \Phi'(x^k) \right\} ,$$

and *Clarke's generalized Jacobian* of Φ at $x \in \mathcal{O}$ [23] takes the form:

$$\partial \Phi(x) = \operatorname{conv} \left\{ \partial_B \Phi(x) \right\} ,$$

where "conv" stands for the convex hull in the usual sense of convex analysis [83]. A function $\Phi : \mathcal{O} \subseteq \mathcal{E} \to \mathcal{E}'$ is said to be *Hadamard directionally differentiable* at $x \in \mathcal{O}$ if the limit

$$\lim_{\substack{t \downarrow 0 \\ h' \to h}} \frac{\Phi(x + th') - \Phi(x)}{t} \quad \text{exists for any } h \in \mathcal{E} . \tag{2.1}$$

It is clear that if Φ is Hadamard directionally differentiable at x, then Φ is directionally differentiable at x, and the limit in (2.1) equals the directional derivative $\Phi'(x; h)$ for any $h \in \mathcal{E}$. Let $\rho > 0$ be given. A function $\Phi : \mathcal{O} \subseteq \mathcal{E} \to \mathcal{E}'$ is said to be *ρ-order B(ouligand)-differentiable* at $x \in \mathcal{O}$ if for any $h \in \mathcal{E}$ with $h \to 0$,

$$\Phi(x+h) - \Phi(x) - \Phi'(x;h) = O(\|h\|^{1+\rho}) . \tag{2.2}$$

Definition 2.1. *Let $\mathcal{E}$ and $\mathcal{E}'$ be two finite dimensional real Euclidean spaces. We say that $\Phi : \mathcal{E} \to \mathcal{E}'$ is (parabolic) second order directionally differentiable at $x \in \mathcal{E}$, if Φ is directionally differentiable at x and for any $h, w \in \mathcal{E}$*

$$\lim_{t \downarrow 0} \frac{\Phi(x + th + \frac{1}{2}t^2 w) - \Phi(x) - t\Phi'(x;h)}{\frac{1}{2}t^2} \quad \text{exists} ;$$

and the above limit is said to be the (parabolic) second order directional derivative of Φ at x along h, w, denoted by $\Phi''(x;h,w)$.

Let $\Phi : \mathcal{O} \subseteq \mathcal{E} \to \mathcal{E}'$ be a locally Lipschitz continuous function on the open set $\mathcal{O}$. The function Φ is said to be G-semismooth at a point $x \in \mathcal{O}$ if for any $y \to x$ and $V \in \partial\Phi(y)$,

$$\Phi(y) - \Phi(x) - V(y-x) = o(||y-x||) .$$

A stronger notion than G-semismoothness is ρ-order G-semismoothness with $\rho > 0$. The function Φ is said to be ρ-order G-semismooth at x if for any $y \to x$ and $V \in \partial\Phi(y)$,

$$\Phi(y) - \Phi(x) - V(y-x) = O(||y-x||^{1+\rho}) .$$

In particular, the function Φ is said to be strongly G-semismooth at x if Φ is 1-order G-semismooth at x. Furthermore, the function Φ is said to be (ρ-order, strongly) semismooth at $x \in \mathcal{O}$ if (i) the directional derivative of Φ at x along any direction $h \in \mathcal{E}$ exists; and (ii) Φ is (ρ-order, strongly) G-semismooth.

The following result taken from [95, Theorem 3.7] provides a convenient tool for proving the G-semismoothness of Lipschitz functions.

Lemma 2.1. *Let $\Phi : \mathcal{O} \subseteq \mathcal{E} \to \mathcal{E}'$ be a locally Lipschitz continuous function on the open set $\mathcal{O}$. Let $\rho > 0$ be a constant. If Z is a set of Lebesgue measure zero in $\mathcal{O}$, then Φ is*

ρ-order G-semismooth (G-semismooth) at x if and only if for any $y \to x$, $y \in \mathcal{D}_\Phi$, and $y \notin Z$,

$$G(y) - G(x) - G'(y)(y-x) = O(||y-x||^{1+\rho}) \quad \big(= o(||y-x||)\big) . \tag{2.3}$$

It is easy to show that if $\Phi : \mathcal{O} \subseteq \mathcal{E} \to \mathcal{E}'$ is locally Lipschitz continuous and directionally differentiable, then the directional derivative is globally Lipschitz continuous (cf. [27] or [82, Theorem A.2(a)]). Therefore, we have the following lemma.

Lemma 2.2. *Suppose that the function $\Phi : \mathcal{O} \subseteq \mathcal{E} \to \mathcal{E}'$ is locally Lipschitz continuous near $x \in \mathcal{E}$ with modulus $L > 0$ and directionally differentiable at x. Then the directional derivative $\Phi'(x; \cdot) : \mathcal{E} \to \mathcal{E}'$ is globally Lipschitz continuous on $\mathcal{E}$ with the same modulus L.*

In the next two subsections, we collect some useful preliminary results on symmetric and non-symmetric matrices, which are important for our subsequent analysis.

2.1 The eigenvalue decomposition of symmetric matrices

Let $\mathcal{S}^n$ be the space of all real $n \times n$ symmetric matrices and $\mathcal{O}^n$ be the set of all $n \times n$ orthogonal matrices. Let $\overline{Y} \in \mathcal{S}^n$ be any given symmetric matrix. We use $\lambda_1(\overline{Y}) \geq \lambda_2(\overline{Y}) \geq \ldots \geq \lambda_n(\overline{Y})$ to denote the real eigenvalues of $\overline{Y}$ (counting multiplicity) being arranged in non-increasing order. Denote $\lambda(\overline{Y}) := (\lambda_1(\overline{Y}), \lambda_2(\overline{Y}), \ldots, \lambda_n(\overline{Y}))^T \in \Re^n$ and $\Lambda(\overline{Y}) := \mathrm{diag}(\lambda(\overline{Y}))$. Let $\overline{P} \in \mathcal{O}^n$ be such that

$$\overline{Y} = \overline{P}\Lambda(\overline{Y})\overline{P}^T . \tag{2.4}$$

We denote the set of such matrices $\overline{P}$ in the eigenvalue decomposition (2.4) by $\mathcal{O}^n(\overline{Y})$. Let $\overline{\mu}_1 > \overline{\mu}_2 > \ldots > \overline{\mu}_r$ be the distinct eigenvalues of $\overline{Y}$. Define

$$\alpha_k := \{i \,|\, \lambda_i(\overline{Y}) = \overline{\mu}_k,\ 1 \leq i \leq n\}, \quad k = 1, \ldots, r . \tag{2.5}$$

For each $i \in \{1, \ldots, n\}$, we define $l_i(\overline{Y})$ to be the number of eigenvalues that are equal to $\lambda_i(\overline{Y})$ but are ranked before i (including i) and $\tilde{l}_i(\overline{Y})$ to be the number of eigenvalues that are equal to $\lambda_i(\overline{Y})$ but are ranked after i (excluding i), respectively, i.e., we define $l_i(\overline{Y})$ and $\tilde{l}_i(\overline{Y})$ such that

$$\lambda_1(\overline{Y}) \geq \ldots \geq \lambda_{i-l_i(\overline{Y})}(\overline{Y}) > \lambda_{i-l_i(\overline{Y})+1}(\overline{Y}) = \ldots = \lambda_i(\overline{Y}) = \ldots = \lambda_{i+\tilde{l}_i(\overline{Y})}(\overline{Y}) > \lambda_{i+\tilde{l}_i(\overline{Y})+1}(\overline{Y}) \geq \ldots \geq \lambda_n(\overline{Y}) . \tag{2.6}$$

In later discussions, when the dependence of l_i and $\tilde{l}_i$, $i = 1, \ldots, n$ on $\overline{Y}$ can be seen clearly from the context, we often drop $\overline{Y}$ from these notations.

The inequality in the following lemma is known as Ky Fan's inequality [34].

Lemma 2.3. *Let A and B be two matrices in $\mathcal{S}^n$. Then*

$$\langle A, B\rangle \leq \lambda(A)^T \lambda(B) , \tag{2.7}$$

where the equality holds if and only if A and B admit a simultaneous ordered eigenvalue decomposition, i.e., there exists an orthogonal matrix $U \in \mathcal{O}^n$ such that

$$A = U\Lambda(A)U^T \quad \text{and} \quad B = U\Lambda(B)U^T .$$

By elementary calculation, one can obtain the following simple observation easily.

Proposition 2.4. *Let $Q \in \mathcal{O}^n$ be an orthogonal matrix such that $Q^T\Lambda(\overline{Y})Q = \Lambda(\overline{Y})$. Then, we have*

$$\begin{cases} Q_{\alpha_k\alpha_l} = 0 , & k, l = 1, \ldots, r, \ k \neq l , \quad (2.8) \\ Q_{\alpha_k\alpha_k} Q^T_{\alpha_k\alpha_k} = Q^T_{\alpha_k\alpha_k} Q_{\alpha_k\alpha_k} = I_{|\alpha_k|} , & k = 1, \ldots, r . \quad (2.9) \end{cases}$$

The following result, which was stated in [96], was essentially proved in the derivation of Lemma 4.12 in [95].

Proposition 2.5. *For any $\mathcal{S}^n \ni H \to 0$, let $Y := \Lambda(\overline{Y}) + H$. Suppose that $P \in \mathcal{O}^n$ satisfies*

$$Y = P\Lambda(Y)P^T .$$

Then, we have

$$\begin{cases} P_{\alpha_k\alpha_l} = O(\|H\|)\,, & k,l = 1,\ldots,r,\ k \neq l\,, \quad (2.10) \\ P_{\alpha_k\alpha_k}P^T_{\alpha_k\alpha_k} = I_{|\alpha_k|} + O(\|H\|^2)\,, & k = 1,\ldots,r\,, \quad (2.11) \end{cases}$$

and there exist $Q_k \in \mathcal{O}^{|\alpha_k|}$, $k = 1,\ldots,r$ *such that*

$$P_{\alpha_k\alpha_k} = Q_k + O(\|H\|^2), \quad k = 1,\ldots,r\,. \tag{2.12}$$

Moreover, we have

$$\Lambda(Y)_{\alpha_k\alpha_k} - \Lambda(\overline{Y})_{\alpha_k\alpha_k} = Q_k^T H_{\alpha_k\alpha_k} Q_k + O(\|H\|^2), \quad k = 1,\ldots,r\,. \tag{2.13}$$

The next proposition follows easily from Proposition 2.5. It has also been proved in [20] based on a so-called "sin(Θ)" theorem in [91, Theorem 3.4].

Proposition 2.6. *For any* $H \in \mathcal{S}^n$*, let* $P \in \mathcal{O}^n$ *be an orthogonal matrix such that* $\overline{Y} + H = P\mathrm{diag}(\lambda(\overline{Y}+H))P^T$*. Then, for any* $\mathcal{S}^n \ni H \to 0$*, we have*

$$\mathrm{dist}(P, \mathcal{O}^n(\overline{Y})) = O(\|H\|)\,.$$

The following proposition about the directional differentiability of the eigenvalue function $\lambda(\cdot)$ is well known. For example, see [51, Theorem 7] and [101, Proposition 1.4].

Proposition 2.7. *Let* $\overline{Y} \in \mathcal{S}^n$ *have the eigenvalue decomposition (2.4). Then, for any* $\mathcal{S}^n \ni H \to 0$*, we have*

$$\lambda_i(\overline{Y}+H) - \lambda_i(\overline{Y}) - \lambda_{l_i}(\overline{P}^T_{\alpha_k} H \overline{P}_{\alpha_k}) = O(\|H\|^2), \quad i \in \alpha_k,\ k = 1,\ldots,r\,, \tag{2.14}$$

where for each $i \in \{1,\ldots,n\}$*,* l_i *is defined in (2.6). Hence, for any given direction* $H \in \mathcal{S}^n$*, the eigenvalue function* $\lambda_i(\cdot)$ *is directionally differentiable at* $\overline{Y}$ *with* $\lambda_i'(\overline{Y};H) = \lambda_{l_i}(\overline{P}^T_{\alpha_k} H \overline{P}_{\alpha_k})$*,* $i \in a_k$*,* $k = 1,\ldots,r$*.*

Next, let us consider the (parabolic) second order directional derivative (Defintion 2.1) of the eigenvalue function $\lambda(\cdot)$. Suppose that $H, W \in \mathcal{S}^n$ are given. Denote

$$Y(t) = \overline{Y} + tH + \frac{1}{2}t^2 W, \quad t > 0\,.$$

Consider the eigenvalue decomposition of $Y(t)$, i.e.,

$$Y(t) = U(t)\Lambda(Y(t))U(t)^T\,,$$

where $U(t) \in \mathcal{O}^n$. Then, we have the following result (see [115, Lemma 2.1]), which can be used to study the second order directional differentiability of the eigenvalue function $\lambda(\cdot)$.

Proposition 2.8. *For each $k \in \{1, \ldots, r\}$, there exists $Q_k(t) \in \mathcal{O}^{|\alpha_k|}$ such that*

$$\begin{aligned} U_{\alpha_k\alpha_l}(t) &= t\frac{H_{\alpha_k\alpha_l}Q_k(t)}{\mu_l - \mu_k} + O(t^2) \quad \text{if } 1 \le l \ne k \le n\,, \\ U_{\alpha_k\alpha_k}(t)^T U_{\alpha_k\alpha_k}(t) &= I_{|\alpha_k|} - t^2\sum_{l\ne k}\frac{Q_k(t)^T H^T_{\alpha_l\alpha_k} H_{\alpha_l\alpha_k} Q_k(t)}{(\mu_l-\mu_k)^2} + O(t^3)\,. \end{aligned}$$

Let $k \in \{1, \ldots, r\}$ be fixed. Consider the symmetric matrix $P^T_{\alpha_k} H P_{\alpha_k} \in \mathcal{S}^{|\alpha_k|}$. Let $R \in \mathcal{O}^{|\alpha_k|}$ be such that

$$P^T_{\alpha_k} H P_{\alpha_k} = R\Lambda(P^T_{\alpha_k} H P_{\alpha_k})R^T\,. \tag{2.15}$$

Denote the distinct eigenvalues of $P^T_{\alpha_k} H P_{\alpha_k}$ by $\tilde\mu_1 > \tilde\mu_2 > \ldots > \tilde\mu_{\tilde r}$. Define

$$\tilde\alpha_j := \{i \,|\, \lambda_i(P^T_{\alpha_k} H P_{\alpha_k}) = \tilde\mu_j, 1 \le i \le |\alpha_k|\}, \quad j = 1, \ldots, \tilde r\,. \tag{2.16}$$

For each $i \in \alpha_k$, let $\tilde l_i \in \{1, \ldots, |\alpha_k|\}$ and $\tilde k \in \{1, \ldots, \tilde r\}$ be such that

$$\tilde l_i := l_{l_i}(P^T_{\alpha_k} H P_{\alpha_k}) \quad \text{and} \quad \tilde l_i \in \tilde\alpha_{\tilde k}\,, \tag{2.17}$$

where l_i is defined by (2.6).

Then Proposition 2.8 leads to the following well known result.

Proposition 2.9 (e.g., [101]). *For any given $H, W \in \mathcal{S}^n$, denote $Y(t) := \overline{Y} + tH + \frac{1}{2}t^2 W$, $t > 0$. Then for any $i \in \alpha_k$, $k = 1, \ldots, r$, we have for any $t \downarrow 0$,*

$$\begin{aligned} \lambda_i(Y(t)) &= \lambda_i(\overline{Y}) + t\lambda_{l_i}(P^T_{\alpha_k} H P_{\alpha_k}) \\ &\quad + \frac{t^2}{2}\lambda_{\tilde l_i}\left(R^T_{\tilde\alpha_{\tilde k}} P^T_{\alpha_k}\left[W - 2H(X - \lambda_i I_n)^\dagger H\right] P_{\alpha_k} R_{\tilde\alpha_{\tilde k}}\right) + O(t^3)\,. \end{aligned}$$

Hence, the eigenvalue function $\lambda(\cdot)$ is second order directionally differentiable at $\overline{Y}$ with

$$\lambda_i''(\overline{Y}; H, W) = \lambda_{\tilde{l}_i}\left(R_{\tilde{\alpha}_{\tilde{k}}}^T P_{\alpha_k}^T \left[W - 2H(\overline{Y} - \lambda_i I_n)^\dagger H\right] P_{\alpha_k} R_{\tilde{\alpha}_{\tilde{k}}}\right) .$$

Suppose that $\overline{Y} \in \mathcal{S}^n$ has the eigenvalue decomposition (2.4). Let $f : \Re \to \Re$ be a scalar function. As we mentioned in Section 1.2, the corresponding Löwner's operator is defined by [61]

$$F(\overline{Y}) := \overline{P}\,\mathrm{diag}(f(\lambda_1(\overline{Y})), f(\lambda_2(\overline{Y})), \ldots, f(\lambda_n(\overline{Y})))\,\overline{P}^T = \sum_{i=1}^n f(\lambda_i(\overline{Y}))\bar{p}_i\bar{p}_i^T . \qquad (2.18)$$

Let $D := \mathrm{diag}(d)$, where $d \in \Re^n$ is a given vector. Assume that the scalar function f is differentiable at each d_i with the derivatives $f'(d_i)$, $i = 1, \ldots, n$. Let $f^{[1]}(D) \in \mathcal{S}^n$ be the first divided difference matrix whose (i, j)-th entry is given by

$$(f^{[1]}(D))_{ij} = \begin{cases} \dfrac{f(d_i) - f(d_j)}{d_i - d_j} & \text{if } d_i \neq d_j, \\ f'(d_i) & \text{if } d_i = d_j, \end{cases} \qquad i, j = 1, \ldots, n.$$

The following result for the differentiability of Löwner's operator F defined in (2.18) can be largely derived from [31] or [49]. Actually, Proposition 4.3 of [19] shows that F is differentiable at $\overline{Y}$ if and only if f is differentiable at every eigenvalue of $\overline{Y}$. This result is also implied in [56, Theorem 3.3] for the case that $f = \nabla h$ for some differentiable function $h : \Re \to \Re$. Lemma 4 of [20] and Proposition 4.4 of [19] show that F is continuously differentiable at $\overline{Y}$ if and only if f is continuously differentiable at every eigenvalue of $\overline{Y}$. For the related directional differentiability of F, one may refer to [89] for a nice derivation.

Proposition 2.10. *Let $\overline{Y} \in \mathcal{S}^n$ be given and have the eigenvalue decomposition (2.4). Then, Löwner's operator F is (continuously) differentiable at $\overline{Y}$ if and only if for each $i \in \{1, \ldots, n\}$, f is (continuously) differentiable at $\lambda_i(\overline{Y})$. In this case, the (Fréchet) derivative of F at $\overline{Y}$ is given by*

$$F'(\overline{Y})H = \overline{P}\left[f^{[1]}(\Lambda(\overline{Y})) \circ (\overline{P}^T H \overline{P})\right]\overline{P}^T \quad \forall\, H \in \mathcal{S}^n . \qquad (2.19)$$

The following second order differentiability of Löwner's operator F can be derived as in [3, Exercise V.3.9].

Proposition 2.11. *Let $\overline{Y} \in \mathcal{S}^n$ have the eigenvalue decomposition (2.4). If the scalar function f is twice continuously differentiable at each $\lambda_i(\overline{Y})$, $i = 1, \ldots, n$, then Löwner's operator F is twice continuously differentiable at $\overline{Y}$.*

Let $\overline{Y} \in \mathcal{S}^n$ be given. For each $k \in \{1, \ldots, r\}$, there exists $\delta_k > 0$ such that $|\bar{\mu}_l - \bar{\mu}_k| > \delta_k, \forall\, 1 \le l \neq k \le r$. Define a scalar function $g_k(\cdot) : \Re \to \Re$ by

$$g_k(t) = \begin{cases} -\dfrac{6}{\delta_k}(t - \bar{\mu}_k - \dfrac{\delta_k}{2}) & \text{if } t \in (\bar{\mu}_k + \frac{\delta_k}{3}, \bar{\mu}_k + \frac{\delta_k}{2}], \\ 1 & \text{if } t \in [\bar{\mu}_k - \frac{\delta_k}{3}, \bar{\mu}_k + \frac{\delta_k}{3}], \\ \dfrac{6}{\delta_k}(t - \bar{\mu}_k + \dfrac{\delta_k}{2}) & \text{if } t \in [\bar{\mu}_k - \frac{\delta_k}{2}, \bar{\mu}_k - \frac{\delta_k}{3}), \\ 0 & \text{otherwise.} \end{cases} \tag{2.20}$$

For each $k \in \{1, \ldots, r\}$, define $\mathcal{P}_k : \mathcal{S}^n \to \mathcal{S}^n$ by

$$\mathcal{P}_k(Y) := \sum_{i \in \alpha_k} p_i p_i^T, \quad Y \in \mathcal{S}^n, \tag{2.21}$$

where $P \in \mathcal{O}^n$ is an orthogonal matrix such that $Y = P\mathrm{diag}(\lambda(Y))P^T$. For each $k \in \{1, \ldots, r\}$, we know that there exists an open neighborhood $\mathcal{N}$ of $\overline{Y}$ such that $\mathcal{P}_k$ is at least twice continuously differentiable on $\mathcal{N}$. By shrinking $\mathcal{N}$ if necessary, we may assume that for any $Y \in \mathcal{N}$ and $k, l \in \{1, \ldots, r\}$,

$$\lambda_i(Y) \neq \lambda_j(Y) \quad \forall\, i \in \alpha_k,\ j \in \alpha_l \text{ and } k \neq l\,.$$

Define $\Omega_k(Y) \in \mathcal{S}^n$, $k = 1, \ldots, r$ by

$$(\Omega_k(Y))_{ij} = \begin{cases} \dfrac{1}{\lambda_i(Y) - \lambda_j(Y)} & \text{if } i \in \alpha_k,\ j \in \alpha_l,\ k \neq l,\ l = 1, \ldots, r\,, \\ \dfrac{-1}{\lambda_i(Y) - \lambda_j(Y)} & \text{if } i \in \alpha_l,\ j \in \alpha_k,\ k \neq l,\ l = 1, \ldots, r\,, \\ 0 & \text{otherwise}\,. \end{cases} \tag{2.22}$$

Then, the following proposition follows from Proposition 2.10 and Proposition 2.11, directly.

Proposition 2.12. *For each $k = 1, \ldots, r$, there exists an open neighborhood $\mathcal{N}$ of $\overline{Y}$ such that $\mathcal{P}_k$ is at least twice continuously differentiable on $\mathcal{N}$, and for any $H \in \mathcal{S}^n$, the first order derivative of $\mathcal{P}_k$ at $Y \in \mathcal{N}$ is given by*

$$\mathcal{P}_k'(Y)H = P[\Omega_k(Y) \circ (P^T H P)]P^T \,, \tag{2.23}$$

where $P \in \mathcal{O}^n$ is any orthogonal matrix such that $Y = P\Lambda(Y)P^T$.

2.2 The singular value decomposition of matrices

From now on, without loss of generality, we always assume that $m \leq n$ in this thesis. Let $\overline{Z} \in \Re^{m\times n}$ be any given matrix. We use $\sigma_1(\overline{Z}) \geq \sigma_2(\overline{Z}) \geq \ldots \geq \sigma_m(\overline{Z})$ to denote the singular values of $\overline{Z}$ (counting multiplicity) being arranged in non-increasing order. Let $\sigma(\overline{Z}) := (\sigma_1(\overline{Z}), \sigma_2(\overline{Z}), \ldots, \sigma_m(\overline{Z}))^T \in \Re^m$ and $\Sigma(\overline{Z}) := \mathrm{diag}(\sigma(\overline{Z}))$. Let $\overline{Z} \in \Re^{m\times n}$ admit the following singular value decomposition (SVD):

$$\overline{Z} = \overline{U}\left[\Sigma(\overline{Z}) \;\; 0\right]\overline{V}^T = \overline{U}\left[\Sigma(\overline{Z}) \;\; 0\right]\left[\overline{V}_1 \;\; \overline{V}_2\right]^T = \overline{U}\Sigma(\overline{Z})\overline{V}_1^T \,, \tag{2.24}$$

where $\overline{U} \in \mathcal{O}^m$ and $\overline{V} = \left[\overline{V}_1 \;\; \overline{V}_2\right] \in \mathcal{O}^n$ with $\overline{V}_1 \in \Re^{n\times m}$ and $\overline{V}_2 \in \Re^{n\times(n-m)}$. The set of such matrices $(\overline{U}, \overline{V})$ in the SVD (2.24) is denoted by $\mathcal{O}^{m,n}(\overline{Z})$, i.e.,

$$\mathcal{O}^{m,n}(\overline{Z}) := \{(\overline{U}, \overline{V}) \in \mathcal{O}^m \times \mathcal{O}^n \mid \overline{Z} = \overline{U}\left[\Sigma(\overline{Z}) \;\; 0\right]\overline{V}^T\}\,.$$

Define the three index sets a, b and c by

$$a := \{i \mid \sigma_i(\overline{Z}) > 0,\ 1 \leq i \leq m\},\ b := \{i \mid \sigma_i(\overline{Z}) = 0,\ 1 \leq i \leq m\} \text{ and } c := \{m+1, \ldots, n\}\,. \tag{2.25}$$

We use $\overline{\nu}_1 > \overline{\nu}_2 > \ldots > \overline{\nu}_r$ to denote the nonzero distinct singular values of $\overline{Z}$. Define

$$a_k := \{i \mid \sigma_i(\overline{Z}) = \overline{\nu}_k,\ 1 \leq i \leq m\}, \quad k = 1, \ldots, r\,. \tag{2.26}$$

For notational convenience, let $a_{r+1} := b$. For each $i \in \{1, \ldots, m\}$, we also define $l_i(\overline{Z})$ to be the number of singular values that are equal to $\sigma_i(\overline{Z})$ but are ranked before i

(including i) and $\tilde{l}_i(\overline{Z})$ to be the number of singular values that are equal to $\sigma_i(\overline{Z})$ but are ranked after i (excluding i), respectively, i.e., we define $l_i(\overline{Z})$ and $\tilde{l}_i(\overline{Z})$ such that

$$\begin{aligned}&\sigma_1(\overline{Z}) \geq \ldots \geq \sigma_{i-l_i(\overline{Z})}(\overline{Z}) > \sigma_{i-l_i(\overline{Z})+1}(\overline{Z}) = \ldots = \sigma_i(\overline{Z}) = \ldots = \sigma_{i+\tilde{l}_i(\overline{Z})}(\overline{Z}) \\ &> \sigma_{i+\tilde{l}_i(\overline{Z})+1}(\overline{Z}) \geq \ldots \geq \sigma_m(\overline{Z}) . \end{aligned} \tag{2.27}$$

In later discussions, when the dependence of l_i and $\tilde{l}_i$, $i = 1, \ldots, m$, on $\overline{Z}$ can be seen clearly from the context, we often drop $\overline{Z}$ from these notations.

Let $\mathcal{B} : \Re^{m \times n} \to \mathcal{S}^{m+n}$ be the linear operator defined by

$$\mathcal{B}(Z) := \begin{bmatrix} 0 & Z \\ Z^T & 0 \end{bmatrix}, \quad Z \in \Re^{m \times n}. \tag{2.28}$$

We use $I_p^{\uparrow}$ to denote the p by p anti-diagonal matrix whose anti-diagonal entries are all ones and other entries are zeros. Denote

$$U_a^{\uparrow} = U_a I_{|a|}^{\uparrow} \quad \text{and} \quad V_a^{\uparrow} = V_a I_{|a|}^{\uparrow} .$$

Let

$$P := \frac{1}{\sqrt{2}} \begin{bmatrix} U_a & U_b & 0 & U_b & U_a^{\uparrow} \\ V_a & V_b & \sqrt{2}\, V_2 & -V_b & -V_a^{\uparrow} \end{bmatrix} \in \mathcal{O}^{m+n} . \tag{2.29}$$

It is well-known [42, Theorem 7.3.7] that

$$P^T \mathcal{B}(Z) P = \Lambda(\mathcal{B}(Z)) = \begin{bmatrix} \Sigma(Z) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\Sigma(Z)^{\uparrow} \end{bmatrix} . \tag{2.30}$$

For notational convenience, we define two linear operators $S : \Re^{p \times p} \to \mathcal{S}^p$ and $T : \Re^{p \times p} \to \Re^{p \times p}$ by

$$S(X) := \frac{1}{2}(X + X^T) \quad \text{and} \quad T(X) := \frac{1}{2}(X - X^T) \quad \forall\, X \in \Re^{p \times p} . \tag{2.31}$$

The inequality in the following lemma is known as von Neumann's trace inequality [108].

Lemma 2.13. *Let Y and Z be two matrices in $\Re^{m\times n}$. Then*

$$\langle Y, Z\rangle \leq \sigma(Y)^T\sigma(Z)\,, \tag{2.32}$$

where the equality holds if Y and Z admit a simultaneous ordered singular value decomposition, i.e., there exist orthogonal matrices $U \in \mathcal{O}^m$ and $V \in \mathcal{O}^n$ such that

$$Y = U[\Sigma(Y) \;\; 0]V^T \quad \text{and} \quad Z = U[\Sigma(Z) \;\; 0]V^T\,.$$

Similar as the symmetric case (Proposition 2.4), we have the following simple observation.

Proposition 2.14. *Let $\overline{\Sigma} := \Sigma(\overline{Z})$. Then, the two orthogonal matrices $P \in \mathcal{O}^m$ and $W \in \mathcal{O}^n$ satisfy*

$$P\left[\overline{\Sigma} \;\; 0\right] = \left[\overline{\Sigma} \;\; 0\right]W \tag{2.33}$$

if and only if there exist $Q \in \mathcal{O}^{|a|}$, $Q' \in \mathcal{O}^{|b|}$ and $Q'' \in \mathcal{O}^{n-|a|}$ such that

$$P = \begin{bmatrix} Q & 0 \\ 0 & Q' \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} Q & 0 \\ 0 & Q'' \end{bmatrix},$$

where $Q = \mathrm{diag}(Q_1, Q_2, \ldots, Q_r)$ is a block diagonal orthogonal matrix with the k-th diagonal block given by $Q_k \in \mathcal{O}^{|a_k|}$, $k = 1, \ldots, r$.

Proof. "$\Longleftarrow$" Obvious.

"$\Longrightarrow$" Define $\Sigma_+ := \Sigma_{aa}$. Let $\bar{a} := \{1, \ldots, n\} \setminus a$. From (2.33), we obtain that

$$\begin{bmatrix} P_{aa} & P_{ab} \\ P_{ba} & P_{bb} \end{bmatrix}\begin{bmatrix} \Sigma_+ & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \Sigma_+ & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} W_{aa} & W_{a\bar{a}} \\ W_{\bar{a}a} & W_{\bar{a}\bar{a}} \end{bmatrix},$$

which, implies

$$P_{aa}\Sigma_+ = \Sigma_+ W_{aa}, \quad \Sigma_+ W_{a\bar{a}} = 0 \quad \text{and} \quad P_{ba}\Sigma_+ = 0\,.$$

Since Σ_+ is nonsingular, we know that $W_{a\bar{a}} = 0$ and $P_{ba} = 0$. Then, since W and P are two orthogonal matrices, we also have

$$P^T\begin{bmatrix} \Sigma_+ & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \Sigma_+ & 0 \\ 0 & 0 \end{bmatrix}W^T,$$

which, implies $W_{\bar{a}a} = 0$ and $P_{ab} = 0$. Therefore, we know that

$$P = \begin{bmatrix} P_{aa} & 0 \\ 0 & P_{bb} \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} W_{aa} & 0 \\ 0 & W_{\bar{a}\bar{a}} \end{bmatrix},$$

where $W_{aa}, P_{aa} \in \mathcal{O}^{|a|}$, $P_{bb} \in \mathcal{O}^{m-|a|}$ and $W_{\bar{a}\bar{a}} \in \mathcal{O}^{n-|a|}$. By noting that

$$\Sigma_+ = \begin{bmatrix} \overline{\mu}_1 I_{|a_1|} & 0 & \cdots & 0 \\ 0 & \overline{\mu}_2 I_{|a_2|} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \overline{\mu}_r I_{|a_r|} \end{bmatrix},$$

from $P_{aa}\Sigma_+ = \Sigma_+ W_{aa}$, we obtain that

$$\begin{bmatrix} \overline{\mu}_1 P_{a_1a_1} & \overline{\mu}_2 P_{a_1a_2} & \cdots & \overline{\mu}_r P_{a_1a_r} \\ \overline{\mu}_1 P_{a_2a_1} & \overline{\mu}_2 P_{a_2a_2} & \cdots & \overline{\mu}_r P_{a_2a_r} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\mu}_1 P_{a_ra_1} & \overline{\mu}_2 P_{a_ra_2} & \cdots & \overline{\mu}_r P_{a_ra_r} \end{bmatrix} = \begin{bmatrix} \overline{\mu}_1 W_{a_1a_1} & \overline{\mu}_1 W_{a_1a_2} & \cdots & \overline{\mu}_1 W_{a_1a_r} \\ \overline{\mu}_2 W_{a_2a_1} & \overline{\mu}_2 W_{a_2a_2} & \cdots & \overline{\mu}_2 W_{a_2a_r} \\ \vdots & \vdots & \ddots & \vdots \\ \overline{\mu}_r W_{a_ra_1} & \overline{\mu}_r W_{a_ra_2} & \cdots & \overline{\mu}_r W_{a_ra_r} \end{bmatrix}. \tag{2.34}$$

By using the fact that $\overline{\mu}_k > 0$, $k = 1, \ldots, r$, we obtain from (2.34) that

$$\begin{cases} P_{a_ka_k} = W_{a_ka_k}, & k = 1, \ldots, r, \quad (2.35) \\ P_{a_ka_l} = \overline{\mu}_l^{-1}\overline{\mu}_k W_{a_ka_l}, & k, l = 1, \ldots, r, \ k \neq l. \quad (2.36) \end{cases}$$

Next, we shall show by induction that for each $k \in \{1, \ldots, r\}$,

$$P_{a_ka_l} = W_{a_ka_l} = 0 \quad \text{and} \quad P_{a_la_k} = W_{a_la_k} = 0 \ \ \forall\, l = 1, \ldots, r, \ l \neq k. \tag{2.37}$$

First for $k = 1$, since P and W are orthogonal matrices, we have

$$I_{|a_1|} = \sum_{l=1}^{r} P_{a_1a_l} P_{a_1a_l}^T = \sum_{l=1}^{r} W_{a_1a_l} W_{a_1a_l}^T.$$

Therefore, by further using (2.35) and (2.36), we obtain that

$$\sum_{l=2}^{r} (1 - (\overline{\mu}_l^{-1}\overline{\mu}_1)^2) W_{a_1a_l} W_{a_1a_l}^T = 0.$$

Since for each $l \in \{2, 3, \ldots, r\}$, $\overline{\mu}_l^{-1}\overline{\mu}_1 > 1$ and $W_{a_1a_l}W_{a_1a_l}^T$ is symmetric and positive semidefinite, we can easily conclude that

$$W_{a_1a_l} = 0 \quad \forall\, l = 2, 3, \ldots, r \quad \text{and} \quad W_{a_1a_1}^{-1} = W_{a_1a_1}^T\,.$$

From the condition that $W^TW = I_m$, we also have

$$I_{|a_1|} = W_{a_1a_1}^T W_{a_1a_1} + \sum_{l=2}^{r} W_{a_la_1}^T W_{a_la_1}\,.$$

Then, $W_{a_1a_1}^T W_{a_1a_1} = I_{|a_1|}$ implies that

$$\sum_{l=2}^{r} W_{a_la_1}^T W_{a_la_1} = 0\,.$$

Therefore, we have $W_{a_la_1} = 0$, for each $l \in \{2, 3, \ldots, r\}$. By (2.36), we know that (2.37) holds for $k = 1$.

Now, suppose that for some $p \in \{1, \ldots, r-1\}$, (2.37) holds for any $k \leq p$. We will show that (2.37) also holds for $k = p+1$. Since P and W are orthogonal matrices, from the induction assumption we know that

$$I_{|a_{p+1}|} = \sum_{l=p+1}^{r} P_{a_{p+1}a_l}P_{a_{p+1}a_l}^T = \sum_{l=p+1}^{r} W_{a_{p+1}a_l}W_{a_{p+1}a_l}^T\,.$$

From (2.35) and (2.36), we obtain that

$$\sum_{l=p+2}^{r} (1 - (\overline{\mu}_l^{-1}\overline{\mu}_{p+1})^2)W_{a_{p+1}a_l}W_{a_{p+1}a_l}^T = 0\,.$$

Since $\overline{\mu}_l^{-1}\overline{\mu}_{p+1} > 1$ for each $l \in \{p+2, \ldots, r\}$, it can then be checked easily that

$$W_{a_{p+1}a_l} = 0 \quad \forall\, l \in \{p+2, \ldots, r\} \quad \text{and} \quad W_{a_{p+1}a_{p+1}}^{-1} = W_{a_{p+1}a_{p+1}}^T\,.$$

So we have

$$I_{|a_{p+1}|} = W_{a_{p+1}a_{p+1}}^T W_{a_{p+1}a_{p+1}} + \sum_{l=p+2}^{r} W_{a_la_{p+1}}^T W_{a_la_{p+1}}\,,$$

which, together with $W_{a_{p+1}a_{p+1}}^T W_{a_{p+1}a_{p+1}} = I_{|a_{p+1}|}$, implies that

$$\sum_{l=p+2}^{r} W_{a_la_{p+1}}^T W_{a_la_{p+1}} = 0\,.$$

Therefore, we have $W_{a_l a_{p+1}} = 0$ for all $l \in \{p+2, \ldots, r\}$. From (2.36), we know that (2.37) holds for $k = p+1$.

Since (2.37) holds for all $k \in \{1, \ldots, r\}$, we obtain from (2.35) that $P_{aa} = W_{aa}$. Let $Q := P_{aa} = W_{aa}$, $Q' := P_{bb}$ and $Q'' := W_{\bar{a}\bar{a}}$. Then,

$$P = \begin{bmatrix} Q & 0 \\ 0 & Q' \end{bmatrix} \quad \text{and} \quad W = \begin{bmatrix} Q & 0 \\ 0 & Q'' \end{bmatrix},$$

where $Q = \mathrm{diag}(Q_1, Q_2, \ldots, Q_r)$ is a block diagonal orthogonal matrix with the k-th diagonal block given by $Q_k = P_{a_k a_k} \in \mathcal{O}^{|a_k|}$, $k = 1, \ldots, r$. The proof is completed. □

By using (2.30), one can derive the following proposition on the directional derivative of the singular value function $\sigma(\cdot)$ directly from (2.14). For more details, see [57, Section 5.1].

Proposition 2.15. *Suppose that $\overline{Z} \in \Re^{m\times n}$ has the singular value decomposition (2.24). For any $\Re^{m\times n} \ni H \to 0$, we have*

$$\sigma_i(\overline{Z} + H) - \sigma_i(\overline{Z}) - \sigma_i'(\overline{Z}; H) = O(\|H\|^2), \quad i = 1, \ldots, m, \tag{2.38}$$

where

$$\sigma_i'(\overline{Z}; H) = \begin{cases} \lambda_{l_i}\left(S(\overline{U}_{a_k}^T H \overline{V}_{a_k})\right) & \text{if} \quad i \in a_k, \ k = 1, \ldots, r, \\ \sigma_{l_i}\left(\begin{bmatrix}\overline{U}_b^T H \overline{V}_b & \overline{U}_b^T H \overline{V}_2\end{bmatrix}\right) & \text{if} \quad i \in b, \end{cases} \tag{2.39}$$

where for each $i \in \{1, \ldots, m\}$, l_i is defined in (2.27).

The following proposition plays an important role of our study on spectral operators. It also can be regarded as the nonsymmetric analogue to Proposition 2.5 for symmetric matrices.

Proposition 2.16. *For any $\Re^{m\times n} \ni H \to 0$, let $Z := \begin{bmatrix}\Sigma(\overline{Z}) & 0\end{bmatrix} + H$. Suppose that $U \in \mathcal{O}^m$ and $V = [V_1 \ \ V_2] \in \mathcal{O}^n$ with $V_1 \in \Re^{n\times m}$ and $V_2 \in \Re^{n\times(n-m)}$ satisfy*

$$\begin{bmatrix}\Sigma(\overline{Z}) & 0\end{bmatrix} + H = U\,[\Sigma(Z) \ \ 0]\,V^T = U\,[\Sigma(Z) \ \ 0]\,[V_1 \ \ V_2]^T.$$

Then, there exist $Q \in \mathcal{O}^{|a|}$, $Q' \in \mathcal{O}^{|b|}$ *and* $Q'' \in \mathcal{O}^{n-|a|}$ *such that*

$$U = \begin{bmatrix} Q & 0 \\ 0 & Q' \end{bmatrix} + O(\|H\|) \quad \text{and} \quad V = \begin{bmatrix} Q & 0 \\ 0 & Q'' \end{bmatrix} + O(\|H\|)\,, \tag{2.40}$$

where $Q = \mathrm{diag}(Q_1, Q_2, \ldots, Q_r)$ *is a block diagonal orthogonal matrix with the* k*-th diagonal block given by* $Q_k \in \mathcal{O}^{|a_k|}$, $k = 1, \ldots, r$. *Furthermore, we have*

$$\Sigma(Z)_{a_k a_k} - \Sigma(\overline{Z})_{a_k a_k} = Q_k^T S(H_{a_k a_k}) Q_k + O(\|H\|^2), \quad k = 1, \ldots, r \tag{2.41}$$

and

$$\big[\Sigma(Z)_{bb} - \Sigma(\overline{Z})_{bb} \;\; 0\big] = Q'^T \left[H_{bb} \;\; H_{bc}\right] Q'' + O(\|H\|^2)\,. \tag{2.42}$$

Proof. Let $\widehat{Z} := \big[\Sigma(\overline{Z}) \;\; 0\big]$. Let $H \in \Re^{m \times n}$ be given. We use $I_p^{\uparrow}$ to denote the p by p anti-diagonal matrix whose anti-diagonal entries are all ones and other entries are zeros. Denote

$$U_a^{\uparrow} = U_a I_{|a|}^{\uparrow} \quad \text{and} \quad V_a^{\uparrow} = V_a I_{|a|}^{\uparrow}\,.$$

Let

$$P^{\uparrow} := \frac{1}{\sqrt{2}} \begin{bmatrix} U_a & U_b & 0 & U_b & U_a^{\uparrow} \\ V_a & V_b & \sqrt{2}\, V_2 & -V_b & -V_a^{\uparrow} \end{bmatrix} \in \Re^{(m+n)\times(m+n)}\,. \tag{2.43}$$

Then, from (2.30), we have

$$\mathcal{B}(Z) = \mathcal{B}(\widehat{Z}) + \mathcal{B}(H) = P^{\uparrow} \Lambda(\mathcal{B}(Z))(P^{\uparrow})^T\,.$$

By Proposition 2.6, we know that for any $H \to 0$, there exists $P' \in \mathcal{O}^{m+n}(\mathcal{B}(\widehat{Z}))$ such that

$$P^{\uparrow} - P' = O(\|\mathcal{B}(H)\|) = O(\|H\|)\,. \tag{2.44}$$

On the other hand, suppose that $\widehat{U} \in \mathcal{O}^m$ and $\widehat{V} \in \mathcal{O}^n$ are two arbitrary orthogonal matrices such that

$$\widehat{Z} = [\Sigma(\overline{Z}) \;\; 0] = \widehat{U}[\Sigma(\overline{Z}) \;\; 0]\widehat{V}^T\,.$$

From Proposition 2.14, we know that

$$\widehat{U}_a = \begin{bmatrix} \widehat{U}_{aa} \\ 0 \end{bmatrix} \quad \text{and} \quad \widehat{V}_a = \begin{bmatrix} \widehat{U}_{aa} \\ 0 \end{bmatrix}, \tag{2.45}$$

where $\widehat{U}_{aa} = \mathrm{diag}(\widehat{U}_{a_1a_1}, \widehat{U}_{a_2a_2}, \ldots, \widehat{U}_{a_ra_r})$ is a block diagonal orthogonal matrix with the k-th diagonal block given by $\widehat{U}_{a_ka_k} \in \mathcal{O}^{|a_k|}$, $k = 1, \ldots, r$. Let

$$\widehat{P}^{\uparrow} := \frac{1}{\sqrt{2}} \begin{bmatrix} \widehat{U}_a & \widehat{U}_b & 0 & \widehat{U}_b & \widehat{U}_a^{\uparrow} \\ \widehat{V}_a & \widehat{V}_b & \sqrt{2}\,\widehat{V}_2 & -\widehat{V}_b & -\widehat{V}_a^{\uparrow} \end{bmatrix} \in \Re^{(m+n)\times(m+n)},$$

where

$$\widehat{U}_a^{\uparrow} = \widehat{U}_a I_{|a|}^{\uparrow} \quad \text{and} \quad \widehat{V}_a^{\uparrow} = \widehat{V}_a I_{|a|}^{\uparrow}.$$

Then, from (2.30), we know that the orthogonal matrix $\widehat{P}^{\uparrow} \in \mathcal{O}^{m+n}(\mathcal{B}(\widehat{Z}))$. By Proposition 2.4, we know that there exist orthogonal matrices $N_k, N_k' \in \mathcal{O}^{|a_k|}$, $k = 1, \ldots, r$ and $M \in \mathcal{O}^{2|b|+n-m}$ such that

$$P' = \widehat{P}^{\uparrow}\mathrm{diag}(N_1, \ldots, N_r, M, N_r', \ldots, N_1').$$

Therefore, from (2.44), we obtain that

$$\begin{bmatrix} U_a \\ V_a \end{bmatrix} = \begin{bmatrix} \widehat{U}_a\mathrm{diag}(N_1, N_2, \ldots, N_r) \\ \widehat{V}_a\mathrm{diag}(N_1, N_2, \ldots, N_r) \end{bmatrix} + O(\|H\|). \tag{2.46}$$

Denote

$$Q := \widehat{U}_{aa}\mathrm{diag}(N_1, N_2, \ldots, N_r).$$

Then, we know that $Q = \mathrm{diag}(Q_1, Q_2, \ldots, Q_r)$ is a block diagonal orthogonal matrix with the k-th diagonal block given by $Q_k = \widehat{U}_{a_ka_k}N_k \in \mathcal{O}^{|a_k|}$, $k = 1, \ldots, r$. Thus, from (2.45) and (2.46), we obtain that

$$U_a = \begin{bmatrix} Q \\ 0 \end{bmatrix} + O(\|H\|) \quad \text{and} \quad V_a = \begin{bmatrix} Q \\ 0 \end{bmatrix} + O(\|H\|).$$

Since U and Q are orthogonal matrices, from $0 = U_a^T U_b = Q^T U_{ab} + O(\|H\|)$, we obtain that

$$U_{ab} = O(\|H\|)\,.$$

Therefore, we have

$$I_{|b|} = U_{ab}^T U_{ab} + U_{bb}^T U_{bb} = U_{bb}^T U_{bb} + O(\|H\|^2)\,.$$

By considering the singular value decomposition of U_{bb}, we know that there exists an orthogonal matrix $Q' \in \mathcal{O}^{|b|}$ such that

$$U_{bb} = Q' + O(\|H\|^2)\,.$$

Similarly, since V and Q are orthogonal matrices, from $0 = V_a^T V_{\bar{a}} = Q^T V_{a\bar{a}} + O(\|H\|)$, we know that

$$V_{a\bar{a}} = O(\|H\|)\,,$$

where $\bar{a} = \{1, \ldots, n\} \setminus a$. Therefore, we have

$$I_{|\bar{a}|} = V_{a\bar{a}}^T V_{a\bar{a}} + V_{\bar{a}\bar{a}}^T V_{\bar{a}\bar{a}} = V_{\bar{a}\bar{a}}^T V_{\bar{a}\bar{a}} + O(\|H\|^2)\,.$$

By considering the singular value decomposition of $V_{\bar{a}\bar{a}}$, we know that there exists an orthogonal matrix $Q'' \in \mathcal{O}^{n-|a|}$ such that

$$V_{\bar{a}\bar{a}} = Q'' + O(\|H\|^2)\,.$$

Thus,

$$U = \begin{bmatrix} Q & 0 \\ 0 & Q' \end{bmatrix} + O(\|H\|) \quad \text{and} \quad V = \begin{bmatrix} Q & 0 \\ 0 & Q'' \end{bmatrix} + O(\|H\|)\,. \tag{2.47}$$

Hence, (2.40) is proved.

From $\mathcal{B}(\widehat{Z}) + \mathcal{B}(H) = P^{\uparrow}\Lambda(\mathcal{B}(Z))(P^{\uparrow})^T$ and $\widehat{P}^{\uparrow} \in \mathcal{O}^{m+n}(\mathcal{B}(\widehat{Z}))$, we obtain that

$$\Lambda(\mathcal{B}(\widehat{Z})) + (\widehat{P}^{\uparrow})^T \mathcal{B}(H)\widehat{P}^{\uparrow} = (\widehat{P}^{\uparrow})^T P^{\uparrow}\Lambda(\mathcal{B}(Z))(P^{\uparrow})^T \widehat{P}^{\uparrow}\,. \tag{2.48}$$

Let $\widetilde{P} := (\widehat{P}^{\uparrow})^T P^{\uparrow}$ and $\widetilde{\mathcal{B}}(H) := (\widehat{P}^{\uparrow})^T \mathcal{B}(H) \widehat{P}^{\uparrow}$. Then, we can re-write (2.48) as

$$\widetilde{P}^T(\Lambda(\mathcal{B}(\widehat{Z})) + \widetilde{\mathcal{B}}(H))\widetilde{P} = \Lambda(\mathcal{B}(Z)) . \tag{2.49}$$

By comparing both sides of (2.49), we obtain that

$$\widetilde{P}_{a_k}^T \Lambda(\mathcal{B}(\widehat{Z}))\widetilde{P}_{a_k} + (P_{a_k}^{\uparrow})^T \mathcal{B}(H) P_{a_k}^{\uparrow} = \Lambda(B(Z))_{a_k a_k}, \quad k = 1, \ldots, r . \tag{2.50}$$

From (2.10) in Proposition 2.5, we know that

$$\widetilde{P}_{a_k}^T \Lambda(\mathcal{B}(\widehat{Z}))\widetilde{P}_{a_k} = \widetilde{P}_{a_k a_k}^T \Lambda(\mathcal{B}(\widehat{Z}))_{a_k a_k} \widetilde{P}_{a_k a_k} + O(\|H\|^2) .$$

By noting that for each $k \in \{1, \ldots, r\}$, $\Lambda(\mathcal{B}(\widehat{Z}))_{a_k a_k} = \Sigma(\overline{Z})_{a_k a_k} = \overline{\mu}_k I_{|a_k|}$ and $\Lambda(B(Z))_{a_k a_k} = \Sigma(Z)_{a_k a_k}$, we obtain from (2.50) that

$$\overline{\mu}_k \widetilde{P}_{a_k a_k}^T \widetilde{P}_{a_k a_k} + (P_{a_k}^{\uparrow})^T \mathcal{B}(H) P_{a_k}^{\uparrow} = \Sigma(Z)_{a_k a_k} + O(\|H\|^2), \quad k = 1, \ldots, r .$$

By (2.11) in Proposition 2.5, we know that $\widetilde{P}_{a_k a_k}^T \widetilde{P}_{a_k a_k} = I_{|a_k|} + O(\|H\|^2)$, $k = 1, \ldots, r$. Therefore, from (2.43), we obtain that for each $k \in \{1, \ldots, r\}$,

$$S(U_{a_k}^T H V_{a_k}) = \Sigma(Z)_{a_k a_k} - \overline{\mu}_k I_{|a_k|} + O(\|H\|^2) = \Sigma(Z)_{a_k a_k} - \Sigma(\overline{Z})_{a_k a_k} + O(\|H\|^2) .$$

By (2.47), we know that

$$U_{a_k}^T H V_{a_k} = Q_k^T H_{a_k a_k} Q_k + O(\|H\|^2) .$$

Therefore, we have

$$Q_k^T S(H_{a_k a_k}) Q_k = \Sigma(Z)_{a_k a_k} - \Sigma(\overline{Z})_{a_k a_k} + O(\|H\|^2), \quad k = 1, \ldots, r .$$

Hence (2.41) is proved.

Next, we shall show that (2.42) holds. Since $\left[\Sigma(\overline{Z}) \ \ 0\right] + H = U\left[\Sigma(Z) \ \ 0\right] V^T$, we know that

$$U_b^T(\left[\Sigma(\overline{Z}) \ \ 0\right] + H)V_{\bar{a}} = \left[\Sigma(Z)_{bb} \ \ 0\right] . \tag{2.51}$$

Again, from (2.47), we know that

$$U_b = \begin{bmatrix} O(\|H\|) \\ U_{bb} \end{bmatrix} \quad \text{and} \quad V_{\bar{a}} = \begin{bmatrix} O(\|H\|) \\ V_{\bar{a}\bar{a}} \end{bmatrix}.$$

By comparing both sides of (2.51), we obtain that

$$U_{bb}^T \left[\Sigma(\overline{Z})_{bb} \ \ 0\right] V_{\bar{a}\bar{a}} + U_{bb}^T \left[H_{bb} \ \ H_{bc}\right] V_{\bar{a}\bar{a}} + O(\|H\|^2) = \left[\Sigma(Z)_{bb} \ \ 0\right].$$

Since $\Sigma(\overline{Z})_{bb} = 0$, we have

$$U_{bb}^T \left[H_{bb} \ \ H_{bc}\right] V_{\bar{a}\bar{a}} = \left[\Sigma(Z)_{bb} - \Sigma(\overline{Z})_{bb} \ \ 0\right] + O(\|H\|^2).$$

From (2.47), we know that

$$U_{bb}^T \left[H_{bb} \ \ H_{bc}\right] V_{\bar{a}\bar{a}} = Q'^T \left[H_{bb} \ \ H_{bc}\right] Q'' + O(\|H\|^2).$$

Therefore,

$$Q'^T \left[H_{bb} \ \ H_{bc}\right] Q'' = \left[\Sigma(Z)_{bb} - \Sigma(\overline{Z})_{bb} \ \ 0\right] + O(\|H\|^2).$$

Hence (2.42) is proved. The proof is completed. □

Let $\overline{Z} \in \Re^{m\times n}$ be given. For each $k \in \{1,\ldots,r\}$, define the mapping $\mathcal{U}_k : \Re^{m\times n} \to \Re^{m\times n}$ by

$$\mathcal{U}_k(Z) = \sum_{i\in a_k} u_i v_i^T, \quad Z \in \Re^{m\times n}, \tag{2.52}$$

where $U \in \mathcal{O}^m$ and $V \in \mathcal{O}^n$ are such that $Z = U\left[\Sigma(Z) \ \ 0\right]V^T$. For each $k \in \{1,\ldots,r\}$, by constructing the similar scalar function $g_k(\cdot)$ in (2.20), we can show that there exists an open neighborhood $\mathcal{N}$ of $\overline{Z}$ such that $\mathcal{U}_k$ is continuously differentiable in $\mathcal{N}$ (see [30, pp. 14–15] for details). By shrinking $\mathcal{N}$ if necessary, we may assume that for any $k, l \in \{1,\ldots,r\}$,

$$\sigma_i(Z) > 0, \quad \sigma_i(Z) \neq \sigma_j(Z) \quad \forall\, i \in a_k,\ j \in a_l \text{ and } k \neq l,$$

For any fixed $Z \in \mathcal{N}$, define $\Gamma_k(Z)$ and $\Xi_k(Z) \in \Re^{m\times m}$ and $\Upsilon_k(Z) \in \Re^{m\times(n-m)}$, $k = 1,\ldots,r$ by

$$(\Gamma_k(Z))_{ij} = \begin{cases} \dfrac{1}{\sigma_i(Z)-\sigma_j(Z)} & \text{if } i\in a_k,\ j\in a_l,\ k\neq l,\ l=1,\ldots,r+1\,, \\ \dfrac{-1}{\sigma_i(Z)-\sigma_j(Z)} & \text{if } i\in a_l,\ j\in a_k,\ k\neq l,\ l=1,\ldots,r+1\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{2.53}$$

$$(\Xi_k(Z))_{ij} = \begin{cases} \dfrac{1}{\sigma_i(Z)+\sigma_j(Z)} & \text{if } i\in a_k,\ j\in a_l,\ k\neq l,\ l=1,\ldots,r+1\,, \\ \dfrac{1}{\sigma_i(Z)+\sigma_j(Z)} & \text{if } i\in a_l,\ j\in a_k,\ k\neq l,\ l=1,\ldots,r+1\,, \\ \dfrac{2}{\sigma_i(Z)+\sigma_j(Z)} & \text{if } i,j\in a_k, \\ 0 & \text{otherwise} \end{cases} \tag{2.54}$$

and

$$(\Upsilon_k(Z))_{ij} = \begin{cases} \dfrac{1}{\sigma_i(Z)} & \text{if } i\in a_k\,, \\ 0 & \text{otherwise}\,, \end{cases} \qquad j=1,\ldots,n-m\,. \tag{2.55}$$

Therefore, by Proposition 2.12 and (2.28), we are able to show that the following proposition holds, i.e., there exists an open neighborhood $\mathcal{N}$ such that for each $k \in \{1,\ldots,r\}$, $\mathcal{U}_k$ is at least twice continuously differentiable in $\mathcal{N}$. See [30, Proposition 2.11] for more details.

Proposition 2.17. *Let $\mathcal{U}_k$, $k = 1,\ldots,r$ be defined by (2.52). Then, there exists an open neighborhood $\mathcal{N}$ of $\overline{Z}$ such that for each $k \in \{1,\ldots,r\}$, $\mathcal{U}_k$ is at least twice continuously differentiable in $\mathcal{N}$, and for each $k \in \{1,\ldots,r\}$ and any $H \in \Re^{m\times n}$, the first order derivative of $\mathcal{U}_k$ at $Z \in \mathcal{N}$ is given by*

$$\mathcal{U}_k'(Z)H = U[\Gamma_k(Z)\circ S(U^THV_1) + \Xi_k(Z)\circ T(U^THV_1)]V_1^T + U(\Upsilon_k(Z)\circ U^THV_2)V_2^T\,, \tag{2.56}$$

where $(U,V) \in \mathcal{O}^{m,n}(Z)$ and the two linear operators S and T are defined by (2.31).

Finally, let us consider the (parabolic) second order directional derivative of the singular value function $\sigma(\cdot)$. Let $\overline{Z} \in \Re^{m\times n}$ be given. Since $\sigma_i(\overline{Z}) = \lambda_i(\mathcal{B}(\overline{Z}))$, $i =$

$1,\ldots,m$, we know from (2.39) that for any given direction $H, W \in \Re^{m\times n}$, the second order directional derivatives of the singular value function $\sigma_i(\cdot)$, $i = 1,\ldots,m$ are given by

$$\sigma_i''(\overline{Z}; H, W) = \lambda_i''(\mathcal{B}(\overline{Z}); \mathcal{B}(H), \mathcal{B}(W)), \quad i = 1,\ldots,m\,. \tag{2.57}$$

Therefore, from (2.30), we know that the corresponding index sets α_k of $\mathcal{B}(\overline{Z})$, $k = 1,\ldots,r+1$ are given by

$$\alpha_k = a_k, \quad k = 1,\ldots,r \quad \text{and} \quad \alpha_{r+1} = \{|a|+1,\ldots,|a|+2|b|+n-m\}\,.$$

Then, we know from (2.43) that

$$P_{\alpha_k} = \frac{1}{\sqrt{2}}\begin{bmatrix} U_{a_k} \\ V_{a_k}\end{bmatrix}, \quad k = 1,\ldots,r \quad \text{and} \quad P_{\alpha_{r+1}} = \frac{1}{\sqrt{2}}\begin{bmatrix} U_b & 0 & U_b \\ V_b & \sqrt{2}\,V_2 & -V_b \end{bmatrix}.$$

For any $i \in \{1,\ldots,m\}$, consider the following two cases.

Case 1. $i \in a_k$, $1 \le k \le r$. Consider the eigenvalue decomposition of the symmetric matrix $P_{\alpha_k}^T \mathcal{B}(H) P_{\alpha_k} = S(U_{a_k}^T H V_{a_k}) \in \mathcal{S}^{|\alpha_k|}$, i.e.,

$$S(U_{a_k}^T H V_{a_k}) = R\Lambda(S(U_{a_k}^T H V_{a_k}))R^T\,,$$

where $R \in \mathcal{O}^{|\alpha_k|}$. Let $\{\tilde{\alpha}_j\}_{j=1}^{\tilde{r}}$ and $\tilde{l}_i$, $\tilde{k}$ be defined by (2.16) and (2.17) respectively for $P_{\alpha_k}^T \mathcal{B}(H) P_{\alpha_k}$. From (2.57) and by Proposition 2.9, we have

$$\sigma_i''(\overline{Z}; H, W) = \lambda_{\tilde{l}_i}\left(R_{\tilde{a}_{\tilde{k}}}^T P_{\alpha_k}^T \left[\mathcal{B}(W) - 2\mathcal{B}(H)\left(\mathcal{B}(\overline{Z}) - \sigma_i(\overline{Z}) I_{m+n}\right)^{\dagger} \mathcal{B}(H)\right] P_{\alpha_k} R_{\tilde{a}_{\tilde{k}}}\right)\,.$$

Case 2. $i \in b$. Since $(\mathcal{B}(\overline{Z}))^{\dagger} = \mathcal{B}((\overline{Z}^{\dagger})^T)$, we have $\mathcal{B}(W) - 2\mathcal{B}(H)(\mathcal{B}(\overline{Z}))^{\dagger}\mathcal{B}(H) = \mathcal{B}(Y)$, where $Y := W - 2H\overline{Z}^{\dagger}H \in \Re^{m\times n}$. Next, consider the eigenvalue decomposition of the symmetric matrix $P_{\alpha_{r+1}}^T \mathcal{B}(H) P_{\alpha_{r+1}}$, i.e., let $R \in \mathcal{O}^{2|b|+n-m}$ such that

$$P_{\alpha_{r+1}}^T \mathcal{B}(H) P_{\alpha_{r+1}} = R\Lambda(P_{\alpha_{r+1}}^T \mathcal{B}(H) P_{\alpha_{r+1}})R^T\,.$$

On the other hand, it is easy to verify that

$$\begin{aligned} P_{\alpha_{r+1}}^T \mathcal{B}(H) P_{\alpha_{r+1}} &= \frac{1}{2}\begin{bmatrix} A^T + A & \sqrt{2}\,B & A^T - A \\ \sqrt{2}\,B^T & 0 & \sqrt{2}\,B^T \\ -A^T + A & \sqrt{2}\,B & -A^T - A \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} I & I & 0 \\ 0 & 0 & \sqrt{2}\,I \\ I & -I & 0 \end{bmatrix}\begin{bmatrix} 0 & A & B \\ A^T & 0 & 0 \\ B^T & 0 & 0 \end{bmatrix}\begin{bmatrix} I & 0 & I \\ I^T & 0 & -I^T \\ 0 & \sqrt{2}\,I^T & 0 \end{bmatrix}, \end{aligned}$$

where $A := U_b^T H V_b \in \Re^{|b|\times|b|}$ and $B := U_b^T H V_2 \in \Re^{|b|\times(n-m)}$. Denote $K := [A \;\; B] \in \Re^{|b|\times(2|b|+n-m)}$. Let $E \in \mathcal{O}^{|b|}$, $F = [F_1 \;\; F_2] \in \mathcal{O}^{|b|+(n-m)}$ with $F_1 \in \Re^{|b|+(n-m)\times|b|}$ and $F_2 \in \Re^{|b|+(n-m)\times(n-m)}$ be such that

$$K = [A \;\; B] = E[\Sigma(K) \;\; 0]F^T .$$

Let $\tilde{\nu}_1 > \tilde{\nu}_2 > \ldots > \tilde{\nu}_{\tilde{r}}$ be the nonzero distinct singular values of K. Denote

$$\tilde{a} := \{i \,|\, \sigma_i(K) > 0,\ 1 \le i \le |b|\}\,,$$

$$\tilde{a}_j := \{i \,|\, \sigma_i(K) = \tilde{\nu}_j,\ 1 \le i \le |b|\}, \quad j = 1, \ldots, \tilde{r}\,, \tag{2.58}$$

$$\tilde{b} := \{i \,|\, \sigma_i(K) = 0,\ 1 \le i \le |b|\}\,. \tag{2.59}$$

Therefore, by [42, Theorem 7.3.7], we know that

$$R = J \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} E & 0 & E^{\uparrow} \\ F_1 & \sqrt{2}\,F_2 & -F_1^{\uparrow} \end{bmatrix},$$

where $J = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I & 0 \\ 0 & 0 & \sqrt{2}\,I \\ I & -I & 0 \end{bmatrix} \in \mathcal{O}^{2|b|+n-m}$, $E^{\uparrow} = E I_{|b|}^{\uparrow}$ and $F_1^{\uparrow} = F_1 I_{|b|}^{\uparrow}$. Therefore, for $Y = W - 2H\overline{Z}^{\dagger}H \in \Re^{m\times n}$, we have

$$\begin{aligned}
& R^T P_{\alpha_{r+1}}^T \mathcal{B}(Y) P_{\alpha_{r+1}} R \\
= & \ \frac{1}{2}\begin{bmatrix} E^T & F_1^T \\ 0 & \sqrt{2}\,F_2^T \\ (E^{\uparrow})^T & (-F_1^{\uparrow})^T \end{bmatrix} J^T P_{\alpha_{r+1}}^T \mathcal{B}(Y) P_{\alpha_{r+1}} J \begin{bmatrix} E & 0 & E^{\uparrow} \\ F_1 & \sqrt{2}\,F_2 & -F_1^{\uparrow} \end{bmatrix} \\
= & \ \frac{1}{2}\begin{bmatrix} E^T & F_1^T \\ 0 & \sqrt{2}\,F_2^T \\ (E^{\uparrow})^T & (-F_1^{\uparrow})^T \end{bmatrix} \begin{bmatrix} 0 & A' & B' \\ A'^T & 0 & 0 \\ B'^T & 0 & 0 \end{bmatrix} \begin{bmatrix} E & 0 & E^{\uparrow} \\ F_1 & \sqrt{2}\,F_2 & -F_1^{\uparrow} \end{bmatrix}, \qquad (2.60)
\end{aligned}$$

where $[A' \ \ B'] := [U_b^T Y V_b \ \ U_b^T Y V_2] \in \Re^{|b|\times(|b|+n-m)}$.

If $l_i \in \tilde{a}$, i.e., there exists a positive integer $\tilde{k} \in \{1,\ldots,\tilde{r}\}$ such that $l_i \in \tilde{a}_{\tilde{k}}$. Then, from (2.60), we have

$$\sigma_i''(\overline{Z}; H, W) = \lambda_{\tilde{l}_i}(S(E_{\tilde{a}_{\tilde{k}}}^T [A' \ \ B'] F_{\tilde{a}_{\tilde{k}}}))\,,$$

where $\tilde{l}_i$ is defined by (2.17).

If $l_i \in \tilde{b}$, then $\tilde{\alpha}_{\tilde{r}+1} = \{|\tilde{a}|+1,\ldots,|\tilde{a}|+2|\tilde{b}|+n-m\}$ and

$$R_{\tilde{\alpha}_{\tilde{r}+1}} = J \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} E_{\tilde{b}} & 0 & E_{\tilde{b}} \\ F_{\tilde{b}} & \sqrt{2}\,F_2 & -F_{\tilde{b}} \end{bmatrix}.$$

Let $K' = [A' \ \ B'] \in \Re^{|b|\times(|b|+n-m)}$. Then, from (2.60), we obtain that

$$\begin{aligned}
& R_{\tilde{\alpha}_{\tilde{r}+1}}^T P_{\alpha_{r+1}}^T \mathcal{B}(Y) P_{\alpha_{r+1}} R_{\tilde{\alpha}_{\tilde{r}+1}} \\
= & \ \frac{1}{2}\begin{bmatrix} E_{\tilde{b}}^T & F_{\tilde{b}}^T \\ 0 & \sqrt{2}\,F_2^T \\ (E_{\tilde{b}})^T & (-F_{\tilde{b}})^T \end{bmatrix} \begin{bmatrix} 0 & K' \\ K'^T & 0 \end{bmatrix} \begin{bmatrix} E_{\tilde{b}} & 0 & E_{\tilde{b}} \\ F_{\tilde{b}} & \sqrt{2}\,F_2 & -F_{\tilde{b}} \end{bmatrix} \\
= & \ \frac{1}{2}\begin{bmatrix} I & I & 0 \\ 0 & 0 & \sqrt{2}\,I \\ I & -I & 0 \end{bmatrix} \begin{bmatrix} 0 & A'' & B'' \\ A''^T & 0 & 0 \\ B''^T & 0 & 0 \end{bmatrix} \begin{bmatrix} I & 0 & I \\ I^T & 0 & -I^T \\ 0 & \sqrt{2}\,I^T & 0 \end{bmatrix},
\end{aligned}$$

where $[A'' \;\; B''] := [E_{\tilde b}^T K' F_{\tilde b} \;\; E_{\tilde b}^T K' F_2] \in \Re^{|\tilde b|\times(|\tilde b|+n-m)}$. Therefore, we know that

$$\sigma_i''(\overline{Z}; H, W) = \sigma_{\tilde l_i}\left([E_{\tilde b}^T K' F_{\tilde b} \;\; E_{\tilde b}^T K' F_2]\right),$$

where $K' = [A' \;\; B'] = [U_b^T Y V_b \;\; U_b^T Y V_2] \in \Re^{|b|\times(|b|+n-m)}$ and $\tilde l_i = l_{l_i}$ is defined by (2.27). Finally, we have the following proposition.

Proposition 2.18. *Let $\overline{Z} \in \Re^{m\times n}$ have the singular value decomposition (2.24). Suppose that the direction $H, W \in \Re^{m\times n}$ are given. Denote $Y = W - 2H\overline{Z}^\dagger H \in \Re^{m\times n}$.*

(i) If $\sigma_i(\overline{Z}) > 0$, then

$$\sigma_i''(\overline{Z}; H, W) = \lambda_{\tilde l_i}\left(R_{\tilde a_{\tilde k}}^T P_{\alpha_k}^T \left[\mathcal{B}(W) - 2\mathcal{B}(H)\left(\mathcal{B}(\overline{Z}) - \sigma_i(\overline{Z}) I_{m+n}\right)^\dagger \mathcal{B}(H)\right] P_{\alpha_k} R_{\tilde a_{\tilde k}}\right),$$

where $R \in \mathcal{O}^{|\alpha_k|}$ satisfies

$$S(U_{a_k}^T H V_{a_k}) = R\Lambda(S(U_{a_k}^T H V_{a_k}))R^T,$$

and $\{\tilde\alpha_j\}_{j=1}^{\tilde r}$ and $\tilde l_i$, $\tilde k$ be defined by (2.16) and (2.17) respectively for $S(U_{a_k}^T H V_{a_k})$.

(ii) If $\sigma_i(\overline{Z}) = 0$ and $\sigma_{l_i}([U_b^T H V_b \;\; U_b^T H V_2]) > 0$, then

$$\sigma_i''(\overline{Z}; H, W) = \lambda_{\tilde l_i}(S(E_{\tilde a_{\tilde k}}^T [U_b^T Y V_b \;\; U_b^T Y V_2] F_{\tilde a_{\tilde k}})),$$

where $E \in \mathcal{O}^{|b|}$, $F = [F_1 \;\; F_2] \in \mathcal{O}^{|b|+(n-m)}$ satisfy

$$K = [U_b^T H V_b \;\; U_b^T H V_2] = E[\Sigma(K) \;\; 0]F^T,$$

$\tilde a_{\tilde k}$ is defined by (2.58) and $\tilde l_i = l_{l_i}$ is defined by (2.27).

(iii) If $\sigma_i(\overline{Z}) = 0$ and $\sigma_{l_i}([U_b^T H V_b \;\; U_b^T H V_2]) = 0$, then

$$\sigma_i''(\overline{Z}; H, W) = \sigma_{\tilde l_i}\left([E_{\tilde b}^T K' F_{\tilde b} \;\; E_{\tilde b}^T K' F_2]\right),$$

where the index set $\tilde b$ is defined by (2.59), $K' = [A' \;\; B'] = [U_b^T Y V_b \;\; U_b^T Y V_2] \in \Re^{|b|\times(|b|+n-m)}$ and $\tilde l_i = l_{l_i}$ is defined by (2.27).

Chapter 3

Spectral operator of matrices

3.1 The well-definiteness

Let $\mathcal{X}$ be the Euclidean space defined by (1.1) in Chapter 1, i.e.,

$$\mathcal{X} := \mathcal{S}^{m_1} \times \ldots \times \mathcal{S}^{m_{s_0}} \times \Re^{m_{s_0+1} \times n_{s_0+1}} \times \ldots \times \Re^{m_s \times n_s} .$$

Denote $m_0 := \sum_{k=1}^{s_0} m_k$, $m = \sum_{k=s_0+1}^{s} m_k$, and $n := \sum_{k=s_0+1}^{s} n_k$. For any $\boldsymbol{X} := (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{s_0}, \boldsymbol{X}_{s_0+1}, \ldots, \boldsymbol{X}_s) \in \mathcal{X}$, define $\boldsymbol{\kappa}(\boldsymbol{X}) \in \Re^{m_0+m}$ by

$$\boldsymbol{\kappa}(\boldsymbol{X}) := (\lambda(\boldsymbol{X}_1), \ldots, \lambda(\boldsymbol{X}_{s_0}), \sigma(\boldsymbol{X}_{s_0+1}), \ldots, \sigma(\boldsymbol{X}_s)) .$$

A matrix $Q \in \Re^{p \times p}$ is said to be a *signed permutation matrix* if each element of Q has exactly one nonzero entry in each row and each column, that entry being ± 1. For the Euclidean space $\mathcal{X}$, define the set $\mathcal{Q}$ by

$$\mathcal{Q} := \{\boldsymbol{Q} := (\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_s) \mid \mathbf{Q}_k \in \mathbb{P}^{m_k},\ 1 \le k \le s_0 \text{ and } \boldsymbol{Q}_k \in |\mathbb{P}|^{m_k},\ s_0 + 1 \le k \le s\} , \tag{3.1}$$

where $\mathbb{P}^{m_k}$, $1 \le k \le s_0$ are the sets of the permutation matrices in $\Re^{m_k \times m_k}$, and $|\mathbb{P}|^{m_k}$, $s_0 + 1 \le k \le s$ are the sets of the signed permutation matrices in $\Re^{m_k \times m_k}$. For any

$\boldsymbol{Q} \in \mathcal{Q}$, the transpose of $\boldsymbol{Q}$ is defined by

$$\boldsymbol{Q}^T := \left(\boldsymbol{Q}_1^T, \ldots, \boldsymbol{Q}_s^T\right) \in \mathcal{Q}\,.$$

For any $\boldsymbol{x} \in \Re^{m_0+m}$ and $\boldsymbol{Q} \in \mathcal{Q}$, write $\boldsymbol{x}$ as the form $\boldsymbol{x} := (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_s)$, where $\boldsymbol{x}_k \in \Re^{m_k}$, $k = 1, \ldots, s$. Then, for any $\boldsymbol{x} \in \Re^{m_0+m}$ and $\boldsymbol{Q} \in \mathcal{Q}$, define the product $\boldsymbol{Qx} \in \Re^{m_0+m}$ by

$$\boldsymbol{Qx} := (\boldsymbol{Q}_1\boldsymbol{x}_1, \ldots, \ldots, \boldsymbol{Q}_s\boldsymbol{x}_s)\ .$$

For any given $\boldsymbol{x} \in \Re^{m_0+m}$, define a subset $\mathcal{Q}_{\boldsymbol{x}} \subseteq \mathcal{Q}$ by

$$\mathcal{Q}_{\boldsymbol{x}} := \{\boldsymbol{Q} \in \mathcal{Q} \,|\, \boldsymbol{x} = \boldsymbol{Qx}\}\ . \tag{3.2}$$

Let $\boldsymbol{g} : \Re^{m_0+m} \to \Re^{m_0+m}$ be given. For any $\boldsymbol{x} \in \Re^{m_0+m}$, re-write the function value $\boldsymbol{g}(\boldsymbol{x})$ as the following form

$$\boldsymbol{g}(\boldsymbol{x}) = (\boldsymbol{g}_1(\boldsymbol{x}), \ldots, \boldsymbol{g}_s(\boldsymbol{x}))\ ,$$

where $\boldsymbol{g}_k(\boldsymbol{x}) \in \Re^{m_k}$, $k = 1, \ldots, s$. The so-called *(mixed) symmetric property* of the function $\boldsymbol{g}$ is defined as follows.

Definition 3.1. *A vector valued function $\boldsymbol{g} : \Re^{m_0+m} \to \Re^{m_0+m}$ is said to be (mixed) symmetric with respect to $\mathcal{X}$ if*

$$\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{Q}^T\boldsymbol{g}(\boldsymbol{Qx}) \quad \forall\, \boldsymbol{Q} \in \mathcal{Q} \text{ and } \boldsymbol{x} \in \Re^{m_0+m}\,, \tag{3.3}$$

where the set $\mathcal{Q}$ is defined by (3.1).

For a given symmetric function $\boldsymbol{g}$, the corresponding spectral operator $\boldsymbol{G} : \mathcal{X} \to \mathcal{X}$ is defined as follows.

Definition 3.2. *The spectral operator $\boldsymbol{G} : \mathcal{X} \to \mathcal{X}$ with respect to the symmetric function $\boldsymbol{g}$ is defined by*

$$\boldsymbol{G}(\boldsymbol{X}) := (\boldsymbol{G}_1(\boldsymbol{X}), \ldots, \boldsymbol{G}_s(\boldsymbol{X}))\,, \quad \boldsymbol{X} \in \mathcal{X}\,,$$

where

$$G_k(X) := \begin{cases} P_k \mathrm{diag}\big(g_k(\kappa(X))\big) P_k^T & \text{if } 1 \le k \le s_0, \\ U_k \left[\mathrm{diag}\big(g_k(\kappa(X))\big) \;\; 0\right] V_k^T & \text{if } s_0+1 \le k \le s, \end{cases}$$

and $P_k \in \mathcal{O}^{m_k}(X_k)$, $1 \le k \le s_0$, $(U_k, V_k) \in \mathcal{O}^{m_k,n_k}(X_k)$, $s_0+1 \le k \le s$, *i.e.,*

$$X_k = \begin{cases} P_k \Lambda(X_k) P_k^T & \text{if } 1 \le k \le s_0, \\ U_k [\Sigma(X_k) \;\; 0] V_k^T & \text{if } s_0+1 \le k \le s. \end{cases}$$

Theorem 3.1. *If* g *is symmetric, then the corresponding spectral operator* $G : \mathcal{X} \to \mathcal{X}$ *is well-defined.*

Proof. For any given $x = (x_1, \ldots, x_s) \in \Re^{m_0+m}$, we know from (3.3) that for each $k \in \{1, \ldots, s\}$, if $(x_k)_i = (x_k)_j$, $1 \le i, j \le m_k$, then

$$(g_k(x))_i = (g_k(x))_j \,, \tag{3.4}$$

and for each $k \in \{s_0+1, \ldots, s\}$, if $(x_k)_i = 0$, $1 \le i \le m_k$, then

$$(g_k(x))_i = 0 \,. \tag{3.5}$$

For the well-definiteness of G, it is sufficient to prove that for any given X, the function value $G(X)$ is independent of the choice of the orthogonal matrices $P_k \in \mathcal{O}^{m_k}(X_k)$, $1 \le k \le s_0$ and $(U_k, V_k) \in \mathcal{O}^{m_k,n_k}(X_k)$, $s_0+1 \le k \le s$. By using (3.4) and (3.5), we can prove this directly from Proposition 2.4 and Proposition 2.14. □

Next, consider the Moreau-Yosida regularization $\psi_{f,\eta} : \mathcal{X} \to \Re$ and the proximal point mapping $P_{f,\eta} : \mathcal{X} \to \mathcal{X}$ of the unitarily invariant closed proper convex function $f : \mathcal{X} \to (-\infty, \infty]$ with respect to $\eta > 0$, which are introduced in Section 1.2. Firstly, it is well-known [108, 25] (see e.g., [42]) that if the closed proper convex function $f : \mathcal{X} \to (-\infty, \infty]$ is unitarily invariant, then there exists a closed proper convex function $g : \Re^{m_0+m} \to (-\infty, \infty]$ such that for any $X \in \mathcal{X}$,

$$f(X) = (g \circ \kappa)(X) \,. \tag{3.6}$$

Moreover, it is easy to see that the closed proper convex function $g : \Re^{m_0+m} \to (-\infty, \infty]$ in (3.6) is invariant under permutations, i.e., for any $\boldsymbol{x} \in \Re^{m_0+m}$,

$$g(\boldsymbol{x}) = g(\boldsymbol{Q}\boldsymbol{x}) \quad \forall\, \boldsymbol{Q} \in \mathcal{Q}\,, \tag{3.7}$$

where the set $\mathcal{Q}$ is defined by (3.1). Since g is a closed proper convex function in $\Re^{m_0+m}$, we know that for the given $\eta > 0$, the Moreau-Yosida regularization $\psi_{g,\eta}$ and the proximal mapping $P_{g,\eta}$ of g with respect to η are well-defined. The relationship between $\psi_{f,\eta}$ and $\psi_{g,\eta}$ is established in the following proposition. Moreover, we show that the proximal point mapping $P_{f,\eta} : \mathcal{X} \to \mathcal{X}$ is the spectral operator with respect to the proximal point mapping $P_{g,\eta} : \Re^{m_0+m} \to \Re^{m_0+m}$.

Proposition 3.2. *Let $f : \mathcal{X} \to (-\infty, \infty]$ be a closed proper convex function. Let $\eta > 0$ be given. If f is unitarily invariant and $g : \Re^{m_0+m} \to (-\infty, \infty]$ is the closed proper convex function which satisfies the condition (3.6), then the Moreau-Yosida regularization function $\psi_{f,\eta}$ of f is also unitarily invariant. Moreover, for any $\boldsymbol{X} \in \mathcal{X}$, we have*

$$\psi_{f,\eta}(\boldsymbol{X}) = \psi_{g,\eta}(\boldsymbol{\kappa}(\boldsymbol{X}))\,. \tag{3.8}$$

Denote $\boldsymbol{G}(\boldsymbol{X}) := P_{f,\eta}(\boldsymbol{X})$, $\boldsymbol{X} \in \mathcal{X}$ and $\boldsymbol{g}(\boldsymbol{x}) := P_{g,\eta}(\boldsymbol{x})$, $\boldsymbol{x} \in \Re^{m_0+m}$. Then, the vector valued function $\boldsymbol{g}$ satisfies the condition

$$\boldsymbol{g}(\boldsymbol{x}) = \boldsymbol{Q}^T \boldsymbol{g}(\boldsymbol{Q}\boldsymbol{x}) \quad \forall\, \boldsymbol{Q} \in \mathcal{Q} \text{ and } \boldsymbol{x} \in \Re^{m_0+m}\,, \tag{3.9}$$

where $\mathcal{Q}$ is defined in (3.1). Furthermore, we have

$$\boldsymbol{G}(\boldsymbol{X}) = (\boldsymbol{G}_1(\boldsymbol{X}), \ldots, \boldsymbol{G}_s(\boldsymbol{X}))\,, \quad \boldsymbol{X} \in \mathcal{X}\,, \tag{3.10}$$

where

$$\boldsymbol{G}_k(\boldsymbol{X}) := \begin{cases} P_k \mathrm{diag}\big(\boldsymbol{g}_k(\boldsymbol{\kappa}(\boldsymbol{X}))\big) P_k^T & k = 1, \ldots, s_0\,, \\ U_k \left[\mathrm{diag}\big(\boldsymbol{g}_k(\boldsymbol{\kappa}(\boldsymbol{X}))\big) \;\; 0\right] V_k^T & k = s_0 + 1, \ldots, s\,, \end{cases}$$

and $P_k \in \mathcal{O}^{m_k}(\boldsymbol{X}_k)$, $1 \le k \le s_0$, $(U_k, V_k) \in \mathcal{O}^{m_k,n_k}(\boldsymbol{X}_k)$, $s_0 + 1 \le k \le s$, i.e.,

$$\boldsymbol{X}_k = \begin{cases} P_k \Lambda(\boldsymbol{X}_k) P_k^T & k = 1, \ldots, s_0\,, \\ U_k [\Sigma(\boldsymbol{X}_k) \;\; 0] V_k^T & k = s_0 + 1, \ldots, s\,. \end{cases}$$

Proof. From the definitions of $\psi_{f,\eta}$ and $P_{g,\eta}$, it is easy to see that $\psi_{f,\eta}$ is unitarily invariant and (3.9) holds. Next, we will show that both (3.8) and (3.10) hold.

Firstly, assume that $\boldsymbol{X} := (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{s_0}, \boldsymbol{X}_{s_0+1}, \ldots, \boldsymbol{X}_s) \in \mathcal{X}$ satisfies

$$\boldsymbol{X}_k = \begin{cases} \Lambda(\boldsymbol{X}_k) & k = 1, \ldots, s_0\,, \\ [\Sigma(\boldsymbol{X}_k) \;\; 0] & k = s_0 + 1, \ldots, s\,. \end{cases}$$

For any $\boldsymbol{Z} \in \mathcal{X}$, by considering the corresponding eigenvalue and single value decompositions of $\boldsymbol{Z}_k$, $k = 1, \ldots, s$, we have

$$\begin{aligned} f(\boldsymbol{Z}) + \frac{1}{2\eta}\|\boldsymbol{Z} - \boldsymbol{X}\|^2 &= (g \circ \boldsymbol{\kappa})(\boldsymbol{Z}) + \frac{1}{2\eta}\|\boldsymbol{Z} - \boldsymbol{X}\|^2 \\ = \quad & (g \circ \boldsymbol{\kappa})(\boldsymbol{Z}) + \frac{1}{2\eta}\sum_{k=1}^{s_0}\|\boldsymbol{Z}_k - \boldsymbol{X}_k\|^2 + \frac{1}{2\eta}\sum_{k=s_0+1}^{s}\|\boldsymbol{Z}_k - \boldsymbol{X}_k\|^2 \end{aligned}$$

For each $k \in \{1, \ldots, s_0\}$, by Ky Fan's inequality (Lemma 2.3), we know that

$$\|\boldsymbol{Z}_k - \boldsymbol{X}_k\| \geq \|\lambda(\boldsymbol{Z}_k) - \lambda(\boldsymbol{X}_k)\|\,.$$

Also, for each $k \in \{s_0 + 1, \ldots, s\}$, by von Neumann's trace inequality (Lemma 2.13), we have

$$\|\boldsymbol{Z}_k - \boldsymbol{X}_k\| \geq \|\sigma(\boldsymbol{Z}_k) - \sigma(\boldsymbol{X}_k)\|$$

Then, we know that

$$f(\boldsymbol{Z}) + \frac{1}{2\eta}\|\boldsymbol{Z} - \boldsymbol{X}\|^2 \geq g(\boldsymbol{\kappa}(\boldsymbol{Z})) + \frac{1}{2\eta}\|\boldsymbol{\kappa}(\boldsymbol{Z}) - \boldsymbol{\kappa}(\boldsymbol{X})\|^2 \quad \forall\, \boldsymbol{Z} \in \mathcal{X}\,,$$

which means that

$$\psi_{f,\eta}(\boldsymbol{X}) \geq \psi_{g,\eta}(\boldsymbol{\kappa}(\boldsymbol{X}))\,.$$

On the other hand, since $\boldsymbol{g} \equiv P_{g,\eta}$, if choose $\boldsymbol{Z}^* = \boldsymbol{d}iag(\boldsymbol{g}(\boldsymbol{\kappa}(\boldsymbol{X}))) \in \mathcal{X}$, i.e.,

$$\boldsymbol{Z}^* = (\boldsymbol{Z}_1^*, \ldots, \boldsymbol{Z}_s^*)$$

with

$$\boldsymbol{Z}_k^* = \begin{cases} \operatorname{diag}\big(\boldsymbol{g}_k(\boldsymbol{\kappa}(\boldsymbol{X}))\big) & k = 1, \ldots, s_0\,, \\ \big[\operatorname{diag}\big(\boldsymbol{g}_k(\boldsymbol{\kappa}(\boldsymbol{X}))\big) \;\; 0\big] & k = s_0 + 1, \ldots, s\,, \end{cases}$$

then, we have

$$f(\boldsymbol{Z}^*) + \frac{1}{2\eta}\|\boldsymbol{Z}^* - \boldsymbol{X}\|^2 = \psi_{g,\eta}(\boldsymbol{\kappa}(\boldsymbol{X}))\,.$$

Therefore, $\boldsymbol{Z}^*$ is one optimal solution of the following problem

$$\min_{\boldsymbol{Z}\in\mathcal{X}} \left\{ f(\boldsymbol{Z}) + \frac{1}{2\eta}\|\boldsymbol{Z} - \boldsymbol{X}\|^2 \right\}\,.$$

By the uniqueness of $P_{f,\eta}(\boldsymbol{X})$, we know that

$$P_{f,\eta}(\boldsymbol{X}) = \boldsymbol{Z}^* \quad \text{and} \quad \psi_{f,\eta}(\boldsymbol{X}) = \psi_{g,\eta}(\boldsymbol{\kappa}(\boldsymbol{X}))\,. \tag{3.11}$$

For the general $\boldsymbol{X} = (\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{s_0}, \boldsymbol{X}_{s_0+1}, \ldots, \boldsymbol{X}_s) \in \mathcal{X}$, let $P_k \in \mathcal{O}^{m_k}(\boldsymbol{X}_k)$, $1 \le k \le s_0$ and $(U_k, V_k) \in \mathcal{O}^{m_k,n_k}(\boldsymbol{X}_k)$, $s_0+1 \le k \le s$, i.e.,

$$\boldsymbol{X}_k = \begin{cases} P_k \Lambda(\boldsymbol{X}_k) P_k^T & k = 1, \ldots, s_0\,, \\ U_k[\Sigma(\boldsymbol{X}_k) \ \ 0]V_k^T & k = s_0+1, \ldots, s\,. \end{cases}$$

Define $\boldsymbol{D} := (\boldsymbol{D}_1, \ldots, \boldsymbol{D}_s) \in \mathcal{X}$ by

$$\boldsymbol{D}_k = \begin{cases} \Lambda(\boldsymbol{X}_k) & k = 1, \ldots, s_0\,, \\ [\Sigma(\boldsymbol{X}_k) \ \ 0] & k = s_0+1, \ldots, s\,. \end{cases}$$

Since $\psi_{f,\eta}$ is unitarily invariant, we know from (3.11) that

$$\psi_{f,\eta}(\boldsymbol{X}) = \psi_{f,\eta}(\boldsymbol{D}) = \psi_{g,\eta}(\boldsymbol{\kappa}(\boldsymbol{X}))\,.$$

Also, since f is unitarily invariant, we have for any $\boldsymbol{Z} \in \mathcal{X}$,

$$f(\boldsymbol{Z}) + \frac{1}{2\eta}\|\boldsymbol{Z} - \boldsymbol{X}\|^2 = f(\widetilde{\boldsymbol{Z}}) + \frac{1}{2\eta}\|\widetilde{\boldsymbol{Z}} - \boldsymbol{D}\|^2\,,$$

where $\widetilde{\boldsymbol{Z}} = (\widetilde{\boldsymbol{Z}}_1 \ldots, \widetilde{\boldsymbol{Z}}_s) \in \mathcal{X}$ satisfies

$$\widetilde{\boldsymbol{Z}}_k = \begin{cases} P_k^T \boldsymbol{Z}_k P_k & k = 1, \ldots, s_0\,, \\ U_k^T \boldsymbol{Z}_k V_k & k = s_0+1, \ldots, s\,. \end{cases}$$

Therefore, from (3.11), we know that

$$\boldsymbol{G}(\boldsymbol{X}) = P_{f,\eta}(\boldsymbol{X}) = \widetilde{P}_{f,\eta}(\boldsymbol{D}) = (\boldsymbol{G}_1(\boldsymbol{X}), \ldots, \boldsymbol{G}_s(\boldsymbol{X}))\ ,$$

where

$$\boldsymbol{G}_k(\boldsymbol{X}) := \begin{cases} P_k \operatorname{diag}\big(\boldsymbol{g}_k(\boldsymbol{\kappa}(\boldsymbol{X}))\big) P_k^T & k = 1, \ldots, s_0 \,, \\ U_k \left[\operatorname{diag}\big(\boldsymbol{g}_k(\boldsymbol{\kappa}(\boldsymbol{X}))\big) \;\; 0\right] V_k^T & k = s_0 + 1, \ldots, s \,. \end{cases}$$

The proof is completed. □

Next, we study several important properties of general spectral operators, including the well-definiteness, the directional differentiability, the differentiability, the locally Lipschitz continuity, the ρ-order B(ouligand)-differentiability ($0 < \rho \leq 1$), the ρ-order G-semismooth ($0 < \rho \leq 1$) and the characterization of Clarke's generalized Jacobian. Without loss of generality, from now on, we just consider the case that $\mathcal{X} = \mathcal{S}^{m_0} \times \Re^{m \times n}$. For any given $\boldsymbol{X} := (Y, Z) \in \mathcal{X}$, let $\boldsymbol{\kappa} := \boldsymbol{\kappa}(\boldsymbol{X}) = (\lambda(Y), \sigma(Z))$. Denote

$$\mathcal{I}_1 := \{1, \ldots, m_0\} \quad \text{and} \quad \mathcal{I}_2 := \{m_0 + 1, \ldots, m_0 + m\} \,.$$

Then, the given symmetric function $\boldsymbol{g} : \Re^{m_0+m} \to \Re^{m_0+m}$ can be written as

$$\boldsymbol{g}(\boldsymbol{x}) = (\boldsymbol{g}_1(\boldsymbol{x}), \boldsymbol{g}_2(\boldsymbol{x})), \quad \boldsymbol{x} \in \Re^{m_0+m} \,.$$

Define the matrices $\mathcal{A}(\boldsymbol{\kappa}) \in \mathcal{S}^{m_0}$, $\mathcal{E}_1(\boldsymbol{\kappa}), \mathcal{E}_2(\boldsymbol{\kappa}) \in \Re^{m \times m}$ and $\mathcal{F}(\boldsymbol{\kappa}) \in \Re^{m \times (n-m)}$ (depending on $\boldsymbol{X} \in \mathcal{X}$) by

$$(\mathcal{A}(\boldsymbol{\kappa}))_{ij} := \begin{cases} \dfrac{(\boldsymbol{g}_1(\boldsymbol{\kappa}))_i - (\boldsymbol{g}_1(\boldsymbol{\kappa}))_j}{\lambda_i(Y) - \lambda_j(Y)} & \text{if } \lambda_i(Y) \neq \lambda_j(Y) \,, \\ 0 & \text{otherwise} \,, \end{cases} \qquad i, j \in \{1, \ldots, m_0\} \,, \tag{3.12}$$

$$(\mathcal{E}_1(\boldsymbol{\kappa}))_{ij} := \begin{cases} \dfrac{(\boldsymbol{g}_2(\boldsymbol{\kappa}))_i - (\boldsymbol{g}_2(\boldsymbol{\kappa}))_j}{\sigma_i(Z) - \sigma_j(Z)} & \text{if } \sigma_i(Z) \neq \sigma_j(Z) \,, \\ 0 & \text{otherwise} \,, \end{cases} \qquad i, j \in \{1, \ldots, m\} \,, \tag{3.13}$$

$$(\mathcal{E}_2(\boldsymbol{\kappa}))_{ij} := \begin{cases} \dfrac{(\boldsymbol{g}_2(\boldsymbol{\kappa}))_i + (\boldsymbol{g}_2(\boldsymbol{\kappa}))_j}{\sigma_i(Z) + \sigma_j(Z)} & \text{if } \sigma_i(Z) + \sigma_j(Z) \neq 0 \,, \\ 0 & \text{otherwise} \,, \end{cases} \qquad i, j \in \{1, \ldots, m\} \,, \tag{3.14}$$

and

$$(\mathcal{F}(\boldsymbol{\kappa}))_{ij} := \begin{cases} \dfrac{(\boldsymbol{g}_2(\boldsymbol{\kappa}))_i}{\sigma_i(Z)} & \text{if } \sigma_i(Z) \neq 0\,, \\ 0 & \text{otherwise.} \end{cases} \qquad i \in \{1,\ldots,m\}, \quad j \in \{1,\ldots,n-m\}\,. \tag{3.15}$$

In later discussions, when the dependence of $\mathcal{A}(\boldsymbol{\kappa})$, $\mathcal{E}_1(\boldsymbol{\kappa})$, $\mathcal{E}_2(\boldsymbol{\kappa})$ and $\mathcal{F}(\boldsymbol{\kappa})$ on $\boldsymbol{X}$ can be seen clearly from the context, we often drop $\boldsymbol{\kappa}$ from these notations.

Let $\overline{\boldsymbol{X}} := (\overline{Y}, \overline{Z}) \in \mathcal{X}$ be given. Consider the eigenvalue decomposition (2.4) of $\overline{Y} \in \mathcal{S}^{m_0}$ and the singular value decomposition (2.24) of $\overline{Z} \in \Re^{m\times n}$, respectively, i.e.,

$$\overline{Y} = \overline{P}\Lambda(\overline{Y})\overline{P}^T \quad \text{and} \quad \overline{Z} = \overline{U}\left[\Sigma(\overline{Z}) \;\; 0\right]\overline{V}^T\,, \tag{3.16}$$

where $\overline{P} \in \mathcal{O}^{m_0}$, $\overline{U} \in \mathcal{O}^m$ and $\overline{V} = \left[\overline{V}_1 \;\; \overline{V}_2\right] \in \mathcal{O}^n$ with $\overline{V}_1 \in \Re^{n\times m}$ and $\overline{V}_2 \in \Re^{n\times(n-m)}$. Let

$$\overline{\boldsymbol{\kappa}} := \boldsymbol{\kappa}(\overline{\boldsymbol{X}}) = (\lambda(\overline{Y}), \sigma(\overline{Z})) \in \Re^{m_0} \times \Re^m\,.$$

We use $\overline{\mu}_1 > \ldots > \overline{\mu}_{r_0}$ to denote the distinct eigenvalues of $\overline{Y}$ and $\overline{\nu}_1 > \ldots > \overline{\nu}_r$ to denote the nonzero distinct singular values of $\overline{Z}$. Let α_k, $k = 1,\ldots,r_0$ be the index sets defined by (2.5) for $\overline{Y}$, and a, b, c, a_l, $l = 1,\ldots,r$ be the index sets defined by (2.25) and (2.26) for $\overline{Z}$. Denote $\bar{a} := \{1,\ldots,n\} \setminus a$. For notational convenience, define the index sets

$$\alpha_{r_0+l} := \{j \,|\, j = m_0 + i, i \in a_l\}, \quad l = 1,\ldots,r \quad \text{and} \quad \alpha_{r_0+r+1} := \{j \,|\, j = m_0 + i, i \in b\}\,. \tag{3.17}$$

Since $\boldsymbol{g}$ is symmetric, we may define the vector $\bar{g} \in \Re^{r_0+r+1}$ by

$$\bar{g}_k := \begin{cases} (\boldsymbol{g}_1(\overline{\boldsymbol{\kappa}}))_{i\in\alpha_k} & \text{if } 1 \leq k \leq r_0\,, \\ (\boldsymbol{g}_2(\overline{\boldsymbol{\kappa}}))_{i\in a_l} & \text{if } r_0 + 1 \leq k = r_0 + l \leq r_0 + r + 1\,. \end{cases}$$

Moreover, let $\overline{\mathcal{A}} \in \mathcal{S}^{m_0}$, $\overline{\mathcal{E}}_1$, $\overline{\mathcal{E}}_2 \in \Re^{m\times m}$ and $\overline{\mathcal{F}} \in \Re^{m\times(n-m)}$ be the matrices defined by (3.12)-(3.15) with respect to $\overline{\boldsymbol{X}}$. Hence, for the given $\overline{\boldsymbol{X}}$, define a linear operator $\boldsymbol{T} : \mathcal{X} \to \mathcal{X}$ by for any $\boldsymbol{Z} := (\boldsymbol{Z}_1, \boldsymbol{Z}_2) = (\boldsymbol{Z}_1, [\boldsymbol{Z}_{21} \;\; \boldsymbol{Z}_{22}]) \in \mathcal{X}$,

$$\boldsymbol{T}(\boldsymbol{Z}) := (\boldsymbol{T}_1(\boldsymbol{Z}_1), \boldsymbol{T}_2(\boldsymbol{Z}_2)) = \left(\overline{\mathcal{A}} \circ \boldsymbol{Z}_1, \left[\overline{\mathcal{E}}_1 \circ S(\boldsymbol{Z}_{21}) + \overline{\mathcal{E}}_2 \circ T(\boldsymbol{Z}_{21}) \;\; \overline{\mathcal{F}} \circ \boldsymbol{Z}_{22}\right]\right)\,. \tag{3.18}$$

For any $\boldsymbol{X} = (Y, Z) \in \mathcal{X}$, define

$$\boldsymbol{G}_S(\boldsymbol{X}) := \Big((\boldsymbol{G}_1)_S(Y), (\boldsymbol{G}_2)_S(Z)\Big) = \Big(\sum_{k=1}^{r_0} \bar{g}_k \mathcal{P}_k(Y), \sum_{l=1}^{r} \bar{g}_{r_0+l} \mathcal{U}_l(Z)\Big), \tag{3.19}$$

and

$$\boldsymbol{G}_R(\boldsymbol{X}) := \boldsymbol{G}(\boldsymbol{X}) - \boldsymbol{G}_S(\boldsymbol{X}), \tag{3.20}$$

where $\mathcal{P}_k(Y)$, $k = 1, \ldots, r_0$ and $\mathcal{U}_l(Z)$, $l = 1, \ldots, r$ are given by (2.21) and (2.52), respectively. Therefore, the following lemma follows from Proposition 2.12 and Proposition 2.17, directly.

Lemma 3.3. *Let $\boldsymbol{G}_S : \mathcal{X} \to \mathcal{X}$ be defined by (3.19). Then, there exists an open neighborhood $\mathcal{N}$ of $\overline{\boldsymbol{X}} = \big(\overline{Y}, \overline{Z}\big)$ in $\mathcal{X}$ such that $\boldsymbol{G}_S$ is twice continuously differentiable on $\mathcal{N}$, and for any $\mathcal{X} \ni \boldsymbol{H} = (A, B) \to 0$,*

$$\boldsymbol{G}_S(\overline{\boldsymbol{X}} + \boldsymbol{H}) - \boldsymbol{G}_S(\overline{\boldsymbol{X}}) = \boldsymbol{G}_S'(\overline{\boldsymbol{X}})\boldsymbol{H} + O(\|\boldsymbol{H}\|^2).$$

with

$$\begin{aligned} \boldsymbol{G}_S'(\overline{\boldsymbol{X}})\boldsymbol{H} &= \Big(\sum_{k=1}^{r_0} \bar{g}_k \mathcal{P}_k'(\overline{Y})A, \sum_{l=1}^{r} \bar{g}_{r_0+l} \mathcal{U}_l'(\overline{Z})B\Big) \\ &= \Big(\overline{\mathcal{A}} \circ \widetilde{A}, \Big[\overline{\mathcal{E}}_1 \circ S(\widetilde{B}_1) + \overline{\mathcal{E}}_2 \circ T(\widetilde{B}_1) \quad \overline{\mathcal{F}} \circ (\widetilde{B}_2)\Big]\Big) = \Big(\boldsymbol{T}_1(\widetilde{A}), \boldsymbol{T}_2(\widetilde{B})\Big) = \boldsymbol{T}(\widetilde{\boldsymbol{H}}), \end{aligned}$$

where $\widetilde{\boldsymbol{H}} = (\widetilde{A}, \widetilde{B})$, $\widetilde{A} = \overline{P}^T A \overline{P}$, $\widetilde{B} = \begin{bmatrix}\widetilde{B}_1 & \widetilde{B}_2\end{bmatrix} = \begin{bmatrix}\overline{U}^T B \overline{V}_1 & \overline{U}^T B \overline{V}_2\end{bmatrix}$; and the linear operator $\boldsymbol{T} : \mathcal{X} \to \mathcal{X}$ is defined in (3.18).

3.2 The directional differentiability

Firstly, if we assume that the symmetric function $\boldsymbol{g}$ is directionally differentiable at $\overline{\boldsymbol{\kappa}}$, then, from the definition of directional derivative of $\boldsymbol{g}$ at $\overline{\boldsymbol{\kappa}}$ and the condition (3.3), it is easy to see that the directional derivative $\phi := \boldsymbol{g}'(\overline{\boldsymbol{\kappa}}; \cdot) : \Re^{m_0+m} \to \Re^{m_0+m}$ satisfies

$$\phi(\boldsymbol{h}) = \boldsymbol{Q}^T \phi(\boldsymbol{Q}\boldsymbol{h}) \quad \forall\, \boldsymbol{Q} \in \mathcal{Q}_{\overline{\boldsymbol{\kappa}}} \quad \text{and} \quad \forall\, \boldsymbol{h} \in \Re^{m_0+m}, \tag{3.21}$$

where $\mathcal{Q}_{\overline{\boldsymbol{\kappa}}}$ is the subset defined for $\overline{\boldsymbol{\kappa}}$ in (3.2). Note that $\boldsymbol{Q} = (\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_{r_0+r}, \boldsymbol{Q}_{r_0+r+1}) \in \mathcal{Q}_{\overline{\boldsymbol{\kappa}}}$ if and only if $\boldsymbol{Q}_k \in \mathbb{P}^{|\alpha_k|}$, $1 \leq k \leq r_0$, $\boldsymbol{Q}_{r_0+l} \in \mathbb{P}^{|a_l|}$, $1 \leq l \leq r$ and $\boldsymbol{Q}_{r_0+r+1} \in |\mathbb{P}|^{|b|}$. For any $\boldsymbol{h} \in \Re^{m_0+m}$, write $\phi(\boldsymbol{h})$ as the form

$$\phi(\boldsymbol{h}) = (\phi_1(\boldsymbol{h}), \ldots, \phi_{r_0+r}(\boldsymbol{h}), \phi_{r_0+r+1}(\boldsymbol{h}))\ .$$

Denote the Euclidean space $\mathcal{W}$ by

$$\mathcal{W} := \mathcal{S}^{|\alpha_1|} \times \ldots \times \mathcal{S}^{|\alpha_{r_0}|} \times \mathcal{S}^{|a_1|} \times \ldots \times \mathcal{S}^{|a_r|} \times \Re^{|b| \times (n-|a|)}\ .$$

Let $\Phi : \mathcal{W} \to \mathcal{W}$ be the spectral operator with respect to the symmetric function ϕ, i.e., for any $\boldsymbol{W} = (\boldsymbol{W}_1, \ldots, \boldsymbol{W}_{r_0+r}, \boldsymbol{W}_{r_0+r+1}) \in \mathcal{W}$,

$$\Phi(\boldsymbol{W}) = \Big(\Phi_1(\boldsymbol{W}), \ldots, \Phi_{r_0+r}(\boldsymbol{W}), \Phi_{r_0+r+1}(\boldsymbol{W})\Big) \tag{3.22}$$

with

$$\Phi_k(\boldsymbol{W}) = \begin{cases} \widetilde{Q}_k \mathrm{diag}(\phi_k(\boldsymbol{\kappa}(\boldsymbol{W})))\widetilde{Q}_k^T & \text{if } 1 \leq k \leq r_0 + r, \\ \widetilde{M} \mathrm{diag}(\phi_{r_0+r+1}(\boldsymbol{\kappa}(\boldsymbol{W})))\widetilde{N}_1^T & \text{if } k = r_0 + r + 1, \end{cases} \quad k = 1, \ldots, r_0 + r + 1\,,$$

where $\boldsymbol{\kappa}(\boldsymbol{W}) = (\lambda(\boldsymbol{W}_1), \ldots, \lambda(\boldsymbol{W}_{r_0+r}), \sigma(\boldsymbol{W}_{r_0+r+1})) \in \Re^{m_0+m}$; $\widetilde{Q}_k \in \mathcal{O}^{|\alpha_k|}(\boldsymbol{W}_k)$, $1 \leq k \leq r_0$, $\widetilde{Q}_k \in \mathcal{O}^{|a_l|}(\boldsymbol{W}_{r_0+l})$, $r_0+1 \leq k = r_0+l \leq r_0+r$; and $(\widetilde{M}, \widetilde{N}) \in \mathcal{O}^{|b|, n-|a|}(\boldsymbol{W}_{r_0+r+1})$, $\widetilde{N} := \begin{bmatrix} \widetilde{N}_1 & \widetilde{N}_2 \end{bmatrix}$ with $\widetilde{N}_1 \in \Re^{(n-|a|) \times |b|}$, $\widetilde{N}_2 \in \Re^{(n-|a|) \times (n-m)}$. By Theorem 3.1, we know from (3.21) that the spectral operator $\Phi : \mathcal{W} \to \mathcal{W}$ is well-defined.

Define the first divided directional difference $\boldsymbol{g}^{[1]}(\overline{\boldsymbol{X}}; \widetilde{\boldsymbol{H}}) \in \mathcal{X}$ of $\boldsymbol{g}$ at $\overline{\boldsymbol{X}}$ along the direction $\boldsymbol{H} = (A, B) \in \mathcal{X}$ by

$$\boldsymbol{g}^{[1]}(\overline{\boldsymbol{X}}; \widetilde{\boldsymbol{H}}) := \Big(\boldsymbol{g}_1^{[1]}(\overline{\boldsymbol{X}}; \widetilde{\boldsymbol{H}}), \boldsymbol{g}_2^{[1]}(\overline{\boldsymbol{X}}; \widetilde{\boldsymbol{H}})\Big)\ ,$$

with

$$\boldsymbol{g}_1^{[1]}(\overline{\boldsymbol{X}}; \widetilde{\boldsymbol{H}}) = \boldsymbol{T}_1(\widetilde{A}) + \begin{bmatrix} \Phi_1(\boldsymbol{D}(\widetilde{\boldsymbol{H}})) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Phi_{r_0}(\boldsymbol{D}(\widetilde{\boldsymbol{H}})) \end{bmatrix} \in \mathcal{S}^{m_0} \tag{3.23}$$

and

$$\boldsymbol{g}_2^{[1]}(\overline{\boldsymbol{X}};\widetilde{\boldsymbol{H}}) = \boldsymbol{T}_2(\widetilde{B}) + \begin{bmatrix} \Phi_{r_0+1}(\boldsymbol{D}(\widetilde{\boldsymbol{H}})) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \Phi_{r_0+r}(\boldsymbol{D}(\widetilde{\boldsymbol{H}})) & 0 \\ 0 & \cdots & 0 & \Phi_{r_0+r+1}(\boldsymbol{D}(\widetilde{\boldsymbol{H}})) \end{bmatrix} \in \Re^{m\times n}, \tag{3.24}$$

where the linear operator $\boldsymbol{T} : \mathcal{X} \to \mathcal{X}$ is defined in (3.18),

$$\boldsymbol{D}(\widetilde{\boldsymbol{H}}) := \left(\widetilde{A}_{\alpha_1\alpha_1}, \ldots, \widetilde{A}_{\alpha_{r_0}\alpha_{r_0}}, S(\widetilde{B}_{a_1a_1}), \ldots, S(\widetilde{B}_{a_ra_r}), \widetilde{B}_{b\bar{a}}\right) \in \mathcal{W} \tag{3.25}$$

and $\widetilde{\boldsymbol{H}} = (\widetilde{A}, \widetilde{B}) = \left(\overline{P}^T A\overline{P}, [\overline{U}^T B\overline{V}_1 \;\; \overline{U}^T B\overline{V}_2]\right)$. Therefore, we have the following result on the directional differentiability of spectral operators.

Theorem 3.4. *Let $\overline{\boldsymbol{X}} = (\overline{Y}, \overline{Z}) \in \mathcal{S}^{m_0} \times \Re^{m\times n} = \mathcal{X}$ be given. Suppose that $\overline{Y}$ and $\overline{Z}$ have the decompositions (3.16). The spectral operator $\boldsymbol{G}$ is Hadamard directionally differentiable at $\overline{\boldsymbol{X}}$ if and only if the symmetric function $\boldsymbol{g}$ is Hadamard directionally differentiable at $\boldsymbol{\kappa}(\overline{\boldsymbol{X}})$. In particular, $\boldsymbol{G}$ is directionally differentiable at $\overline{\boldsymbol{X}}$ and the directional derivative at $\overline{\boldsymbol{X}}$ along any direction $\boldsymbol{H} \in \mathcal{X}$ is given by*

$$\boldsymbol{G}'(\overline{\boldsymbol{X}};\boldsymbol{H}) = \left(\overline{P}\boldsymbol{g}_1^{[1]}(\overline{\boldsymbol{X}};\widetilde{\boldsymbol{H}})\overline{P}^T, \overline{U}\boldsymbol{g}_2^{[1]}(\overline{\boldsymbol{X}};\widetilde{\boldsymbol{H}})\overline{V}^T\right). \tag{3.26}$$

Proof. " $\Longleftarrow$ " Let $\boldsymbol{H} = (A, B) \in \mathcal{X}$ be any given direction. For any $\mathcal{X} \ni \boldsymbol{H}' \to \boldsymbol{H}$ and $\tau > 0$, let $\boldsymbol{X} := \overline{\boldsymbol{X}} + \tau\boldsymbol{H}' = (\overline{Y} + \tau A', \overline{Z} + \tau B') = (Y, Z)$. Consider the eigenvalue decomposition of Y and the singular value decomposition of Z, i.e.,

$$Y = P\Lambda(Y)P^T \quad \text{and} \quad Z = U[\Sigma(Z) \;\; 0]V^T. \tag{3.27}$$

Denote $\boldsymbol{\kappa} := \boldsymbol{\kappa}(\boldsymbol{X})$. Let $\boldsymbol{G}_S$ and $\boldsymbol{G}_R$ be defined by (3.19) and (3.20), respectively. Therefore, by Lemma 3.3, we know that

$$\lim_{\substack{\tau\downarrow 0\\ \boldsymbol{H}'\to\boldsymbol{H}}} \frac{1}{\tau}(\boldsymbol{G}_S(\boldsymbol{X}) - \boldsymbol{G}_S(\overline{\boldsymbol{X}})) = \boldsymbol{G}_S'(\overline{\boldsymbol{X}})\boldsymbol{H} = \left(\boldsymbol{T}_1(\widetilde{A}), \boldsymbol{T}_2(\widetilde{B})\right) = \boldsymbol{T}(\widetilde{\boldsymbol{H}}), \tag{3.28}$$

where $\widetilde{\boldsymbol{H}} = (\widetilde{A}, \widetilde{B})$ with $\widetilde{A} = \overline{P}^T A \overline{P}$, $\widetilde{B} = \begin{bmatrix}\widetilde{B}_1 & \widetilde{B}_2\end{bmatrix} = \begin{bmatrix}\overline{U}^T B \overline{V}_1 & \overline{U}^T B \overline{V}_2\end{bmatrix}$, and the linear operator $\boldsymbol{T} : \mathcal{X} \to \mathcal{X}$ is given by (3.18).

On the other hand, for τ and $\boldsymbol{H}'$ sufficiently close to 0 and $\boldsymbol{H}$, we have $\mathcal{P}_k(Y) = \sum_{i\in\alpha_k} p_i p_i^T$, $k = 1,\ldots,r_0$ and $\mathcal{U}_l(Z) = \sum_{i\in a_l} u_i v_i^T$, $l = 1,\ldots,r$. Therefore, we know that

$$\begin{aligned}
&\boldsymbol{G}_R(\boldsymbol{X}) = \boldsymbol{G}(\boldsymbol{X}) - \boldsymbol{G}_S(\boldsymbol{X}) = \Big((\boldsymbol{G}_1)_R(\boldsymbol{X}), (\boldsymbol{G}_2)_R(\boldsymbol{X})\Big)\\
= \;& (\boldsymbol{G}_1(\boldsymbol{X}) - (\boldsymbol{G}_1)_S(Y), \boldsymbol{G}_2(\boldsymbol{X}) - (\boldsymbol{G}_2)_S(Z))\\
= \;& \left(\sum_{k=1}^{r_0}\sum_{i\in\alpha_k}[(\boldsymbol{g}_1(\boldsymbol{\kappa}))_i - (\boldsymbol{g}_1(\overline{\boldsymbol{\kappa}}))_i] p_i p_i^T,\right.\\
&\left.\sum_{l=1}^{r}\sum_{i\in a_l}[(\boldsymbol{g}_2(\boldsymbol{\kappa}))_i - (\boldsymbol{g}_2(\overline{\boldsymbol{\kappa}}))_i] u_i v_i^T + \sum_{i\in b}(\boldsymbol{g}_2(\boldsymbol{\kappa}))_i u_i v_i^T\right).
\end{aligned} \tag{3.29}$$

For any $\tau > 0$ and $\boldsymbol{H}'$, let

$$\Delta_k(\tau, \boldsymbol{H}') = \begin{cases} \dfrac{1}{\tau}\displaystyle\sum_{i\in\alpha_k}[(\boldsymbol{g}_1(\boldsymbol{\kappa}))_i - (\boldsymbol{g}_1(\overline{\boldsymbol{\kappa}}))_i] p_i p_i^T & \text{if } 1 \le k \le r_0,\\ \dfrac{1}{\tau}\displaystyle\sum_{i\in a_l}[(\boldsymbol{g}_2(\boldsymbol{\kappa}))_i - (\boldsymbol{g}_2(\overline{\boldsymbol{\kappa}}))_i] u_i v_i^T & \text{if } r_0 + 1 \le k = r_0 + l \le r_0 + r \end{cases}$$

and

$$\Delta_{r_0+r+1}(\tau, \boldsymbol{H}') = \sum_{i\in b}(\boldsymbol{g}_2(\boldsymbol{\kappa}))_i u_i v_i^T .$$

We first consider the case that $\overline{\boldsymbol{X}} = (\overline{Y}, \overline{Z}) = \big(\Lambda(\overline{Y}), [\Sigma(\overline{Z}) \;\; 0]\big)$. Then, from (2.14), (2.38) and (2.39), for any τ and $\boldsymbol{H}' \in \mathcal{X}$ sufficiently close to 0 and $\boldsymbol{H}$, we have

$$\lambda(Y) = \lambda(\overline{Y}) + \tau\lambda'(\overline{Y}; A') + O(\tau^2\|\boldsymbol{H}'\|^2) \quad \text{and} \quad \sigma(Z) = \sigma(\overline{Z}) + \tau\sigma'(\overline{Z}; B') + O(\tau^2\|\boldsymbol{H}'\|^2)\,, \tag{3.30}$$

where $\lambda'(\overline{Y}; A') = (\lambda(A'_{\alpha_1\alpha_1}), \ldots, \lambda(A'_{\alpha_{r_0}\alpha_{r_0}})) \in \Re^{m_0}$ and $\sigma'(\overline{Z}; B') \in \Re^m$ with

$$(\sigma'(\overline{Z}; B'))_{a_l} = \lambda(S(B'_{a_l a_l})),\; l = 1,\ldots,r \quad \text{and} \quad (\sigma'(\overline{Z}; B'))_b = \sigma([B'_{bb} \;\; B'_{bc}])\,.$$

Denote $\boldsymbol{h}' := (\lambda'(\overline{Y}; A'), \sigma'(\overline{Z}; B'))$ and $\boldsymbol{h} := (\lambda'(\overline{Y}; A), \sigma'(\overline{Z}; B))$. Since the functions $\lambda(\cdot)$ and $\sigma(\cdot)$ are globally Lipschitz continuous, we know that

$$\lim_{\substack{\tau\downarrow 0\\ \boldsymbol{H}'\to\boldsymbol{H}}} \boldsymbol{h}' + O(\tau\|\boldsymbol{H}'\|^2) = \boldsymbol{h}\,. \tag{3.31}$$

Since $\boldsymbol{g}$ is Hadamard directionally differentiable at $\overline{\boldsymbol{\kappa}}$, we know that

$$\lim_{\substack{\tau\downarrow 0\\ \boldsymbol{H}'\to \boldsymbol{H}}} \frac{1}{\tau}(\boldsymbol{g}(\boldsymbol{\kappa}(\boldsymbol{X}))-\boldsymbol{g}(\overline{\boldsymbol{\kappa}})) = \lim_{\substack{\tau\downarrow 0\\ \boldsymbol{H}'\to \boldsymbol{H}}} \frac{1}{\tau}[\boldsymbol{g}(\overline{\boldsymbol{\kappa}}+\tau(\boldsymbol{h}'+O(\tau\|\boldsymbol{H}'\|^2)))-\boldsymbol{g}(\overline{\boldsymbol{\kappa}})] = g'(\overline{\boldsymbol{\kappa}};\boldsymbol{h}) = \phi(\boldsymbol{h}),$$

where $\phi \equiv g'(\overline{\boldsymbol{\kappa}};\cdot) : \Re^{m_0+m} \to \Re^{m_0+m}$ satisfies (3.21). Since $p_ip_i^T$, $i = 1,\ldots,m_0$ and $u_iv_i^T$, $i = 1,\ldots,m$ are uniformly bounded, we know that for τ and $\boldsymbol{H}'$ sufficiently close to 0 and $\boldsymbol{H}$,

$$\Delta_k(\tau,\boldsymbol{H}') = \begin{cases} P_{\alpha_k}\mathrm{diag}(\phi_k(\boldsymbol{h}))P_{\alpha_k}^T + o(1) & \text{if } 1\le k\le r_0,\\ U_{a_l}\mathrm{diag}(\phi_k(\boldsymbol{h}))V_{a_l}^T + o(1) & \text{if } r_0+1\le k = r_0+l\le r_0+r \end{cases}$$

and

$$\Delta_{r_0+r+1}(\tau,\boldsymbol{H}') = U_b\mathrm{diag}(\phi_{r_0+r+1}(\boldsymbol{h}))V_b^T + o(1)\,.$$

By (2.10) and (2.12) in Proposition 2.5, we know that there exist $Q_k \in \mathcal{O}^{|\alpha_k|}$, $k = 1,\ldots,r_0$ and $Q_{r_0+l} \in \mathcal{O}^{|a_l|}$, $l = 1,\ldots,r$ (depending on τ and $\boldsymbol{H}'$) such that for each $i \in \alpha_k$,

$$P_{\alpha_k} = \begin{bmatrix} O(\tau\|\boldsymbol{H}'\|)\\ Q_k + O(\tau\|\boldsymbol{H}'\|)\\ O(\tau\|\boldsymbol{H}'\|)\end{bmatrix},\ k = 1,\ldots,r_0\,,$$

$$U_{a_l} = \begin{bmatrix} O(\tau\|\boldsymbol{H}'\|)\\ Q_{r_0+l} + O(\tau\|\boldsymbol{H}'\|)\\ O(\tau\|\boldsymbol{H}'\|)\end{bmatrix} \quad\text{and}\quad V_{a_l} = \begin{bmatrix} O(\tau\|\boldsymbol{H}'\|)\\ Q_{r_0+l} + O(\tau\|\boldsymbol{H}'\|)\\ O(\tau\|\boldsymbol{H}'\|)\end{bmatrix},\ l = 1,\ldots,r\,.$$

Therefore, we have

$$\begin{aligned}\Delta_k(\tau,\boldsymbol{H}') &= \begin{bmatrix} O(\tau^2\|\boldsymbol{H}'\|^2) & O(\tau\|\boldsymbol{H}'\|) & O(\tau^2\|\boldsymbol{H}'\|^2)\\ O(\tau\|\boldsymbol{H}'\|) & Q_k\mathrm{diag}(\phi_k(\boldsymbol{h}))Q_k^T + O(\tau\|\boldsymbol{H}'\|) & O(\tau\|\boldsymbol{H}'\|)\\ O(\tau^2\|\boldsymbol{H}'\|^2) & O(\tau\|\boldsymbol{H}'\|) & O(\tau^2\|\boldsymbol{H}'\|^2)\end{bmatrix} + o(1)\\ &= \begin{bmatrix} 0 & 0 & 0\\ 0 & Q_k\mathrm{diag}(\phi_k(\boldsymbol{h}))Q_k^T & 0\\ 0 & 0 & 0\end{bmatrix} + O(\tau\|\boldsymbol{H}'\|) + o(1), \quad 1\le k\le r_0+r\end{aligned} \tag{3.32}$$

Meanwhile, by (2.40), we know that there exist $M \in \mathcal{O}^{|b|}$ and $N = [N_1 \ \ N_2] \in \mathcal{O}^{n-|a|}$ with $N_1 \in \Re^{(n-|a|)\times|b|}$ and $N_2 \in \Re^{(n-|a|)\times(n-m)}$ such that

$$U_b = \begin{bmatrix} O(\tau\|\boldsymbol{H}'\|) \\ \\ M + O(\tau\|\boldsymbol{H}'\|) \end{bmatrix} \quad \text{and} \quad [V_b \ \ V_c] = \begin{bmatrix} O(\tau\|\boldsymbol{H}'\|) \\ \\ N + O(\tau\|\boldsymbol{H}'\|) \end{bmatrix}.$$

Therefore, we obtain that

$$\Delta_{r_0+r+1}(\tau, \boldsymbol{H}') = \begin{bmatrix} 0 & 0 \\ 0 & M\mathrm{diag}(\phi_{r_0+r+1}(\boldsymbol{h}))N_1^T \end{bmatrix} + O(\tau\|\boldsymbol{H}'\|) + o(1)\,. \tag{3.33}$$

On the other hand, from (2.13), we know that if $1 \le k \le r_0$,

$$A_{\alpha_k\alpha_k} + o(1) = A'_{\alpha_k\alpha_k} = \frac{1}{\tau}Q_k(\Lambda(Y)_{\alpha_k\alpha_k} - \overline{\mu}_k I_{|\alpha_k|})Q_k^T + O(\tau\|\boldsymbol{H}'\|^2)\,, \tag{3.34}$$

if $r_0 + 1 \le k = r_0 + l \le r_0 + r$,

$$S(B_{a_la_l}) + o(1) = S(B'_{a_la_l}) = \frac{1}{\tau}Q_{r_0+l}(\Sigma(Z)_{a_la_l} - \overline{\nu}_l I_{|a_l|})Q_{r_0+l}^T + O(\tau\|\boldsymbol{H}'\|^2) \tag{3.35}$$

and

$$[B_{bb} \ \ B_{bc}] + o(1) = [B'_{bb} \ \ B'_{bc}] = \frac{1}{\tau}M(\Sigma(Z)_{bb} - \overline{\nu}_{r+1} I_{|b|})N_1^T + O(\tau\|\boldsymbol{H}'\|^2)\,. \tag{3.36}$$

Since Q_k, $k = 1, \ldots, r_0 + r$, M and N are uniformly bounded, by taking a subsequence if necessary, we assume that when $\tau \downarrow 0$ and $\boldsymbol{H}' \to \boldsymbol{H}$, Q_k, $k = 1, \ldots, r_0 + r$, M and N converge to the orthogonal matrices $\widetilde{Q}_k$, $k = 1, \ldots, r_0 + r$, $\widetilde{M}$ and $\widetilde{N}$, respectively. Therefore, by taking limits in (3.34), (3.35) and (3.36), we obtain from (3.30) and (3.31) that

$$\begin{aligned} A_{\alpha_k\alpha_k} &= \widetilde{Q}_k\Lambda(A_{\alpha_k\alpha_k})\widetilde{Q}_k^T \qquad \text{if } 1 \le k \le r_0\,, \\ S(B_{a_la_l}) &= \widetilde{Q}_k\Lambda(S(B_{a_la_l}))\widetilde{Q}_k^T \qquad \text{if } r_0 + 1 \le k = r_0 + l \le r_0 + r \end{aligned}$$

and

$$[B_{bb} \ \ B_{bc}] = \widetilde{M}\,[\Sigma([B_{bb} \ \ B_{bc}]) \ \ 0]\,\widetilde{N}^T = \widetilde{M}\Sigma([B_{bb} \ \ B_{bc}])\widetilde{N}_1^T$$

Hence, by using the notation (3.22), we know from (3.32) and (3.33) that

$$\Upsilon_1(\boldsymbol{H}) = \lim_{\substack{\tau\downarrow 0\\ \boldsymbol{H}'\to\boldsymbol{H}}} \sum_{k=1}^{r_0} \Delta_k(\tau,\boldsymbol{H}') = \begin{bmatrix} \Phi_1(\boldsymbol{D}(\boldsymbol{H})) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Phi_{r_0}(\boldsymbol{D}(\boldsymbol{H})) \end{bmatrix} \in \mathcal{S}^{m_0}$$

and

$$\begin{aligned}\Upsilon_2(\boldsymbol{H}) &= \lim_{\substack{\tau\downarrow 0\\ \boldsymbol{H}'\to\boldsymbol{H}}} \sum_{k=r_0+1}^{r_0+r+1} \Delta_k(\tau,\boldsymbol{H}') \\ &= \begin{bmatrix} \Phi_{r_0+1}(\boldsymbol{D}(\boldsymbol{H})) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \Phi_{r_0+r}(\boldsymbol{D}(\boldsymbol{H})) & 0 \\ 0 & \cdots & 0 & \Phi_{r_0+r+1}(\boldsymbol{D}(\boldsymbol{H})) \end{bmatrix} \in \Re^{m\times n},\end{aligned}$$

where $\boldsymbol{D}(\boldsymbol{H}) = \big(A_{\alpha_1\alpha_1},\ldots,A_{\alpha_{r_0}\alpha_{r_0}}, S(B_{a_1a_1}),\ldots,S(B_{a_ra_r}), B_{b\bar{a}}\big)$. Therefore, by (3.29), we obtain that

$$\begin{aligned}\lim_{\substack{\tau\downarrow 0\\ \boldsymbol{H}'\to\boldsymbol{H}}} \frac{1}{\tau}\boldsymbol{G}_R(\boldsymbol{X}) &= \lim_{\substack{\tau\downarrow 0\\ \boldsymbol{H}'\to\boldsymbol{H}}} (\sum_{k=1}^{r_0}\Delta_k(\tau,\boldsymbol{H}'), \sum_{k=r_0+1}^{r_0+r+1}\Delta_k(\tau,\boldsymbol{H}')) \\ &= (\Upsilon_1(\boldsymbol{H}),\Upsilon_2(\boldsymbol{H}))\,. \qquad (3.37)\end{aligned}$$

Next, consider the general case for $\overline{\boldsymbol{X}} = (\overline{Y},\overline{Z}) \in \mathcal{X}$. For any $\mathcal{X} \ni \boldsymbol{H}' \to \boldsymbol{H}$ and $\tau > 0$, re-write (3.27) as

$$\Lambda(\overline{Y}) + \overline{P}^T A'\overline{P} = \overline{P}^T P\Lambda(Y)P^T\overline{P} \quad \text{and} \quad [\Sigma(\overline{Z}) \;\; 0] + \overline{U}^T B'\overline{V} = \overline{U}^T U[\Sigma(Z) \;\; 0]V^T\overline{V}\,.$$

Let $\widetilde{P} = \overline{P}^T P$, $\widetilde{U} := \overline{U}^T U$ and $\widetilde{V} := \overline{V}^T V$. Let $\widetilde{\boldsymbol{X}} := (\widetilde{Y},\widetilde{Z}) \in \mathcal{X}$ with

$$\widetilde{Y} := \Lambda(\overline{Y}) + \overline{P}^T A'\overline{P} \quad \text{and} \quad \widetilde{Z} := [\Sigma(\overline{Z}) \;\; 0] + \overline{U}^T B'\overline{V}\,.$$

Then, we have

$$\boldsymbol{G}_R(\boldsymbol{X}) = \Big(\overline{P}(\boldsymbol{G}_1)_R(\widetilde{\boldsymbol{X}})\overline{P}^T, \overline{U}(\boldsymbol{G}_2)_R(\widetilde{\boldsymbol{X}})\overline{V}^T\Big)\,.$$

Therefore, by (3.37), we know that

$$\lim_{\substack{\tau\downarrow 0\\ \boldsymbol{H}'\to\boldsymbol{H}}} \frac{1}{\tau}\boldsymbol{G}_R(\boldsymbol{X}) = \left(\overline{P}\Upsilon_1(\widetilde{\boldsymbol{H}})\overline{P}^T, \overline{U}\Upsilon_2(\widetilde{\boldsymbol{H}})\overline{V}^T\right). \tag{3.38}$$

Thus, by combining (3.28) and (3.38) and noting that $\boldsymbol{G}(\overline{\boldsymbol{X}}) = \boldsymbol{G}_S(\overline{\boldsymbol{X}})$, we obtain that for any given $\boldsymbol{H} \in \mathcal{X}$,

$$\begin{aligned}\lim_{\substack{\tau\downarrow 0\\ \boldsymbol{H}'\to\boldsymbol{H}}} \frac{1}{\tau}(\boldsymbol{G}(\boldsymbol{X}) - \boldsymbol{G}(\overline{\boldsymbol{X}})) &= \lim_{\substack{\tau\downarrow 0\\ \boldsymbol{H}'\to\boldsymbol{H}}} \frac{1}{\tau}(\boldsymbol{G}_S(\boldsymbol{X}) - \boldsymbol{G}_S(\overline{\boldsymbol{X}}) + \boldsymbol{G}_R(\boldsymbol{X})) \\ &= \left(\overline{P}[\boldsymbol{g}_1^{[1]}(\overline{\boldsymbol{X}};\widetilde{\boldsymbol{H}})]\overline{P}^T, \overline{U}[\boldsymbol{g}_2^{[1]}(\overline{\boldsymbol{X}};\widetilde{\boldsymbol{H}})]\overline{V}^T\right),\end{aligned}$$

where $\boldsymbol{g}_1^{[1]}(\overline{\boldsymbol{X}};\widetilde{\boldsymbol{H}})$ and $\boldsymbol{g}_2^{[1]}(\overline{\boldsymbol{X}};\widetilde{\boldsymbol{H}})$ are given by (3.23) and (3.24). This implies that $\boldsymbol{G}$ is Hadamard directionally differentiable at $\overline{\boldsymbol{X}}$ and (3.26) holds.

" $\Longrightarrow$ " Suppose that $\boldsymbol{G}$ is Hadamard directionally differentiable at $\overline{\boldsymbol{X}} = (\overline{Y}, \overline{Z})$. Let $\overline{P} \in \mathcal{O}^{m_0}(\overline{Y})$ and $(\overline{U}, \overline{V}) \in \mathcal{O}^{m\times n}(\overline{Z})$ be fixed. For any given direction $\boldsymbol{h} := (h_1, h_2) \in \Re^{m_0} \times \Re^m$, suppose that $\Re^{m_0} \times \Re^m \ni \boldsymbol{h}' = (h_1', h_2') \to h$. Let $\boldsymbol{H}' = (A', B') \in \mathcal{X}$ with $A' := \overline{P}\mathrm{diag}(h_1')\overline{P}^T$ and $B' := \overline{U}[\mathrm{diag}(h_2') \quad 0]\overline{V}^T$. Denote $A := \overline{P}\mathrm{diag}(h_1)\overline{P}^T$ and $B := \overline{U}[\mathrm{diag}(h_2) \quad 0]\overline{V}^T$. Then, we have $\boldsymbol{H}' \to \boldsymbol{H} := (A, B)$ as $\boldsymbol{h}' \to \boldsymbol{h}$. By the assumption, we know that

$$\begin{aligned}&\boldsymbol{G}'(\overline{\boldsymbol{X}};\boldsymbol{H}) = \lim_{\substack{\tau\downarrow 0\\ \boldsymbol{H}'\to\boldsymbol{H}}} \frac{1}{\tau}(\boldsymbol{G}(\overline{\boldsymbol{X}} + \tau\boldsymbol{H}') - \boldsymbol{G}(\overline{\boldsymbol{X}})) \\ &= \lim_{\substack{\tau\downarrow 0\\ \boldsymbol{h}'\to\boldsymbol{h}}} \frac{1}{\tau}\left(\overline{P}\mathrm{diag}(\boldsymbol{g}_1(\overline{\boldsymbol{\kappa}} + \tau\boldsymbol{h}') - \boldsymbol{g}_1(\overline{\boldsymbol{\kappa}}))\overline{P}^T, \overline{U}[\mathrm{diag}(\boldsymbol{g}_2(\overline{\boldsymbol{\kappa}} + \tau\boldsymbol{h}') - \boldsymbol{g}_2(\overline{\boldsymbol{\kappa}})) \quad 0]\overline{V}^T\right).\end{aligned}$$

This implies that $\boldsymbol{g}(\cdot) = (\boldsymbol{g}_1(\cdot), \boldsymbol{g}_2(\cdot)) : \Re^{m_0} \times \Re^m \to \Re^{m_0} \times \Re^m$ is Hadamard directionally differentiable at $\overline{\boldsymbol{\kappa}}$. Hence, the proof is completed. □

Remark 3.1. *Note that for general spectral operator $\boldsymbol{G}$, we can not obtain the directional differentiability at $\overline{\boldsymbol{X}}$ if we only assume that $\boldsymbol{g}$ is directionally differentiable at $\boldsymbol{\kappa}(\overline{X})$. In fact, for the case that $\mathcal{X} \equiv \mathcal{S}^{m_0}$, a counterexample can be found in [54]. However, since $\mathcal{X}$ is a finite dimensional Euclidean space, it is well-known that for locally Lipschitz continuous functions, the directional differentiability in sense of Hadamard and Gâteaux* *are equivalent (see e.g., [67, Theorem 1.13], [27, Lemma 3.2], [36, p.259]). Therefore, if the spectral operator* $\boldsymbol{G}$ *is locally Lipschitz continuous near* $\overline{\boldsymbol{X}}$ *(e.g., the proximal point mapping* $P_{f,\eta}$*), then* $\boldsymbol{G}$ *is directionally differentiable at* $\overline{\boldsymbol{X}}$ *if and only if the corresponding symmetric function* $\boldsymbol{g}$ *is directionally differentiable at* $\boldsymbol{\kappa}(\overline{\boldsymbol{X}})$.

3.3 The Fréchet differentiability

For any $\boldsymbol{X} = (Y, Z) \in \mathcal{X}$, let

$$\boldsymbol{\kappa} = (\lambda(Y), \sigma(Z)) \in \Re^{m_0+m} . \tag{3.39}$$

Suppose that the symmetric mapping $\boldsymbol{g}$ with respect to $\mathcal{X}$ is F-differentiable at $\boldsymbol{\kappa}$. Then, by using the symmetric property of $\boldsymbol{g}$, we obtain that the Jacobian matrix $\boldsymbol{g}'(\boldsymbol{\kappa})$ is symmetric and

$$\boldsymbol{g}'(\boldsymbol{\kappa})\boldsymbol{h} = \boldsymbol{Q}^T\boldsymbol{g}'(\boldsymbol{\kappa})\boldsymbol{Q}\boldsymbol{h} \quad \forall\, \boldsymbol{Q} \in \mathcal{Q}_{\boldsymbol{\kappa}} \quad \text{and} \quad \forall\, \boldsymbol{h} \in \Re^{m_0+m} . \tag{3.40}$$

Moreover, by using the block structure of $\boldsymbol{Q} \in \mathcal{Q}_{\boldsymbol{\kappa}}$, we can derive the following lemma easily.

Lemma 3.5. *For any* $\boldsymbol{X} \in \mathcal{X}$*, let* $\boldsymbol{\kappa}$ *be given by (3.39). Suppose that the function* $\boldsymbol{g}$ *is symmetric with respect to* $\mathcal{X}$ *and F-differentiable at* $\boldsymbol{\kappa}$*. Then, the Jacobian matrix* $\boldsymbol{g}'(\boldsymbol{\kappa}) \in \mathcal{S}^{m_0+m}$ *satisfies*

$$\begin{cases} (\boldsymbol{g}'(\boldsymbol{\kappa}))_{ii} = (\boldsymbol{g}'(\boldsymbol{\kappa}))_{i'i'} & \text{if } \boldsymbol{\kappa}_i = \boldsymbol{\kappa}_{i'}, \\ (\boldsymbol{g}'(\boldsymbol{\kappa}))_{ij} = (\boldsymbol{g}'(\boldsymbol{\kappa}))_{i'j'} & \text{if } \boldsymbol{\kappa}_i = \boldsymbol{\kappa}_{i'},\ \boldsymbol{\kappa}_j = \boldsymbol{\kappa}_{j'},\ i \neq j \text{ and } i' \neq j', \\ (\boldsymbol{g}'(\boldsymbol{\kappa}))_{ij} = (\boldsymbol{g}'(\boldsymbol{\kappa}))_{ji} = 0 & \text{if } \boldsymbol{\kappa}_i = 0,\ i \in \{m_0+1, \ldots, m_0+m\} \text{ and } i \neq j. \end{cases}$$

Define the matrices $\mathcal{A}^D(\boldsymbol{\kappa}) \in \mathcal{S}^{m_0}$, $\mathcal{E}_1^D(\boldsymbol{\kappa})$, $\mathcal{E}_2^D(\boldsymbol{\kappa}) \in \Re^{m\times m}$ and $\mathcal{F}^D(\boldsymbol{\kappa}) \in \Re^{m\times(n-m)}$ (depending on $\boldsymbol{X} \in \mathcal{X}$) by

$$(\mathcal{A}^D(\boldsymbol{\kappa}))_{ij} := \begin{cases} \dfrac{(\boldsymbol{g}_1(\boldsymbol{\kappa}))_i - (\boldsymbol{g}_1(\boldsymbol{\kappa}))_j}{\lambda_i(Y) - \lambda_j(Y)} & \text{if } \lambda_i(Y) \neq \lambda_j(Y) , \\ (\boldsymbol{g}'(\boldsymbol{\kappa}))_{ii} - (\boldsymbol{g}'(\boldsymbol{\kappa}))_{ij} & \text{otherwise} , \end{cases} \quad i,j \in \{1, \ldots, m_0\} , \tag{3.41}$$

$$(\mathcal{E}_1^D(\boldsymbol{\kappa}))_{ij} := \begin{cases} \dfrac{(\boldsymbol{g}_2(\boldsymbol{\kappa}))_i - (\boldsymbol{g}_2(\boldsymbol{\kappa}))_j}{\sigma_i(Z) - \sigma_j(Z)} & \text{if } \sigma_i(Z) \neq \sigma_j(Z)\,, \\ (\boldsymbol{g}'(\boldsymbol{\kappa}))_{ii} - (\boldsymbol{g}'(\boldsymbol{\kappa}))_{ij} & \text{otherwise}\,, \end{cases} \quad i,j \in \{1,\ldots,m\}\,, \tag{3.42}$$

$$(\mathcal{E}_2^D(\boldsymbol{\kappa}))_{ij} := \begin{cases} \dfrac{(\boldsymbol{g}_2(\boldsymbol{\kappa}))_i + (\boldsymbol{g}_2(\boldsymbol{\kappa}))_j}{\sigma_i(Z) + \sigma_j(Z)} & \text{if } \sigma_i(Z) + \sigma_j(Z) \neq 0\,, \\ (\boldsymbol{g}'(\boldsymbol{\kappa}))_{ii} - (\boldsymbol{g}'(\boldsymbol{\kappa}))_{ij} & \text{otherwise}\,, \end{cases} \quad i,j \in \{1,\ldots,m\}\,, \tag{3.43}$$

and

$$(\mathcal{F}^D(\boldsymbol{\kappa}))_{ij} := \begin{cases} \dfrac{(\boldsymbol{g}_2(\boldsymbol{\kappa}))_i}{\sigma_i(Z)} & \text{if } \sigma_i(Z) \neq 0\,, \\ (\boldsymbol{g}'(\boldsymbol{\kappa}))_{ii} - (\boldsymbol{g}'(\boldsymbol{\kappa}))_{ij} & \text{otherwise.} \end{cases} \quad i \in \{1,\ldots,m\}, \quad j \in \{1,\ldots,n{-}m\}\,. \tag{3.44}$$

In later discussions, when the dependence of $\mathcal{A}^D$, $\mathcal{E}_1^D$, $\mathcal{E}_2^D$ and $\mathcal{F}^D$ on $\boldsymbol{X}$ can be seen clearly from the context, we often drop $\boldsymbol{\kappa}$ from these notations.

Let $\overline{\boldsymbol{X}} \in \mathcal{X}$ be given. By Lemma 3.5, we know that the Jacobian matrix $\boldsymbol{g}'(\overline{\boldsymbol{\kappa}}) \in \mathcal{S}^{m_0+m}$ can be written as

$$\boldsymbol{g}'(\overline{\boldsymbol{\kappa}}) = \begin{bmatrix} \bar{c}_{11}E_{|\alpha_1||\alpha_1|} & \cdots & \bar{c}_{1(r_0+r)}E_{|\alpha_1||a_r|} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \bar{c}_{(r_0+r)1}E_{|a_r||\alpha_1|} & \cdots & \bar{c}_{(r_0+r)(r_0+r)}E_{|a_r||a_r|} & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix} + \begin{bmatrix} \bar{\eta}_1 I_{|\alpha_1|} & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \bar{\eta}_{r_0+r} I_{|a_r|} & 0 \\ 0 & \cdots & 0 & \bar{\eta}_{r_0+r+1} I_{|b|} \end{bmatrix}, \tag{3.45}$$

where $\bar{c} \in \mathcal{S}^{r_0+r}$ is a real symmetric matrix and $\bar{\eta} \in \Re^{r_0+r+1}$ is a real vector with the elements

$$\bar{\eta}_k = \begin{cases} (\boldsymbol{g}'(\overline{\boldsymbol{\kappa}}))_{ii} & \text{if } |\alpha_k| = 1,\ i \in \alpha_k, \\ (\boldsymbol{g}'(\overline{\boldsymbol{\kappa}}))_{ii} - (\boldsymbol{g}'(\overline{\boldsymbol{\kappa}}))_{ij} & \text{if } |\alpha_k| > 1, \text{ for any } i \neq j \in \alpha_k\,, \end{cases} \quad k = 1,\ldots,r_0 + r + 1\,. \tag{3.46}$$

Moreover, let $\overline{\mathcal{A}}^D \in \mathcal{S}^{m_0}$, $\overline{\mathcal{E}}_1^D, \overline{\mathcal{E}}_2^D \in \Re^{m\times m}$ and $\overline{\mathcal{F}}^D \in \Re^{m\times(n-m)}$ be the matrices defined in (3.41)-(3.44) with respect to $\overline{\boldsymbol{X}}$. Therefore, for the given $\overline{\boldsymbol{X}}$, define a linear operator $\boldsymbol{L}(\overline{\boldsymbol{\kappa}},\cdot) := (\boldsymbol{L}_1(\overline{\boldsymbol{\kappa}},\cdot), \boldsymbol{L}_2(\overline{\boldsymbol{\kappa}},\cdot)) : \mathcal{X} \to \mathcal{X}$ by

$$\boldsymbol{L}_1(\overline{\boldsymbol{\kappa}},\boldsymbol{Z}) := \begin{bmatrix} \theta_1(\overline{\boldsymbol{\kappa}},\boldsymbol{Z})I_{|\alpha_1|} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \theta_{r_0}(\overline{\boldsymbol{\kappa}},\boldsymbol{Z})I_{|\alpha_{r_0}|} \end{bmatrix} \in \mathcal{S}^{m_0} \tag{3.47a}$$

and

$$\boldsymbol{L}_2(\overline{\boldsymbol{\kappa}},\boldsymbol{Z}) := \begin{bmatrix} \theta_{r_0+1}(\overline{\boldsymbol{\kappa}},\boldsymbol{Z})I_{|a_1|} & \cdots & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & \theta_{r_0+r}(\overline{\boldsymbol{\kappa}},\boldsymbol{Z})I_{|a_r|} & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 \end{bmatrix} \in \Re^{m\times n}, \quad \boldsymbol{Z} = (A,B) \in \mathcal{X}, \tag{3.47b}$$

where $\theta_k(\overline{\boldsymbol{\kappa}},\cdot) : \mathcal{X} \to \Re$, $k = 1,\ldots,r_0+r$ are given by

$$\theta_k(\overline{\boldsymbol{\kappa}},\boldsymbol{Z}) := \sum_{k'=1}^{r_0} \bar{c}_{kk'}\operatorname{tr}(A_{\alpha_{k'}\alpha_{k'}}) + \sum_{k'=r_0+l=r_0+1}^{r_0+r} \bar{c}_{kk'}\operatorname{tr}(S(B_{a_l a_l}))\,. \tag{3.48}$$

For the given $\overline{\boldsymbol{X}}$, define a linear operator $\mathcal{T}(\overline{\boldsymbol{\kappa}},\cdot) : \Re^{m\times n} \to \Re^{m\times n}$ by

$$\mathcal{T}(\overline{\boldsymbol{\kappa}},B) := \left[\overline{\mathcal{E}}_1^D \circ S(B_1) + \overline{\mathcal{E}}_2^D \circ T(B_1) \quad \overline{\mathcal{F}}^D \circ B_2\right] \in \Re^{m\times n}, \quad B = [B_1 \quad B_2] \in \Re^{m\times n}\,. \tag{3.49}$$

Now, we are ready to state the result on the F-differentiability of spectral operators in the following theorem.

Theorem 3.6. *Let $\overline{\boldsymbol{X}} = (\overline{Y},\overline{Z}) \in \mathcal{S}^{m_0} \times \Re^{m\times n} = \mathcal{X}$ be given. Suppose that $\overline{Y}$ and $\overline{Z}$ have the decompositions (3.16). The spectral operator $\boldsymbol{G}$ is F-differentiable at $\overline{\boldsymbol{X}}$ if and only if the symmetric mapping $\boldsymbol{g}$ is F-differentiable at $\overline{\boldsymbol{\kappa}}$. In that case, the derivative of $\boldsymbol{G}$ at $\overline{\boldsymbol{X}}$ is given by for any $\boldsymbol{H} = (A,B) \in \mathcal{X}$,*

$$\boldsymbol{G}'(\overline{\boldsymbol{X}})\boldsymbol{H} = \left(\overline{P}[\boldsymbol{L}_1(\overline{\boldsymbol{\kappa}},\widetilde{\boldsymbol{H}}) + \overline{\mathcal{A}}^D \circ \widetilde{A}]\overline{P}^T, \overline{U}[\boldsymbol{L}_2(\overline{\boldsymbol{\kappa}},\widetilde{\boldsymbol{H}}) + \mathcal{T}(\overline{\boldsymbol{\kappa}},\widetilde{B})]\overline{V}^T\right)\,, \tag{3.50}$$

where $\widetilde{\boldsymbol{H}} = (\widetilde{A}, \widetilde{B}) = (\overline{P}^T A\overline{P}, \overline{U}^T B\overline{V})$*, and* $\boldsymbol{L}(\overline{\boldsymbol{\kappa}}, \cdot)$ *and* $\mathcal{T}(\overline{\boldsymbol{\kappa}}, \cdot)$ *are defined in (3.47) and (3.48), respectively.*

Proof. " $\Longleftarrow$ " For any $\boldsymbol{H} = (A, B) \in \mathcal{X}$, let $\boldsymbol{X} = \overline{\boldsymbol{X}} + \boldsymbol{H} = (\overline{Y} + A, \overline{Z} + B) = (Y, Z)$. Let $P \in \mathcal{O}^{m_0}$, $U \in \mathcal{O}^m$ and $V \in \mathcal{O}^n$ be such that

$$Y = P\Lambda(Y)P^T \quad \text{and} \quad Z = U[\Sigma(Z) \ \ 0]V^T . \tag{3.51}$$

Denote $\boldsymbol{\kappa} = \boldsymbol{\kappa}(\boldsymbol{X})$. Let $\boldsymbol{G}_S$ and $\boldsymbol{G}_R$ be defined by (3.19) and (3.20), respectively. Therefore, by Lemma 3.3, we know that for any $\mathcal{X} \ni \boldsymbol{H} \to 0$,

$$\boldsymbol{G}_S(\boldsymbol{X}) - \boldsymbol{G}_S(\overline{\boldsymbol{X}}) = \boldsymbol{G}_S'(\overline{\boldsymbol{X}})\boldsymbol{H} + O(\|\boldsymbol{H}\|^2) = \Big(\boldsymbol{T}_1(\widetilde{A}), \boldsymbol{T}_2(\widetilde{B})\Big) + O(\|\boldsymbol{H}\|^2) , \tag{3.52}$$

where $\widetilde{\boldsymbol{H}} = (\widetilde{A}, \widetilde{B})$ with $\widetilde{A} = \overline{P}^T A\overline{P}$, $\widetilde{B} = \Big[\widetilde{B}_1 \ \ \widetilde{B}_2\Big] = \Big[\overline{U}^T B\overline{V}_1 \ \ \overline{U}^T B\overline{V}_2\Big]$, and the linear operator $\boldsymbol{T}(\cdot) = (\boldsymbol{T}_1(\cdot), \boldsymbol{T}_2(\cdot)) : \mathcal{X} \to \mathcal{X}$ is given by (3.18).

On the other hand, for $\boldsymbol{H} \in \mathcal{X}$ sufficiently close to zero, we have $\mathcal{P}_k(Y) = \sum_{i \in \alpha_k} p_i p_i^T$, $k = 1, \ldots, r_0$ and $\mathcal{U}_l(Z) = \sum_{i \in a_l} u_i v_i^T$, $l = 1, \ldots, r$. Therefore, we know that

$$\begin{aligned} & \boldsymbol{G}_R(\boldsymbol{X}) = \boldsymbol{G}(\boldsymbol{X}) - \boldsymbol{G}_S(\boldsymbol{X}) \\ = \ & ((\boldsymbol{G}_1)_R(\boldsymbol{X}), (\boldsymbol{G}_2)_R(\boldsymbol{X})) = (\boldsymbol{G}_1(\boldsymbol{X}) - (\boldsymbol{G}_1)_S(Y), \boldsymbol{G}_2(\boldsymbol{X}) - (\boldsymbol{G}_2)_S(Z)) \\ = \ & \left(\sum_{k=1}^{r_0} \Delta_k(\boldsymbol{H}), \sum_{k=r_0+1}^{r_0+r+1} \Delta_k(\boldsymbol{H}) \right) , \end{aligned} \tag{3.53}$$

where

$$\Delta_k(\boldsymbol{H}) = \begin{cases} \sum_{i \in \alpha_k} [(\boldsymbol{g}_1(\boldsymbol{\kappa}))_i - (\boldsymbol{g}_1(\overline{\boldsymbol{\kappa}}))_i] p_i p_i^T & \text{if } 1 \le k \le r_0, \\ \sum_{i \in a_l} [(\boldsymbol{g}_2(\boldsymbol{\kappa}))_i - (\boldsymbol{g}_2(\overline{\boldsymbol{\kappa}}))_i] u_i v_i^T & \text{if } r_0 + 1 \le k = r_0 + l \le r_0 + r \end{cases}$$

and

$$\Delta_{r_0+r+1}(\boldsymbol{H}) = \sum_{i \in b} (\boldsymbol{g}_2(\boldsymbol{\kappa}))_i u_i v_i^T .$$

Firstly, we consider the case that $\overline{\boldsymbol{X}} = (\overline{Y}, \overline{Z}) = \big(\Lambda(\overline{Y}), [\Sigma(\overline{Z}) \ \ 0]\big)$. Then, from (2.14), (2.38) and (2.39), for any $\boldsymbol{H} \in \mathcal{X}$ sufficiently close to 0, we know that

$$\boldsymbol{\kappa} = \boldsymbol{\kappa}(\boldsymbol{X}) = \overline{\boldsymbol{\kappa}} + \boldsymbol{h} + O(\|\boldsymbol{H}\|^2) , \tag{3.54}$$

where $\boldsymbol{h} := (\lambda'(\overline{Y}; A), \sigma'(\overline{Z}; B)) \in \Re^{m_0} \times \Re^m$ with $(\lambda'(\overline{Y}; A))_{\alpha_k} = \lambda(A_{\alpha_k \alpha_k})$, $k = 1, \ldots, r_0$,

$$(\sigma'(\overline{Z}; B))_{a_l} = \lambda(S(B_{a_l a_l})), \ l = 1, \ldots, r \quad \text{and} \quad (\sigma'(\overline{Z}; B))_b = \sigma([B_{bb} \ \ B_{bc}]) .$$

Since $\boldsymbol{g}$ is F-differentiable at $\overline{\boldsymbol{\kappa}}$, we know that for any $\boldsymbol{H} \in \mathcal{X}$ sufficiently close to 0,

$$\begin{aligned} \boldsymbol{g}(\boldsymbol{\kappa}) - \boldsymbol{g}(\overline{\boldsymbol{\kappa}}) &= \boldsymbol{g}(\boldsymbol{\kappa} + \boldsymbol{h} + O(\|\boldsymbol{H}\|^2)) - g(\overline{\boldsymbol{\kappa}}) \\ &= \boldsymbol{g}'(\overline{\boldsymbol{\kappa}})(\boldsymbol{h} + O(\|\boldsymbol{H}\|^2)) + o(\|\boldsymbol{h}\|) \\ &= \boldsymbol{g}'(\overline{\boldsymbol{\kappa}})\boldsymbol{h} + o(\|\boldsymbol{H}\|) . \end{aligned}$$

Since $p_i p_i^T$, $i = 1, \ldots, m_0$ and $u_i v_i^T$, $i = 1, \ldots, m$ are uniformly bounded, we know that for $\boldsymbol{H}$ sufficiently close to 0,

$$\Delta_k(\boldsymbol{H}) = \begin{cases} P_{\alpha_k} \mathrm{diag}((\boldsymbol{g}'(\overline{\boldsymbol{\kappa}})\boldsymbol{h})_{\alpha_k}) P_{\alpha_k}^T + o(\|\boldsymbol{H}\|) & \text{if } 1 \leq k \leq r_0, \\ U_{a_l} \mathrm{diag}((\boldsymbol{g}'(\overline{\boldsymbol{\kappa}})\boldsymbol{h})_{\alpha_k}) V_{a_l}^T + o(\|\boldsymbol{H}\|) & \text{if } r_0 + 1 \leq k = r_0 + l \leq r_0 + r \end{cases}$$

and

$$\Delta_{r_0+r+1}(\boldsymbol{H}) = U_b \mathrm{diag}((\boldsymbol{g}'(\overline{\boldsymbol{\kappa}})\boldsymbol{h})_{\alpha_{r_0+r+1}}) V_b^T + o(\|\boldsymbol{H}\|) .$$

By (2.10) and (2.12) in Proposition 2.5, we know that there exist $Q_k \in \mathcal{O}^{|\alpha_k|}$, $k = 1, \ldots, r_0$ and $Q_{r_0+l} \in \mathcal{O}^{|a_l|}$, $l = 1, \ldots, r$ (depending on $\boldsymbol{H}$) such that for each $i \in \alpha_k$,

$$P_{\alpha_k} = \begin{bmatrix} O(\|\boldsymbol{H}\|) \\ Q_k + O(\|\boldsymbol{H}\|) \\ O(\|\boldsymbol{H}\|) \end{bmatrix}, \ k = 1, \ldots, r_0 ,$$

$$U_{a_l} = \begin{bmatrix} O(\|\boldsymbol{H}\|) \\ Q_{r_0+l} + O(\|\boldsymbol{H}\|) \\ O(\|\boldsymbol{H}\|) \end{bmatrix} \quad \text{and} \quad V_{a_l} = \begin{bmatrix} O(\|\boldsymbol{H}\|) \\ Q_{r_0+l} + O(\|\boldsymbol{H}\|) \\ O(\|\boldsymbol{H}\|) \end{bmatrix}, \ l = 1, \ldots, r .$$

Therefore, since $\|\boldsymbol{g}'(\overline{\boldsymbol{\kappa}})\boldsymbol{h}\| = O(\|\boldsymbol{H}\|)$, we obtain that

$$\Delta_k(\boldsymbol{H}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & Q_k \mathrm{diag}((\boldsymbol{g}'(\overline{\boldsymbol{\kappa}})\boldsymbol{h})_{\alpha_k}) Q_k^T & 0 \\ 0 & 0 & 0 \end{bmatrix} + o(\|\boldsymbol{H}\|), \quad 1 \leq k \leq r_0 + r . \tag{3.55}$$

Meanwhile, by (2.40), we know that there exist $M \in \mathcal{O}^{|b|}$ and $N = [N_1 \ \ N_2] \in \mathcal{O}^{n-|a|}$ (depending on $\boldsymbol{H}$) with $N_1 \in \Re^{(n-|a|)\times|b|}$ and $N_2 \in \Re^{(n-|a|)\times(n-m)}$ such that

$$U_b = \begin{bmatrix} O(\|\boldsymbol{H}\|) \\ \\ M + O(\|\boldsymbol{H}\|) \end{bmatrix} \quad \text{and} \quad [V_b \ \ V_c] = \begin{bmatrix} O(\|\boldsymbol{H}\|) \\ \\ N + O(\|\boldsymbol{H}\|) \end{bmatrix}.$$

Therefore, we obtain that

$$\Delta_{r_0+r+1}(\boldsymbol{H}) = \begin{bmatrix} 0 & 0 \\ 0 & M\mathrm{diag}((\boldsymbol{g}'(\overline{\boldsymbol{\kappa}})\boldsymbol{h})_{\alpha_{r_0+r+1}})N_1^T \end{bmatrix} + o(\|\boldsymbol{H}\|)\,. \tag{3.56}$$

By (3.45), we know that

$$\big(\boldsymbol{g}'(\overline{\boldsymbol{\kappa}})\boldsymbol{h}\big)_{\alpha_k} = \begin{cases} \theta_k(\overline{\boldsymbol{\kappa}}, \boldsymbol{H})e_{|\alpha_k|} + \bar{\eta}_k\lambda(A_{\alpha_k\alpha_k}) & \text{if } 1 \le k \le r_0 + r, \\ \bar{\eta}_{r_0+r+1}\sigma([B_{bb} \ \ B_{bc}]) & \text{if } k = r_0 + r + 1\,, \end{cases}$$

where $\theta_k(\overline{\boldsymbol{\kappa}}, \cdot) : \mathcal{X} \to \Re$, $k = 1, \ldots, r_0 + r$ are given by (3.48). On the other hand, from (2.13), (2.41) and (2.42), we know that for $\boldsymbol{H}$ sufficiently close to 0,

$$\begin{aligned} A_{\alpha_k\alpha_k} &= Q_k(\Lambda(Y)_{\alpha_k\alpha_k} - \overline{\mu}_k I_{|\alpha_k|})Q_k^T + O(\|\boldsymbol{H}\|^2) \\ &= Q_k\Lambda(A_{\alpha_k\alpha_k})Q_k^T + O(\|\boldsymbol{H}\|^2), \quad 1 \le k \le r_0\,, \\ S(B_{a_la_l}) &= Q_k(\Sigma(Z)_{a_la_l} - \overline{\nu}_l I_{|a_l|})Q_k^T + O(\|\boldsymbol{H}\|^2) \\ &= Q_k\Lambda(S(B_{a_la_l}))Q_k^T + O(\|\boldsymbol{H}\|^2), \quad r_0 + 1 \le k = r_0 + l \le r_0 + r\,, \\ [B_{bb} \ \ B_{bc}] &= M(\Sigma(Z)_{bb} - \overline{\nu}_{r+1}I_{|b|})N_1^T + O(\|\boldsymbol{H}\|^2) \\ &= M\Sigma([B_{bb} \ \ B_{bc}])N_1 + O(\|\boldsymbol{H}\|^2)\,. \end{aligned}$$

Therefore, from (3.55) and (3.56), we obtain that

$$\Delta_k(\boldsymbol{H}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \theta_k(\overline{\boldsymbol{\kappa}}, \boldsymbol{H})I_{|\alpha_k|} + \bar{\eta}_k A_{\alpha_k\alpha_k} & 0 \\ 0 & 0 & 0 \end{bmatrix} + o(\|\boldsymbol{H}\|), \quad 1 \le k \le r_0\,,$$

$$\Delta_k(\boldsymbol{H}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \theta_k(\overline{\boldsymbol{\kappa}}, \boldsymbol{H})I_{|a_l|} + \bar{\eta}_k S(B_{a_la_l}) & 0 \\ 0 & 0 & 0 \end{bmatrix} + o(\|\boldsymbol{H}\|), \quad r_0+1 \le k = r_0+l \le r_0+r\,,$$

$$\Delta_{r_0+r+1}(\boldsymbol{H}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \bar{\eta}_{r_0+r+1}B_{bb} & \bar{\eta}_{r_0+r+1}B_{bc} \end{bmatrix} + o(\|\boldsymbol{H}\|)\,.$$

Thus, from (3.85), we have for any $\boldsymbol{H}$ sufficiently close to 0,

$$\begin{aligned} &\boldsymbol{G}_R(\boldsymbol{X}) \\ = &\left(\boldsymbol{L}_1(\overline{\boldsymbol{\kappa}},\boldsymbol{H}) + \begin{bmatrix} \bar{\eta}_1 A_{\alpha_1\alpha_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \bar{\eta}_{r_0}A_{\alpha_{r_0}\alpha_{r_0}} \end{bmatrix}, \boldsymbol{L}_2(\overline{\boldsymbol{\kappa}},\boldsymbol{H}) \right. \\ &\left. + \begin{bmatrix} \bar{\eta}_{r_0+1}S(B_{a_1a_1}) & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & \bar{\eta}_{r_0+r}S(B_{a_ra_r}) & 0 & 0 \\ 0 & 0 & 0 & \bar{\eta}_{r_0+r+1}B_{bb} & \bar{\eta}_{r_0+r+1}B_{bc} \end{bmatrix} \right) + o(\|\boldsymbol{H}\|)\,, \end{aligned} \tag{3.57}$$

where the linear operator $\boldsymbol{L}(\overline{\boldsymbol{\kappa}},\cdot) := (\boldsymbol{L}_1(\overline{\boldsymbol{\kappa}},\cdot),\boldsymbol{L}_2(\overline{\boldsymbol{\kappa}},\cdot)) : \mathcal{X} \to \mathcal{X}$ is given by (3.47).

Next, consider the general $\overline{\boldsymbol{X}} = (\overline{Y},\overline{Z}) \in \mathcal{X}$. For any $\boldsymbol{H} \in \mathcal{X}$, re-write (3.51) as

$$\Lambda(\overline{Y}) + \overline{P}^T A'\overline{P} = \overline{P}^T P\Lambda(Y)P^T\overline{P} \quad \text{and} \quad [\Sigma(\overline{Z}) \;\; 0] + \overline{U}^T B'\overline{V} = \overline{U}^T U[\Sigma(Z) \;\; 0]V^T\overline{V}\,.$$

Let $\widetilde{P} = \overline{P}^T P$, $\widetilde{U} := \overline{U}^T U$ and $\widetilde{V} := \overline{V}^T V$. Let $\widetilde{\boldsymbol{X}} := (\widetilde{Y},\widetilde{Z}) \in \mathcal{X}$ with

$$\widetilde{Y} := \Lambda(\overline{Y}) + \overline{P}^T A'\overline{P} \quad \text{and} \quad \widetilde{Z} := [\Sigma(\overline{Z}) \;\; 0] + \overline{U}^T B'\overline{V}\,.$$

Then, since $\overline{P}$, $\overline{U}$ and $\overline{V}$ are bounded, we know from (3.92) that

$$
\begin{aligned}
&\boldsymbol{G}_R(\boldsymbol{X}) \\
= \; & \left(\overline{P}(\boldsymbol{G}_1)_R(\widetilde{\boldsymbol{X}})\overline{P}^T, \overline{U}(\boldsymbol{G}_2)_R(\widetilde{\boldsymbol{X}})\overline{V}^T\right) + o(\|\boldsymbol{H}\|)\,. \\
= \; & \left(\boldsymbol{L}_1(\overline{\boldsymbol{\kappa}}, \widetilde{\boldsymbol{H}}) + \begin{bmatrix} \overline{\eta}_1 \widetilde{A}_{\alpha_1\alpha_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \overline{\eta}_{r_0}\widetilde{A}_{\alpha_{r_0}\alpha_{r_0}} \end{bmatrix}, \boldsymbol{L}_2(\overline{\boldsymbol{\kappa}}, \widetilde{\boldsymbol{H}}) \right. \\
& \left. + \begin{bmatrix} \overline{\eta}_{r_0+1} S(\widetilde{B}_{a_1a_1}) & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & \overline{\eta}_{r_0+r} S(\widetilde{B}_{a_ra_r}) & 0 & 0 \\ 0 & 0 & 0 & \overline{\eta}_{r_0+r+1}\widetilde{B}_{bb} & \overline{\eta}_{r_0+r+1}\widetilde{B}_{bc} \end{bmatrix} \right) + o(\|\boldsymbol{H}\|)\,,
\end{aligned}
\tag{3.58}
$$

Thus, by combining (3.52) and (3.58) and noting that $\boldsymbol{G}(\overline{\boldsymbol{X}}) = \boldsymbol{G}_S(\overline{\boldsymbol{X}})$, we obtain that for any $\boldsymbol{H} \in \mathcal{X}$ sufficiently close to 0,

$$
\boldsymbol{G}(\boldsymbol{X}) - \boldsymbol{G}(\overline{\boldsymbol{X}}) = \left(\overline{P}[\boldsymbol{L}_1(\overline{\boldsymbol{\kappa}}, \widetilde{\boldsymbol{H}}) + \overline{\mathcal{A}}^D \circ \widetilde{A}]\overline{P}^T, \overline{U}[\boldsymbol{L}_2(\overline{\boldsymbol{\kappa}}, \widetilde{\boldsymbol{H}}) + \mathcal{T}(\overline{\boldsymbol{\kappa}}, \widetilde{B})]\overline{V}^T\right) + o(\|\boldsymbol{H}\|)\,.
$$

Therefore, we know that $\boldsymbol{G}$ is F-differentiable at $\overline{\boldsymbol{X}}$ and (3.50) holds.

" $\Longrightarrow$ " Let $\overline{P} \in \mathcal{O}^{m_0}(\overline{Y})$ and $(\overline{U}, \overline{V}) \in \mathcal{O}^{m\times n}(\overline{Z})$ be fixed. For any $\boldsymbol{h} := (h_1, h_2) \in \Re^{m_0} \times \Re^m$, let $\boldsymbol{H} = (A, B) \in \mathcal{X}$, where $A := \overline{P}\text{diag}(h_1)\overline{P}^T$ and $B := \overline{U}[\text{diag}(h_2) \;\; 0]\overline{V}^T$. Then, by the assumption, we know that for $\boldsymbol{h}$ sufficiently close to 0,

$$
\begin{aligned}
& \left(\overline{P}\text{diag}(\boldsymbol{g}_1(\overline{\boldsymbol{\kappa}} + \boldsymbol{h}) - \boldsymbol{g}_1(\overline{\boldsymbol{\kappa}}))\overline{P}^T, \overline{U}\text{diag}(\boldsymbol{g}_2(\overline{\boldsymbol{\kappa}} + \boldsymbol{h}) - \boldsymbol{g}_2(\overline{\boldsymbol{\kappa}}))\overline{V}_1^T\right) \\
= \; & \boldsymbol{G}(\overline{\boldsymbol{X}} + \boldsymbol{H}) - \boldsymbol{G}(\overline{\boldsymbol{X}}) = \boldsymbol{G}'(\overline{\boldsymbol{X}})\boldsymbol{H} + o(\|\boldsymbol{H}\|)\,.
\end{aligned}
$$

Hence, for $\boldsymbol{h}$ sufficiently close to 0,

$$
\begin{aligned}
\boldsymbol{g}(\overline{\boldsymbol{\kappa}} + \boldsymbol{h}) - \boldsymbol{g}(\overline{\boldsymbol{\kappa}}) &= (\boldsymbol{g}_1(\overline{\boldsymbol{\kappa}} + \boldsymbol{h}) - \boldsymbol{g}_1(\overline{\boldsymbol{\kappa}}), \boldsymbol{g}_2(\overline{\boldsymbol{\kappa}} + \boldsymbol{h}) - \boldsymbol{g}_2(\overline{\boldsymbol{\kappa}})) \\
&= \boldsymbol{g}'(\overline{\boldsymbol{\kappa}})\boldsymbol{h} + o(\|\boldsymbol{h}\|)\,.
\end{aligned}
$$

The proof is competed. □

Remark 3.2. *It is easy to see that the formula (3.50) is independent of the choice of the orthogonal matrices* $\overline{P} \in \mathcal{O}^{m_0}(\overline{Y})$ *and* $(\overline{U}, \overline{V}) \in \mathcal{O}^{m,n}(\overline{Z})$ *in (3.16).*

Finally, let us consider the continuous differentiability of spectral operators as follows.

Theorem 3.7. *Let* $\overline{\boldsymbol{X}} = (\overline{Y}, \overline{Z}) \in \mathcal{S}^{m_0} \times \Re^{m\times n} = \mathcal{X}$ *be given. Suppose that* $\overline{Y}$ *and* $\overline{Z}$ *have the decompositions (3.16). The spectral operator* $\boldsymbol{G}$ *is continuously differentiable at* $\overline{\boldsymbol{X}}$ *if and only if the symmetric mapping* $\boldsymbol{g}$ *is continuously differentiable at* $\boldsymbol{\kappa}(\overline{\boldsymbol{X}})$*.*

Proof. " $\Longleftarrow$ " By the assumption, we know from Theroem 3.6 that there exists an open neighborhood $\mathcal{N}$ of $\overline{\boldsymbol{X}}$ such that $\boldsymbol{G}$ is differentiable on $\mathcal{N}$, and for any $\boldsymbol{X} := (Y, Z) \in \mathcal{N}$, the derivative of $\boldsymbol{G}$ at $\boldsymbol{X}$ is given by

$$\boldsymbol{G}'(\boldsymbol{X})\boldsymbol{H} = \Big(P[\boldsymbol{L}_1(\boldsymbol{\kappa}, \widehat{\boldsymbol{H}}) + \mathcal{A}^D \circ \widehat{A}]P^T, U[\boldsymbol{L}_2(\boldsymbol{\kappa}, \widehat{\boldsymbol{H}}) + \mathcal{T}(\boldsymbol{\kappa}, \widehat{B})]V^T\Big), \quad \boldsymbol{H} = (A, B) \in \mathcal{X}, \tag{3.59}$$

where $P \in \mathcal{O}^{m_0}$, $U \in \mathcal{O}^m$ and $V \in \mathcal{O}^n$ satisfy

$$Y = P\Lambda(Y)P^T \quad \text{and} \quad Z = U[\Sigma(Z) \ \ 0]V^T,$$

$\boldsymbol{\kappa} = (\lambda(Y), \sigma(Z)) \in \Re^{m_0} \times \Re^m$, $\widehat{\boldsymbol{H}} = (\widehat{A}, \widehat{B}) = (P^TAP, U^TBV)$, and $\boldsymbol{L}(\overline{\boldsymbol{\kappa}}, \cdot)$ and $\mathcal{T}(\overline{\boldsymbol{\kappa}}, \cdot)$ are defined in (3.47) and (3.48) with respect to $\boldsymbol{X}$. We shall prove that

$$\lim_{\boldsymbol{X} \to \overline{\boldsymbol{X}}} \boldsymbol{G}'(\boldsymbol{X})\boldsymbol{H} \to G'(\overline{\boldsymbol{X}})\boldsymbol{H} \quad \forall\, \boldsymbol{H} \in \mathcal{X}. \tag{3.60}$$

Firstly, we will show that (3.60) holds for the special case that $\overline{\boldsymbol{X}} = (\Lambda(\overline{Y}), [\Sigma(\overline{Z}) \ \ 0])$ and $\boldsymbol{X} = (\Lambda(Y), [\Sigma(Z) \ \ 0])$. In this case, we may assume that $\overline{P} = P \equiv I_{m_0}$, $\overline{U} = U \equiv I_m$ and $\overline{V} = V \equiv I_n$. Let $\{\boldsymbol{E}^{(ij)}\} \cup \{\boldsymbol{F}^{(ij)}\}$ be the standard basis of $\mathcal{X}$, i.e.,

$$\boldsymbol{E}^{(ij)} = (E^{(ij)}, \boldsymbol{0}), \quad 1 \le i \le j \le m_0 \quad \text{and} \quad \boldsymbol{F}^{(ij)} = (\boldsymbol{0}, F^{(ij)}), \quad 1 \le i \le m, \ 1 \le j \le n, \tag{3.61}$$

where for each $1 \le i \le j \le m_0$, $E^{(ij)} \in \mathcal{S}^{m_0}$ is a matrix whose entries are zeros, except the (i, j)-th and (j, i)-th entries are ones; For each $1 \le i \le m$, $1 \le j \le n$, $F^{(ij)} \in \Re^{m\times n}$ is a matrix whose entries are zeros, except the (i, j)-th entry is one. Therefore, we only need to show (3.60) holds for all $E^{(ij)}$ and $F^{(ij)}$. Since $\lambda(\cdot)$ and $\sigma(\cdot)$ are globally Lipchitz continuous, we know that for $\boldsymbol{X}$ sufficiently close to $\overline{\boldsymbol{X}}$,

$$\begin{cases} \lambda_i(Y) \neq \lambda_j(Y) & \text{if } i \in \alpha_k,\ j \in \alpha_{k'} \text{ and } 1 \leq k \neq k' \leq r_0, \\ \sigma_i(Z) \neq \sigma_j(Z) & \text{if } i \in a_l,\ j \in a_{l'} \text{ and } 1 \leq l \neq l' \leq r+1. \end{cases}$$

Without loss of generality, we only prove (3.60) holds for any $\boldsymbol{F}^{(ij)}$, $1 \leq i \leq m$, $1 \leq j \leq n$. Write $F^{(ij)}$ as the form

$$F^{(ij)} = \begin{bmatrix} F_1^{(ij)} & F_2^{(ij)} \end{bmatrix}$$

with $F_1^{(ij)} \in \Re^{m\times m}$ and $F_2^{(ij)} \in \Re^{m\times(n-m)}$. Next, we consider the following several cases.

Case 1: $1 \leq i = j \leq m$. In this case, since $\boldsymbol{g}'$ is continuous at $\overline{\boldsymbol{\kappa}}$, we know that

$$\lim_{\boldsymbol{X}\to\overline{\boldsymbol{X}}} \boldsymbol{G}'(\boldsymbol{X})\boldsymbol{F}^{(ij)} = \lim_{\boldsymbol{X}\to\overline{\boldsymbol{X}}} \left(\boldsymbol{0}, \begin{bmatrix}\mathrm{diag}(\boldsymbol{g}'(\boldsymbol{\kappa})e_i) & 0\end{bmatrix}\right) = \left(\boldsymbol{0}, \begin{bmatrix}\mathrm{diag}(\boldsymbol{g}'(\overline{\boldsymbol{\kappa}})e_i) & 0\end{bmatrix}\right) = \boldsymbol{G}'(\overline{\boldsymbol{X}})\boldsymbol{F}^{(ij)}\,,$$

where for each $1 \leq i \leq m$, e_i is a vector whose entries are zeros, except the i-th entry is one.

Case 2: $1 \leq i \neq j \leq m$, $\sigma_i(Z) = \sigma_j(Z)$ and $\sigma_i(\overline{Z}) = \sigma_j(\overline{Z}) > 0$. Therefore, we know that there exists $l \in \{1, \ldots, r\}$ such that $i, j \in a_l$. Since $\boldsymbol{g}'$ is continuous at $\overline{\boldsymbol{\kappa}}$, we know from (3.46) that

$$\begin{aligned} & \lim_{\boldsymbol{X}\to\overline{\boldsymbol{X}}} \boldsymbol{G}'(\boldsymbol{X})\boldsymbol{F}^{(ij)} \\ = & \lim_{\boldsymbol{X}\to\overline{\boldsymbol{X}}} \left(\boldsymbol{0}, \begin{bmatrix}\left((\boldsymbol{g}'(\boldsymbol{\kappa}))_{ii} - (\boldsymbol{g}'(\boldsymbol{\kappa}))_{ij}\right) S(F_1^{(ij)}) + \dfrac{\boldsymbol{g}_i(\boldsymbol{\kappa}) + \boldsymbol{g}_j(\boldsymbol{\kappa})}{\sigma_i(Z) + \sigma_j(Z)} T(F_1^{(ij)}) & 0\end{bmatrix}\right) \\ = & \left(\boldsymbol{0}, \begin{bmatrix}\left((\boldsymbol{g}'(\overline{\boldsymbol{\kappa}}))_{ii} - (\boldsymbol{g}'(\overline{\boldsymbol{\kappa}}))_{ij}\right) S(F_1^{(ij)}) + \dfrac{\boldsymbol{g}_i(\overline{\boldsymbol{\kappa}}) + \boldsymbol{g}_j(\overline{\boldsymbol{\kappa}})}{\sigma_i(\overline{Z}) + \sigma_j(\overline{Z})} T(F_1^{(ij)}) & 0\end{bmatrix}\right) \\ = & \boldsymbol{G}'(\overline{\boldsymbol{X}})\boldsymbol{F}^{(ij)}\,. \end{aligned}$$

Case 3: $1 \leq i \neq j \leq m$ and $\sigma_i(Z) \neq \sigma_j(Z)$ but $\sigma_i(\overline{Z}) = \sigma_j(\overline{Z}) > 0$. In this case, we know that

$$\boldsymbol{G}'(\boldsymbol{X})\boldsymbol{F}^{(ij)} = \left(\boldsymbol{0}, \begin{bmatrix}\dfrac{\boldsymbol{g}_i(\boldsymbol{\kappa}) - \boldsymbol{g}_j(\boldsymbol{\kappa})}{\sigma_i(Z) - \sigma_j(Z)} S(F_1^{(ij)}) + \dfrac{\boldsymbol{g}_i(\boldsymbol{\kappa}) + \boldsymbol{g}_j(\boldsymbol{\kappa})}{\sigma_i(Z) + \sigma_j(Z)} T(F_1^{(ij)}) & 0\end{bmatrix}\right)$$

and

$$G'(\overline{X})F^{(ij)} = \left(\mathbf{0}, \left[\left((g'(\overline{\kappa}))_{ii} - (g'(\overline{\kappa}))_{ij}\right) S(F_1^{(ij)}) + \frac{g_i(\overline{\kappa}) + g_j(\overline{\kappa})}{\sigma_i(\overline{Z}) + \sigma_j(\overline{Z})} T(F_1^{(ij)}) \;\; 0\right]\right) .$$

Let $s, t \in \Re^m$ be two vectors defined by

$$s_p := \begin{cases} \sigma_p(Z) & \text{if } p \neq i, \\ \sigma_j(Z) & \text{if } p = i \end{cases} \quad \text{and} \quad t_p := \begin{cases} \sigma_p(Z) & \text{if } p \neq i, j, \\ \sigma_j(Z) & \text{if } p = i\,, \\ \sigma_i(Z) & \text{if } p = j\,, \end{cases} \qquad p = 1, \ldots, m\,.$$

Define $\boldsymbol{s}, \boldsymbol{t} \in \Re^{m_0} \times \Re^m$ as follows

$$\boldsymbol{s} := (\lambda(Y), s) \quad \text{and} \quad \boldsymbol{t} := (\lambda(Y), t)\,. \tag{3.62}$$

It is clear that both $\boldsymbol{s}$ and $\boldsymbol{t}$ converge to $\overline{\kappa}$ as $X \to \overline{X}$. By noting that g is symmetric, we know from (3.1) that $g_i(\boldsymbol{t}) = g_j(\kappa)$, since the vector t is obtained from $\sigma(Z)$ by swapping the i-th and the j-th components. By the mean value theorem, we have

$$\begin{aligned} \frac{g_i(\kappa) - g_j(\kappa)}{\sigma_i(Z) - \sigma_j(Z)} &= \frac{g_i(\kappa) - g_i(\boldsymbol{s}) + g_i(\boldsymbol{s}) - g_j(\kappa)}{\sigma_i(Z) - \sigma_j(Z)} \\ &= \frac{\dfrac{\partial g_i(\xi)}{\partial \mu_i}(\sigma_i(Z) - \sigma_j(Z)) + g_i(\boldsymbol{s}) - g_j(\kappa)}{\sigma_i(Z) - \sigma_j(Z)} \\ &= \frac{\partial g_i(\xi)}{\partial \mu_i} + \frac{g_i(\boldsymbol{s}) - g_i(\boldsymbol{t}) + g_i(\boldsymbol{t}) - g_j(\kappa)}{\sigma_i(Z) - \sigma_j(Z)} \\ &= \frac{\partial g_i(\xi)}{\partial \mu_i} + \frac{\dfrac{\partial g_i(\hat{\xi})}{\partial \mu_j}(\sigma_j(Z) - \sigma_i(Z)) + g_i(\boldsymbol{t}) - g_j(\kappa)}{\sigma_i(Z) - \sigma_j(Z)} \\ &= \frac{\partial g_i(\xi)}{\partial \mu_i} - \frac{\partial g_i(\hat{\xi})}{\partial \mu_j}\,, \end{aligned} \tag{3.63}$$

where $\xi \in \Re^{m_0} \times \Re^m$ lies between κ and $\boldsymbol{s}$ and $\hat{\xi} \in \Re^{m_0} \times \Re^m$ is between $\boldsymbol{s}$ and $\boldsymbol{t}$. Consequently, we have $\xi \to \overline{\kappa}$ and $\widehat{\xi} \to \overline{\kappa}$ as $X \to \overline{X}$. By the continuity of g', we know that

$$\lim_{X \to \overline{X}} \frac{g_i(\kappa) - g_j(\kappa)}{\sigma_i(Z) - \sigma_j(Z)} = (g'(\overline{\kappa}))_{ii} - (g'(\overline{\kappa}))_{ij}\,.$$

Therefore, we have

$$\lim_{X \to \overline{X}} G'(X)F^{(ij)} = G'(\overline{X})F^{(ij)}\,.$$

Case 4: $1 \leq i \neq j \leq m$ and $\sigma_i(\overline{Z}) > 0$ or $\sigma_j(\overline{Z}) > 0$ and $\sigma_i(\overline{Z}) \neq \sigma_j(\overline{Z})$. Then, we have $\sigma_i(Z) > 0$ or $\sigma_j(Z) > 0$ and $\sigma_i(Z) \neq \sigma_j(Z)$. Since $\boldsymbol{g}'$ is continuous at $\overline{\boldsymbol{\kappa}}$, we know that

$$\begin{aligned}
\lim_{\boldsymbol{X}\to\overline{\boldsymbol{X}}} \boldsymbol{G}'(\boldsymbol{X})\boldsymbol{F}^{(ij)} &= \lim_{\boldsymbol{X}\to\overline{\boldsymbol{X}}} \left(\boldsymbol{0}, \left[\frac{\boldsymbol{g}_i(\boldsymbol{\kappa}) - \boldsymbol{g}_j(\boldsymbol{\kappa})}{\sigma_i(Z) - \sigma_j(Z)} S(F_1^{(ij)}) + \frac{\boldsymbol{g}_i(\boldsymbol{\kappa}) + \boldsymbol{g}_j(\boldsymbol{\kappa})}{\sigma_i(Z) + \sigma_j(Z)} T(F_1^{(ij)}) \ \ 0\right]\right) \\
&= \left(\boldsymbol{0}, \left[\frac{\boldsymbol{g}_i(\overline{\boldsymbol{\kappa}}) - \boldsymbol{g}_j(\overline{\boldsymbol{\kappa}})}{\sigma_i(\overline{Z}) - \sigma_j(\overline{Z})} S(F_1^{(ij)}) + \frac{\boldsymbol{g}_i(\overline{\boldsymbol{\kappa}}) + \boldsymbol{g}_j(\overline{\boldsymbol{\kappa}})}{\sigma_i(\overline{Z}) + \sigma_j(\overline{Z})} T(F_1^{(ij)}) \ \ 0\right]\right) \\
&= \boldsymbol{G}'(\overline{\boldsymbol{X}})\boldsymbol{F}^{(ij)} .
\end{aligned}$$

Case 5: $m+1 \leq j \leq n$, $\sigma_i(\overline{Z}) > 0$. Since $\boldsymbol{g}'$ is continuous at $\overline{\boldsymbol{\kappa}}$, we obtain that

$$\lim_{\boldsymbol{X}\to\overline{\boldsymbol{X}}} \boldsymbol{G}'(\boldsymbol{X})\boldsymbol{F}^{(ij)} = \lim_{\boldsymbol{X}\to\overline{\boldsymbol{X}}} \left(\boldsymbol{0}, \left[0 \ \ \frac{\boldsymbol{g}_i(\boldsymbol{\kappa})}{\sigma_i(Z)} F_2^{(ij)}\right]\right) = \left(\boldsymbol{0}, \left[0 \ \ \frac{\boldsymbol{g}_i(\overline{\boldsymbol{\kappa}})}{\sigma_i(\overline{Z})} F_2^{(ij)}\right]\right) = \boldsymbol{G}'(\overline{\boldsymbol{X}})\boldsymbol{F}^{(ij)} .$$

Case 6: $1 \leq i \neq j \leq m$, $\sigma_i(\overline{Z}) = \sigma_j(\overline{Z}) = 0$ and $\sigma_i(Z) = \sigma_j(Z) > 0$. Therefore, we know that

$$\boldsymbol{G}'(\boldsymbol{X})\boldsymbol{F}^{(ij)} = \left(\boldsymbol{0}, \left[\left((\boldsymbol{g}'(\boldsymbol{\kappa}))_{ii} - (\boldsymbol{g}'(\boldsymbol{\kappa}))_{ij}\right) S(F_1^{(ij)}) + \frac{\boldsymbol{g}_i(\boldsymbol{\kappa}) + \boldsymbol{g}_j(\boldsymbol{\kappa})}{\sigma_i(Z) + \sigma_j(Z)} T(F_1^{(ij)}) \ \ 0\right]\right) .$$

Since $\boldsymbol{g}'$ is continuous, we know from (3.45) that

$$\lim_{\boldsymbol{X}\to\overline{\boldsymbol{X}}} (\boldsymbol{g}'(\boldsymbol{\kappa}))_{ii} = (\boldsymbol{g}'(\overline{\boldsymbol{\kappa}}))_{ii} = \bar{\eta}_{r_0+r+1} \quad \text{and} \quad \lim_{\boldsymbol{X}\to\overline{\boldsymbol{X}}} (\boldsymbol{g}'(\boldsymbol{\kappa}))_{ij} \to 0 . \tag{3.64}$$

Let $\hat{s}, \hat{t} \in \Re^m$ be two vectors defined by

$$\hat{s}_p := \begin{cases} \sigma_p(Z) & \text{if } p \neq i, \\ -\sigma_j(Z) & \text{if } p = i \end{cases} \quad \text{and} \quad \hat{t}_p := \begin{cases} \sigma_p(Z) & \text{if } p \neq i, j, \\ -\sigma_j(Z) & \text{if } p = i\,, \\ -\sigma_i(Z) & \text{if } p = j\,, \end{cases} \quad p = 1, \ldots, m .$$

Define $\hat{\boldsymbol{s}}, \hat{\boldsymbol{t}} \in \Re^{m_0} \times \Re^m$ as follows

$$\hat{\boldsymbol{s}} := (\lambda(Y), \hat{s}) \quad \text{and} \quad \hat{\boldsymbol{t}} := (\lambda(Y), \hat{t}\,) . \tag{3.65}$$

Also, it clear that both $\hat{\boldsymbol{s}}$ and $\hat{\boldsymbol{t}}$ converge to $\overline{\boldsymbol{\kappa}}$ as $\boldsymbol{X} \to \overline{\boldsymbol{X}}$. Again, by noting that $\boldsymbol{g}$ is (mixed) symmetric, we know from (3.1) that

$$\boldsymbol{g}_j(\boldsymbol{\kappa}) = -\boldsymbol{g}_i(\hat{\boldsymbol{t}}) \quad \text{and} \quad \boldsymbol{g}_i(\boldsymbol{\kappa}) = -\boldsymbol{g}_j(\hat{\boldsymbol{t}}) .$$

By the mean value theorem, we have

$$\begin{aligned}
\frac{\boldsymbol{g}_i(\boldsymbol{\kappa})+\boldsymbol{g}_j(\boldsymbol{\kappa})}{\sigma_i(Z)+\sigma_j(Z)} &= \frac{\boldsymbol{g}_i(\boldsymbol{\kappa})-\boldsymbol{g}_i(\hat{\boldsymbol{s}})+\boldsymbol{g}_i(\hat{\boldsymbol{s}})+\boldsymbol{g}_j(\boldsymbol{\kappa})}{\sigma_i(Z)+\sigma_j(Z)} \\
&= \frac{\dfrac{\partial \boldsymbol{g}_i(\zeta)}{\partial \mu_i}(\sigma_i(Z)+\sigma_j(Z))+\boldsymbol{g}_i(\hat{\boldsymbol{s}})+\boldsymbol{g}_j(\boldsymbol{\kappa})}{\sigma_i(Z)+\sigma_j(Z)} \\
&= \frac{\partial \boldsymbol{g}_i(\zeta)}{\partial \mu_i}+\frac{\boldsymbol{g}_i(\hat{\boldsymbol{s}})-\boldsymbol{g}_i(\hat{\boldsymbol{t}})+\boldsymbol{g}_i(\hat{\boldsymbol{t}})+\boldsymbol{g}_j(\boldsymbol{\kappa})}{\sigma_i(Z)+\sigma_j(Z)} \\
&= \frac{\partial \boldsymbol{g}_i(\zeta)}{\partial \mu_i}+\frac{\dfrac{\partial \boldsymbol{g}_i(\hat{\zeta})}{\partial \mu_j}(\sigma_j(Z)+\sigma_i(Z))+\boldsymbol{g}_i(\hat{\boldsymbol{t}})+\boldsymbol{g}_j(\boldsymbol{\kappa})}{\sigma_i(Z)+\sigma_j(Z)} \\
&= \frac{\partial \boldsymbol{g}_i(\zeta)}{\partial \mu_i}+\frac{\partial \boldsymbol{g}_i(\hat{\zeta})}{\partial \mu_j}\,, \qquad (3.66)
\end{aligned}$$

where $\zeta \in \Re^{m_0}\times\Re^m$ is between $\boldsymbol{\kappa}$ and $\hat{\boldsymbol{s}}$ and $\hat{\zeta}\in\Re^{m_0}\times\Re^m$ is between $\hat{\boldsymbol{s}}$ and $\hat{\boldsymbol{t}}$. Consequently, we know that $\zeta,\hat{\zeta}\to\overline{\boldsymbol{\kappa}}$ as $\boldsymbol{X}\to\overline{\boldsymbol{X}}$. By the continuity of $\boldsymbol{g}'$, we know from (3.45) that

$$\lim_{\boldsymbol{X}\to\overline{\boldsymbol{X}}}\frac{\boldsymbol{g}_i(\boldsymbol{\kappa})+\boldsymbol{g}_j(\boldsymbol{\kappa})}{\sigma_i(Z)+\sigma_j(Z)} = (\boldsymbol{g}'(\overline{\boldsymbol{\kappa}}))_{ii}+(\boldsymbol{g}'(\overline{\boldsymbol{\kappa}}))_{ij} = \bar{\eta}_{r_0+r+1}\,. \qquad (3.67)$$

Therefore, from (3.64) and (3.67), we have

$$\lim_{\boldsymbol{X}\to\overline{\boldsymbol{X}}}\boldsymbol{G}'(\boldsymbol{X})\boldsymbol{F}^{(ij)} = \left(\boldsymbol{0},\left[\bar{\eta}_{r_0+r+1}F_1^{(ij)}\ \ 0\right]\right) = \boldsymbol{G}'(\overline{\boldsymbol{X}})\boldsymbol{F}^{(ij)}\,.$$

Case 7: $1\le i\ne j\le m$, $\sigma_i(\overline{Z})=\sigma_j(\overline{Z})=0$, $\sigma_i(Z)\ne\sigma_j(Z)$ and $\sigma_i(Z)>0$ or $\sigma_j(Z)>0$. By using $\boldsymbol{s},\boldsymbol{t}$ and $\hat{\boldsymbol{s}},\hat{\boldsymbol{t}}$ defined in (3.62) and (3.65), respectively, since $\boldsymbol{g}'$ is continuous at $\overline{\boldsymbol{\kappa}}$, we know from (3.63) and (3.66) that

$$\begin{aligned}
\lim_{\boldsymbol{X}\to\overline{\boldsymbol{X}}}\boldsymbol{G}'(\boldsymbol{X})\boldsymbol{F}^{(ij)} &= \lim_{\boldsymbol{X}\to\overline{\boldsymbol{X}}}\left(\boldsymbol{0},\left[\frac{\boldsymbol{g}_i(\boldsymbol{\kappa})-\boldsymbol{g}_j(\boldsymbol{\kappa})}{\sigma_i(Z)-\sigma_j(Z)}S(F_1^{(ij)})+\frac{\boldsymbol{g}_i(\boldsymbol{\kappa})+\boldsymbol{g}_j(\boldsymbol{\kappa})}{\sigma_i(Z)+\sigma_j(Z)}T(F_1^{(ij)})\ \ 0\right]\right) \\
&= \left(\boldsymbol{0},\left[\bar{\eta}_{r_0+r+1}S(F_1^{(ij)})+\bar{\eta}_{r_0+r+1}T(F_1^{(ij)})\ \ 0\right]\right) \\
&= \left(\boldsymbol{0},\left[\bar{\eta}_{r_0+r+1}F_1^{(ij)}\ \ 0\right]\right) = \boldsymbol{G}'(\overline{\boldsymbol{X}})\boldsymbol{F}^{(ij)}\,.
\end{aligned}$$

Case 8: $1\le i\ne j\le m$, $\sigma_i(\overline{Z})=\sigma_j(\overline{Z})=0$ and $\sigma_i(Z)=\sigma_j(Z)=0$. By the continuity of $\boldsymbol{g}'$, we obtain that

$$\begin{aligned}
\lim_{\boldsymbol{X}\to\overline{\boldsymbol{X}}}\boldsymbol{G}'(\boldsymbol{X})\boldsymbol{F}^{(ij)} &= \lim_{\boldsymbol{X}\to\overline{\boldsymbol{X}}}\left(\boldsymbol{0},\left[(\boldsymbol{g}'(\boldsymbol{\kappa}))_{ii}F_1^{(ij)}\ \ 0\right]\right) = \left(\boldsymbol{0},\left[(\boldsymbol{g}'(\overline{\boldsymbol{\kappa}}))_{ii}F_1^{(ij)}\ \ 0\right]\right) \\
&= \left(\boldsymbol{0},\left[\bar{\eta}_{r_0+r+1}F_1^{(ij)}\ \ 0\right]\right) = \boldsymbol{G}'(\overline{\boldsymbol{X}})\boldsymbol{F}^{(ij)}\,.
\end{aligned}$$

Case 9: $m+1 \le j \le n$, $\sigma_i(\overline{Z}) = 0$ and $\sigma_i(Z) > 0$. We know that

$$\boldsymbol{G}'(\boldsymbol{X})\boldsymbol{F}^{(ij)} = \left(\boldsymbol{0}, \left[0 \;\; \frac{\boldsymbol{g}_i(\boldsymbol{\kappa})}{\sigma_i(Z)} F_2^{(ij)}\right]\right) .$$

Let $\tilde{s} \in \Re^m$ be a vector given by

$$\tilde{s}_p := \begin{cases} \sigma_p(Z) & \text{if } p \neq i, \\ 0 & \text{if } p = i, \end{cases} \qquad p = 1, \ldots, m .$$

Define $\tilde{\boldsymbol{s}} = (\lambda(Y), \tilde{s}) \in \Re^{m_0} \times \Re^m$. Therefore, we have $\tilde{\boldsymbol{s}}$ converges to $\overline{\boldsymbol{\kappa}}$ as $\boldsymbol{X} \to \overline{\boldsymbol{X}}$. Since $\boldsymbol{g}$ is symmetric, we know that $\boldsymbol{g}_i(\tilde{\boldsymbol{s}}) = 0$. By the mean value theorem, we have

$$\frac{\boldsymbol{g}_i(\boldsymbol{\kappa})}{\sigma_i(Z)} = \frac{\boldsymbol{g}_i(\boldsymbol{\kappa}) - \boldsymbol{g}_i(\tilde{\boldsymbol{s}})}{\sigma_i(Z)} = \frac{\partial \boldsymbol{g}_i(\rho)}{\partial \mu_i} ,$$

where $\rho \in \Re^{m_0} \times \Re^m$ is between $\boldsymbol{\kappa}$ and $\tilde{\boldsymbol{s}}$. Consequently, we have ρ converges to $\overline{\boldsymbol{\kappa}}$ as $\boldsymbol{X} \to \overline{\boldsymbol{X}}$. By the continuity of $\boldsymbol{g}'$, we know from (3.45) that

$$\lim_{\boldsymbol{X} \to \overline{\boldsymbol{X}}} \frac{\boldsymbol{g}_i(\boldsymbol{\kappa})}{\sigma_i(Z)} = (\boldsymbol{g}'(\overline{\boldsymbol{\kappa}}))_{ii} = \bar{\eta}_{r_0+r+1} .$$

Thus,

$$\lim_{\boldsymbol{X} \to \overline{\boldsymbol{X}}} \boldsymbol{G}'(\boldsymbol{X})\boldsymbol{F}^{(ij)} = \lim_{\boldsymbol{X} \to \overline{\boldsymbol{X}}} \left(\boldsymbol{0}, \left[0 \;\; \frac{\boldsymbol{g}_i(\boldsymbol{\kappa})}{\sigma_i(Z)} F_2^{(ij)}\right]\right) = \left(\boldsymbol{0}, \left[0 \;\; \bar{\eta}_{r_0+r+1} F_2^{(ij)}\right]\right) = \boldsymbol{G}'(\overline{\boldsymbol{X}})\boldsymbol{F}^{(ij)} .$$

Case 10: $m+1 \le j \le n$, $\sigma_i(\overline{Z}) = 0$ and $\sigma_i(Z) = 0$. By the continuity of $\boldsymbol{g}'$, we know that

$$\lim_{\boldsymbol{X} \to \overline{\boldsymbol{X}}} \boldsymbol{G}'(\boldsymbol{X})\boldsymbol{F}^{(ij)} = \lim_{\boldsymbol{X} \to \overline{\boldsymbol{X}}} \left(\boldsymbol{0}, \left[0 \;\; (\boldsymbol{g}'(\boldsymbol{\kappa}))_{ii} F_2^{(ij)}\right]\right) = \left(\boldsymbol{0}, \left[0 \;\; (\boldsymbol{g}'(\overline{\boldsymbol{\kappa}}))_{ii} F_2^{(ij)}\right]\right) = \boldsymbol{G}'(\overline{\boldsymbol{X}})\boldsymbol{F}^{(ij)} .$$

Finally, we consider the general case that

$$\boldsymbol{X} = \left(P\Lambda(Y)P^T, U\,[\Sigma(Z) \;\; 0]\, V^T\right) \quad \text{and} \quad \overline{\boldsymbol{X}} = \left(\overline{P}\Lambda(\overline{Y})\overline{P}^T, \overline{U}\,[\Sigma(\overline{Z}) \;\; 0]\, \overline{V}^T\right) .$$

We know that for any given $\boldsymbol{H} \in \mathcal{X}$, any accumulation point of $\boldsymbol{G}'(\boldsymbol{X})\boldsymbol{H}$ as $\boldsymbol{X} \to \overline{\boldsymbol{X}}$ can be written as $\boldsymbol{G}'(\overline{\boldsymbol{X}})\boldsymbol{H}$, since the derivative formula is independent of the choice of the orthogonal matrices $\overline{P}$, $\overline{U}$ and $\overline{V}$.

" $\Longrightarrow$ " From the proof of the second part of Theorem 3.6, it is easy to see that if $\boldsymbol{G}$ is continuously differentiable at $\overline{\boldsymbol{X}}$, then the symmetric mapping $\boldsymbol{g}$ is continuously differentiable at $\overline{\boldsymbol{\kappa}}$. $\square$

3.4 The Lipschitz continuity

In this section, we consider the local Lipschitz continuity of the spectral operator $\boldsymbol{G}$. Firstly, by using the systemic property of $\boldsymbol{g}$, we can obtain the following proposition.

Proposition 3.8. *Let $\overline{\boldsymbol{X}} = (\overline{Y}, \overline{Z}) \in \mathcal{S}^{m_0} \times \Re^{m\times n} = \mathcal{X}$ be given. Suppose that the symmetric mapping $\boldsymbol{g}$ is locally Lipschitz continuous near $\overline{\boldsymbol{\kappa}} = \boldsymbol{\kappa}(\overline{X})$ with module $L > 0$, i.e., there exists a positive constant $\delta_0 > 0$ such that*

$$\|\boldsymbol{g}(\boldsymbol{\kappa}) - \boldsymbol{g}(\boldsymbol{\kappa}')\| \leq L\|\boldsymbol{\kappa} - \boldsymbol{\kappa}'\| \quad \forall\, \boldsymbol{\kappa}, \boldsymbol{\kappa}' \in B(\overline{\boldsymbol{\kappa}}, \delta_0)\,.$$

Then, there exist a positive constant $L' > 0$ and a positive constant $\delta > 0$ such that for any $\boldsymbol{\kappa} \in B(\overline{\boldsymbol{\kappa}}, \delta)$,

$$|\boldsymbol{g}_i(\boldsymbol{\kappa}) - \boldsymbol{g}_j(\boldsymbol{\kappa})| \leq L'|\boldsymbol{\kappa}_i - \boldsymbol{\kappa}_j| \quad \forall\, 1 \leq i \neq j \leq m_0 + m \text{ and } \boldsymbol{\kappa}_i \neq \boldsymbol{\kappa}_j\,, \tag{3.68}$$

$$|\boldsymbol{g}_i(\boldsymbol{\kappa}) + \boldsymbol{g}_j(\boldsymbol{\kappa})| \leq L'|\boldsymbol{\kappa}_i + \boldsymbol{\kappa}_j| \quad \forall\, m_0 + 1 \leq i, j \leq m_0 + m \text{ and } \boldsymbol{\kappa}_i + \boldsymbol{\kappa}_j > 0\,, \tag{3.69}$$

$$|\boldsymbol{g}_i(\boldsymbol{\kappa})| \leq L'|\boldsymbol{\kappa}_i| \quad \forall\, m_0 + 1 \leq i \leq m_0 + m \text{ and } \boldsymbol{\kappa}_i > 0\,. \tag{3.70}$$

Proof. For the convenience, let $\alpha_{r_0+l} = \{j \,|\, j = m_0 + i, i \in a_l\}$, $1 \leq l \leq r$ and $\alpha_{r_0+r+1} = \{j \,|\, j = m_0 + i, i \in b\}$. We know that there exists a positive constant $\delta_1 > 0$ such that for any $\boldsymbol{\kappa} \in B(\overline{\boldsymbol{\kappa}}, \delta_1)$,

$$|\boldsymbol{\kappa}_i - \boldsymbol{\kappa}_j| \geq \delta_1 > 0 \quad \forall\, 1 \leq i \neq j \leq m_0 + m \text{ and } \overline{\boldsymbol{\kappa}}_i \neq \overline{\boldsymbol{\kappa}}_j\,, \tag{3.71}$$

$$|\boldsymbol{\kappa}_i + \boldsymbol{\kappa}_j| = \boldsymbol{\kappa}_i + \boldsymbol{\kappa}_j \geq \delta_1 > 0 \quad \forall\, m_0 + 1 \leq i, j \leq m_0 + m \text{ and } \overline{\boldsymbol{\kappa}}_i + \overline{\boldsymbol{\kappa}}_j > 0\,. \tag{3.72}$$

and

$$|\boldsymbol{\kappa}_i| = \boldsymbol{\kappa}_i \geq \delta_1 > 0 \quad \forall\, m_0 + 1 \leq i \leq m_0 + m \text{ and } \overline{\boldsymbol{\kappa}}_i > 0\,. \tag{3.73}$$

Let $\delta := \min\{\delta_0, \delta_1\} > 0$. Denote $\nu := \max\limits_{i,j}\{|\boldsymbol{g}_i(\overline{\boldsymbol{\kappa}}) - \boldsymbol{g}_j(\overline{\boldsymbol{\kappa}})|, |\boldsymbol{g}_i(\overline{\boldsymbol{\kappa}}) + \boldsymbol{g}_j(\overline{\boldsymbol{\kappa}})|, |\boldsymbol{g}_i(\overline{\boldsymbol{\kappa}})|\}$, $L_1 := (2L\delta + \nu)/\delta$ and $L' := \max\{L_1, \sqrt{2}L\}$. Let $\boldsymbol{\kappa}$ be any fixed vector in $B(\overline{\boldsymbol{\kappa}}, \delta)$.

Firstly, we consider the case that $i \neq j \in \{1, \ldots, m_0 + m\}$ and $\boldsymbol{\kappa}_i \neq \boldsymbol{\kappa}_j$. If $\overline{\boldsymbol{\kappa}}_i \neq \overline{\boldsymbol{\kappa}}_j$, then from (3.71), we know that

$$\begin{aligned} |\boldsymbol{g}_i(\boldsymbol{\kappa}) - \boldsymbol{g}_j(\boldsymbol{\kappa})| &= |\boldsymbol{g}_i(\boldsymbol{\kappa}) - \boldsymbol{g}_i(\overline{\boldsymbol{\kappa}}) + \boldsymbol{g}_i(\overline{\boldsymbol{\kappa}}) - \boldsymbol{g}_j(\overline{\boldsymbol{\kappa}}) + \boldsymbol{g}_j(\overline{\boldsymbol{\kappa}}) - \boldsymbol{g}_j(\boldsymbol{\kappa})| \\ &\leq 2\|\boldsymbol{g}(\boldsymbol{\kappa}) - \boldsymbol{g}(\overline{\boldsymbol{\kappa}})\| + \nu \\ &\leq \frac{2L\delta + \nu}{\delta}|\boldsymbol{\kappa}_i - \boldsymbol{\kappa}_j| \\ &= L_1|\boldsymbol{\kappa}_i - \boldsymbol{\kappa}_j|\,. \end{aligned} \tag{3.74}$$

If $\overline{\boldsymbol{\kappa}}_i = \overline{\boldsymbol{\kappa}}_j$, consider the vector $\boldsymbol{t} \in \Re^{m_0+m}$ defined by

$$\boldsymbol{t}_p := \begin{cases} \boldsymbol{\kappa}_p & \text{if } p \neq i, j, \\ \boldsymbol{\kappa}_j & \text{if } p = i\,, \\ \boldsymbol{\kappa}_i & \text{if } p = j\,, \end{cases} \qquad p = 1, \ldots, m_0 + m\,.$$

It is easy to see that $\|\boldsymbol{t} - \overline{\boldsymbol{\kappa}}\| = \|\boldsymbol{\kappa} - \overline{\boldsymbol{\kappa}}\| \leq \delta$. Moreover, since $\boldsymbol{g}$ is symmetric, we know that

$$\boldsymbol{g}_i(\boldsymbol{t}) = \boldsymbol{g}_j(\boldsymbol{\kappa})\,.$$

Therefore, for such i, j, we have

$$\begin{aligned} |\boldsymbol{g}_i(\boldsymbol{\kappa}) - \boldsymbol{g}_j(\boldsymbol{\kappa})| &= |\boldsymbol{g}_i(\boldsymbol{\kappa}) - \boldsymbol{g}_i(\boldsymbol{t}) + \boldsymbol{g}_i(\boldsymbol{t}) - \boldsymbol{g}_j(\boldsymbol{\kappa})| \\ &\leq |\boldsymbol{g}_i(\boldsymbol{\kappa}) - \boldsymbol{g}_i(\boldsymbol{t})| \leq L\|\boldsymbol{\kappa} - \boldsymbol{t}\| = \sqrt{2}L|\boldsymbol{\kappa}_i - \boldsymbol{\kappa}_j|\,. \end{aligned} \tag{3.75}$$

Thus, the inequality (3.68) follows from (3.74) and (3.75).

Secondly, we consider the case $i, j \in \{m_0 + 1, \ldots, m_0 + m\}$ and $\boldsymbol{\kappa}_i + \boldsymbol{\kappa}_j > 0$. If $\overline{\boldsymbol{\kappa}}_i + \overline{\boldsymbol{\kappa}}_j > 0$, then we know from (3.72) that

$$\begin{aligned} |\boldsymbol{g}_i(\boldsymbol{\kappa}) + \boldsymbol{g}_j(\boldsymbol{\kappa})| &= |\boldsymbol{g}_i(\boldsymbol{\kappa}) - \boldsymbol{g}_i(\overline{\boldsymbol{\kappa}}) + \boldsymbol{g}_i(\overline{\boldsymbol{\kappa}}) + \boldsymbol{g}_j(\overline{\boldsymbol{\kappa}}) - \boldsymbol{g}_j(\overline{\boldsymbol{\kappa}}) + \boldsymbol{g}_j(\boldsymbol{\kappa})| \\ &\leq 2\|\boldsymbol{g}(\boldsymbol{\kappa}) - \boldsymbol{g}(\overline{\boldsymbol{\kappa}})\| + \nu \\ &\leq \frac{2L\delta + \nu}{\delta}|\boldsymbol{\kappa}_i + \boldsymbol{\kappa}_j| \\ &= L_1|\boldsymbol{\kappa}_i + \boldsymbol{\kappa}_j|\,. \end{aligned} \tag{3.76}$$

If $\overline{\boldsymbol{\kappa}}_i + \overline{\boldsymbol{\kappa}}_j = 0$, i.e., $\overline{\boldsymbol{\kappa}}_i = \overline{\boldsymbol{\kappa}}_j = 0$, consider the vector $\hat{\boldsymbol{t}} \in \Re^{m_0+m}$ defined by

$$\hat{\boldsymbol{t}}_p := \begin{cases} \boldsymbol{\kappa}_p & \text{if } p \neq i, j, \\ -\boldsymbol{\kappa}_j & \text{if } p = i\,, \\ -\boldsymbol{\kappa}_i & \text{if } p = j\,, \end{cases} \qquad p = 1, \ldots, m_0 + m\,.$$

By noting that $\overline{\boldsymbol{\kappa}}_i = \overline{\boldsymbol{\kappa}}_j = 0$, we obtain that $\|\hat{\boldsymbol{t}} - \overline{\boldsymbol{\kappa}}\| = \|\boldsymbol{\kappa} - \overline{\boldsymbol{\kappa}}\| \leq \delta$. Moreover, since $\boldsymbol{g}$ is symmetric, we know that

$$\boldsymbol{g}_i(\hat{\boldsymbol{t}}) = -\boldsymbol{g}_j(\boldsymbol{\kappa})\,.$$

Therefore, for such i, j, we have

$$\begin{aligned} |\boldsymbol{g}_i(\boldsymbol{\kappa}) + \boldsymbol{g}_j(\boldsymbol{\kappa})| &= |\boldsymbol{g}_i(\boldsymbol{\kappa}) - \boldsymbol{g}_i(\hat{\boldsymbol{t}}) + \boldsymbol{g}_i(\hat{\boldsymbol{t}}) + \boldsymbol{g}_j(\boldsymbol{\kappa})| \leq |\boldsymbol{g}_i(\boldsymbol{\kappa}) - \boldsymbol{g}_i(\hat{\boldsymbol{t}})| \\ &\leq \|\boldsymbol{g}(\boldsymbol{\kappa}) - \boldsymbol{g}(\hat{\boldsymbol{t}})\| \leq L\|\boldsymbol{\kappa} - \hat{\boldsymbol{t}}\| = \sqrt{2}L|\boldsymbol{\kappa}_i + \boldsymbol{\kappa}_j|\,. \end{aligned} \tag{3.77}$$

Then, the inequality (3.69) follows from (3.76) and (3.77).

Finally, we consider the case that $i \in \{m_0 + 1, \ldots, m_0 + m\}$ and $\boldsymbol{\kappa}_i > 0$. If $\overline{\boldsymbol{\kappa}}_i > 0$, then we know from (3.73) that

$$\begin{aligned} |\boldsymbol{g}_i(\boldsymbol{\kappa})| &= |\boldsymbol{g}_i(\boldsymbol{\kappa}) - \boldsymbol{g}_i(\overline{\boldsymbol{\kappa}}) + \boldsymbol{g}_i(\overline{\boldsymbol{\kappa}})| \leq |\boldsymbol{g}_i(\boldsymbol{\kappa}) - \boldsymbol{g}_i(\overline{\boldsymbol{\kappa}})| + |\boldsymbol{g}_i(\overline{\boldsymbol{\kappa}})| \\ &\leq \|\boldsymbol{g}(\boldsymbol{\kappa}) - \boldsymbol{g}(\overline{\boldsymbol{\kappa}})\| + \nu \leq \frac{2L\delta + \nu}{\delta}|\boldsymbol{\kappa}_i| \leq L_1|\boldsymbol{\kappa}_i|\,. \end{aligned} \tag{3.78}$$

If $\overline{\boldsymbol{\kappa}}_i = 0$, consider the vector $\boldsymbol{s} \in \Re^{m_0+m}$ defined by

$$\boldsymbol{s}_p := \begin{cases} \boldsymbol{\kappa}_p & \text{if } p \neq i, \\ 0 & \text{if } p = i \end{cases} \qquad p = 1, \ldots, m_0 + m\,.$$

Then, since $\boldsymbol{\kappa}_i > 0$, we know that $\|\boldsymbol{s} - \overline{\boldsymbol{\kappa}}\| < \|\boldsymbol{\kappa} - \overline{\boldsymbol{\kappa}}\| \leq \delta$. Moreover, since $\boldsymbol{g}$, we know that

$$\boldsymbol{g}_i(\boldsymbol{s}) = 0\,.$$

Therefore, for such i, we have

$$|\boldsymbol{g}_i(\boldsymbol{\kappa})| = |\boldsymbol{g}_i(\boldsymbol{\kappa}) - \boldsymbol{g}_i(\boldsymbol{s})| \leq \|\boldsymbol{g}(\boldsymbol{\kappa}) - \boldsymbol{g}(\boldsymbol{s})\| \leq L\|\boldsymbol{\kappa} - \boldsymbol{s}\| \leq L|\boldsymbol{\kappa}_i| \leq \sqrt{2}L|\boldsymbol{\kappa}_i|\,. \tag{3.79}$$

Thus, the inequality (3.68) follows from (3.78) and (3.79). This completed the proof. □

Suppose that $\boldsymbol{g}$ is locally Lipschitz continuous near $\overline{\boldsymbol{\kappa}}$ with the module $L > 0$. For any fixed $0 < \eta \leq \delta_0/\sqrt{n}$ and $\boldsymbol{y} \in B_\infty(\overline{\boldsymbol{\kappa}}, \delta_0/(2\sqrt{n})) := \{\|\boldsymbol{y} - \overline{\boldsymbol{\kappa}}\|_\infty \leq \delta_0/(2\sqrt{n})\}$, the function $\boldsymbol{g}$ is integrable on $V_\eta(\boldsymbol{y}) := \{\boldsymbol{z} \in \Re^n \mid \|\boldsymbol{y}-\boldsymbol{z}\|_\infty \leq \eta/2\}$ (in the sense of Lebesgue). Therefore, we know that the function

$$\boldsymbol{g}(\eta, \boldsymbol{y}) := \frac{1}{\eta^n} \int_{V_\eta(\boldsymbol{y})} g(\boldsymbol{y}) d\boldsymbol{y} \tag{3.80}$$

is well-defined on $(0, \delta_0/\sqrt{n}\,] \times B_\infty(\overline{\boldsymbol{\kappa}}, \delta_0/(2\sqrt{n}))$ and is said to be Steklov averaged function of $\boldsymbol{g}$. For convenience of discussion, we always define $\boldsymbol{g}(0, \boldsymbol{y}) = \boldsymbol{g}(\boldsymbol{y})$. Since $\boldsymbol{g}$ is symmetric, it is easy to check that for each fixed $0 < \eta \leq \delta_0/\sqrt{n}$, the function $\boldsymbol{g}(\eta, \cdot)$ is also symmetric on $B_\infty(\overline{\boldsymbol{\kappa}}, \delta_0/(2\sqrt{n}))$. By the definition, we know that $\boldsymbol{g}(\cdot, \cdot)$ is locally Lipschitz continuous on $(0, \delta_0/\sqrt{n}\,] \times B_\infty(\overline{\boldsymbol{\kappa}}, \delta_0/(2\sqrt{n}))$ with the module L. Meanwhile, by elementary calculation, we know that $\boldsymbol{g}(\cdot, \cdot)$ is continuously differentiable on $(0, \delta_0/\sqrt{n}\,] \times B_\infty(\overline{\boldsymbol{\kappa}}, \delta_0/(2\sqrt{n}))$ and for any fixed $\eta \in (0, \delta_0/\sqrt{n}\,]$ and $\boldsymbol{y} \in B_\infty(\overline{\boldsymbol{\kappa}}, \delta_0/(2\sqrt{n}))$,

$$\|\boldsymbol{g}'_{\boldsymbol{y}}(\eta, \boldsymbol{y})\| \leq L\,. \tag{3.81}$$

Moreover, we know that $\boldsymbol{g}(\eta, \cdot)$ converges to $\boldsymbol{g}$ uniformly on the compact set $B_\infty(\overline{\boldsymbol{\kappa}}, \delta_0/(2\sqrt{n}))$ as $\eta \downarrow 0$. By using the formula (3.50), the following results can be obtained from Theorem 3.7 and Proposition 3.8 directly.

Proposition 3.9. *Suppose that the symmetric mapping $\boldsymbol{g}$ is locally Lipschitz continuous near $\overline{\boldsymbol{\kappa}}$, Let $\boldsymbol{g}(\cdot, \cdot)$ be the corresponding Steklov averaged function defined in (3.80). Then, for any given $\eta \in (0, \delta_0/\sqrt{n}\,]$, the spectral operator $\boldsymbol{G}(\eta, \cdot) : \mathcal{X} \to \mathcal{X}$ with respect to the symmetric mapping $\boldsymbol{g}(\eta, \cdot)$ is continuously differentiable on $B_*(\overline{\boldsymbol{X}}, \delta_0/(2\sqrt{n})) := \{\boldsymbol{X} \in \mathcal{X} \mid \|\boldsymbol{\kappa}(\boldsymbol{X}) - \overline{\boldsymbol{\kappa}}\|_\infty \leq \delta_0/(2\sqrt{n})\}$, and there exist two positive constants $\delta_1 > 0$ and $\overline{L} > 0$ such that*

$$\|\boldsymbol{G}'(\eta, \boldsymbol{X})\| \leq \overline{L} \quad \forall\, 0 < \eta \leq \min\{\delta_0/\sqrt{n}, \delta_1\} \text{ and } \boldsymbol{X} \in B_*(\overline{\boldsymbol{X}}, \delta_0/(2\sqrt{n}))\,. \tag{3.82}$$

Moreover, $\boldsymbol{G}(\eta, \cdot)$ converges to $\boldsymbol{G}$ uniformly in the compact set $B_*(\overline{\boldsymbol{X}}, \delta_0/(2\sqrt{n}))$ as $\eta \downarrow 0$.

We state the main result of this section in the following theorem.

Theorem 3.10. *Let $\overline{\boldsymbol{X}} = (\overline{Y}, \overline{Z}) \in \mathcal{S}^{m_0} \times \Re^{m\times n} = \mathcal{X}$ be given. Suppose that $\overline{Y}$ and $\overline{Z}$ have the decompositions (3.16). The spectral operator $\boldsymbol{G}$ is locally Lipschitz continuous near $\overline{\boldsymbol{X}}$ if and only if the symmetric mapping $\boldsymbol{g}$ is locally Lipschitz continuous near $\overline{\boldsymbol{\kappa}} = \boldsymbol{\kappa}(\overline{X})$.*

Proof. " $\Longleftarrow$ " Suppose that the symmetric mapping $\boldsymbol{g}$ is locally Lipschitz continuous near $\overline{\boldsymbol{\kappa}} = \boldsymbol{\kappa}(\overline{\boldsymbol{X}})$ with module $L > 0$, i.e., there exists a positive constant $\delta_0 > 0$ such that

$$\|\boldsymbol{g}(\boldsymbol{\kappa}) - \boldsymbol{g}(\boldsymbol{\kappa}')\| \leq L\|\boldsymbol{\kappa} - \boldsymbol{\kappa}'\| \quad \forall\, \boldsymbol{\kappa}, \boldsymbol{\kappa}' \in B(\overline{\boldsymbol{\kappa}}, \delta_0)\,.$$

By Proposition 3.9, for any given $\eta \in (0, \delta_0/\sqrt{n}\,]$, we may consider the continuously differentiable spectral operator $\boldsymbol{G}(\eta, \cdot) : \mathcal{X} \to \mathcal{X}$ with respect to the Steklov averaged function $\boldsymbol{g}(\eta, \cdot)$ of $\boldsymbol{g}$. Since $\boldsymbol{G}(\eta, \cdot)$ converges to $\boldsymbol{G}$ uniformly in the compact set $B_*(\overline{\boldsymbol{X}}, \delta_0/(2\sqrt{n}))$ as $\eta \downarrow 0$, we know that for any $\varepsilon > 0$, there exists a constant $\delta_2 > 0$ such that for any $0 < \eta \leq \delta_2$

$$\|\boldsymbol{G}(\eta, \boldsymbol{X}) - \boldsymbol{G}(\boldsymbol{X})\| \leq \varepsilon \quad \forall\, \boldsymbol{X} \in B_*(\overline{\boldsymbol{X}}, \delta_0/(2\sqrt{n}))\,.$$

Fix any $\boldsymbol{X}, \boldsymbol{X}' \in \mathcal{B}_*(\overline{\boldsymbol{X}}, \delta_0/(2\sqrt{n}))$ with $\boldsymbol{X} \neq \boldsymbol{X}'$. Meanwhile, by Proposition 3.9, we know that there exists $\delta_1 > 0$ such that (3.82) holds. Let $\bar{\delta} := \min\{\delta_1, \delta_2, \delta_0/\sqrt{n}\}$. Then, by the mean value theorem, we know that

$$\begin{aligned}
\|\boldsymbol{G}(\boldsymbol{X}) - \boldsymbol{G}(\boldsymbol{X}')\| &= \|\boldsymbol{G}(\boldsymbol{X}) - \boldsymbol{G}(\eta, \boldsymbol{X}) + \boldsymbol{G}(\eta, \boldsymbol{X}) - \boldsymbol{G}(\eta, \boldsymbol{X}') + \boldsymbol{G}(\eta, \boldsymbol{X}') - \boldsymbol{G}(\boldsymbol{X}')\| \\
&\leq 2\varepsilon + \|\int_0^1 \boldsymbol{G}'(\eta, \boldsymbol{X} + t(\boldsymbol{X} - \boldsymbol{X}'))dt\| \\
&\leq \overline{L}\|\boldsymbol{X} - \boldsymbol{X}'\| + 2\varepsilon \quad \forall\, 0 < \eta < \bar{\delta}\,.
\end{aligned}$$

Since $\boldsymbol{X}, \boldsymbol{X}' \in \mathcal{B}_*(\overline{\boldsymbol{X}}, \delta_0/(2\sqrt{n}))$ and $\varepsilon > 0$ are arbitrary, by letting $\varepsilon \downarrow 0$, we obtain that

$$\|\boldsymbol{G}(\boldsymbol{X}) - \boldsymbol{G}(\boldsymbol{X}')\| \leq \overline{L}\|\boldsymbol{X} - \boldsymbol{X}'\| \quad \forall\, \boldsymbol{X}, \boldsymbol{X}' \in B_*(\overline{\boldsymbol{X}}, \delta_0/(2\sqrt{n}))\,.$$

Thus $\boldsymbol{G}$ is locally Lipchitz continuous near $\overline{\boldsymbol{X}}$.

" $\Longrightarrow$ " Suppose that $\boldsymbol{G}$ is locally Lipschitz continuous near $\overline{\boldsymbol{X}}$. For any $\boldsymbol{y} = (\boldsymbol{y}_1, \boldsymbol{y}_2) \in \Re^{m_0} \times \Re^m$, we may define $\boldsymbol{Y} := (\mathrm{diag}(\boldsymbol{y}_1), [\mathrm{diag}(\boldsymbol{y}_2) \ \ 0]) \in \mathcal{X}$. Then, since $\boldsymbol{g}$ is symmetric, we have $\boldsymbol{G}(\boldsymbol{Y}) = (\mathrm{diag}(\boldsymbol{g}_1(\boldsymbol{y})), [\mathrm{diag}(\boldsymbol{g}_2(\boldsymbol{y})) \ \ 0])$. Therefore, we obtain that there exist a positive number $\boldsymbol{\kappa} > 0$ and a open neighborhood $\mathcal{N}_{\overline{\boldsymbol{\kappa}}}$ such that

$$\|\boldsymbol{g}(\boldsymbol{y}) - \boldsymbol{g}(\boldsymbol{y}')\| = \|\boldsymbol{G}(\boldsymbol{Y}) - \boldsymbol{G}(\boldsymbol{Y}')\| \le L\|\boldsymbol{Y} - \boldsymbol{Y}'\| = L\|\boldsymbol{y} - \boldsymbol{y}'\| \quad \forall\, \boldsymbol{y}, \boldsymbol{y}' \in \mathcal{N}_{\overline{\boldsymbol{\kappa}}}\,.$$

This completed the proof. □

3.5 The ρ-order Bouligand-differentiability

For the ρ-order B(ouligand)-differentiability of spectral operators, we have the following result.

Theorem 3.11. *Let $\overline{\boldsymbol{X}} = (\overline{Y}, \overline{Z}) \in \mathcal{S}^{m_0} \times \Re^{m\times n} = \mathcal{X}$ be given. Suppose that $\overline{Y}$ and $\overline{Z}$ have the decompositions (3.16). Let $0 < \rho \le 1$ be given. If the symmetric function $\boldsymbol{g}$ is locally Lipschitz continuous near $\boldsymbol{\kappa}(\overline{\boldsymbol{X}})$, then the spectral operator $\boldsymbol{G}$ is ρ-order B-differentiable at $\overline{\boldsymbol{X}}$ if and only if the symmetric mapping $\boldsymbol{g}$ is ρ-order B-differentiable at $\boldsymbol{\kappa}(\overline{\boldsymbol{X}})$.*

Proof. Without loss of generality, we just prove the results for the case $\rho = 1$.

" $\Longleftarrow$ " For any $\boldsymbol{H} = (A, B) \in \mathcal{X}$, let $\boldsymbol{X} = \overline{\boldsymbol{X}} + \boldsymbol{H} = (\overline{Y} + A, \overline{Z} + B) = (Y, Z)$. Let $P \in \mathcal{O}^{m_0}$, $U \in \mathcal{O}^m$ and $V \in \mathcal{O}^n$ be such that

$$Y = P\Lambda(Y)P^T \quad \text{and} \quad Z = U[\Sigma(Z) \ \ 0]V^T\,. \tag{3.83}$$

Denote $\boldsymbol{\kappa} = \boldsymbol{\kappa}(\boldsymbol{X})$. Let $\boldsymbol{G}_S$ and $\boldsymbol{G}_R$ be defined by (3.19) and (3.20), respectively. Therefore, by Lemma 3.3, we know that for any $\mathcal{X} \ni \boldsymbol{H} \to 0$,

$$\boldsymbol{G}_S(\boldsymbol{X}) - \boldsymbol{G}_S(\overline{\boldsymbol{X}}) = \boldsymbol{G}_S'(\overline{\boldsymbol{X}})\boldsymbol{H} + O(\|\boldsymbol{H}\|^2) = \Big(\boldsymbol{T}_1(\widetilde{A}), \boldsymbol{T}_2(\widetilde{B})\Big) + O(\|\boldsymbol{H}\|^2)\,, \tag{3.84}$$

where $\widetilde{\boldsymbol{H}} = (\widetilde{A}, \widetilde{B})$ with $\widetilde{A} = \overline{P}^T A \overline{P}$, $\widetilde{B} = \begin{bmatrix}\widetilde{B}_1 & \widetilde{B}_2\end{bmatrix} = \begin{bmatrix}\overline{U}^T B \overline{V}_1 & \overline{U}^T B \overline{V}_2\end{bmatrix}$, and the linear operator $\boldsymbol{T}(\cdot) = (\boldsymbol{T}_1(\cdot), \boldsymbol{T}_2(\cdot)) : \mathcal{X} \to \mathcal{X}$ is given by (3.18).

On the other hand, for $\boldsymbol{H} \in \mathcal{X}$ sufficiently close to zero, we know that $\mathcal{P}_k(Y) = \sum_{i\in\alpha_k} p_i p_i^T$, $k = 1, \ldots, r_0$ and $\mathcal{U}_l(Z) = \sum_{i\in a_l} u_i v_i^T$, $l = 1, \ldots, r$. Therefore,

$$
\begin{aligned}
& \boldsymbol{G}_R(\boldsymbol{X}) = \boldsymbol{G}(\boldsymbol{X}) - \boldsymbol{G}_S(\boldsymbol{X}) \\
= \; & ((\boldsymbol{G}_1)_R(\boldsymbol{X}), (\boldsymbol{G}_2)_R(\boldsymbol{X})) = (\boldsymbol{G}_1(\boldsymbol{X}) - (\boldsymbol{G}_1)_S(Y), \boldsymbol{G}_2(\boldsymbol{X}) - (\boldsymbol{G}_2)_S(Z)) \\
= \; & \left(\sum_{k=1}^{r_0} \Delta_k(\boldsymbol{H}), \sum_{k=r_0+1}^{r_0+r+1} \Delta_k(\boldsymbol{H}) \right), \qquad (3.85)
\end{aligned}
$$

where

$$
\Delta_k(\boldsymbol{H}) = \begin{cases} \sum_{i\in\alpha_k} [(\boldsymbol{g}_1(\boldsymbol{\kappa}))_i - (\boldsymbol{g}_1(\overline{\boldsymbol{\kappa}}))_i] p_i p_i^T & \text{if } 1 \leq k \leq r_0, \\ \sum_{i\in a_l} [(\boldsymbol{g}_2(\boldsymbol{\kappa}))_i - (\boldsymbol{g}_2(\overline{\boldsymbol{\kappa}}))_i] u_i v_i^T & \text{if } r_0 + 1 \leq k = r_0 + l \leq r_0 + r \end{cases}
$$

and

$$
\Delta_{r_0+r+1}(\boldsymbol{H}) = \sum_{i\in b} (\boldsymbol{g}_2(\boldsymbol{\kappa}))_i u_i v_i^T .
$$

Firstly, we consider the case that $\overline{\boldsymbol{X}} = (\overline{Y}, \overline{Z}) = \big(\Lambda(\overline{Y}), [\Sigma(\overline{Z}) \;\; 0]\big)$. Then, from (2.14), (2.38) and (2.39), for any $\boldsymbol{H} \in \mathcal{X}$ sufficiently close to 0, we know that

$$
\boldsymbol{\kappa} = \boldsymbol{\kappa}(\boldsymbol{X}) = \overline{\boldsymbol{\kappa}} + \boldsymbol{h} + O(\|\boldsymbol{H}\|^2), \qquad (3.86)
$$

where $\boldsymbol{h} := (\lambda'(\overline{Y}; A), \sigma'(\overline{Z}; B)) \in \Re^{m_0} \times \Re^m$ with $(\lambda'(\overline{Y}; A))_{\alpha_k} = \lambda(A_{\alpha_k\alpha_k})$, $k = 1, \ldots, r_0$,

$$
(\sigma'(\overline{Z}; B))_{a_l} = \lambda(S(B_{a_l a_l})), \; l = 1, \ldots, r \quad \text{and} \quad (\sigma'(\overline{Z}; B))_b = \sigma([B_{bb} \;\; B_{bc}]) .
$$

Since $\boldsymbol{g}$ is locally Lipschitz continuous near $\overline{\boldsymbol{\kappa}}$ and 1-order B-differentiable at $\overline{\boldsymbol{\kappa}}$, we know that for any $\boldsymbol{H}$ sufficiently close to 0,

$$
\begin{aligned}
\boldsymbol{g}(\boldsymbol{\kappa}) - \boldsymbol{g}(\overline{\boldsymbol{\kappa}}) &= \boldsymbol{g}(\boldsymbol{\kappa} + \boldsymbol{h} + O(\|\boldsymbol{H}\|^2)) - g(\overline{\boldsymbol{\kappa}}) \\
&= \boldsymbol{g}(\boldsymbol{\kappa} + \boldsymbol{h}) - \boldsymbol{g}(\overline{\boldsymbol{\kappa}}) + O(\|\boldsymbol{H}\|^2) \\
&= \boldsymbol{g}'(\overline{\boldsymbol{\kappa}}; \boldsymbol{h}) + O(\|\boldsymbol{H}\|^2) = \phi(\boldsymbol{h}) + O(\|\boldsymbol{H}\|^2) .
\end{aligned}
$$

Since $p_i p_i^T$, $i = 1, \ldots, m_0$ and $u_i v_i^T$, $i = 1, \ldots, m$ are uniformly bounded, we know that for $\boldsymbol{H}$ sufficiently close to 0,

$$\Delta_k(\boldsymbol{H}) = \begin{cases} P_{\alpha_k}\mathrm{diag}(\phi_k(\boldsymbol{h}))P_{\alpha_k}^T + O(\|\boldsymbol{H}\|^2) & \text{if } 1 \le k \le r_0, \\ U_{a_l}\mathrm{diag}(\phi_k(\boldsymbol{h}))V_{a_l}^T + O(\|\boldsymbol{H}\|^2) & \text{if } r_0 + 1 \le k = r_0 + l \le r_0 + r \end{cases}$$

and

$$\Delta_{r_0+r+1}(\boldsymbol{H}) = U_b \mathrm{diag}(\phi_{r_0+r+1}(\boldsymbol{h}))V_b^T + O(\|\boldsymbol{H}\|^2)\,.$$

By (2.10) and (2.12) in Proposition 2.5, we know that there exist $Q_k \in \mathcal{O}^{|\alpha_k|}$, $k = 1, \ldots, r_0$ and $Q_{r_0+l} \in \mathcal{O}^{|a_l|}$, $l = 1, \ldots, r$ (depending on $\boldsymbol{H}$) such that for each $i \in \alpha_k$,

$$P_{\alpha_k} = \begin{bmatrix} O(\|\boldsymbol{H}\|) \\ Q_k + O(\|\boldsymbol{H}\|) \\ O(\|\boldsymbol{H}\|) \end{bmatrix}, \ k = 1, \ldots, r_0\,,$$

$$U_{a_l} = \begin{bmatrix} O(\|\boldsymbol{H}\|) \\ Q_{r_0+l} + O(\|\boldsymbol{H}\|) \\ O(\|\boldsymbol{H}\|) \end{bmatrix} \quad \text{and} \quad V_{a_l} = \begin{bmatrix} O(\|\boldsymbol{H}\|) \\ Q_{r_0+l} + O(\|\boldsymbol{H}\|) \\ O(\|\boldsymbol{H}\|) \end{bmatrix}, \ l = 1, \ldots, r\,.$$

Since $\boldsymbol{g}$ is locally Lipchitz continuous near $\overline{\boldsymbol{\kappa}}$ and directionally differentiable at $\overline{\boldsymbol{\kappa}}$, we know from Lemma 2.2 that for $\boldsymbol{H}$ sufficiently close to 0,

$$\|\phi(\boldsymbol{h})\| = \|\boldsymbol{g}'(\overline{\boldsymbol{\kappa}}; \boldsymbol{h})\| = O(\|\boldsymbol{H}\|)\,.$$

Therefore, we have

$$\Delta_k(\boldsymbol{H}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & Q_k\mathrm{diag}(\phi_k(\boldsymbol{h}))Q_k^T & 0 \\ 0 & 0 & 0 \end{bmatrix} + O(\|\boldsymbol{H}\|^2), \quad 1 \le k \le r_0 + r\,. \tag{3.87}$$

Meanwhile, by (2.40), we know that there exist $M \in \mathcal{O}^{|b|}$ and $N = [N_1 \ \ N_2] \in \mathcal{O}^{n-|a|}$ (depending on $\boldsymbol{H}$) with $N_1 \in \Re^{(n-|a|)\times|b|}$ and $N_2 \in \Re^{(n-|a|)\times(n-m)}$ such that

$$U_b = \begin{bmatrix} O(\|\boldsymbol{H}\|) \\ M + O(\|\boldsymbol{H}\|) \end{bmatrix} \quad \text{and} \quad [V_b \ \ V_c] = \begin{bmatrix} O(\|\boldsymbol{H}\|) \\ N + O(\|\boldsymbol{H}\|) \end{bmatrix}.$$

Therefore, we obtain that

$$\Delta_{r_0+r+1}(\boldsymbol{H}) = \begin{bmatrix} 0 & 0 \\ 0 & M\mathrm{diag}(\phi_{r_0+r+1}(\boldsymbol{h}))N_1^T \end{bmatrix} + O(\|\boldsymbol{H}\|^2)\,. \tag{3.88}$$

On the other hand, from (2.13), we know that

$$A_{\alpha_k\alpha_k} = Q_k(\Lambda(Y)_{\alpha_k\alpha_k} - \overline{\mu}_k I_{|\alpha_k|})Q_k^T + O(\|\boldsymbol{H}\|^2), \quad 1 \le k \le r_0\,, \tag{3.89}$$

$$S(B_{a_la_l}) = Q_k(\Sigma(Z)_{a_la_l} - \overline{\nu}_l I_{|a_l|})Q_k^T + O(\|\boldsymbol{H}\|^2), \quad r_0+1 \le k = r_0+l \le r_0+r \tag{3.90}$$

and

$$[B_{bb} \;\; B_{bc}] = M(\Sigma(Z)_{bb} - \overline{\nu}_{r+1} I_{|b|})N_1^T + O(\|\boldsymbol{H}\|^2)\,. \tag{3.91}$$

Since the symmetric mapping $\phi(\cdot) = g'(\overline{\boldsymbol{\kappa}};\cdot)$ is globally Lipschitz continuous on $\Re^{m_0}\times\Re^m$, by Theorem 3.10, we know that the corresponding spectral operator Φ defined by (3.22) is globally Lipchitz continuous. Hence, we know from (3.85) that for $\boldsymbol{H}$ sufficiently close to 0,

$$\boldsymbol{G}_R(\boldsymbol{X}) = (\Upsilon_1(\boldsymbol{H}), \Upsilon_2(\boldsymbol{H})) + O(\|\boldsymbol{H}\|^2)\,, \tag{3.92}$$

where

$$\Upsilon_1(\boldsymbol{H}) = \begin{bmatrix} \Phi_1(\boldsymbol{D}(\boldsymbol{H})) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \Phi_{r_0}(\boldsymbol{D}(\boldsymbol{H})) \end{bmatrix} \in \mathcal{S}^{m_0}\,,$$

$$\Upsilon_2(\boldsymbol{H}) = \begin{bmatrix} \Phi_{r_0+1}(\boldsymbol{D}(\boldsymbol{H})) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \Phi_{r_0+r}(\boldsymbol{D}(\boldsymbol{H})) & 0 \\ 0 & \cdots & 0 & \Phi_{r_0+r+1}(\boldsymbol{D}(\boldsymbol{H})) \end{bmatrix} \in \Re^{m\times n}\,,$$

and $\boldsymbol{D}(\boldsymbol{H}) = \big(A_{\alpha_1\alpha_1}, \ldots, A_{\alpha_{r_0}\alpha_{r_0}}, S(B_{a_1a_1}), \ldots, S(B_{a_ra_r}), B_{b\bar{a}}\big)$.

Next, consider the general case for $\overline{\boldsymbol{X}} = (\overline{Y}, \overline{Z}) \in \mathcal{X}$. For any $\boldsymbol{H} \in \mathcal{X}$, re-write (3.83) as

$$\Lambda(\overline{Y}) + \overline{P}^T A'\overline{P} = \overline{P}^T P\Lambda(Y)P^T\overline{P} \quad \text{and} \quad [\Sigma(\overline{Z}) \;\; 0] + \overline{U}^T B'\overline{V} = \overline{U}^T U[\Sigma(Z) \;\; 0]V^T\overline{V}\,.$$

Let $\widetilde{P} = \overline{P}^T P$, $\widetilde{U} := \overline{U}^T U$ and $\widetilde{V} := \overline{V}^T V$. Let $\widetilde{\boldsymbol{X}} := (\widetilde{Y}, \widetilde{Z}) \in \mathcal{X}$ with

$$\widetilde{Y} := \Lambda(\overline{Y}) + \overline{P}^T A' \overline{P} \quad \text{and} \quad \widetilde{Z} := [\Sigma(\overline{Z}) \ \ 0] + \overline{U}^T B' \overline{V} \,.$$

Then, since $\overline{P}$, $\overline{U}$ and $\overline{V}$ are bounded, we know from (3.92) that

$$\boldsymbol{G}_R(\boldsymbol{X}) = \left(\overline{P}(\boldsymbol{G}_1)_R(\widetilde{\boldsymbol{X}})\overline{P}^T, \overline{U}(\boldsymbol{G}_2)_R(\widetilde{\boldsymbol{X}})\overline{V}^T\right) = \left(\overline{P}\Upsilon_1(\widetilde{\boldsymbol{H}})\overline{P}^T, \overline{U}\Upsilon_2(\widetilde{\boldsymbol{H}})\overline{V}^T\right) + O(\|\boldsymbol{H}\|^2) \,. \tag{3.93}$$

Thus, by combining (3.84) and (3.93) and noting that $\boldsymbol{G}(\overline{\boldsymbol{X}}) = \boldsymbol{G}_S(\overline{\boldsymbol{X}})$, we obtain that for any $\boldsymbol{H} \in \mathcal{X}$ sufficiently close to 0,

$$\boldsymbol{G}(\boldsymbol{X}) - \boldsymbol{G}(\overline{\boldsymbol{X}}) - \boldsymbol{G}'(\overline{\boldsymbol{X}}; \boldsymbol{H}) = O(\|\boldsymbol{H}\|^2) \,,$$

where $\boldsymbol{G}'(\overline{\boldsymbol{X}}; \boldsymbol{H})$ is given by (3.26). This implies that $\boldsymbol{G}$ is 1-order B-differentiable at $\overline{\boldsymbol{X}}$.

" $\Longrightarrow$ " Suppose that $\boldsymbol{G}$ is 1-order B-differentiable at $\overline{\boldsymbol{X}} = (\overline{Y}, \overline{Z})$. Let $\overline{P} \in \mathcal{O}^{m_0}(\overline{Y})$ and $(\overline{U}, \overline{V}) \in \mathcal{O}^{m \times n}(\overline{Z})$ be fixed. For any $\boldsymbol{h} := (h_1, h_2) \in \Re^{m_0} \times \Re^m$, let $\boldsymbol{H} = (A, B) \in \mathcal{X}$, where $A := \overline{P}\mathrm{diag}(h_1)\overline{P}^T$ and $B := \overline{U}[\mathrm{diag}(h_2) \ \ 0]\overline{V}^T$. Then, by the assumption, we know that for $\boldsymbol{h}$ sufficiently close to 0,

$$\begin{aligned} & \left(\overline{P}\mathrm{diag}(\boldsymbol{g}_1(\overline{\boldsymbol{\kappa}} + \boldsymbol{h}) - \boldsymbol{g}_1(\overline{\boldsymbol{\kappa}}))\overline{P}^T, \overline{U}\mathrm{diag}(\boldsymbol{g}_2(\overline{\boldsymbol{\kappa}} + \boldsymbol{h}) - \boldsymbol{g}_2(\overline{\boldsymbol{\kappa}}))\overline{V}_1^T\right) \\ = \ & \boldsymbol{G}(\overline{\boldsymbol{X}} + \boldsymbol{H}) - \boldsymbol{G}(\overline{\boldsymbol{X}}) = \boldsymbol{G}'(\overline{\boldsymbol{X}}; \boldsymbol{H}) + O(\|\boldsymbol{H}\|^2) \,. \end{aligned}$$

Hence, for $\boldsymbol{h}$ sufficiently close to 0,

$$\begin{aligned} \boldsymbol{g}(\overline{\boldsymbol{\kappa}} + \boldsymbol{h}) - \boldsymbol{g}(\overline{\boldsymbol{\kappa}}) &= (\boldsymbol{g}_1(\overline{\boldsymbol{\kappa}} + \boldsymbol{h}) - \boldsymbol{g}_1(\overline{\boldsymbol{\kappa}}), \boldsymbol{g}_2(\overline{\boldsymbol{\kappa}} + \boldsymbol{h}) - \boldsymbol{g}_2(\overline{\boldsymbol{\kappa}})) \\ &= \boldsymbol{g}'(\overline{\boldsymbol{\kappa}}; \boldsymbol{h}) + O(\|\boldsymbol{h}\|^2) \,. \end{aligned}$$

The proof is competed. □

3.6 The ρ-order G-semismoothness

In this section, we consider the ρ-order G-semismoothness of spectral operators.

Theorem 3.12. *Let* $\overline{\boldsymbol{X}} = (\overline{Y}, \overline{Z}) \in \mathcal{S}^{m_0} \times \Re^{m\times n} = \mathcal{X}$ *be given. Suppose that* $\overline{Y}$ *and* $\overline{Z}$ *have the decompositions (3.16). Let* $0 < \rho \leq 1$ *be given. If the symmetric mapping* $\boldsymbol{g}$ *is locally Lipschitz continuous near* $\boldsymbol{\kappa}(\overline{\boldsymbol{X}})$*, then the corresponding spectral operator* $\boldsymbol{G}$ *is* ρ*-order G-semismooth at* $\overline{\boldsymbol{X}}$ *if and only if* $\boldsymbol{g}$ *is* ρ*-order G-semismooth at* $\boldsymbol{\kappa}(\overline{\boldsymbol{X}})$*.*

Proof. Let $\overline{\boldsymbol{\kappa}} = \boldsymbol{\kappa}(\overline{\boldsymbol{X}})$. Without loss of generality, we consider the case that $\rho = 1$.

" $\Longleftarrow$ " For any $\boldsymbol{H} = (A, B) \in \mathcal{X}$, let $\boldsymbol{X} := \overline{\boldsymbol{X}} + \boldsymbol{H} = (\overline{Y} + A, \overline{Z} + B) = (Y, Z)$, where $Y \in \mathcal{S}^{m_0}$ and $Z \in \Re^{m\times n}$. Let $P \in \mathcal{O}^{m_0}$, $U \in \mathcal{O}^m$ and $V \in \mathcal{O}^n$ be such that

$$Y = P\Lambda(Y)P^T \quad \text{and} \quad Z = U[\Sigma(Z) \ \ 0]V^T . \tag{3.94}$$

Denote $\boldsymbol{\kappa} = \boldsymbol{\kappa}(\boldsymbol{X})$. Let $\boldsymbol{G}_S$ and $\boldsymbol{G}_R$ be defined by (3.19) and (3.20), respectively. Therefore, by Lemma 3.3, we know that there exists an open neighborhood $\widehat{\mathcal{N}}$ of $\overline{\boldsymbol{X}}$ such that $\boldsymbol{G}_S$ twice continuously differentiable on $\widehat{\mathcal{N}}$, and

$$\begin{aligned}
& \boldsymbol{G}_S(\boldsymbol{X}) - \boldsymbol{G}_S(\overline{\boldsymbol{X}}) = \boldsymbol{G}_S'(\boldsymbol{X})\boldsymbol{H} + O(\|\boldsymbol{H}\|^2) \\
= \ & \Big(\sum_{k=1}^{r_0} \bar{g}_k \mathcal{P}_k'(Y)A, \sum_{l=1}^{r} \bar{g}_{r_0+l}\, \mathcal{U}_l'(Z)B\Big) + O(\|\boldsymbol{H}\|^2) \\
= \ & \Bigg(\sum_{k=1}^{r_0} \bar{g}_k P[\Omega_k(Y) \circ \widehat{A}]P^T, \\
& \sum_{l=1}^{r} \bar{g}_{r_0+l} \Big\{U[\Gamma_l(Z) \circ S(\widehat{B}_1) + \Xi_l(Z) \circ T(\widehat{B}_1)]V_1^T + U(\Upsilon_l(Z) \circ \widehat{B}_2)V_2^T\Big\}\Bigg) + O(\|\boldsymbol{H}\|^2) ,
\end{aligned} \tag{3.95}$$

where $\left(\widehat{A}, \widehat{B}\right) = \left(\widehat{A}, \left[\widehat{B}_1 \ \ \widehat{B}_2\right]\right) = \left(P^TAP, \left[U^TBV_1 \ \ U^TBV_2\right]\right) = \widehat{\boldsymbol{H}}$; $\Omega_k(Y) \in \mathcal{S}^{m_0}$, $k = 1, \ldots, r_0$ is given by (2.22); $\Gamma_l(Z)$, $\Xi_l(Z) \in \Re^{m\times m}$ and $\Upsilon_l(Z) \in \Re^{m\times(n-m)}$, $l = 1, \ldots, r$ are given by (2.53), (2.54) and (2.55) respectively. Since $\boldsymbol{g}$ s locally Lipschitz continuous near $\overline{\boldsymbol{\kappa}}$, we know that for any $\boldsymbol{X} \in \mathcal{X}$ converging to $\overline{\boldsymbol{X}}$,

$$\bar{g}_k = \begin{cases} (\boldsymbol{g}_1(\boldsymbol{\kappa}))_i + O(\|\boldsymbol{H}\|) \ \forall\, i \in \alpha_k & \text{if } 1 \leq k \leq r_0, \\ (\boldsymbol{g}_2(\boldsymbol{\kappa}))_j + O(\|\boldsymbol{H}\|) \ \forall\, j \in a_l & \text{if } r_0 + 1 \leq k = r_0 + l \leq r_0 + r. \end{cases}$$

Let $\mathcal{A} \in \mathcal{S}^{m_0}$, $\mathcal{E}_1$, $\mathcal{E}_2 \in \Re^{m\times m}$ and $\mathcal{F} \in \Re^{m\times(n-m)}$ (depending on $\boldsymbol{X} \in \mathcal{X}$) be the matrices defined by (3.12)-(3.15). Since $\boldsymbol{g}$ s locally Lipschitz continuous near $\overline{\boldsymbol{\kappa}}$, we know that $\mathcal{A}$,

$\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{F}$ are uniformly bounded on $\widehat{\mathcal{N}}$. Therefore, since $P \in \mathcal{O}^{m_0}$, $U \in \mathcal{O}^m$ and $V \in \mathcal{O}^n$ are also uniformly bounded, by shrinking $\widehat{\mathcal{N}}$ if necseeary, we know that for any $\boldsymbol{X} \in \widehat{\mathcal{N}}$,

$$\boldsymbol{G}_S(\boldsymbol{X})-\boldsymbol{G}_S(\overline{\boldsymbol{X}}) = \left(P(\mathcal{A} \circ \widehat{A})P^T, U\left[\mathcal{E}_1 \circ S(\widehat{B}_1) + \mathcal{E}_2 \circ T(\widehat{B}_1) \quad \mathcal{F} \circ \widehat{B}_2\right] V^T\right)+O(\|\boldsymbol{H}\|^2) \,. \tag{3.96}$$

Let $\boldsymbol{X} \in \mathcal{D}_{\boldsymbol{G}} \cap \widehat{\mathcal{N}}$, where $\mathcal{D}_{\boldsymbol{G}}$ is the set of points in $\mathcal{X}$, where $\boldsymbol{G}$ is (F-)differentiable. Let $\mathcal{A}^D \in \mathcal{S}^{m_0}$, $\mathcal{E}_1^D$, $\mathcal{E}_2^D \in \Re^{m\times m}$ and $\mathcal{F}^D \in \Re^{m\times(n-m)}$ be the matrices defined in (3.41)-(3.44), respectively. Since $\boldsymbol{G}$ is differentiable at $\boldsymbol{X}$, by Theorem 3.6, we know that

$$\boldsymbol{G}'(\boldsymbol{X})\boldsymbol{H} = \left(P[\boldsymbol{L}_1(\boldsymbol{\kappa}, \widetilde{\boldsymbol{H}}) + \mathcal{A}^D \circ \widetilde{A}]P^T, U[\boldsymbol{L}_2(\boldsymbol{\kappa}, \widetilde{\boldsymbol{H}}) + \mathcal{T}(\boldsymbol{\kappa}, \widetilde{B})]V^T\right) \,, \tag{3.97}$$

where $\boldsymbol{L}(\boldsymbol{\kappa}, \cdot) = (\boldsymbol{L}_1(\boldsymbol{\kappa}, \cdot), \boldsymbol{L}_2(\boldsymbol{\kappa}, \cdot))$ and $\mathcal{T}(\boldsymbol{\kappa}, \cdot)$ are given by (3.47) and (3.49), respectively with $\overline{\boldsymbol{\kappa}}$ being replaced by $\boldsymbol{\kappa}$. Denote

$$\Delta(\boldsymbol{H}) = (\Delta_1(\boldsymbol{H}), \Delta_2(\boldsymbol{H})) = \boldsymbol{G}'(\boldsymbol{X})\boldsymbol{H} - (\boldsymbol{G}_S(\boldsymbol{X}) - \boldsymbol{G}_S(\overline{\boldsymbol{X}}))\,.$$

From (3.96) and (3.97), we obtain that

$$\Delta_1(\boldsymbol{H}) = P \begin{bmatrix} R_1(\boldsymbol{H}) & 0 & \cdots & 0 \\ 0 & R_2(\boldsymbol{H}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_{r_0}(\boldsymbol{H}) \end{bmatrix} P^T + O(\|\boldsymbol{H}\|^2) \tag{3.98}$$

and

$$\Delta_2(\boldsymbol{H}) = U \begin{bmatrix} R_{r_0+1}(\boldsymbol{H}) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & R_{r_0+r}(\boldsymbol{H}) & 0 \\ 0 & \cdots & 0 & R_{r_0+r+1}(\boldsymbol{H}) \end{bmatrix} V^T + O(\|\boldsymbol{H}\|^2)\,, \tag{3.99}$$

where

$$R_k(\boldsymbol{H}) = \mathrm{diag}\left((\theta(\boldsymbol{\kappa}, \widetilde{\boldsymbol{H}}))_{\alpha_k}\right) + (\mathcal{A}^D)_{\alpha_k\alpha_k} \circ \widetilde{A}_{\alpha_k\alpha_k}, \quad 1 \le k \le r_0\,, \tag{3.100}$$

$$R_{r_0+l}(\boldsymbol{H}) = \operatorname{diag}\Big((\theta(\boldsymbol{\kappa}, \widetilde{\boldsymbol{H}}))_{\alpha_k}\Big) + (\mathcal{E}_1^D)_{a_l a_l} \circ S(\widetilde{B}_{a_l a_l}) + (\mathcal{E}_2^D)_{a_l a_l} \circ T(\widetilde{B}_{a_l a_l}),\ 1 \le l \le r_0\,, \tag{3.101}$$

$$R_{r_0+r+1}(\boldsymbol{H}) = \operatorname{diag}\Big((\theta(\boldsymbol{\kappa}, \widetilde{\boldsymbol{H}}))_{\alpha_{r_0+r+1}}\Big) + \Big[(\mathcal{E}_1^D)_{bb} \circ S(\widetilde{B}_{bb}) + (\mathcal{E}_2^D)_{bb} \circ T(\widetilde{B}_{bb}) \ \ (\mathcal{F}^D)_b \circ \widetilde{B}_{bc}\Big]\,. \tag{3.102}$$

By (3.16), we obtain from (3.94) that

$$\Lambda(\overline{Y}) + \overline{P}^T A \overline{P} = \overline{P}^T P \Lambda(Y) P^T \overline{P} \quad \text{and} \quad \big[\Sigma(\overline{Z}) \ \ 0\big] + \overline{U}^T B \overline{V} = \overline{U}^T U\, [\Sigma(Z) \ \ 0]\, V^T \overline{V}\,.$$

Let $\widehat{\boldsymbol{H}} := (\widehat{A}, \widehat{B}) = (\overline{P}^T A \overline{P}, \overline{U}^T B \overline{V})$, $\widehat{P} = \overline{P}^T P$, $\widehat{U} := \overline{U}^T U$ and $\widehat{V} := \overline{V}^T V$. Then,

$$P^T A P = \widehat{P}^T \overline{P}^T A \overline{P} \widehat{P} = \widehat{P}^T \widehat{A} \widehat{P} \quad \text{and} \quad U^T B V = \widehat{U}^T \overline{U}^T B \overline{V} \widehat{V} = \widehat{U}^T \widehat{B} \widehat{V}\,.$$

From (2.10), (2.12) and (2.40), we know that there exist $Q_k \in \mathcal{O}^{|\alpha_k|}$, $k = 1, \ldots, r_0$, $Q_{r_0+l} \in \mathcal{O}^{|a_l|}$, $l = 1, \ldots, r$ and $M \in \mathcal{O}^{|b|}$, $N \in \mathcal{O}^{n-|a|}$ such that

$$P_{\alpha_k}^T A P_{\alpha_k} = \widehat{P}_{\alpha_k}^T \widehat{A} \widehat{P}_{\alpha_k} = Q_k^T \widehat{A}_{\alpha_k \alpha_k} Q_k + O(\|\boldsymbol{H}\|^2), \quad 1 \le k \le r_0\,,$$

$$U_{a_l}^T B V_{a_l} = \widehat{U}_{a_l}^T \widehat{B} \widehat{V}_{a_l} = Q_{r_0+l}^T \widehat{B}_{a_l a_l} Q_{r_0+l} + O(\|\boldsymbol{H}\|^2), \quad 1 \le l \le r$$

and

$$\big[U_b^T B V_b \ \ U_b^T B V_2\big] = \Big[\widehat{U}_b^T \widehat{B} \widehat{V}_b \ \ \widehat{U}_b^T \widehat{B} \widehat{V}_2\Big] = M^T \Big[\widehat{B}_{bb} \ \ \widehat{B}_{bc}\Big] N + O(\|\boldsymbol{H}\|^2)\,.$$

From (2.13), (2.41) and (2.42), we obtain that

$$P_{\alpha_k}^T A P_{\alpha_k} = \Lambda(Y)_{\alpha_k \alpha_k} - \Lambda(\overline{Y})_{\alpha_k \alpha_k} + O(\|\boldsymbol{H}\|^2), \quad 1 \le k \le r_0\,,$$

$$S(U_{a_l}^T B V_{a_l}) = Q_{r_0+l}^T S(\widehat{B}_{a_l a_l}) Q_{r_0+l} + O(\|\boldsymbol{H}\|^2) = \Sigma(Z)_{a_l a_l} - \Sigma(\overline{Z})_{a_l a_l} + O(\|\boldsymbol{H}\|^2), \quad 1 \le l \le r$$

and

$$\big[U_b^T B V_b \ \ U_b^T B V_2\big] = M^T \Big[\widehat{B}_{bb} \ \ \widehat{B}_{bc}\Big] N = \big[\Sigma(Z)_{bb} - \Sigma(\overline{Z})_{bb} \ \ 0\big] + O(\|\boldsymbol{H}\|^2)\,.$$

Let $\boldsymbol{h} := (\boldsymbol{h}_1, \boldsymbol{h}_2) = (\lambda'(Y; A), \sigma'(Z; B)) \in \Re^{m_0} \times \Re^m$. Since $\lambda(\cdot)$ and $\sigma(\cdot)$ are strongly semismooth [96], we know that

$$\begin{aligned} \widetilde{A}_{\alpha_k \alpha_k} = P_{\alpha_k}^T A P_{\alpha_k} &= \operatorname{diag}(\lambda_i'(Y; A) : \ i \in \alpha_k) + O(\|\boldsymbol{H}\|^2) \\ &= \operatorname{diag}((\boldsymbol{h}_1)_{\alpha_k}) + O(\|\boldsymbol{H}\|^2)\,, \quad 1 \le k \le r_0\,, \end{aligned} \tag{3.103}$$

$$\begin{aligned} S(\widetilde{B}_{a_l a_l}) = S(U_{a_l}^T B V_{a_l}) &= \operatorname{diag}(\sigma_i'(Z;B):\ i \in a_l) + O(\|\boldsymbol{H}\|^2) \\ &= \operatorname{diag}((\boldsymbol{h}_2)_{a_l}) + O(\|\boldsymbol{H}\|^2)\,, \quad 1 \le l \le r \qquad (3.104) \end{aligned}$$

and

$$\begin{aligned} \begin{bmatrix} \widetilde{B}_{bb} & \widetilde{B}_{bc} \end{bmatrix} = \begin{bmatrix} U_b^T B V_b & U_b^T B V_2 \end{bmatrix} &= \begin{bmatrix} \operatorname{diag}(\sigma_i'(Z;B):\ i \in b) & 0 \end{bmatrix} + O(\|\boldsymbol{H}\|^2) \\ &= [\operatorname{diag}((\boldsymbol{h}_2)_b) \quad 0] + O(\|\boldsymbol{H}\|^2)\,. \qquad (3.105) \end{aligned}$$

Therefore, by (3.100), (3.101) and (3.102), we obtain from (3.98) and (3.99) that

$$\Delta(\boldsymbol{H}) = \left(P\operatorname{diag}\left((\boldsymbol{g}'(\boldsymbol{\kappa})\boldsymbol{h})_{\mathcal{I}_1}\right)P^T, U\left[\operatorname{diag}\left((\boldsymbol{g}'(\boldsymbol{\kappa})\boldsymbol{h})_{\mathcal{I}_2}\right) \quad 0\right]V^T\right) + O(\|\boldsymbol{H}\|^2)\,. \quad (3.106)$$

On the other hand, for $\boldsymbol{H} \in \mathcal{X}$ sufficiently close to 0, we have $\mathcal{P}_k(Y) = \sum_{i\in\alpha_k} p_i p_i^T$, $k = 1,\ldots,r_0$ and $\mathcal{U}_l(Z) = \sum_{i\in a_l} u_i v_i^T$, $l = 1,\ldots,r$. Therefore,

$$\begin{aligned} \boldsymbol{G}_R(\boldsymbol{X}) &= \boldsymbol{G}(\boldsymbol{X}) - \boldsymbol{G}_S(\boldsymbol{X}) \\ &= (\sum_{k=1}^{r_0}\sum_{i\in\alpha_k}[(\boldsymbol{g}_1(\boldsymbol{\kappa}))_i - (\boldsymbol{g}_1(\overline{\boldsymbol{\kappa}}))_i]p_i p_i^T, \sum_{k=r_0+1}^{r_0+r+1}\sum_{i\in a_l}[(\boldsymbol{g}_2(\boldsymbol{\kappa}))_i - (\boldsymbol{g}_2(\overline{\boldsymbol{\kappa}}))_i]u_i v_i^T). \quad (3.107) \end{aligned}$$

Note that by Theorem 3.6, we know that $\boldsymbol{G}$ is F-differentiable at $\boldsymbol{X}$ if and only if $\boldsymbol{g}$ is F-differentiable at $\boldsymbol{\kappa}$. Since $\boldsymbol{g}$ is 1-order G-semismooth at $\overline{\boldsymbol{\kappa}}$, $\lambda(\cdot)$ and $\sigma(\cdot)$ are strongly semismooth at $\overline{Y}$ and $\overline{Z}$ [96], we obtain that for any $\boldsymbol{Y} \in \mathcal{D}_{\boldsymbol{G}} \cap \widehat{\mathcal{N}}$ (shrinking $\widehat{\mathcal{N}}$ if necessary),

$$\begin{aligned} \boldsymbol{g}(\boldsymbol{\kappa}) - \boldsymbol{g}(\overline{\boldsymbol{\kappa}}) &= \boldsymbol{g}'(\boldsymbol{\kappa})(\boldsymbol{\kappa} - \overline{\boldsymbol{\kappa}}) + O(\|\boldsymbol{H}\|^2) \\ &= \boldsymbol{g}'(\boldsymbol{\kappa})(\boldsymbol{h} + O(\|\boldsymbol{H}\|^2)) + O(\|\boldsymbol{H}\|^2) \\ &= \left((\boldsymbol{g}'(\boldsymbol{\kappa})\boldsymbol{h})_{\mathcal{I}_1}, (\boldsymbol{g}'(\boldsymbol{\kappa})\boldsymbol{h})_{\mathcal{I}_2}\right) + O(\|\boldsymbol{H}\|^2)\,. \end{aligned}$$

Then, since $P \in \mathcal{O}^{m_0}$, $U \in \mathcal{O}^m$ and $U \in \mathcal{O}^n$ are uniformly bounded, we obtain from (3.107) that

$$\boldsymbol{G}_R(\boldsymbol{X}) = \left(P\operatorname{diag}\left((\boldsymbol{g}'(\boldsymbol{\kappa})\boldsymbol{h})_{\mathcal{I}_1}\right)P^T, U\left[\operatorname{diag}\left((\boldsymbol{g}'(\boldsymbol{\kappa})\boldsymbol{h})_{\mathcal{I}_2}\right) \quad 0\right]V^T\right) + O(\|\boldsymbol{H}\|^2)\,.$$

Thus, from (3.106), we obtain that

$$\Delta(\boldsymbol{H}) = \boldsymbol{G}_R(\boldsymbol{X}) + O(\|\boldsymbol{H}\|^2)\,.$$

That is, for any $\boldsymbol{X} \in \mathcal{D}_{\boldsymbol{G}}$ converging to $\overline{\boldsymbol{X}}$,

$$\begin{aligned} \boldsymbol{G}(\boldsymbol{X}) - \boldsymbol{G}(\overline{\boldsymbol{X}}) - \boldsymbol{G}'(\boldsymbol{X})\boldsymbol{H} &= \boldsymbol{G}_S(\boldsymbol{X}) - \boldsymbol{G}_S(\overline{\boldsymbol{X}}) - \boldsymbol{G}'(\boldsymbol{X})\boldsymbol{H} + \boldsymbol{G}_R(\boldsymbol{X}) \\ &= -\Delta(\boldsymbol{H}) + \boldsymbol{G}_R(\boldsymbol{X}) = O(\|\boldsymbol{H}\|^2)\,. \end{aligned}$$

" $\Longrightarrow$ " Let $\overline{P} \in \mathcal{O}^{m_0}(\overline{Y})$ and $(\overline{U}, \overline{V}) \in \mathcal{O}^{m\times n}(\overline{Z})$ be fixed. Assume that $\boldsymbol{\kappa} = (\lambda, \sigma) = \overline{\boldsymbol{\kappa}} + \boldsymbol{h} \in \mathcal{D}_{\boldsymbol{g}}$ and $\boldsymbol{h} = (\boldsymbol{h}_1, \boldsymbol{h}_2) \in \Re^{m_0} \times \Re^m_+$ sufficiently small. Let $\boldsymbol{X} = \left(\overline{P}\mathrm{diag}(\lambda)\overline{P}^T, \overline{U}\,[\mathrm{diag}(\sigma)\ \ 0]\,\overline{V}^T\right)$ and $\boldsymbol{H} := \left(\overline{P}\mathrm{diag}(\boldsymbol{h}_1)\overline{P}^T, \overline{U}\,[\mathrm{diag}(\sigma)\ \ 0]\,\overline{V}^T\right)$. Then, we know that $\boldsymbol{X} \in \mathcal{D}_{\boldsymbol{G}}$ and converges to $\overline{\boldsymbol{X}}$. Therefore, we have

$$\boldsymbol{G}(\boldsymbol{X}) - \boldsymbol{G}(\overline{\boldsymbol{X}}) = \left(\overline{P}\mathrm{diag}(\boldsymbol{g}_1(\overline{\boldsymbol{\kappa}} + \boldsymbol{h}) - \boldsymbol{g}_1(\overline{\boldsymbol{\kappa}}))\overline{P}^T, \overline{U}\mathrm{diag}(\boldsymbol{g}_2(\overline{\boldsymbol{\kappa}} + \boldsymbol{h}) - \boldsymbol{g}_2(\overline{\boldsymbol{\kappa}}))\overline{V}_1^T\right)$$

and

$$\boldsymbol{G}'(\boldsymbol{X})\boldsymbol{H} = \left(P\mathrm{diag}\left((\boldsymbol{g}'(\boldsymbol{\kappa})\boldsymbol{h})_{\mathcal{I}_1}\right)P^T, U\left[\mathrm{diag}\left((\boldsymbol{g}'(\boldsymbol{\kappa})\boldsymbol{h})_{\mathcal{I}_2}\right)\ \ 0\right]V^T\right)\,.$$

Then, from the 1-order G-semismoothness of $\boldsymbol{G}$ at $\overline{\boldsymbol{X}}$, we know that $\boldsymbol{g}$ is 1-order G-semismooth at $\overline{\boldsymbol{\kappa}}$. □

3.7 The characterization of Clarke's generalized Jacobian

Let $\overline{\boldsymbol{X}} = (\overline{Y}, \overline{Z}) \in \mathcal{S}^{m_0} \times \Re^{m\times n} = \mathcal{X}$ be given. In this section, we also assume that $\boldsymbol{g}$ is Lipschitz continuous on an open neighborhood $\mathcal{N}_{\overline{\boldsymbol{\kappa}}} \subseteq \Re^{m_0} \times \Re^m$ of $\overline{\boldsymbol{\kappa}} = \boldsymbol{\kappa}(\overline{\boldsymbol{X}})$. Therefore, we know from Theorem 3.10 that the corresponding spectral operator $\boldsymbol{G}$ is locally Lipschitz continuous near $\overline{\boldsymbol{X}}$. In order to characterize the B-subdifferential and Clarke's generalized Jacobian of spectral operators, we first introduce some notations. Define a subset $\mathcal{D}_{\boldsymbol{g}}^{\downarrow} \subseteq \mathcal{N}_{\overline{\boldsymbol{\kappa}}}$ by

$$\mathcal{D}_{\boldsymbol{g}}^{\downarrow} := \left\{(\boldsymbol{y}_1, \boldsymbol{y}_2) \in \mathcal{N}_{\overline{\boldsymbol{\kappa}}} \,|\, \boldsymbol{g} \text{ is F-differentiable at } \boldsymbol{y}, \text{ and } \boldsymbol{y}_1,\ \boldsymbol{y}_2 \text{ are in non-increasing order}\right\}.$$

For any $\boldsymbol{\kappa} \in \mathcal{D}_{\boldsymbol{g}}^{\downarrow}$, let $\boldsymbol{J}(\boldsymbol{\kappa}, \cdot) : \mathcal{X} \to \mathcal{X}$ be the linear operator given by

$$\boldsymbol{J}(\boldsymbol{\kappa}, \boldsymbol{Z}) := (\boldsymbol{J}_1(\boldsymbol{\kappa}, A), \boldsymbol{J}_2(\boldsymbol{\kappa}, B)), \quad \boldsymbol{Z} = (A, B) \in \mathcal{X}\,, \tag{3.108}$$

with

$$\boldsymbol{J}_1(\boldsymbol{\kappa}, A) = \begin{bmatrix} \left(\mathcal{A}^D(\boldsymbol{\kappa})\right)_{\alpha_1\alpha_1} \circ A_{\alpha_1\alpha_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \left(\mathcal{A}^D(\boldsymbol{\kappa})\right)_{\alpha_{r_0}\alpha_{r_0}} \circ A_{\alpha_{r_0}\alpha_{r_0}} \end{bmatrix} \in \mathcal{S}^{m_0}$$

and

$$\boldsymbol{J}_2(\boldsymbol{\kappa}, B) = \begin{bmatrix} \left(\mathcal{E}_1^D(\boldsymbol{\kappa})\right)_{a_1a_1} \circ S(B_{a_1a_1}) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \left(\mathcal{E}_1^D(\boldsymbol{\kappa})\right)_{a_ra_r} \circ S(B_{a_ra_r}) & 0 \\ 0 & \cdots & 0 & (\mathcal{T}(\boldsymbol{\kappa}, B))_{b\bar{a}} \end{bmatrix} \in \Re^{m\times n},$$

where $\mathcal{A}^D(\boldsymbol{\kappa}) \in \mathcal{S}^{m_0}$, $\mathcal{E}_1^D(\boldsymbol{\kappa})$, $\mathcal{E}_2^D(\boldsymbol{\kappa}) \in \Re^{m\times m}$ and $\mathcal{F}^D(\boldsymbol{\kappa}) \in \Re^{m\times(n-m)}$ are the matrices given by (3.41)-(3.44), respectively, and $\mathcal{T}(\boldsymbol{\kappa}, \cdot)$ are given by (3.49). Denote

$$\mathcal{V}_{\overline{\boldsymbol{\kappa}}} := \left\{ \boldsymbol{V}(\cdot) = (\boldsymbol{V}_1(\cdot), \boldsymbol{V}_2(\cdot)) : \mathcal{X} \to \mathcal{X} \mid \boldsymbol{V}(\cdot) = \lim_{\mathcal{D}_{\boldsymbol{g}}^{\downarrow} \ni \boldsymbol{\kappa} \to \overline{\boldsymbol{\kappa}}} \boldsymbol{L}(\boldsymbol{\kappa}, \cdot) + \boldsymbol{J}(\boldsymbol{\kappa}, \cdot) \right\}, \tag{3.109}$$

where for each $\boldsymbol{\kappa} \in \mathcal{D}_{\boldsymbol{g}}^{\downarrow}$, the linear operator $\boldsymbol{L}(\boldsymbol{\kappa}, \cdot) : \mathcal{X} \to \mathcal{X}$ is given by (3.47). Let $\mathcal{K}_{\overline{\boldsymbol{\kappa}}}$ be the set of linear operators such that $\boldsymbol{K}(\cdot) = (\boldsymbol{K}_1(\cdot), \boldsymbol{K}_2(\cdot)) \in \mathcal{K}_{\overline{\boldsymbol{\kappa}}}$ if and only if there exist $Q_k \in \mathcal{O}^{|\alpha_k|}$, $k = 1, \ldots, r_0 + r$, $Q' \in \mathcal{O}^{|b|}$, $Q'' \in \mathcal{O}^{n-|a|}$ and $\boldsymbol{V} = (\boldsymbol{V}_1, \boldsymbol{V}_2) \in \mathcal{V}_{\overline{\boldsymbol{\kappa}}}^{g}$ such that

$$\boldsymbol{K}(\boldsymbol{Z}) = (\boldsymbol{K}_1(\boldsymbol{Z}), \boldsymbol{K}_2(\boldsymbol{Z})) = \left(Q\boldsymbol{V}_1(\widehat{\boldsymbol{Z}})Q^T, M\boldsymbol{V}_2(\widehat{\boldsymbol{Z}})N^T\right) \in \mathcal{X}, \quad \boldsymbol{Z} = (A, B) \in \mathcal{X}\,, \tag{3.110}$$

where $Q = \mathrm{diag}(Q_1, \ldots, Q_{r_0}) \in \mathcal{O}^{m_0}$,

$$M = \mathrm{diag}(Q_{r_0+1}, \ldots, Q_{r_0+r}, Q') \in \mathcal{O}^m \quad \text{and} \quad N = \mathrm{diag}(Q_{r_0+1}, \ldots, Q_{r_0+r}, Q'') \in \mathcal{O}^n\,,$$

and $\widehat{\boldsymbol{Z}} = (Q^TAQ, M^TBN) \in \mathcal{X}$. Therefore, we obtain the following characterization of the B(ouligand)-subdifferential $\partial_B \boldsymbol{G}(\overline{\boldsymbol{X}})$ of the spectral operator $\boldsymbol{G}$ at $\overline{\boldsymbol{X}}$.

Theorem 3.13. *Let $\overline{\boldsymbol{X}} = (\overline{Y}, \overline{Z}) \in \mathcal{S}^{m_0} \times \Re^{m\times n} = \mathcal{X}$ be given. Suppose that $\overline{Y}$ and $\overline{Z}$ have the decomposition (3.16). Assume that the symmetric mapping $\boldsymbol{g}$ is locally Lipschitz continuous near $\overline{\boldsymbol{\kappa}} = \boldsymbol{\kappa}(\overline{\boldsymbol{X}})$. Then, $\mathcal{U} \in \partial_B \boldsymbol{G}(\overline{\boldsymbol{X}})$ if and only if there exists $\boldsymbol{K} = (\boldsymbol{K}_1, \boldsymbol{K}_2) \in \mathcal{K}_{\overline{\boldsymbol{\kappa}}}$ such that*

$$\mathcal{U}(\boldsymbol{H}) = \left(\overline{P}\left(\boldsymbol{K}_1(\widetilde{\boldsymbol{H}}) + \boldsymbol{T}_1(\widetilde{A})\right)\overline{P}^T, \overline{U}\left(\boldsymbol{K}_2(\widetilde{\boldsymbol{H}}) + \boldsymbol{T}_2(\widetilde{B})\right)\overline{V}^T\right) \quad \forall\, \boldsymbol{H} = (A,B) \in \mathcal{X}\,, \tag{3.111}$$

where the linear operator $\boldsymbol{T}(\cdot) = (\boldsymbol{T}_1(\cdot), \boldsymbol{T}_2(\cdot)) : \mathcal{X} \to \mathcal{X}$ is defined in (3.18) and $\widetilde{\boldsymbol{H}} = (\widetilde{A}, \widetilde{B}) = \left(\overline{P}^T A \overline{P}, \overline{U}^T B \overline{V}\right)$.

Proof. " $\Longrightarrow$ " By the definition of $\partial_B \boldsymbol{G}(\overline{\boldsymbol{X}})$, we know that there exists a sequence $\{\boldsymbol{X}^t\}$ in $\mathcal{D}_{\boldsymbol{G}}$ converging to $\overline{\boldsymbol{X}}$ such that

$$\mathcal{U} = \lim_{t\to\infty} \boldsymbol{G}'(\boldsymbol{X}^t)\,.$$

For each $\boldsymbol{X}^t = (Y^t, Z^t)$, let $P^t \in \mathcal{O}^{m_0}$, $U^t \in \mathcal{O}^m$ and $V^t \in \mathcal{O}^n$ be the orthogonal matrices such that

$$Y^t = P^t \Lambda(Y^t)(P^t)^T \quad \text{and} \quad Z^t = U^t[\Sigma(Z^t) \;\; 0](V^t)^T\,.$$

For each t, let $\boldsymbol{\kappa}^t = \boldsymbol{\kappa}(\boldsymbol{X}^t)$. Let $\boldsymbol{G}_S$ and $\boldsymbol{G}_R$ be defined by (3.19) and (3.20), respectively. Therefore, by taking the subsequence if necessary, we know from Lemma 3.3 that for each t, $\boldsymbol{G}_S$ is twice continuously differentiable at $\boldsymbol{X}^t$ and

$$\lim_{t\to\infty} \boldsymbol{G}_S'(\boldsymbol{X}^t) = \boldsymbol{G}_S'(\overline{X})\,.$$

Hence, we know that

$$\lim_{t\to\infty} \boldsymbol{G}_S'(\boldsymbol{X}^t)\boldsymbol{H} = \boldsymbol{G}_S'(\overline{X})\boldsymbol{H} = \left(\boldsymbol{T}_1(\widetilde{A}), \boldsymbol{T}_2(\widetilde{B})\right) = \boldsymbol{T}(\widetilde{\boldsymbol{H}}), \quad \boldsymbol{H} = (A,B) \in \mathcal{X}\,, \tag{3.112}$$

where $\widetilde{\boldsymbol{H}} = (\widetilde{A}, \widetilde{B})$ with $\widetilde{A} = \overline{P}^T A \overline{P}$, $\widetilde{B} = \begin{bmatrix}\widetilde{B}_1 & \widetilde{B}_2\end{bmatrix} = \begin{bmatrix}\overline{U}^T B \overline{V}_1 & \overline{U}^T B \overline{V}_2\end{bmatrix}$, and the linear operator $\boldsymbol{T}(\cdot) = (\boldsymbol{T}_1(\cdot), \boldsymbol{T}_2(\cdot)) : \mathcal{X} \to \mathcal{X}$ is given by (3.18).

Next, consider the function $\boldsymbol{G}_R(\cdot) = \boldsymbol{G}(\cdot) - \boldsymbol{G}_S(\cdot)$. By the assumption, we know that $\boldsymbol{G}_R$ is differentiable at each $\boldsymbol{X}^t$. Furthermore, since $\lambda(\cdot)$ and $\sigma(\cdot)$ are globally Lipschitz continuous, we may also assume that for each $\boldsymbol{X}^t$,

$$\begin{cases} \lambda_i(Y^t) \neq \lambda_j(Y^t) & \text{if } i \in \alpha_k,\ j \in \alpha_{k'} \text{ and } 1 \leq k \neq k' \leq r_0, \\ \sigma_i(Z^t) \neq \sigma_j(Z^t) & \text{if } i \in a_l,\ j \in a_{l'} \text{ and } 1 \leq l \neq l' \leq r+1. \end{cases}$$

Therefore, by (3.50) in Theorem 3.6 and (2.23) and (2.56), we obtain that for each t and $\boldsymbol{H} \in \mathcal{X}$,

$$\begin{aligned} \boldsymbol{G}_R'(\boldsymbol{X}^t)\boldsymbol{H} &= \boldsymbol{G}'(\boldsymbol{X}^t)\boldsymbol{H} - \boldsymbol{G}_S'(\boldsymbol{X}^t)\boldsymbol{H} \\ &= \boldsymbol{G}'(\boldsymbol{X}^t)\boldsymbol{H} - \Big(\sum_{k=1}^{r_0} \bar{g}_k \mathcal{P}_k(Y^t), \sum_{l=1}^{r} \bar{g}_{r_0+l}\,\mathcal{U}_l(Z^t)\Big) \\ &= \Big(P^t(\boldsymbol{L}_1(\boldsymbol{\kappa}^t, \widehat{\boldsymbol{H}}^t) + \boldsymbol{J}_1(\boldsymbol{\kappa}^t, \widehat{A}^t) + \Theta_1(\boldsymbol{\kappa}^t, \widehat{A}^t))(P^t)^T, \\ &\qquad U^t(\boldsymbol{L}_2(\boldsymbol{\kappa}^t, \widehat{\boldsymbol{H}}^t) + \boldsymbol{J}_2(\boldsymbol{\kappa}^t, \widehat{B}^t) + \Theta_2(\boldsymbol{\kappa}^t, \widehat{B}^t))(V^t)^T\Big)\,, \end{aligned} \tag{3.113}$$

where $\widehat{\boldsymbol{H}}^t = \Big(\widehat{A}^t, \widehat{B}^t\Big) = \big((P^t)^T A P^t, (U^t)^T B V^t\big)$, and for each t, $\Theta_1(\boldsymbol{\kappa}^t, \widehat{A}^t) \in \mathcal{S}^{m_0}$ and $\Theta_2(\boldsymbol{\kappa}^t, \widehat{B}^t) \in \Re^{m\times n}$ are given by

$$\Theta_1(\boldsymbol{\kappa}^t, \widehat{A}^t) = \widetilde{\mathcal{A}}(\boldsymbol{\kappa}^t)\circ\widehat{A}^t \quad \text{and} \quad \Theta_2(\boldsymbol{\kappa}^t, \widehat{B}^t) = \Big[\widetilde{\mathcal{E}}_1(\boldsymbol{\kappa}^t)\circ S(\widehat{B}_1^t) + \widetilde{\mathcal{E}}_2(\boldsymbol{\kappa}^t)\circ T(\widehat{B}_1^t) \ \ \widetilde{\mathcal{F}}(\boldsymbol{\kappa}^t)\circ \widehat{B}_2^t\Big]\,,$$

with $\widetilde{\mathcal{A}}(\boldsymbol{\kappa}^t) \in \mathcal{S}^{m_0}$, $\widetilde{\mathcal{E}}_1(\boldsymbol{\kappa}^t), \widetilde{\mathcal{E}}_2(\boldsymbol{\kappa}^t) \in \Re^{m\times m}$ and $\widetilde{\mathcal{F}}(\boldsymbol{\kappa}^t) \in \Re^{m\times(n-m)}$ by

$$(\widetilde{\mathcal{A}}(\boldsymbol{\kappa}^t))_{ij} := \begin{cases} \dfrac{\boldsymbol{g}_i(\boldsymbol{\kappa}^t) - \bar{g}_k - \boldsymbol{g}_j(\boldsymbol{\kappa}^t) + \bar{g}_{k'}}{\lambda_i(Y^t) - \lambda_j(Y^t)} & \text{if } i \in \alpha_k,\ j \in \alpha_{k'} \text{ and } 1 \leq k \neq k' \leq r_0, \\ 0 & \text{if } i, j \in \alpha_k \text{ and } 1 \leq k \leq r_0\,, \end{cases} \tag{3.114}$$

$$(\widetilde{\mathcal{E}}_1(\boldsymbol{\kappa}^t))_{ij} := \begin{cases} \dfrac{\boldsymbol{g}_i(\boldsymbol{\kappa}^t) - \bar{g}_{r_0+l} - \boldsymbol{g}_j(\boldsymbol{\kappa}^t) + \bar{g}_{r_0+l'}}{\sigma_i(Z^t) - \sigma_j(Z^t)} & \text{if } i \in a_l,\ j \in a_{l'} \text{ and } 1 \leq l \neq l' \leq r+1, \\ 0 & \text{if } i, j \in a_l \text{ and } 1 \leq l \leq r+1\,, \end{cases} \tag{3.115}$$

$$(\widetilde{\mathcal{E}}_2(\boldsymbol{\kappa}^t))_{ij} := \begin{cases} \dfrac{\boldsymbol{g}_i(\boldsymbol{\kappa}^t) - \bar{g}_{r_0+l} + \boldsymbol{g}_j(\boldsymbol{\kappa}^t) - \bar{g}_{r_0+l'}}{\sigma_i(Z^t) + \sigma_j(Z^t)} & \text{if } i \text{ or } j \notin b \\ 0 & \text{if } i, j \in b\,, \end{cases} \tag{3.116}$$

$$(\widetilde{\mathcal{F}}(\boldsymbol{\kappa}^t))_{ij} := \begin{cases} \dfrac{\boldsymbol{g}_i(\boldsymbol{\kappa}^t) - \bar{g}_{r_0+l}}{\sigma_i(Z^t)} & \text{if } i \notin b, \\ 0 & \text{otherwise.} \end{cases} \tag{3.117}$$

Since $\boldsymbol{\kappa}^t$ converges to $\overline{\boldsymbol{\kappa}}$ and by the continuity of $\boldsymbol{g}$, we know that

$$\lim_{t\to\infty} \widetilde{\mathcal{A}}(\boldsymbol{\kappa}^t) = 0, \quad \lim_{t\to\infty} \widetilde{\mathcal{E}}_1(\boldsymbol{\kappa}^t) = 0, \quad \lim_{t\to\infty} \widetilde{\mathcal{E}}_2(\boldsymbol{\kappa}^t) = 0, \quad \text{and} \lim_{t\to\infty} \widetilde{\mathcal{F}}(\boldsymbol{\kappa}^t) = 0\,. \tag{3.118}$$

Denote the linear operator $\boldsymbol{L}(\boldsymbol{\kappa}^t,\cdot) + \boldsymbol{J}(\boldsymbol{\kappa}^t,\cdot) : \mathcal{X} \to \mathcal{X}$ by

$$\boldsymbol{L}(\boldsymbol{\kappa}^t,\cdot) + \boldsymbol{J}(\boldsymbol{\kappa}^t,\cdot) := \left(\boldsymbol{L}_1(\boldsymbol{\kappa}^t,\cdot) + \boldsymbol{J}_1(\boldsymbol{\kappa}^t,\cdot), \boldsymbol{L}_2(\boldsymbol{\kappa}^t,\cdot) + \boldsymbol{J}_2(\boldsymbol{\kappa}^t,\cdot)\right)\,.$$

By taking subsequence if necessary, we may assume that the sequence of linear operators $\{\boldsymbol{L}(\boldsymbol{\kappa}^t,\cdot) + \boldsymbol{J}(\boldsymbol{\kappa}^t,\cdot)\}$ converges. Therefore, by (3.109), we know that there exists $\boldsymbol{V} = (\boldsymbol{V}_1, \boldsymbol{V}_2) \in \mathcal{V}_{\overline{\boldsymbol{\kappa}}}$ such that

$$\lim_{t\to\infty} \boldsymbol{L}(\boldsymbol{\kappa}^t,\cdot) + \boldsymbol{J}(\boldsymbol{\kappa}^t,\cdot) = \boldsymbol{V}(\cdot)\,. \tag{3.119}$$

Since $\{P^t\}$, $\{U^t\}$ and $\{V^t\}$ are uniformly bounded, by taking subsequence if necessary, we may assume that $\{P^t\}$, $\{U^t\}$ and $\{V^t\}$ converge and denote the limits by $P^\infty \in \mathcal{O}^{m_0}$, $U^\infty \in \mathcal{O}^m$ and $V^\infty \in \mathcal{O}^n$, respectively. Then, it is easy to see taht

$$\overline{P}\Lambda(\overline{Y})\overline{P}^T = \overline{Y} = P^\infty \Lambda(\overline{Y})(P^\infty)^T$$

and

$$\overline{U}[\Sigma(\overline{Z}) \;\; 0]\overline{V}^T = \overline{Z} = U^\infty[\Sigma(\overline{Z}) \;\; 0](V^\infty)^T\,.$$

Therefore, from Proposition 2.4 and Proposition 2.14, we know that there exist $Q_k \in \mathcal{O}^{|\alpha_k|}$, $k = 1, \ldots, r_0 + r$, $Q' \in \mathcal{O}^{|b|}$ and $Q'' \in \mathcal{O}^{n-|a|}$ such that

$$P^\infty = \overline{P}Q, \quad U^\infty = \overline{U}M \quad \text{and} \quad V^\infty = \overline{V}N\,,$$

with $Q = \operatorname{diag}(Q_1, \ldots, Q_{r_0}) \in \mathcal{O}^{m_0}$,

$$M = \operatorname{diag}(Q_{r_0+1}, \ldots, Q_{r_0+r}, Q') \in \mathcal{O}^m \quad \text{and} \quad N = \operatorname{diag}(Q_{r_0+1}, \ldots, Q_{r_0+r}, Q'') \in \mathcal{O}^n\,.$$

Therefore, from (3.113), (3.118) and (3.119), we obtain that for any $\boldsymbol{H} \in \mathcal{X}$,

$$\begin{aligned} \lim_{t\to\infty} \boldsymbol{G}'_R(\boldsymbol{X}^t)\boldsymbol{H} &= \left(P^\infty \boldsymbol{V}_1(\widehat{\boldsymbol{H}})(P^\infty)^T, U^\infty \boldsymbol{V}_2(\widehat{\boldsymbol{H}})(V^\infty)^T\right) \\ &= \left(\overline{P}Q\boldsymbol{V}_1(\widehat{\boldsymbol{H}})Q^T\overline{P}^T, \overline{U}M\boldsymbol{V}_2(\widehat{\boldsymbol{H}})N^T\overline{V}^T\right) \\ &= \left(\overline{P}\boldsymbol{K}_1(\widetilde{\boldsymbol{H}})\overline{P}^T, \overline{U}\boldsymbol{K}_2(\widetilde{\boldsymbol{H}})\overline{V}^T\right)\,, \end{aligned} \tag{3.120}$$

where $\widehat{\boldsymbol{H}} = (Q^T\widetilde{A}Q, M^T\widetilde{B}N) = (Q^T\overline{P}^T A\overline{P}Q, M^T\overline{U}^T B\overline{V}N) \in \mathcal{X}$ and

$$\boldsymbol{K}(\widetilde{\boldsymbol{H}}) := \left(\boldsymbol{K}_1(\widetilde{\boldsymbol{H}}), \boldsymbol{K}_2(\widetilde{\boldsymbol{H}})\right) = \left(Q\boldsymbol{V}_1(\widehat{\boldsymbol{H}})Q^T, M\boldsymbol{V}_2(\widehat{\boldsymbol{H}})N^T\right).$$

Finally, since $\boldsymbol{G}(\cdot) = \boldsymbol{G}_S(\cdot) + \boldsymbol{G}_R(\cdot)$, from (3.112) and (3.120), we know that (3.111) holds.

" $\Longleftarrow$ " Suppose that there exists $\boldsymbol{K} = (\boldsymbol{K}_1, \boldsymbol{K}_2) \in \mathcal{K}_{\overline{\boldsymbol{\kappa}}}$ such that for any $\boldsymbol{H} \in \mathcal{X}$, (3.111) holds, i.e., there exist a sequence $\{\boldsymbol{\kappa}^t = (\lambda^t, \sigma^t)\}$ in $\mathcal{D}_{\boldsymbol{g}}^{\downarrow}$ converges to $\overline{\boldsymbol{\kappa}}$ and $Q_k \in \mathcal{O}^{|\alpha_k|}$, $k = 1, \ldots, r_0 + r$, $Q' \in \mathcal{O}^{|b|}$ and $Q'' \in \mathcal{O}^{n-|a|}$ such that for any $\boldsymbol{H} \in \mathcal{X}$,

$$\mathcal{U}(\boldsymbol{H}) = \left(\overline{P}\left(\boldsymbol{K}_1(\widetilde{\boldsymbol{H}}) + \boldsymbol{T}_1(\widetilde{A})\right)\overline{P}^T, \overline{U}\left(\boldsymbol{K}_2(\widetilde{\boldsymbol{H}}) + \boldsymbol{T}_2(\widetilde{B})\right)\overline{V}^T\right),$$

with

$$\begin{aligned} & \boldsymbol{K}(\boldsymbol{Z}) = (\boldsymbol{K}_1(\boldsymbol{Z}), \boldsymbol{K}_2(\boldsymbol{Z})) \\ = & \lim_{t\to\infty}\left(Q(\boldsymbol{L}_1(\boldsymbol{\kappa}^t, \widehat{\boldsymbol{Z}}) + \boldsymbol{J}_1(\boldsymbol{\kappa}^t, \widehat{\boldsymbol{Z}}))Q^T, M(\boldsymbol{L}_2(\boldsymbol{\kappa}^t, \widehat{\boldsymbol{Z}}) + \boldsymbol{J}_2(\boldsymbol{\kappa}^t, \widehat{\boldsymbol{Z}}))N^T\right), \quad \boldsymbol{Z} = (A, B) \in \mathcal{X} \end{aligned}$$

where $Q = \text{diag}(Q_1, \ldots, Q_{r_0}) \in \mathcal{O}^{m_0}$,

$$M = \text{diag}(Q_{r_0+1}, \ldots, Q_{r_0+r}, Q') \in \mathcal{O}^m \quad \text{and} \quad N = \text{diag}(Q_{r_0+1}, \ldots, Q_{r_0+r}, Q'') \in \mathcal{O}^n,$$

and $\widehat{\boldsymbol{Z}} = (Q^TAQ, M^TBN) \in \mathcal{X}$. Denote $P = \overline{P}Q$, $U = \overline{U}M$ and $V = \overline{V}N$. For each t, let

$$\boldsymbol{X}_t = (Y^t, Z^t) := (P\text{diag}(\lambda^t)P^T, U[\text{diag}(\sigma^t) \;\; 0]V^T).$$

Then, we have

$$\lim_{m\to\infty} \boldsymbol{X}^t = \boldsymbol{X}.$$

Moreover, by Theorem 3.6, we know that for each t, $\boldsymbol{G}$ is differentiable at $\boldsymbol{X}^t$. By (3.50), we know that for any $\boldsymbol{H} \in \mathcal{X}$,

$$\lim_{m\to\infty} \boldsymbol{G}'(\boldsymbol{X}^t)\boldsymbol{H} = \mathcal{U}(\boldsymbol{H}).$$

Hence, by the definition, we obtain that $\mathcal{U} \in \partial_B \boldsymbol{G}(\overline{\boldsymbol{X}})$. These complete the proof. $\square$

Remark 3.3. *Let $\overline{\boldsymbol{X}} \in \mathcal{X}$ be given. Note that for the given $\boldsymbol{H} \in \mathcal{X}$, $\overline{P}\boldsymbol{T}_1(\widetilde{A})\overline{P}^T$ and $\overline{U}\boldsymbol{T}_2(\widetilde{B})\overline{V}^T$ are independent of the choice of $\overline{P} \in \mathcal{O}^{m_0}(\overline{Y})$ and $(\overline{U}, \overline{V}) \in \mathcal{O}^{m,n}(\overline{Z})$ in (3.16). Since Clarke's generalized Jacobian $\partial \boldsymbol{G}(\overline{\boldsymbol{X}})$ at $\overline{\boldsymbol{X}}$ takes the form*

$$\partial \boldsymbol{G}(\overline{\boldsymbol{X}}) = \operatorname{conv}\left\{\partial_B \boldsymbol{G}(\overline{\boldsymbol{X}})\right\} ,$$

we know from (3.111) that $\mathcal{U} \in \partial_B \boldsymbol{G}(\overline{\boldsymbol{X}})$ if and only if there exists $\boldsymbol{K} = (\boldsymbol{K}_1, \boldsymbol{K}_2) \in \mathcal{K}_{\overline{\boldsymbol{\kappa}}}$ such that for any $\boldsymbol{H} = (A, B) \in \mathcal{X}$,

$$\mathcal{U}(\boldsymbol{H}) = \left(\overline{P}\left(\widehat{\boldsymbol{K}}_1(\widetilde{\boldsymbol{H}}) + \boldsymbol{T}_1(\widetilde{A})\right)\overline{P}^T, \overline{U}\left(\widehat{\boldsymbol{K}}_2(\widetilde{\boldsymbol{H}}) + \boldsymbol{T}_2(\widetilde{B})\right)\overline{V}^T\right) , \tag{3.121}$$

where $\widehat{\boldsymbol{K}}(\widetilde{\boldsymbol{H}}) = \left(\widehat{\boldsymbol{K}}_1(\widetilde{\boldsymbol{H}}), \widehat{\boldsymbol{K}}_2(\widetilde{\boldsymbol{H}})\right)$ is the convex combination of some $\{\boldsymbol{K}_z(\widetilde{\boldsymbol{H}})\}$ in $\mathcal{K}_{\overline{\boldsymbol{\kappa}}}$ defined by (3.110).

Let $\overline{\boldsymbol{X}} = (\overline{Y}, \overline{Z}) \in \mathcal{S}^{m_0} \times \Re^{m\times n} = \mathcal{X}$ be given. Suppose that the symmetric mapping $\boldsymbol{g}$ is also directionally differentiable at $\overline{\boldsymbol{\kappa}}$. Define $\boldsymbol{d} : \Re^{m_0+m} \to \Re^{m_0+m}$ by

$$\boldsymbol{d}(\boldsymbol{h}) := \boldsymbol{g}(\overline{\boldsymbol{\kappa}} + \boldsymbol{h}) - \boldsymbol{g}(\overline{\boldsymbol{\kappa}}) - \boldsymbol{g}'(\overline{\boldsymbol{\kappa}}; \boldsymbol{h}), \quad \boldsymbol{h} \in \Re^{m_0+m} .$$

Then, by (3.3) and (3.21), we know that $\boldsymbol{d}$ is symmetric, i.e.,

$$\boldsymbol{d}(\boldsymbol{h}) = \boldsymbol{Q}^T\boldsymbol{d}(\boldsymbol{Q}\boldsymbol{h}) \quad \forall\, \boldsymbol{Q} \in \mathcal{Q}_{\overline{\boldsymbol{\kappa}}} \quad \text{and} \quad \boldsymbol{h} \in \Re^{m_0+m} ,$$

where $\mathcal{Q}_{\overline{\boldsymbol{\kappa}}}$ is a subset of $\mathcal{Q}$ defined by (3.2). On the orther hand, by the directional differentiability of $\boldsymbol{g}$, we know that $\boldsymbol{d}$ is differentiable at 0. If $\boldsymbol{d}$ is strictly differentiable at 0, then we have

$$\lim_{\substack{\boldsymbol{w},\boldsymbol{w}'\to 0\\ \boldsymbol{w}\neq\boldsymbol{w}'}} \frac{\boldsymbol{d}(\boldsymbol{w}) - \boldsymbol{d}(\boldsymbol{w}')}{\|\boldsymbol{w} - \boldsymbol{w}'\|} = \boldsymbol{0} . \tag{3.122}$$

Let $\{\boldsymbol{w}^t = (\xi^t, \zeta^t)\} \in \Re^{m_0} \times \Re^m$ be a sequence converging to 0. Suppose that $1 \le i \le m$, $1 \le j \le n$, $i \neq j$.

Case 1: $1 \le i \neq j \le m$ and $\zeta_i^t \neq \zeta_j^t$ for all t. Consider the following sequence $\{\boldsymbol{s}^t = (\xi^t, s^t)\}$ in $\Re^{m_0} \times \Re^m$ where for each $p = 1, \ldots, m$,

$$(s^t)_p := \begin{cases} \zeta_p^t & \text{if } p \neq i, j, \\ \zeta_j^t & \text{if } p = i, \\ \zeta_i^t & \text{if } p = j, \end{cases} \qquad t = 1, 2, \ldots .$$

It is clear that the sequence $\{\boldsymbol{s}^t\}$ converges to 0. By the symmetry of $\boldsymbol{d}$, we know that for each $q = 1, \ldots, m_0 + m$,

$$\boldsymbol{d}_q(\boldsymbol{s}^t) := \begin{cases} \boldsymbol{d}_q(\boldsymbol{w}^t) & \text{if } q \neq m_0 + i, m_0 + j, \\ \boldsymbol{d}_{m_0+j}(\boldsymbol{w}^t) & \text{if } q = m_0 + i, \\ \boldsymbol{d}_{m_0+i}(\boldsymbol{w}^t) & \text{if } q = m_0 + j, \end{cases} \qquad t = 1, 2, \ldots .$$

Therefore, by (3.122), we obtain that for such i, j,

$$\lim_{t \to \infty} \frac{\boldsymbol{d}_{m_0+i}(\boldsymbol{w}^t) - \boldsymbol{d}_{m_0+j}(\boldsymbol{w}^t)}{|\zeta_i^t - \zeta_j^t|} = \lim_{t \to \infty} \sqrt{2}\, \frac{\boldsymbol{d}_{m_0+i}(\boldsymbol{w}^t) - \boldsymbol{d}_{m_0+i}(\boldsymbol{s}^t)}{\|\boldsymbol{w}^t - \boldsymbol{s}^t\|} = 0 . \tag{3.123}$$

Case 2: $i \in b$, $j \in b$ and $\zeta_i^t > 0$ or $\zeta_j^t > 0$ for all t. Consider the following sequence $\{\widehat{\boldsymbol{s}}^t = (\xi^t, \widehat{s}^t)\}$ in $\Re^{m_0} \times \Re^m$ with

$$(\widehat{s}^t)_p := \begin{cases} \zeta_p^t & \text{if } p \neq i, j, \\ -\zeta_j^t & \text{if } p = i, \\ -\zeta_i^t & \text{if } p = j, \end{cases} \qquad t = 1, 2, \ldots .$$

It is easy to see that $\widehat{\boldsymbol{s}}^t \neq \boldsymbol{w}^t$ for all t. Also, we know that $\{\widehat{\boldsymbol{s}}^t\}$ converges to 0. By the symmetry of $\boldsymbol{d}$ (with respect to $\overline{\boldsymbol{\kappa}}$), we know that for each $q = 1, \ldots, m_0 + m$,

$$\boldsymbol{d}_q(\widehat{\boldsymbol{s}}^t) := \begin{cases} \boldsymbol{d}_q(\boldsymbol{w}^t) & \text{if } q \neq m_0 + i, m_0 + j, \\ -\boldsymbol{d}_{m_0+j}(\boldsymbol{w}^t) & \text{if } q = m_0 + i, \\ -\boldsymbol{d}_{m_0+i}(\boldsymbol{w}^t) & \text{if } q = m_0 + j, \end{cases} \qquad t = 1, 2, \ldots .$$

Therefore, by (3.122), we obtain that for such i, j,

$$\begin{aligned} \lim_{t \to \infty} \frac{\boldsymbol{d}_{m_0+i}(\boldsymbol{w}^t) + \boldsymbol{d}_{m_0+j}(\boldsymbol{w}^t)}{\zeta_i^t + \zeta_j^t} &= \lim_{t \to \infty} \frac{\boldsymbol{d}_{m_0+i}(\boldsymbol{w}^t) - (-\boldsymbol{d}_{m_0+j}(\boldsymbol{w}^t))}{\zeta_i^t + \zeta_j^t} \\ &= \lim_{t \to \infty} \sqrt{2}\, \frac{\boldsymbol{d}_{m_0+i}(\boldsymbol{w}^t) - \boldsymbol{d}_{m_0+i}(\widehat{\boldsymbol{s}}^t)}{\|\boldsymbol{w}^t - \widehat{\boldsymbol{s}}^t\|} = 0 . \end{aligned} \tag{3.124}$$

Case 3: $i \in b$ and $\zeta_i^t > 0$ for all t. Consider the following sequence $\{\widetilde{\boldsymbol{s}}^t = (\xi^t, \widetilde{s}^t)\}$ in $\Re^{m_0} \times \Re^m$ with

$$(\widetilde{s}^t)_p := \begin{cases} \zeta_p^t & \text{if } p \neq i, \\ 0 & \text{if } p = i, \end{cases} \qquad t = 1, 2, \ldots .$$

It is easy to see that $\widetilde{\boldsymbol{s}}^t \neq \boldsymbol{w}^t$ for all t. Also, we know that $\{\widetilde{\boldsymbol{s}}^t\}$ converges to 0. By the symmetry of $\boldsymbol{d}$ (with respect to $\overline{\boldsymbol{\kappa}}$), we know that

$$\boldsymbol{d}_{m_0+i}(\widetilde{\boldsymbol{s}}^t) = 0\,.$$

Therefore, by (3.122), we obtain that for such i,

$$\begin{aligned}\lim_{t\to\infty}\frac{\boldsymbol{d}_{m_0+i}(\boldsymbol{w}^t)}{\zeta_i^t} &= \lim_{t\to\infty}\frac{\boldsymbol{d}_{m_0+i}(\boldsymbol{w}^t)-0}{\zeta_i^t-0} \\ &= \lim_{t\to\infty}\frac{\boldsymbol{d}_{m_0+i}(\boldsymbol{w}^t)-\boldsymbol{d}_{m_0+i}(\widetilde{\boldsymbol{s}}^t)}{\|\boldsymbol{w}^t-\widetilde{\boldsymbol{s}}^t\|} = 0\,. \end{aligned} \tag{3.125}$$

As mentioned in Remark 3.1, if the symmetric mapping $\boldsymbol{g}$ is locally Lipschitz continuous near $\overline{\boldsymbol{\kappa}} = \boldsymbol{\kappa}(\overline{\boldsymbol{X}})$ and directionally differentiable at $\overline{\boldsymbol{\kappa}}$, then the corresponding spectral operator $\boldsymbol{G}$ is also directionally differentiable at $\overline{\boldsymbol{X}}$. Moreover, we have the following useful result on $\partial \boldsymbol{G}(\overline{\boldsymbol{X}})$.

Theorem 3.14. *Let $\overline{\boldsymbol{X}} = (\overline{Y}, \overline{Z}) \in \mathcal{X}$ be given. Suppose that $\overline{Y}$ and $\overline{Z}$ have the decomposition (3.16). Assume that the symmetric mapping $\boldsymbol{g}$ is locally Lipschitz continuous near $\overline{\boldsymbol{\kappa}} = \boldsymbol{\kappa}(\overline{\boldsymbol{X}})$. Assume that $\boldsymbol{g}$ is directionally differentiable at $\overline{\boldsymbol{\kappa}}$ and there exists an open neighborhood $\mathcal{N} \subseteq \Re^{m_0+m}$ of zero such that the function $\boldsymbol{d} : \Re^{m_0+m} \to \Re^{m_0+m}$ defined by*

$$\boldsymbol{d}(\boldsymbol{h}) = \boldsymbol{g}(\overline{\boldsymbol{\kappa}}+\boldsymbol{h}) - \boldsymbol{g}(\overline{\boldsymbol{\kappa}}) - \boldsymbol{g}'(\overline{\boldsymbol{\kappa}};\boldsymbol{h}), \quad \boldsymbol{h} \in \Re^{m_0+m}$$

is differentiable on $\mathcal{N}$ and strictly differentiable at 0. Then, we have

$$\partial_B \boldsymbol{G}(\overline{\boldsymbol{X}}) = \partial_B \Psi(\boldsymbol{0})\,,$$

where $\Psi(\cdot) := \boldsymbol{G}'(\overline{\boldsymbol{X}};\cdot) : \mathcal{X} \to \mathcal{X}$ is the directional derivative of $\boldsymbol{G}$ at $\overline{\boldsymbol{X}}$.

Proof. Let $\mathcal{U} \in \partial_B \boldsymbol{G}(\overline{\boldsymbol{X}})$. By Theorem 3.13, we know that there exists $\boldsymbol{K} = (\boldsymbol{K}_1, \boldsymbol{K}_2) \in \mathcal{K}_{\overline{\boldsymbol{\kappa}}}$ such that for any $\boldsymbol{H} \in \mathcal{X}$, (3.111) holds, i.e., there exist a sequence $\{\boldsymbol{\kappa}^t = (\lambda^t, \sigma^t)\} \subset \mathcal{D}_{\boldsymbol{g}}^{\downarrow}$ converges to $\overline{\boldsymbol{\kappa}}$ and $Q_k \in \mathcal{O}^{|\alpha_k|}$, $k = 1, \ldots, r_0 + r$, $Q' \in \mathcal{O}^{|b|}$ and $Q'' \in \mathcal{O}^{n-|a|}$ such that for any $\boldsymbol{H} \in \mathcal{X}$,

$$\mathcal{U}(\boldsymbol{H}) = \left(\overline{P}\left(\boldsymbol{K}_1(\widetilde{\boldsymbol{H}}) + \boldsymbol{T}_1(\widetilde{A})\right)\overline{P}^T, \overline{U}\left(\boldsymbol{K}_2(\widetilde{\boldsymbol{H}}) + \boldsymbol{T}_2(\widetilde{B})\right)\overline{V}^T\right)\,, \tag{3.126}$$

with

$$\begin{aligned} \boldsymbol{K}(\boldsymbol{Z}) &= (\boldsymbol{K}_1(\boldsymbol{Z}), \boldsymbol{K}_2(\boldsymbol{Z})) \\ &= \lim_{t\to\infty} \Big(Q(\boldsymbol{L}_1(\boldsymbol{\kappa}^t, \widehat{\boldsymbol{Z}}) + \boldsymbol{J}_1(\boldsymbol{\kappa}^t, \widehat{\boldsymbol{Z}}))Q^T, M(\boldsymbol{L}_2(\boldsymbol{\kappa}^t, \widehat{\boldsymbol{Z}}) + \boldsymbol{J}_2(\boldsymbol{\kappa}^t, \widehat{\boldsymbol{Z}}))N^T \Big), \quad \boldsymbol{Z} = (A, B) \in \mathcal{X}, \end{aligned} \tag{3.127}$$

where for each $\boldsymbol{\kappa}^t$, the linear operators $\boldsymbol{L}(\boldsymbol{\kappa}^t, \cdot)$ and $\boldsymbol{J}(\boldsymbol{\kappa}^t, \cdot)$ are defined by (3.47) and (3.108), respectively; $Q = \mathrm{diag}(Q_1, \ldots, Q_{r_0}) \in \mathcal{O}^{m_0}$,

$$M = \mathrm{diag}(Q_{r_0+1}, \ldots, Q_{r_0+r}, Q') \in \mathcal{O}^m \quad \text{and} \quad N = \mathrm{diag}(Q_{r_0+1}, \ldots, Q_{r_0+r}, Q'') \in \mathcal{O}^n;$$

$\widehat{\boldsymbol{Z}} = (Q^TAQ, M^TBN) \in \mathcal{X}$. For each t, let $\boldsymbol{w}^t := (\xi^t, \zeta^t) = \boldsymbol{\kappa}^t - \overline{\boldsymbol{\kappa}} \in \Re^{m_0} \times \Re^m$ and

$$\boldsymbol{W}^t := \left(\boldsymbol{W}_1^t, \ldots, \boldsymbol{W}_{r_0+r}^t, \boldsymbol{W}_{r_0+r+1}^t\right) \in \mathcal{S}^{\alpha_1} \times \ldots \times \mathcal{S}^{\alpha_{r_0+r}} \times \Re^{|b|\times(n-|a|)} = \mathcal{W}$$

with

$$\boldsymbol{W}_k^t := \begin{cases} Q_k \mathrm{diag}(\boldsymbol{w}_k^t) Q_k^T & \text{if } 1 \le k \le r_0 + r, \\ Q'[\mathrm{diag}(\boldsymbol{w}_{r_0+r+1}^t) \;\; 0] Q''^T & \text{if } k = r_0 + r + 1. \end{cases}$$

By noting that for each t, $\boldsymbol{w}_{r_0+r+1}^t \in \Re_+^{|b|}$, we know that $\boldsymbol{\kappa}(\boldsymbol{W}^t) = \boldsymbol{w}^t$. Therefore, we have

$$\lim_{t\to\infty} \boldsymbol{W}^t = \boldsymbol{0} \in \mathcal{W}.$$

Moreover, for each t, define $\boldsymbol{C}^t := (\boldsymbol{C}_1^t, \boldsymbol{C}_2^t) \in \mathcal{X}$ by

$$\boldsymbol{C}_1^t = \overline{P} \begin{bmatrix} \boldsymbol{W}_1^t & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \boldsymbol{W}_{r_0}^t \end{bmatrix} \overline{P}^T \in \mathcal{S}^{m_0}$$

and

$$\boldsymbol{C}_2^t = \overline{U} \begin{bmatrix} \boldsymbol{W}_{r_0}^t & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \boldsymbol{W}_{r_0+r}^t & 0 \\ 0 & \cdots & 0 & \boldsymbol{W}_{r_0+r+1}^t \end{bmatrix} \overline{V}^T \in \Re^{m\times n}.$$

Therefore, it is easy to see that

$$\lim_{t\to\infty} \boldsymbol{C}^t = \boldsymbol{0} \in \mathcal{X}\,.$$

By recalling the notation $\boldsymbol{D}$ defined in (3.25), we know that

$$\boldsymbol{D}(\widetilde{\boldsymbol{C}}^t) = \boldsymbol{W}^t \in \mathcal{W} \quad \forall\, t\,,$$

where for each t, $\widetilde{\boldsymbol{C}}^t = \left(\overline{P}^T \boldsymbol{C}_1^t \overline{P}, \overline{U}^T \boldsymbol{C}_2^t \overline{V}\right)$. From the directional derivative formula (3.26), we know that for each t and any $\boldsymbol{H} = (A, B) \in \mathcal{X}$,

$$\Psi(\boldsymbol{C}^t + \boldsymbol{H}) - \Psi(\boldsymbol{C}^t) = \left(\overline{P}[\Delta_1^t + \boldsymbol{T}_1(\widetilde{A})]\overline{P}^T, \overline{U}[\Delta_2^t + \boldsymbol{T}_2(\widetilde{B})]\overline{V}^T\right)\,, \tag{3.128}$$

where $\Delta_1^t \in \mathcal{S}^{m_0}$ and $\Delta_2^t \in \Re^{m\times n}$ are defined by

$$(\Delta_1^t)_{\alpha_k \alpha_{k'}} := \begin{cases} \Phi_k(\boldsymbol{D}(\widetilde{\boldsymbol{C}}^t) + \boldsymbol{D}(\widetilde{\boldsymbol{H}})) - \Phi_k(\boldsymbol{D}(\widetilde{\boldsymbol{C}}^t)) & \text{if } k = k', \\ 0 & \text{otherwise,} \end{cases} \quad k, k' = 1, \ldots, r_0\,, \tag{3.129a}$$

and

$$(\Delta_2^t)_{a_l a_{l'}} := \begin{cases} \Phi_{r_0+l}(\boldsymbol{D}(\widetilde{\boldsymbol{C}}^t) + \boldsymbol{D}(\widetilde{\boldsymbol{H}})) - \Phi_{r_0+l}(\boldsymbol{D}(\widetilde{\boldsymbol{C}}^t)) & \text{if } l = l', \\ 0 & \text{otherwise,} \end{cases} \quad l, l' = 1, \ldots, r{+}1\,, \tag{3.129b}$$

where $\Phi : \mathcal{W} \to \mathcal{W}$ is the spectral operator with respect to the symmetric mapping $\phi(\cdot) := \boldsymbol{g}'(\overline{\boldsymbol{\kappa}}; \cdot)$ defined by (3.22). Since $\boldsymbol{d}(\cdot) = \boldsymbol{g}(\overline{\boldsymbol{\kappa}} + \cdot) - \boldsymbol{g}(\overline{\boldsymbol{\kappa}}) - \boldsymbol{g}'(\overline{\boldsymbol{\kappa}}; \cdot)$ is differentiable on $\mathcal{N}$ and all $\boldsymbol{\kappa}^t \in \mathcal{D}_{\boldsymbol{g}}^{\downarrow}$, we know that for t sufficiently large, ϕ is differentiable at each $\boldsymbol{w}^t$ and

$$\phi'(\boldsymbol{w}^t) = \boldsymbol{g}'(\boldsymbol{\kappa}^t) - \boldsymbol{d}'(\boldsymbol{w}^t)\,. \tag{3.130}$$

Moreover, since $\boldsymbol{d}$ is strictly differentiable at 0 and $\boldsymbol{d}'(0) = 0$ and $\{\boldsymbol{g}'(\boldsymbol{\kappa}^t)\}$ converges as $t \to \infty$, we obtain that

$$\lim_{t\to\infty} \boldsymbol{g}'(\boldsymbol{\kappa}^t) = \lim_{t\to\infty} \phi'(\boldsymbol{w}^t)\,. \tag{3.131}$$

Therefore, we know from Theorem 3.6 that for any t sufficiently large, $\Phi : \mathcal{W} \to \mathcal{W}$ is differentiable at $\boldsymbol{D}(\widetilde{\boldsymbol{C}}^t)$, and by using the formula (3.50), the derivative $\Phi'(\boldsymbol{D}(\widetilde{\boldsymbol{C}}^t))\boldsymbol{D}(\widetilde{\boldsymbol{H}}) \in \mathcal{W}$ can be written as the following form:

$$\Phi'(\boldsymbol{D}(\widetilde{\boldsymbol{C}}^t))\boldsymbol{D}(\widetilde{\boldsymbol{H}}) = \Big(Q_1\boldsymbol{O}_1^t(\widetilde{\boldsymbol{H}})Q_1^T, \ldots, Q_{r_0+r}\boldsymbol{O}_{r_0+r}^t(\widetilde{\boldsymbol{H}})Q_{r_0+r}^T, Q'\boldsymbol{O}_{r_0+r+1}^t(\widetilde{\boldsymbol{H}})Q''^T\Big) \tag{3.132}$$

with

$$\boldsymbol{O}_k^t(\widetilde{\boldsymbol{H}}) = \begin{cases} \boldsymbol{L}_k^\phi(\boldsymbol{w}^t, \boldsymbol{D}(\widetilde{\boldsymbol{H}})) + (\mathcal{A}_\phi^D(\boldsymbol{w}^t))_{\alpha_k\alpha_k} \circ (Q_k^T(\boldsymbol{D}(\widetilde{\boldsymbol{H}}))_k Q_k) & \text{if } 1 \le k \le r_0 + r, \\ \boldsymbol{L}_{r_0+r+1}^\phi(\boldsymbol{w}^t, \boldsymbol{D}(\widetilde{\boldsymbol{H}})) + \mathcal{T}^\phi(\boldsymbol{w}^t, Q'^T(\boldsymbol{D}(\widetilde{\boldsymbol{H}}))_{r_0+r+1}Q'') & \text{if } k = r_0 + r + 1, \end{cases} \tag{3.133}$$

where for each $\boldsymbol{w}^t$, $\mathcal{A}_\phi^D(\boldsymbol{w}^t) \in \mathcal{S}^{m_0}$, $\boldsymbol{L}_\phi(\boldsymbol{w}^t, \cdot) = \big((\boldsymbol{L}_\phi)_1(\boldsymbol{w}^t, \cdot), \ldots, (\boldsymbol{L}_\phi)_{r_0+r+1}(\boldsymbol{w}^t, \cdot)\big) : \mathcal{W} \to \mathcal{W}$ and $\mathcal{T}_\phi(\boldsymbol{w}^t, \cdot) : \Re^{|b|\times(n-|a|)} \to \Re^{|b|\times(n-|a|)}$ are defined by (3.41), (3.47) and (3.49) with respect to the symmetric mapping ϕ. For each t, let

$$\boldsymbol{R}^t(\widetilde{\boldsymbol{H}}) := (\boldsymbol{R}_1^t(\widetilde{\boldsymbol{H}}), \boldsymbol{R}_2^t(\widetilde{\boldsymbol{H}})) \in \mathcal{X} \tag{3.134}$$

with

$$\boldsymbol{R}_1^t(\widetilde{\boldsymbol{H}}) := Q \begin{bmatrix} \boldsymbol{O}_1^t(\widetilde{\boldsymbol{H}}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \boldsymbol{O}_{r_0}^t(\widetilde{\boldsymbol{H}}) \end{bmatrix} Q^T \in \mathcal{S}^{m_0}$$

and

$$\boldsymbol{R}_2^t(\widetilde{\boldsymbol{H}}) = M \begin{bmatrix} \boldsymbol{O}_{r_0}^t(\widetilde{\boldsymbol{H}}) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \boldsymbol{O}_{r_0+r}^t(\widetilde{\boldsymbol{H}}) & 0 \\ 0 & \cdots & 0 & \boldsymbol{O}_{r_0+r+1}^t(\widetilde{\boldsymbol{H}}) \end{bmatrix} N^T \in \Re^{m\times n}.$$

Hence, we know from (3.128) and (3.132) that Ψ is differentiable at each $\boldsymbol{C}^t$ and for any $\boldsymbol{H} \in \mathcal{X}$,

$$\Psi'(\boldsymbol{C}^t)\boldsymbol{H} = \Big(\overline{P}[\boldsymbol{R}_1^t(\widetilde{\boldsymbol{H}}) + \boldsymbol{T}_1(\widetilde{A})]\overline{P}^T, \overline{U}[\boldsymbol{R}_2^t(\widetilde{\boldsymbol{H}}) + \boldsymbol{T}_2(\widetilde{B})]\overline{V}^T\Big)\,. \tag{3.135}$$

By comparing with (3.126), we know that the conclusion then follows if we show that

$$\boldsymbol{K} = \lim_{t\to\infty} \boldsymbol{R}^t \,. \tag{3.136}$$

On the other hand, since the orthogonal matrices $Q \in \mathcal{O}^{m_0}$, $M \in \mathcal{O}^m$ and $N \in \mathcal{O}^n$ are fixed, it is sufficient to prove that

$$\boldsymbol{K}(\boldsymbol{Z}) = \lim_{t\to\infty} \boldsymbol{R}^t(\boldsymbol{Z}) \quad \forall\, \boldsymbol{Z} \in \{\widetilde{\boldsymbol{E}}^{(ij)}\} \cup \{\widetilde{\boldsymbol{F}}^{(ij)}\}\,, \tag{3.137}$$

where

$$\{\widetilde{\boldsymbol{E}}^{(ij)}\} \cup \{\widetilde{\boldsymbol{F}}^{(ij)}\} := \Big\{(Q\boldsymbol{Z}_1 Q^T, M\boldsymbol{Z}_2 N^T) \,:\, \boldsymbol{Z} = (\boldsymbol{Z}_1, \boldsymbol{Z}_2) \in \{\boldsymbol{E}^{(ij)}\} \cup \{\boldsymbol{F}^{(ij)}\}\Big\}$$

$\{\boldsymbol{E}^{(ij)}\}\cup\{\boldsymbol{F}^{(ij)}\}$ is the standard basis of $\mathcal{X}$ defined by (3.61). For simplicity, we only show that (3.137) holds for the case that each $\widetilde{\boldsymbol{F}}^{(ij)} = (\mathbf{0}, \widetilde{F}^{(ij)}) \in \mathcal{X}$, $1 \le i \le m$, $1 \le j \le n$, and the other cases can be shown similarly. Rewrite $\widetilde{F}^{(ij)}$ as the form

$$\widetilde{F}^{(ij)} = \begin{bmatrix} \widetilde{F}_1^{(ij)} & \widetilde{F}_2^{(ij)} \end{bmatrix}$$

with $\widetilde{F}_1^{(ij)} \in \Re^{m\times m}$ and $\widetilde{F}_2^{(ij)} \in \Re^{m\times(n-m)}$. Therefore, we know from (3.127) and (3.133) that for any $1 \le i \le m$, $1 \le j \le n$,

$$\boldsymbol{K}(\widetilde{\boldsymbol{F}}^{(ij)}) = \begin{cases} \lim\limits_{t\to\infty} (\mathbf{0}, M(\boldsymbol{L}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)}) + \boldsymbol{J}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)}))N^T) & \text{if } i,j \in a_l \text{ for some } 1 \le l \le r+1, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

and for each t

$$\boldsymbol{R}^t(\widetilde{\boldsymbol{F}}^{(ij)}) = \begin{cases} (\mathbf{0}, \boldsymbol{R}_2^t(\boldsymbol{F}^{(ij)})) & \text{if } i,j \in a_l \text{ for some } 1 \le l \le r+1, \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

Therefore, without loss of generality, we only need to consider the case that $i,j \in a_l$ for some $1 \le l \le r+1$.

Case 1: $1 \le i = j \le m$. By (3.47), (3.108) and (3.133), we know that

$$\boldsymbol{L}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)}) + \boldsymbol{J}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)}) = \begin{bmatrix} \mathrm{diag}(\boldsymbol{g}'(\boldsymbol{\kappa}^t)e_i) & 0 \end{bmatrix}$$

and

$$\boldsymbol{R}_2^t(\boldsymbol{F}^{(ij)}) = M \begin{bmatrix} \mathrm{diag}(\boldsymbol{\phi}'(\boldsymbol{w}^t)e_i) & 0 \end{bmatrix} N^T\,,$$

where for each $1 \leq i \leq m$, e_i is a vector whose entries are zeros, except the i-th entry is one. Therefore, from (3.131), we know that

$$\begin{aligned} \boldsymbol{K}(\widetilde{\boldsymbol{F}}^{(ij)}) &= \lim_{t\to\infty} (\mathbf{0}, M(\boldsymbol{L}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)}) + \boldsymbol{J}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)}))N^T) \\ &= \lim_{t\to\infty} (\mathbf{0}, \boldsymbol{R}_2^t(\boldsymbol{F}^{(ij)})) = \lim_{t\to\infty} \boldsymbol{R}^t(\widetilde{\boldsymbol{F}}^{(ij)})\,. \end{aligned}$$

Case 2: $i \neq j \in a_l$ for some $1 \leq l \leq r$ and $\sigma_i^t \neq \sigma_j^t$ for any t sufficiently large. By (3.47) and (3.108), we know that for any t,

$$\begin{aligned} &\Big(\boldsymbol{L}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)}) + \boldsymbol{J}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)})\Big)_{pq} \\ = &\begin{cases} \dfrac{\boldsymbol{g}_{m_0+i}(\boldsymbol{\kappa}^t) - \boldsymbol{g}_{m_0+j}(\boldsymbol{\kappa}^t)}{2(\sigma_i^t - \sigma_j^t)} & \text{if } (p,q) = (i,j) \text{ or } (q,p) = (i,j), \\ 0 & \text{otherwise,} \end{cases} \qquad 1 \leq p \leq m,\ 1 \leq q \leq n\,. \end{aligned}$$

Meanwhile, by (3.133), we know that for any t,

$$\begin{aligned} &\Big(\widetilde{\boldsymbol{R}}_2^t(\boldsymbol{F}^{(ij)})\Big)_{pq} = \Big(M^T \boldsymbol{R}_2^t(\boldsymbol{F}^{(ij)})N\Big)_{pq} \\ = &\begin{cases} \dfrac{\phi_{m_0+i}(\boldsymbol{w}^t) - \phi_{m_0+j}(\boldsymbol{w}^t)}{2(\zeta_i^t - \zeta_j^t)} & \text{if } (p,q) = (i,j) \text{ or } (q,p) = (i,j), \\ 0 & \text{otherwise,} \end{cases} \qquad 1 \leq p \leq m,\ 1 \leq q \leq n\,. \end{aligned}$$

For each t, since $\overline{\sigma}_i = \overline{\sigma}_j$ and $\boldsymbol{g}_{m_0+i}(\overline{\boldsymbol{\kappa}}) = \boldsymbol{g}_{m_0+j}(\overline{\boldsymbol{\kappa}})$, we know that

$$\begin{aligned} \frac{\boldsymbol{g}_{m_0+i}(\boldsymbol{\kappa}^t) - \boldsymbol{g}_{m_0+j}(\boldsymbol{\kappa}^t)}{2(\sigma_i^t - \sigma_j^t)} &= \frac{\boldsymbol{g}_{m_0+i}(\overline{\boldsymbol{\kappa}} + \boldsymbol{w}^t) - \boldsymbol{g}_{m_0+j}(\overline{\boldsymbol{\kappa}} + \boldsymbol{w}^t)}{2(\zeta_i^t - \zeta_j^t)} \\ &= \frac{\boldsymbol{g}_{m_0+i}(\overline{\boldsymbol{\kappa}} + \boldsymbol{w}^t) - \boldsymbol{g}_{m_0+i}(\overline{\boldsymbol{\kappa}}) + \boldsymbol{g}_{m_0+j}(\overline{\boldsymbol{\kappa}}) - \boldsymbol{g}_{m_0+j}(\overline{\boldsymbol{\kappa}} + \boldsymbol{w}^t)}{2(\zeta_i^t - \zeta_j^t)} \\ &= \frac{\boldsymbol{d}_{m_0+i}(\boldsymbol{w}^t) - \boldsymbol{d}_{m_0+j}(\boldsymbol{w}^t)}{2(\zeta_i^t - \zeta_j^t)} + \frac{\phi_{m_0+i}(\boldsymbol{w}^t) - \phi_{m_0+j}(\boldsymbol{w}^t)}{2(\zeta_i^t - \zeta_j^t)} \end{aligned} \tag{3.138}$$

Therefore, since $\boldsymbol{d}$ is strictly differentiable at 0, by (3.123), we obtain that

$$\lim_{t\to\infty} \frac{\boldsymbol{g}_{m_0+i}(\boldsymbol{\kappa}^t) - \boldsymbol{g}_{m_0+j}(\boldsymbol{\kappa}^t)}{2(\sigma_i^t - \sigma_j^t)} = \lim_{t\to\infty} \frac{\phi_{m_0+i}(\boldsymbol{w}^t) - \phi_{m_0+j}(\boldsymbol{w}^t)}{2(\zeta_i^t - \zeta_j^t)}\,.$$

Therefore,

$$\begin{aligned} \boldsymbol{K}(\widetilde{\boldsymbol{F}}^{(ij)}) &= \lim_{t\to\infty} (\mathbf{0}, M(\boldsymbol{L}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)}) + \boldsymbol{J}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)}))N^T) \\ &= \lim_{t\to\infty} (\mathbf{0}, \boldsymbol{R}_2^t(\boldsymbol{F}^{(ij)})) = \lim_{t\to\infty} \boldsymbol{R}^t(\widetilde{\boldsymbol{F}}^{(ij)})\,. \end{aligned}$$

Case 3: $i \neq j \in a_l$ for some $1 \leq l \leq r$ and $\sigma_i^t = \sigma_j^t$ for any t sufficiently large. By (3.47) and (3.108), we know that for any t sufficiently large and any $1 \leq p \leq m$, $1 \leq q \leq n$,

$$\begin{aligned} & \Big(\boldsymbol{L}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)}) + \boldsymbol{J}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)})\Big)_{pq} \\ = & \begin{cases} ((\boldsymbol{g}'(\boldsymbol{\kappa}^t))_{(m_0+i)(m_0+i)} - (\boldsymbol{g}'(\boldsymbol{\kappa}^t))_{(m_0+i)(m_0+j)})/2 & \text{if } (p,q) \text{ or } (q,p) = (i,j), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Meanwhile, by (3.133), we know that for any t sufficiently large and any $1 \leq p \leq m$, $1 \leq q \leq n$,

$$\begin{aligned} & \Big(\widetilde{\boldsymbol{R}}_2^t(\boldsymbol{F}^{(ij)})\Big)_{pq} = \Big(M^T \boldsymbol{R}_2^t(\boldsymbol{F}^{(ij)}) N\Big)_{pq} \\ = & \begin{cases} ((\phi'(\boldsymbol{w}^t))_{(m_0+i)(m_0+i)} - (\phi'(\boldsymbol{w}^t))_{(m_0+i)(m_0+j)})/2 & \text{if } (p,q) \text{ or } (q,p) = (i,j), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Therefore, from (3.131), we know that

$$\begin{aligned} \boldsymbol{K}(\widetilde{\boldsymbol{F}}^{(ij)}) &= \lim_{t\to\infty} (\boldsymbol{0}, M(\boldsymbol{L}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)}) + \boldsymbol{J}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)}))N^T) \\ &= \lim_{t\to\infty} (\boldsymbol{0}, \boldsymbol{R}_2^t(\boldsymbol{F}^{(ij)})) = \lim_{t\to\infty} \boldsymbol{R}^t(\widetilde{\boldsymbol{F}}^{(ij)}) \,. \end{aligned}$$

Case 4: $i \neq j \in b$ and $\sigma_i^t = \sigma_j^t > 0$ for any t sufficiently large. By (3.47) and (3.108), we know that for any t sufficiently large,

$$\begin{aligned} & \boldsymbol{L}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)}) + \boldsymbol{J}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)}) \\ = & \left[\big((\boldsymbol{g}'(\boldsymbol{\kappa}^t))_{(m_0+i)(m_0+i)} - (\boldsymbol{g}'(\boldsymbol{\kappa}^t))_{(m_0+i)(m_0+j)}\big) S(F_1^{(ij)}) + \frac{\boldsymbol{g}_{m_0+i}(\boldsymbol{\kappa}^t) + \boldsymbol{g}_{m_0+j}(\boldsymbol{\kappa}^t)}{\sigma_i^t + \sigma_j^t} T(F_1^{(ij)}) \;\; 0 \right]. \end{aligned}$$

Meanwhile, from (3.133), we know that for any t sufficiently large,

$$\begin{aligned} & \widetilde{\boldsymbol{R}}_2^t(\boldsymbol{F}^{(ij)}) = M^T \boldsymbol{R}_2^t(\boldsymbol{F}^{(ij)}) N \\ = & \left[\big((\phi'(\boldsymbol{w}^t))_{(m_0+i)(m_0+i)} - (\phi'(\boldsymbol{w}^t))_{(m_0+i)(m_0+j)}\big) S(F_1^{(ij)}) + \frac{\phi_{m_0+i}(\boldsymbol{w}^t) + \phi_{m_0+j}(\boldsymbol{w}^t)}{\zeta_i^t + \zeta_j^t} T(F_1^{(ij)}) \;\; 0 \right]. \end{aligned}$$

For each t, since $\overline{\sigma}_i = \overline{\sigma}_j = 0$ and $\boldsymbol{g}_{m_0+i}(\overline{\boldsymbol{\kappa}}) = \boldsymbol{g}_{m_0+j}(\overline{\boldsymbol{\kappa}}) = 0$, we know that

$$\begin{aligned}\frac{\boldsymbol{g}_{m_0+i}(\boldsymbol{\kappa}^t) + \boldsymbol{g}_{m_0+j}(\boldsymbol{\kappa}^t)}{\sigma_i^t + \sigma_j^t} &= \frac{\boldsymbol{g}_{m_0+i}(\overline{\boldsymbol{\kappa}} + \boldsymbol{w}^t) + \boldsymbol{g}_{m_0+j}(\overline{\boldsymbol{\kappa}} + \boldsymbol{w}^t)}{\zeta_i^t + \zeta_j^t} \\ &= \frac{\boldsymbol{g}_{m_0+i}(\overline{\boldsymbol{\kappa}} + \boldsymbol{w}^t) - \boldsymbol{g}_{m_0+i}(\overline{\boldsymbol{\kappa}}) - \boldsymbol{g}_{m_0+j}(\overline{\boldsymbol{\kappa}}) + \boldsymbol{g}_{m_0+j}(\overline{\boldsymbol{\kappa}} + \boldsymbol{w}^t)}{\zeta_i^t + \zeta_j^t} \\ &= \frac{\boldsymbol{d}_{m_0+i}(\boldsymbol{w}^t) + \boldsymbol{d}_{m_0+j}(\boldsymbol{w}^t)}{\zeta_i^t + \zeta_j^t} + \frac{\phi_{m_0+i}(\boldsymbol{w}^t) + \phi_{m_0+j}(\boldsymbol{w}^t)}{\zeta_i^t + \zeta_j^t} \quad (3.139)\end{aligned}$$

Therefore, since $\boldsymbol{d}$ is strictly differentiable at 0, by (3.124), we know that

$$\lim_{t\to\infty} \frac{\boldsymbol{g}_{m_0+i}(\boldsymbol{\kappa}^t) + \boldsymbol{g}_{m_0+j}(\boldsymbol{\kappa}^t)}{\sigma_i^t + \sigma_j^t} = \lim_{t\to\infty} \frac{\phi_{m_0+i}(\boldsymbol{w}^t) + \phi_{m_0+j}(\boldsymbol{w}^t)}{\zeta_i^t + \zeta_j^t} .$$

Hence, by (3.131), we obtain that

$$\begin{aligned}\boldsymbol{K}(\widetilde{\boldsymbol{F}}^{(ij)}) &= \lim_{t\to\infty} (\boldsymbol{0}, M(\boldsymbol{L}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)}) + \boldsymbol{J}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)}))N^T) \\ &= \lim_{t\to\infty} (\boldsymbol{0}, \boldsymbol{R}_2^t(\boldsymbol{F}^{(ij)})) = \lim_{t\to\infty} \boldsymbol{R}^t(\widetilde{\boldsymbol{F}}^{(ij)}) .\end{aligned}$$

Case 5: $i \neq j \in b$ and $\sigma_i^t \neq \sigma_j^t$ for any t sufficiently large. By (3.47) and (3.108), we know that for any t sufficiently large,

$$\begin{aligned}&\boldsymbol{L}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)}) + \boldsymbol{J}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)}) \\ = &\left[\frac{\boldsymbol{g}_{m_0+i}(\boldsymbol{\kappa}^t) - \boldsymbol{g}_{m_0+j}(\boldsymbol{\kappa}^t)}{\sigma_i^t - \sigma_j^t} S(F_1^{(ij)}) + \frac{\boldsymbol{g}_{m_0+i}(\boldsymbol{\kappa}^t) + \boldsymbol{g}_{m_0+j}(\boldsymbol{\kappa}^t)}{\sigma_i^t + \sigma_j^t} T(F_1^{(ij)}) \ \ 0\right] .\end{aligned}$$

Meanwhile, from (3.133), we know that for any t sufficiently large,

$$\begin{aligned}&\widetilde{\boldsymbol{R}}_2^t(\boldsymbol{F}^{(ij)}) = M^T \boldsymbol{R}_2^t(\boldsymbol{F}^{(ij)}) N \\ = &\left[\frac{\phi_{m_0+i}(\boldsymbol{w}^t) - \phi_{m_0+j}(\boldsymbol{w}^t)}{\zeta_i^t - \zeta_j^t} S(F_1^{(ij)}) + \frac{\phi_{m_0+i}(\boldsymbol{w}^t) + \phi_{m_0+j}(\boldsymbol{w}^t)}{\zeta_i^t + \zeta_j^t} T(F_1^{(ij)}) \ \ 0\right] .\end{aligned}$$

Therefore, by (3.138) and (3.139), since $\boldsymbol{d}$ is strictly differentiable at 0, we know from (3.123) and (3.124) that

$$\lim_{t\to\infty} \frac{\boldsymbol{g}_{m_0+i}(\boldsymbol{\kappa}^t) - \boldsymbol{g}_{m_0+j}(\boldsymbol{\kappa}^t)}{\sigma_i^t - \sigma_j^t} = \lim_{t\to\infty} \frac{\phi_{m_0+i}(\boldsymbol{w}^t) - \phi_{m_0+j}(\boldsymbol{w}^t)}{\zeta_i^t - \zeta_j^t}$$

and

$$\lim_{t\to\infty} \frac{\boldsymbol{g}_{m_0+i}(\boldsymbol{\kappa}^t) + \boldsymbol{g}_{m_0+j}(\boldsymbol{\kappa}^t)}{\sigma_i^t + \sigma_j^t} = \lim_{t\to\infty} \frac{\phi_{m_0+i}(\boldsymbol{w}^t) + \phi_{m_0+j}(\boldsymbol{w}^t)}{\zeta_i^t + \zeta_j^t} .$$

Hence, we know that

$$\boldsymbol{K}(\widetilde{\boldsymbol{F}}^{(ij)}) = \lim_{t\to\infty}(\mathbf{0}, M(\boldsymbol{L}_2(\boldsymbol{\kappa}^t,\boldsymbol{F}^{(ij)})+\boldsymbol{J}_2(\boldsymbol{\kappa}^t,\boldsymbol{F}^{(ij)}))N^T) = \lim_{t\to\infty}(\mathbf{0},\boldsymbol{R}_2^t(\boldsymbol{F}^{(ij)})) = \lim_{t\to\infty}\boldsymbol{R}^t(\widetilde{\boldsymbol{F}}^{(ij)})\,.$$

Case 6: $i \neq j \in b$ and $\sigma_i^t = \sigma_j^t = 0$ for any t sufficiently large. By (3.47) and (3.108), we know that for any t sufficiently large,

$$\begin{aligned}&\boldsymbol{L}_2(\boldsymbol{\kappa}^t,\boldsymbol{F}^{(ij)}) + \boldsymbol{J}_2(\boldsymbol{\kappa}^t,\boldsymbol{F}^{(ij)})\\ =\;&\Big[\big((\boldsymbol{g}'(\boldsymbol{\kappa}^t))_{(m_0+i)(m_0+i)} - (\boldsymbol{g}'(\boldsymbol{\kappa}^t))_{(m_0+i)(m_0+j)}\big)\,F_1^{(ij)} \quad 0\Big]\,.\end{aligned}$$

Meanwhile, from (3.133), we know that for any t sufficiently large,

$$\begin{aligned}&\widetilde{\boldsymbol{R}}_2^t(\boldsymbol{F}^{(ij)}) = M^T\boldsymbol{R}_2^t(\boldsymbol{F}^{(ij)})N\\ =\;&\Big[\big((\phi'(\boldsymbol{w}^t))_{(m_0+i)(m_0+i)} - (\phi'(\boldsymbol{w}^t))_{(m_0+i)(m_0+j)}\big)\,F_1^{(ij)} \quad 0\Big]\,.\end{aligned}$$

Therefore, by (3.131), we obtain that

$$\boldsymbol{K}(\widetilde{\boldsymbol{F}}^{(ij)}) = \lim_{t\to\infty}(\mathbf{0}, M(\boldsymbol{L}_2(\boldsymbol{\kappa}^t,\boldsymbol{F}^{(ij)})+\boldsymbol{J}_2(\boldsymbol{\kappa}^t,\boldsymbol{F}^{(ij)}))N^T) = \lim_{t\to\infty}(\mathbf{0},\boldsymbol{R}_2^t(\boldsymbol{F}^{(ij)})) = \lim_{t\to\infty}\boldsymbol{R}^t(\widetilde{\boldsymbol{F}}^{(ij)})\,.$$

Case 7: $i \in b$, $j \in c$ and $\sigma_i^t > 0$ for any t sufficiently large. By (3.47) and (3.108), we know that for any t sufficiently large,

$$\boldsymbol{L}_2(\boldsymbol{\kappa}^t,\boldsymbol{F}^{(ij)}) + \boldsymbol{J}_2(\boldsymbol{\kappa}^t,\boldsymbol{F}^{(ij)}) = \left[0 \quad \frac{\boldsymbol{g}_{m_0+i}(\boldsymbol{\kappa}^t)}{\sigma_i^t}F_2^{(ij)}\right]\,.$$

Meanwhile, from (3.133), we know that for any t sufficiently large,

$$\widetilde{\boldsymbol{R}}_2^t(\boldsymbol{F}^{(ij)}) = M^T\boldsymbol{R}_2^t(\boldsymbol{F}^{(ij)})N = \left[0 \quad \frac{\phi_{m_0+i}(\boldsymbol{w}^t)}{\zeta_i^t}F_2^{(ij)}\right]\,.$$

Since $\overline{\sigma}_i = 0$ and $\boldsymbol{g}_{m_0+i}(\overline{\boldsymbol{\kappa}}) = 0$, we have for each t,

$$\frac{\boldsymbol{g}_{m_0+i}(\boldsymbol{\kappa}^t)}{\sigma_i^t} = \frac{\boldsymbol{g}_{m_0+i}(\overline{\boldsymbol{\kappa}}+\boldsymbol{w}^t) - \boldsymbol{g}_{m_0+i}(\overline{\boldsymbol{\kappa}})}{\zeta_i^t} = \frac{\boldsymbol{d}_{m_0+i}(\boldsymbol{w}^t)}{\zeta_i^t} + \frac{\phi_{m_0+i}(\boldsymbol{w}^t)}{\zeta_i^t}\,.$$

Therefore, by (3.125), we obtain that

$$\begin{aligned}\boldsymbol{K}(\widetilde{\boldsymbol{F}}^{(ij)}) &= \lim_{t\to\infty}(\mathbf{0}, M(\boldsymbol{L}_2(\boldsymbol{\kappa}^t,\boldsymbol{F}^{(ij)})+\boldsymbol{J}_2(\boldsymbol{\kappa}^t,\boldsymbol{F}^{(ij)}))N^T)\\ &= \lim_{t\to\infty}(\mathbf{0},\boldsymbol{R}_2^t(\boldsymbol{F}^{(ij)})) = \lim_{t\to\infty}\boldsymbol{R}^t(\widetilde{\boldsymbol{F}}^{(ij)})\,.\end{aligned}$$

Case 8: $i \in b$, $j \in c$ and $\sigma_i^t = 0$ for any t sufficiently large. By (3.47) and (3.108), we know that for any t sufficiently large,

$$\begin{aligned} & \boldsymbol{L}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)}) + \boldsymbol{J}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)}) \\ = & \begin{bmatrix} 0 & \left((\boldsymbol{g}'(\boldsymbol{\kappa}^t))_{(m_0+i)(m_0+i)} - (\boldsymbol{g}'(\boldsymbol{\kappa}^t))_{(m_0+i)(m_0+j)}\right) F_2^{(ij)} \end{bmatrix} . \end{aligned}$$

Meanwhile, from (3.133), we know that for any t sufficiently large,

$$\begin{aligned} & \widetilde{\boldsymbol{R}}_2^t(\boldsymbol{F}^{(ij)}) = M^T \boldsymbol{R}_2^t(\boldsymbol{F}^{(ij)}) N \\ = & \begin{bmatrix} 0 & \left((\phi'(\boldsymbol{w}^t))_{(m_0+i)(m_0+i)} - (\phi'(\boldsymbol{w}^t))_{(m_0+i)(m_0+j)}\right) F_2^{(ij)} \end{bmatrix} . \end{aligned}$$

Therefore, by (3.131), we obtain that

$$\begin{aligned} \boldsymbol{K}(\widetilde{\boldsymbol{F}}^{(ij)}) &= \lim_{t\to\infty} (\boldsymbol{0}, M(\boldsymbol{L}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)}) + \boldsymbol{J}_2(\boldsymbol{\kappa}^t, \boldsymbol{F}^{(ij)}))N^T) \\ &= \lim_{t\to\infty} (\boldsymbol{0}, \boldsymbol{R}_2^t(\boldsymbol{F}^{(ij)})) = \lim_{t\to\infty} \boldsymbol{R}^t(\widetilde{\boldsymbol{F}}^{(ij)}) . \end{aligned}$$

Finally, from (3.126), (3.127) and (3.135), we know that there exists a sequence $\{\boldsymbol{C}^t\} \subset \mathcal{X}$ in $\mathcal{D}_\Psi$ converging to $\boldsymbol{0}$ such that

$$\lim_{t\to\infty} \Psi'(\boldsymbol{C}^t)\boldsymbol{H} = \mathcal{U}(\boldsymbol{H}) \quad \forall\, \boldsymbol{H} \in \mathcal{X} .$$

This implies that

$$\mathcal{U} \in \partial_B \Psi(\boldsymbol{0}) .$$

Conversely, let $\mathcal{U} \in \partial_B\Psi(\boldsymbol{0})$. Then, there exists a sequence $\{\boldsymbol{C}^t := (\boldsymbol{C}_1^t, \boldsymbol{C}_2^t)\} \subset \mathcal{X}$ converging to $\boldsymbol{0}$ such that Ψ is differentiable at each $\boldsymbol{C}^t$ and

$$\mathcal{U} = \lim_{t\to\infty} \Psi'(\boldsymbol{C}^t) .$$

Meanwhile, we know from (3.128) and (3.129) that for each t, Ψ is differentiable at $\boldsymbol{C}^t$ if and only if the spectral operator Φ is differentiable at $\boldsymbol{D}(\widetilde{\boldsymbol{C}}^t)$, where

$$\widetilde{\boldsymbol{C}}^t = \left(\widetilde{\boldsymbol{C}}_1^t, \widetilde{\boldsymbol{C}}_2^t\right) = \left(\overline{P}^T \boldsymbol{C}_1^t \overline{P}, \overline{U}^T \boldsymbol{C}_2^t \overline{V}\right) \in \mathcal{S}^{m_0} \times \Re^{m\times n}, \quad t = 1, 2, \ldots .$$

By (3.25), we know that for each t,

$$\boldsymbol{D}(\widetilde{\boldsymbol{C}}^t) = \left((\widetilde{\boldsymbol{C}}^t_1)_{\alpha_1\alpha_1}, \ldots, (\widetilde{\boldsymbol{C}}^t_1)_{\alpha_{r_0}\alpha_{r_0}}, S((\widetilde{\boldsymbol{C}}^t_2)_{a_1a_1}), \ldots, S((\widetilde{\boldsymbol{C}}^t_2)_{a_ra_r}), (\widetilde{\boldsymbol{C}}^t_2)_{b\bar{a}}\right) .$$

For each t, consider the decompositions

$$(\widetilde{\boldsymbol{C}}^t_1)_{\alpha_k\alpha_k} = Q^t_k \mathrm{diag}(\boldsymbol{w}^t_k)(Q^t_k)^T, \quad k = 1, \ldots, r_0 ,$$

$$S((\widetilde{\boldsymbol{C}}^t_2)_{a_la_l}) = Q^t_{r_0+l} \mathrm{diag}(\boldsymbol{w}^t_{r_0+l})(Q^t_{r_0+l})^T, \quad l = 1, \ldots, r$$

and

$$(\widetilde{\boldsymbol{C}}^t_2)_{b\bar{a}} = Q'^t \left[\mathrm{diag}(\boldsymbol{w}^t_{r_0+r+1}) \;\; 0\right] (Q''^t)^T ,$$

where for each t, $Q^t_k \in \mathcal{O}^{|\alpha_k|}$, $k = 1, \ldots, r_0$, $Q^t_{r_0+l} \in \mathcal{O}^{|a_l|}$, $l = 1, \ldots, r$, $Q'^t \in \mathcal{O}^{|b|}$ and $Q''^t \in \mathcal{O}^{n-|a|}$; $\boldsymbol{w}^t \in \Re^{m_0} \times \Re^m$ satisfies

$$\boldsymbol{w}^t_k = \begin{cases} \lambda((\widetilde{\boldsymbol{C}}^t_1)_{\alpha_k\alpha_k}) & \text{if } 1 \le k \le r_0, \\ \lambda(S((\widetilde{\boldsymbol{C}}^t_2)_{a_la_l})) & \text{if } r_0 + 1 \le k = r_0 + l \le r_0 + r \\ \sigma((\widetilde{\boldsymbol{C}}^t_2)_{b\bar{a}}) & \text{if } k = r_0 + r + 1. \end{cases}$$

For each t, let $\xi^t := (\boldsymbol{w}^t_1, \ldots, \boldsymbol{w}^t_{r_0}) \in \Re^{m_0}$ and $\zeta^t := (\boldsymbol{w}^t_{r_0+1}, \ldots, \boldsymbol{w}^t_{r_0+r}, \boldsymbol{w}^t_{r_0+r+1}) \in \Re^m$. Then, we have $\boldsymbol{w}^t = (\xi^t, \zeta^t)$ for each t. For each t, let $Q^t = \mathrm{diag}(Q^t_1, \ldots, Q^t_{r_0}) \in \mathcal{O}^{m_0}$,

$$M^t = \mathrm{diag}(Q^t_{r_0+1}, \ldots, Q^t_{r_0+r}, Q'^t) \in \mathcal{O}^m \quad \text{and} \quad N^t = \mathrm{diag}(Q^t_{r_0+1}, \ldots, Q^t_{r_0+r}, Q''^t) \in \mathcal{O}^n .$$

Since $\{Q^t\}$, $\{M^t\}$ and $\{N^t\}$ are uniformly bounded, by taking subsequence if necessary, we may assume that

$$\begin{aligned} \lim_{t\to\infty} Q^t &= Q = \mathrm{diag}(Q_1, \ldots, Q_{r_0}) \in \mathcal{O}^{m_0} , \\ \lim_{t\to\infty} M^t &= M = \mathrm{diag}(Q_{r_0+1}, \ldots, Q_{r_0+r}, Q') \in \mathcal{O}^m , \\ \lim_{t\to\infty} N^t &= N = \mathrm{diag}(Q_{r_0+1}, \ldots, Q_{r_0+r}, Q'') \in \mathcal{O}^n . \end{aligned}$$

Since Φ is differentiable at each $\boldsymbol{D}(\widetilde{\boldsymbol{C}}^t)$, we know from Theorem 3.6 that ϕ is differentiable at each $\boldsymbol{w}^t$. Also, by (3.128) and (3.50) in Theorem 3.6, we know that for any $\boldsymbol{H} =$

$(A,B)\in\mathcal{X}$,

$$\begin{aligned}\mathcal{U}(\boldsymbol{H}) &= \lim_{t\to\infty}\Psi'(\boldsymbol{C}^t)\boldsymbol{H}\\ &= \left(\overline{P}[\boldsymbol{R}_1(\widetilde{\boldsymbol{H}})+\boldsymbol{T}_1(\widetilde{A})]\overline{P}^T, \overline{U}[\boldsymbol{R}_2(\widetilde{\boldsymbol{H}})+\boldsymbol{T}_2(\widetilde{B})]\overline{V}^T\right), \end{aligned} \tag{3.140}$$

with

$$\boldsymbol{R}=(\boldsymbol{R}_1,\boldsymbol{R}_2)=\lim_{t\to\infty}\boldsymbol{R}^t\,,$$

where for each t, $\boldsymbol{R}^t(\cdot)=(\boldsymbol{R}_1^t(\cdot),\boldsymbol{R}_2^t(\cdot)):\mathcal{X}\to\mathcal{X}$ is a linear operator defined by (3.134). Denote

$$P=\overline{P}Q\in\mathcal{O}^{m_0},\quad U=\overline{U}M\in\mathcal{O}^m \quad\text{and}\quad V=\overline{V}N\in\mathcal{O}^n\,.$$

For t sufficiently large, we have $\boldsymbol{\kappa}^t:=\overline{\boldsymbol{\kappa}}+\boldsymbol{w}^t=(\lambda^t,\sigma^t)=\Re^{m_0}\times\Re^m_+$. Therefore, for such t, we may define

$$\boldsymbol{X}^t:=(Y^t,Z^t)=\left(P\text{diag}(\lambda^t)P^T, U\left[\text{diag}(\sigma^t)\ \ 0\right]V^T\right)\in\mathcal{X}\,.$$

It is clear that the sequence $\{\boldsymbol{X}^t\}$ converges to $\overline{\boldsymbol{X}}$. Meanwhile, since $\boldsymbol{d}$ is differentiable on some neighborhood $\mathcal{N}$, we know that for t sufficiently large, $\boldsymbol{g}$ is differentiable at each $\boldsymbol{\kappa}^t$ and (3.130) holds. Moreover, since $\boldsymbol{d}$ is strictly differentiable at 0 and $\{\phi'(\boldsymbol{w}^t)\}$ converges, we know that (3.131) holds. Therefore, by Theorem 3.6, we know that for t sufficiently large, $\boldsymbol{G}$ is differentiable at each $\boldsymbol{X}^t$ and for any $\boldsymbol{H}=(A,B)\in\mathcal{X}$,

$$\begin{aligned}\boldsymbol{G}'(\boldsymbol{X}^t)\boldsymbol{H} &= \Big(P(\boldsymbol{L}_1(\boldsymbol{\kappa}^t,\widehat{\boldsymbol{H}})+\boldsymbol{J}_1(\boldsymbol{\kappa}^t,\widehat{A})+\Theta_1(\boldsymbol{\kappa}^t,\widehat{A}))P^T,\\ &\qquad U(\boldsymbol{L}_2(\boldsymbol{\kappa}^t,\widehat{\boldsymbol{H}})+\boldsymbol{J}_2(\boldsymbol{\kappa}^t,\widehat{B})+\Theta_2(\boldsymbol{\kappa}^t,\widehat{B}))V^T\Big), \end{aligned} \tag{3.141}$$

where for each t, $\Theta_1(\boldsymbol{\kappa}^t,\widehat{A})\in\mathcal{S}^{m_0}$ and $\Theta_2(\boldsymbol{\kappa}^t,\widehat{B})\in\Re^{m\times n}$ are given by

$$\Theta_1(\boldsymbol{\kappa}^t,\widehat{A})=\mathcal{A}^D(\boldsymbol{\kappa}^t)\circ\widehat{A}-\boldsymbol{J}_1(\boldsymbol{\kappa}^t,\widehat{A})\quad\text{and}\quad\Theta_2(\boldsymbol{\kappa}^t,\widehat{B})=\mathcal{T}(\boldsymbol{\kappa}^t,\widehat{B})-\boldsymbol{J}_2(\boldsymbol{\kappa}^t,\widehat{B})\,,$$

$\mathcal{A}^D(\boldsymbol{\kappa}^t)$, $\mathcal{T}(\boldsymbol{\kappa}^t,\cdot)$, $\boldsymbol{L}(\boldsymbol{\kappa}^t,\cdot)$ and $\boldsymbol{J}(\boldsymbol{\kappa}^t,\cdot)$ are given by (3.41), (3.49), (3.47) and (3.108), respectively; and $\widehat{\boldsymbol{H}}=(\widehat{A},\widehat{B})=\left(P^TAP,U^TBV\right)=(Q^T\widetilde{A}Q,M^T\widetilde{B}N)$. Therefore, since $\boldsymbol{w}^t$ converges to $\overline{\boldsymbol{\kappa}}$, we know that

$$\lim_{t\to\infty}\left(\Theta_1(\boldsymbol{\kappa}^t,\widehat{A}),\Theta_2(\boldsymbol{\kappa}^t,\widehat{B})\right)=\left(\boldsymbol{T}_1(\widetilde{A}),\boldsymbol{T}_2(\widetilde{B})\right)\,.$$

By taking subsequence if necessary, we may assume that $\{\boldsymbol{G}'(\boldsymbol{X}^t)\}$ converges. Then, from (3.141), we know that for any $\boldsymbol{H} \in \mathcal{X}$,

$$\lim_{t\to\infty} \boldsymbol{G}'(\boldsymbol{X}^t)\boldsymbol{H} = \left(\overline{P}\left(\boldsymbol{K}_1(\widetilde{\boldsymbol{H}}) + \boldsymbol{T}_1(\widetilde{A})\right)\overline{P}^T, \overline{U}\left(\boldsymbol{K}_2(\widetilde{\boldsymbol{H}}) + \boldsymbol{T}_2(\widetilde{B})\right)\overline{V}^T\right), \tag{3.142}$$

with

$$\boldsymbol{K}(\boldsymbol{Z}) = (\boldsymbol{K}_1(\boldsymbol{Z}), \boldsymbol{K}_2(\boldsymbol{Z})) = \lim_{t\to\infty}\left(\boldsymbol{K}_1^t(\boldsymbol{Z}), \boldsymbol{K}_2^t(\boldsymbol{Z})\right), \quad \boldsymbol{Z} = (A, B) \in \mathcal{X},$$

where for each t,

$$\left(\boldsymbol{K}_1^t(\boldsymbol{Z}), \boldsymbol{K}_2^t(\boldsymbol{Z})\right) := \left(Q(\boldsymbol{L}_1(\boldsymbol{\kappa}^t, \widehat{\boldsymbol{Z}}) + \boldsymbol{J}_1(\boldsymbol{\kappa}^t, \widehat{\boldsymbol{Z}}))Q^T, M(\boldsymbol{L}_2(\boldsymbol{\kappa}^t, \widehat{\boldsymbol{Z}}) + \boldsymbol{J}_2(\boldsymbol{\kappa}^t, \widehat{\boldsymbol{Z}}))N^T\right)$$

Similarly as the proof of **Case 1-8** in the first part, by using the properties (3.131), we can prove that

$$\boldsymbol{R} = \lim_{t\to\infty} \boldsymbol{K}^t .$$

Therefore, by (3.140) and (3.142), we know that there exists a sequence $\{\boldsymbol{X}^t\}$ in $\mathcal{D}_{\boldsymbol{G}}$ converging to $\overline{\boldsymbol{X}}$ such that

$$\lim_{t\to\infty} \boldsymbol{G}'(\boldsymbol{X}^t)\boldsymbol{H} = \mathcal{U}(\boldsymbol{H}) \quad \forall\, \boldsymbol{H} \in \mathcal{X} .$$

Then, we have $\mathcal{U} \in \partial_B \boldsymbol{G}(\overline{\boldsymbol{X}})$. Therefore, the proof is completed. □

3.8 An example: the metric projector over the Ky Fan k-norm epigraph cone

In this section, as an example of spectral operators, we study the metric projection operator over the Ky Fan k-norm epigraph cone. Let $\mathcal{K} \in \Re \times \Re^{m\times n}$ be the epigraph of the Ky Fan k-norm, i.e., $\mathcal{K} \equiv \text{epi}\|\cdot\|_{(k)}$. Note that the matrix cone $\mathcal{K} \equiv \text{epi}\|\cdot\|_{(k)}$ includes the epigraphs of the spectral norm $\|\cdot\|_2$ ($k = 1$) and nuclear norm $\|\cdot\|_*$ ($k = m$). Let $\Pi_{\mathcal{K}} : \Re \times \Re^{m\times n} \to \Re \times \Re^{m\times n}$ be the metric projection operator over the epigraph of the Ky Fan k-norm, i.e., for any given $(t,X) \in \Re \times \Re^{m\times n}$, $(\bar{t},\overline{X}) := \Pi_{\mathcal{K}}(t,X)$ is the unique optimal solution of the following convex problem

$$\begin{array}{ll} \min & \frac{1}{2}\big((\tau - t)^2 + \|Y - X\|^2\big) \\ \text{s.t.} & \|Y\|_{(k)} \leq \tau\,. \end{array} \tag{3.143}$$

Therefore, from Proposition 3.2, we know that

$$\Pi_{\mathcal{K}}(t,X) = \Big(\boldsymbol{g}_1(t,\sigma), \overline{U}\,[\mathrm{diag}\,(\boldsymbol{g}_2(t,\sigma))\ \ 0]\,\overline{V}^T\Big)\,,$$

where $\sigma = \sigma(X)$, $(\overline{U},\overline{V}) \in \mathcal{O}^{m,n}(X)$ and $\boldsymbol{g}(t,\sigma) := \big(\boldsymbol{g}_1(t,\sigma), \boldsymbol{g}_2(t,\sigma)\big) \in \Re \times \Re^m$ is the metric projection operator over the polyhedral convex set epi $\|\cdot\|_{(k)} \subseteq \Re \times \Re^m$, i.e., the unique optimal solution of the following convex problem

$$\begin{array}{ll} \min & \frac{1}{2}\big((\tau - t)^2 + \|y - \sigma\|^2\big) \\ \text{s.t.} & \|y\|_{(k)} \leq \tau\,, \end{array} \tag{3.144}$$

where $\|\cdot\|_{(k)} : \Re^m \to \Re$ is the *vector k-norm*, i.e., the sum of the k largest components in absolute value of any vector in $\Re^m$. It is clear that $\boldsymbol{g}$ is a symmetric function. Therefore, the metric projection operator $\Pi_{\mathcal{K}}$ is the spectral operator with respect to $\boldsymbol{g}$.

Another important spectral operator which is closely related to the metric projection operator over the epigraph of the Ky Fan k-norm is the metric projection operator over the epigraph of $s_{(k)}(\cdot) : \mathcal{S}^n \to \Re$, the sum of k largest eigenvalues of the symmetric matrix. Let $\mathcal{M} \equiv \mathrm{epi}\, s_{(k)}(\cdot)$ be the epigraph of the positively homogenous convex function $s_{(k)}(\cdot)$. Let $\Pi_{\mathcal{M}} : \Re \times \mathcal{S}^n \to \Re \times \mathcal{S}^n$ be the metric projection operator over $\mathcal{M}$, i.e., for any given $(t,X) \in \Re \times \mathcal{S}^n$, $(\bar{t},\overline{X}) := \Pi_{\mathcal{M}}(t,X)$ is the unique optimal solution of the following convex problem

$$\begin{array}{ll} \min & \frac{1}{2}\big((\tau - t)^2 + \|Y - X\|^2\big) \\ \text{s.t.} & s_{(k)}(Y) \leq \tau\,. \end{array} \tag{3.145}$$

Therefore, since $s_{(k)}(\cdot)$ is unitarily invariant in $\mathcal{S}^n$, from Proposition 3.2, we know that

$$\Pi_{\mathcal{M}}(t,X) = \Big(\boldsymbol{h}_1(t,\lambda), \overline{P}\mathrm{diag}\,(\boldsymbol{h}_2(t,\lambda))\overline{P}^T\Big)\,,$$

where $\lambda = \lambda(X)$, $\overline{P} \in \mathcal{O}^n(X)$ and $\boldsymbol{h}(t,\sigma) := \big(\boldsymbol{h}_1(t,\lambda), \boldsymbol{h}_2(t,\lambda)\big) \in \Re \times \Re^n$ is the metric projection operator over the polyhedral convex set $\operatorname{epi} s_{(k)}(\cdot) \subseteq \Re \times \Re^n$, i.e., the unique optimal solution of the following convex problem

$$\begin{array}{ll} \min & \frac{1}{2}\big((\tau - t)^2 + \|y - \lambda\|^2\big) \\ \text{s.t.} & s_{(k)}(y) \leq \tau\,, \end{array} \tag{3.146}$$

where $s_{(k)}(\cdot) : \Re^n \to \Re$ is the sum of the k largest components of any vector in $\Re^n$. It is clear that $\boldsymbol{h}$ is a symmetric function with respect to $\Re \times \Re^n$. Similarly, the metric projection operator $\Pi_{\mathcal{M}}$ is the spectral operator with respect to $\boldsymbol{h}$.

For the definitions, it is easy to see that the symmetric functions $\boldsymbol{g}$ and $\boldsymbol{h}$ are similar. In fact, several important properties of $\boldsymbol{g}$ and $\boldsymbol{h}$ have been well studied in [113]. The corresponding properties of the spectral operators $\Pi_{\mathcal{K}}$ and $\Pi_{\mathcal{M}}$ can be obtained by applying the results for the general spectral operator which we obtained before. Therefore, from now on, we mainly focus on the spectral operator $\Pi_{\mathcal{K}}$, and the corresponding properties of $\Pi_{\mathcal{M}}$ can be obtained similarly. Since $\operatorname{epi}\|\cdot\|_{(k)} \in \Re \times \Re^m$ is a polyhedral convex set, we know that the corresponding metric projection operator $\boldsymbol{g}$ is a piecewise linear function (for a short proof, see [87, Chapter 2] or [93, Chapter 5]). By [113, Proposition 4.1], we know that for any given $(t,\sigma) \in \Re \times \Re^m$, the unique optimal solution $(\bar{t}, \overline{\sigma}) := \boldsymbol{g}(t,\sigma) \in \Re \times \Re^m$ of (3.144) can be easily obtained by applying [113, Algorithm 1] and the computational cost is $O(k(m-k+1))$. Moreover, by using [113, Lemma 4.2 & 4.1], we have the following simple fact.

Lemma 3.15. *Let $(t, X) \notin \operatorname{int}\mathcal{K}$ be given. Denote $\sigma = \sigma(X)$. Then, the unique optimal solution $(\bar{t}, \overline{\sigma}) = \boldsymbol{g}(t,\sigma) \in \Re \times \Re^m$ of (3.144) satisfies the following conditions.*

(i) If $\overline{\sigma}_k > 0$, then there exist $\theta > 0$ and $u \in \Re^m_+$ such that

$$\overline{\sigma} = \sigma - \theta u\,, \tag{3.147}$$

with $u_i = 1$, $i = 1, \ldots, k_0$, $u_i = 0$, $i = k_1 + 1, \ldots, m$,

$$u_\alpha = e_\alpha, \quad u_\beta = u_\beta^\downarrow, \quad \sum_{i\in\beta} u_i = k - k_0 \quad \text{and} \quad u_\gamma = 0\,, \tag{3.148}$$

where $0 \le k_0 \le k-1$ *and* $k \le k_1 \le m$ *are two integers such that*

$$\overline{\sigma}_1 \ge \ldots \ge \overline{\sigma}_{k_0} > \overline{\sigma}_{k_0+1} = \ldots = \overline{\sigma}_k = \ldots = \overline{\sigma}_{k_1} > \overline{\sigma}_{k_1+1} \ge \ldots \ge \overline{\sigma}_m \ge 0 \tag{3.149}$$

and

$$\alpha = \{1, \ldots, k_0\}, \quad \beta = \{k_0, \ldots, k_1\} \quad \text{and} \quad \gamma = \{k_1 + 1, \ldots, m\}\,. \tag{3.150}$$

(ii) If $\overline{\sigma}_k = 0$, *then there exist* $\theta > 0$ *and* $u \in \Re^m_+$ *such that*

$$\overline{\sigma} = \sigma - \theta u\,, \tag{3.151}$$

with

$$u_\alpha = e, \quad u_\beta = u_\beta^\downarrow \quad \text{and} \quad \sum_{i\in\beta} u_i \le k - k_0\,, \tag{3.152}$$

where $0 \le k_0 \le k-1$ *is the integer such that*

$$\overline{\sigma}_1 \ge \cdots \ge \overline{\sigma}_{k_0} > \overline{\sigma}_{k_0+1} = \ldots = \overline{\sigma}_k = \ldots = \overline{\sigma}_m = 0 \tag{3.153}$$

and

$$\alpha = \{1, \ldots, k_0\} \quad \text{and} \quad \beta = \{k_0, \ldots, m\}\,. \tag{3.154}$$

Other properties, including the close form solution, the directional differentiability, and the F-differentiability, of the symmetric function $\boldsymbol{g}$ have also been studied in [113]. Therefore, the corresponding properties of the metric projection operator $\Pi_{\mathcal{K}}$ follow from the results obtained in previous sections. Next, we list some of them as follows.

Let $(t, X) \in \Re \times \Re^{m\times n}$ be given. Consider the singular value decomposition of X, i.e.,

$$X = \overline{U}\,[\Sigma(X) \;\; 0]\,\overline{V}^T\,, \tag{3.155}$$

where $(\overline{U},\overline{V}) \in \mathcal{O}^{m,n}(X)$. Let a, b, c and a_l, $l = 1,\ldots,r$ be the index sets defined by (2.25) and (2.26) for X. Since $\boldsymbol{g}$ is globally Lipschitz continuous with modulus 1, directionally differentiable ([113, Theorem 5.1]), we know from Theorem 3.4 that the metric projection operator $\Pi_{\mathcal{K}}$ is directionally differentiable everywhere. Next, we will provide the directional derivative formula $\Pi'_{\mathcal{K}}((t,X);(\cdot,\cdot))$ for the metric projector $\Pi_{\mathcal{K}}$ at any given point $(t,X) \in \Re \times \Re^{m\times n}$. Without lose of generality, we assume that $(t,X) \notin \operatorname{int}\mathcal{K} \cup \operatorname{int}\mathcal{K}^\circ$, since otherwise $\Pi_{\mathcal{K}}$ is continuously differentiable and the derivative $\Pi'_{\mathcal{K}}(t,X)$ is either the identity mapping or the zero mapping. For notational convenience, denote $(\bar{t},\overline{\sigma}) = \boldsymbol{g}(t,\sigma)$. For the given (t,X), let $\mathcal{E}_1$, $\mathcal{E}_2 \in \mathcal{S}^m$ and $\mathcal{F} \in \Re^{m\times(n-m)}$ be the matrices defined by (3.13)-(3.15), i.e.,

$$(\mathcal{E}_1)_{ij} := \begin{cases} \dfrac{\overline{\sigma}_i - \overline{\sigma}_j}{\sigma_i - \sigma_j} & \text{if } \sigma_i \neq \sigma_j\,, \\ 0 & \text{otherwise}\,, \end{cases} \quad i,j \in \{1,\ldots,m\}\,, \tag{3.156}$$

$$(\mathcal{E}_2)_{ij} := \begin{cases} \dfrac{\overline{\sigma}_i + \overline{\sigma}_j}{\sigma_i + \sigma_j} & \text{if } \sigma_i + \sigma_j \neq 0\,, \\ 0 & \text{otherwise}\,, \end{cases} \quad i,j \in \{1,\ldots,m\}\,, \tag{3.157}$$

and

$$(\mathcal{F})_{ij} := \begin{cases} \dfrac{\overline{\sigma}_i}{\sigma_i} & \text{if } \sigma_i \neq 0\,, \\ 0 & \text{otherwise}\,, \end{cases} \quad i \in \{1,\ldots,m\}, \quad j \in \{1,\ldots,n-m\}\,. \tag{3.158}$$

In order to introduce the directional derivative formula of the metric projector $\Pi_{\mathcal{K}}$, we consider the following two cases.

Case 1. $(t,X) \notin \operatorname{int}\mathcal{K} \cup \operatorname{int}\mathcal{K}^\circ$ and $\overline{\sigma}_k > 0$. Then, by the part (i) of Lemma 3.15, we know that there exist two integers $r_0, r_1 \in \{1,\ldots,r\}$ such that

$$\alpha = \bigcup_{l=1}^{r_0} a_l, \quad \beta = \bigcup_{l=r_0+1}^{r_1} a_l \quad \text{and} \quad \gamma = \bigcup_{l=r_1+1}^{r+1} a_l\,,$$

where the index sets α, β and γ are defined by (3.150). Define

$$\beta_1 := \{i \in \beta \,|\, u_i = 1\}, \quad \beta_2 := \{i \in \beta \,|\, 0 < u_i < 1\} \quad \text{and} \quad \beta_3 := \{i \in \beta \,|\, u_i = 0\}\,. \tag{3.159}$$

Then, by (3.147) and (3.148), we know from (3.156) that

$$(\mathcal{E}_1)_{a_l a_{l'}} = E_{a_l a_{l'}}, \quad l \neq l' \text{ and } l, l' \in \{1, \ldots, r_0\} \text{ or } l, l' \in \{r_1+1, \ldots, r+1\},$$

$$(\mathcal{E}_1)_{a_l \beta_1} = E_{a_l \beta_1} \quad \text{and} \quad (\mathcal{E}_1)_{\beta_1 a_l} = E_{\beta_1 a_l}, \quad l = 1, \ldots, r_0,$$

$$(\mathcal{E}_1)_{a_l \beta_3} = E_{a_l \beta_3} \quad \text{and} \quad (\mathcal{E}_1)_{\beta_3 a_l} = E_{\beta_3 a_l}, \quad l = r_1+1, \ldots, r+1,$$

$$(\mathcal{E}_1)_{\beta\beta} = 0.$$

For the given $(t, X) \in \Re \times \Re^{m\times n}$, define a linear operator $\boldsymbol{T} : \Re^{m\times n} \to \Re^{m\times n}$ by for any $Z = [Z_1 \ \ Z_2] \in \Re^{m\times n}$,

$$\boldsymbol{T}(Z) = \begin{bmatrix} (\mathcal{E}_1)_{\overline{\gamma}\,\overline{\gamma}} \circ S(Z_{\overline{\gamma}\,\overline{\gamma}}) + (\mathcal{E}_2)_{\overline{\gamma}\,\overline{\gamma}} \circ T(Z_{\overline{\gamma}\,\overline{\gamma}}) & (\mathcal{E}_1)_{\overline{\gamma}\gamma} \circ S(Z_{\overline{\gamma}\gamma}) + (\mathcal{E}_2)_{\overline{\gamma}\gamma} \circ T(Z_{\overline{\gamma}\gamma}) & \mathcal{F}_{\overline{\gamma}c} \circ Z_{\overline{\gamma}c} \\ (\mathcal{E}_1)_{\gamma\overline{\gamma}} \circ S(Z_{\gamma\overline{\gamma}}) + (\mathcal{E}_2)_{\gamma\overline{\gamma}} \circ T(Z_{\gamma\overline{\gamma}}) & Z_{\gamma\gamma} & Z_{\gamma c} \end{bmatrix}. \tag{3.160}$$

Define the finite dimensional real Euclidean space $\mathcal{W}$ by

$$\mathcal{W} := \Re \times \mathcal{S}^{|a_1|} \times \ldots \times \mathcal{S}^{|a_{r_1}|}.$$

For any $(\zeta, \boldsymbol{W}) \in \mathcal{W}$, let $\boldsymbol{\kappa}(\boldsymbol{W}) := (\lambda(\boldsymbol{W}_1), \ldots, \lambda(\boldsymbol{W}_{r_1})) \in \Re^{k_1}$. Let $\mathcal{C}_1 \subseteq \mathcal{W}$ be the closed subset defined as following, if $(t, X) \in \operatorname{bd}\mathcal{K}$,

$$\mathcal{C}_1 := \left\{ (\zeta, \boldsymbol{W}) \in \mathcal{W} \mid \sum_{l=1}^{r_0} \operatorname{tr}(\boldsymbol{W}_l) + s_{(k-k_0)}(\boldsymbol{\kappa}_\beta(\boldsymbol{W})) \leq \zeta \right\}, \tag{3.161a}$$

if $(t, X) \notin \operatorname{bd}\mathcal{K}$,

$$\mathcal{C}_1 := \left\{ (\zeta, \boldsymbol{W}) \in \mathcal{W} \mid \sum_{l=1}^{r_0} \operatorname{tr}(\boldsymbol{W}_l) + s_{(k-k_0)}(\boldsymbol{\kappa}_\beta(\boldsymbol{W})) \leq \zeta, \ \sum_{l=1}^{r_0} \operatorname{tr}(\boldsymbol{W}_l) + \langle u_\beta, \boldsymbol{\kappa}_\beta(\boldsymbol{W}) \rangle = \zeta \right\}, \tag{3.161b}$$

where $s_{(k-k_0)} : \Re^{|\beta|} \to \Re$ is the positively homogeneous convex function defined by

$$s_{(k-k_0)}(z) = \sum_{i=1}^{k-k_0} z_i^{\downarrow}, \quad z \in \Re^{|\beta|}. \tag{3.162}$$

By (3.147), we know that for any $i, j \in \beta$, $u_i = u_j$ if $\sigma_i = \sigma_j$. Therefore, we know that the closed subset $\mathcal{C}_1$ is convex. Also, it is easy to see that $\mathcal{C}_1$ is a cone.

From Proposition 3.2, since the indicator function $\delta_{\mathcal{C}_1}(\cdot)$ is unitarily invariant, we know that the metric projection operator $\Pi_{\mathcal{C}_1} : \mathcal{W} \to \mathcal{W}$ over the closed convex set $\mathcal{C}_1$ is the spectral operator with respect to the symmetric function $\boldsymbol{\phi} = (\boldsymbol{\phi}_0, \boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_{r_1}) : \Re \times \Re^{|a_1|} \times \ldots \times \Re^{|a_{r_1}|} \to \Re \times \Re^{|a_1|} \times \ldots \times \Re^{|a_{r_1}|}$, i.e.,

$$\Pi_{\mathcal{C}_1}(\zeta, \boldsymbol{W}) = (\boldsymbol{\Phi}_0(\zeta, \boldsymbol{W}), \boldsymbol{\Phi}_1(\zeta, \boldsymbol{W}), \ldots, \boldsymbol{\Phi}_{r_1}(\zeta, \boldsymbol{W})) \tag{3.163}$$

with $\boldsymbol{\Phi}_0(\zeta, \boldsymbol{W}) = \boldsymbol{\phi}_0(\zeta, \boldsymbol{\kappa}(\boldsymbol{W})) \in \Re$ and

$$\boldsymbol{\Phi}_l(\zeta, \boldsymbol{W}) = R_l \mathrm{diag}\,(\boldsymbol{\phi}_l(\zeta, \boldsymbol{\kappa}(\boldsymbol{W}))) R_l^T \in \mathcal{S}^{|a_l|}, \quad l = 1, \ldots, r_1\,,$$

where for each $l \in \{1, \ldots, r_1\}$, $R_l \in \mathcal{O}^{|a_l|}(\boldsymbol{W}_l)$, and for any $(\zeta, \boldsymbol{\kappa}) \in \Re \times \Re^{|a_1|} \times \ldots \times \Re^{|a_{r_1}|}$, $\boldsymbol{\phi}(\zeta, \boldsymbol{\kappa})$ is the unique optimal solution of the following convex problem if $(t, X) \in \mathrm{bd}\,\mathcal{K}$,

$$\begin{array}{ll} \min & \dfrac{1}{2}\big((\eta - \zeta)^2 + \|d - \boldsymbol{\kappa}\|^2\big) \\ \text{s.t.} & \langle e_\alpha, d_\alpha \rangle + s_{(k-k_0)}(d_\beta) \le \eta\,, \end{array} \tag{3.164a}$$

if $(t, X) \notin \mathrm{bd}\,\mathcal{K}$,

$$\begin{array}{ll} \min & \dfrac{1}{2}\big((\eta - \zeta)^2 + \|d - \boldsymbol{\kappa}\|^2\big) \\ \text{s.t.} & \langle e_\alpha, d_\alpha \rangle + s_{(k-k_0)}(d_\beta) \le \eta\,, \\ & \langle e_\alpha, d_\alpha \rangle + \langle u_\beta, d_\beta \rangle = \eta\,. \end{array} \tag{3.164b}$$

Define the first divided directional difference $\boldsymbol{g}^{[1]}((t, X); (\tau, H)) \in \Re \times \Re^{m \times n}$ of $\boldsymbol{g}$ at (t, X) along the direction $(\tau, H) \in \Re \times \Re^{m \times n}$ by

$$\boldsymbol{g}^{[1]}((t, X); (\tau, H)) := \Big(\boldsymbol{g}_1^{[1]}((t, X); (\tau, H)), \boldsymbol{g}_2^{[1]}((t, X); (\tau, H))\Big) \tag{3.165}$$

with

$$\boldsymbol{g}_1^{[1]}((t, X); (\tau, H)) = \boldsymbol{\Phi}_0(\tau, \boldsymbol{D}(\widetilde{H})) \in \Re$$

and

$$\boldsymbol{g}_2^{[1]}((t, X); (\tau, H)) = \boldsymbol{T}(\widetilde{H}) + \begin{bmatrix} \boldsymbol{\Phi}_1(\tau, \boldsymbol{D}(\widetilde{H})) & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{\Phi}_{r_1}(\tau, \boldsymbol{D}(\widetilde{H})) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \Re^{m \times n}\,,$$

where the linear mapping $\boldsymbol{T}$ is defined by (3.160), $\widetilde{H} = [\overline{U}^T H\overline{V}_1 \ \ \overline{U}^T H\overline{V}_2]$, and $(\tau, \boldsymbol{D}(\widetilde{H})) \in \mathcal{W}$ with $\boldsymbol{D}(\widetilde{H}) = \Big(S(\widetilde{H}_{a_1a_1}), \ldots, S(\widetilde{H}_{a_{r_1}a_{r_1}})\Big)$.

Case 2. $(t, X) \notin \operatorname{int}\mathcal{K} \cup \operatorname{int}\mathcal{K}^\circ$ and $\overline{\sigma}_k = 0$. Then, by the part (ii) of Lemma 3.15, we know that there exists an integer $r_0 \in \{1, \ldots, r\}$ such that

$$\alpha = \bigcup_{l=1}^{r_0} a_l, \quad \beta = \bigcup_{l=r_0+1}^{r+1} a_l \quad (\text{where } a_{r+1} = b)\,,$$

where the index sets α and β are given by (3.154). Define

$$\beta_1 := \{i \in \beta \,|\, u_i = 1\}, \quad \beta_2 := \{i \in \beta \,|\, 0 < u_i < 1\} \quad \text{and} \quad \beta_3 := \{i \in \beta \,|\, u_i = 0\}\,. \tag{3.166}$$

Then, by (3.147), we know that

$$\beta_1 \cup \beta_2 = \bigcup_{l=r_0+1}^{r} a_l \quad \text{and} \quad \beta_3 = a_{r+1} = b\,.$$

Since $\overline{\sigma}_i = 0$ for any $i \in \beta$, we know from (3.151) and (3.152) that the corresponding matrices defined by (3.156)-(3.158) satisfy

$$(\mathcal{E}_1)_{a_la_{l'}} = E_{a_la_{l'}} \quad \forall\, l \neq l' \in \{1, \ldots, r_0\}\,,$$

$$(\mathcal{E}_1)_{\beta\beta} = (\mathcal{E}_2)_{\beta\beta} = 0 \quad \text{and} \quad \mathcal{F}_{\beta c} = 0\,.$$

For the given $(t, X) \in \Re \times \Re^{m\times n}$, define a linear operator $\boldsymbol{T} : \Re^{m\times n} \to \Re^{m\times n}$ by for any $Z = [Z_1 \ \ Z_2] \in \Re^{m\times n}$,

$$\boldsymbol{T}(Z) = \begin{bmatrix} (\mathcal{E}_1)_{\alpha\alpha} \circ S(Z_{\alpha\alpha}) + (\mathcal{E}_2)_{\alpha\alpha} \circ T(Z_{\alpha\alpha}) & (\mathcal{E}_1)_{\alpha\beta} \circ S(Z_{\alpha\beta}) + (\mathcal{E}_2)_{\alpha\beta} \circ T(Z_{\alpha\beta}) & \mathcal{F}_{\alpha c} \circ Z_{\alpha c} \\ (\mathcal{E}_1)_{\beta\alpha} \circ S(Z_{\beta\alpha}) + (\mathcal{E}_2)_{\beta\alpha} \circ T(Z_{\beta\alpha}) & 0 & 0 \end{bmatrix}. \tag{3.167}$$

Define the finite dimensional real Euclidean space $\mathcal{W}$ by

$$\mathcal{W} := \Re \times \mathcal{S}^{|a_1|} \times \ldots \times \mathcal{S}^{|a_r|} \times \Re^{|b|\times(|b|+n-m)}\,.$$

For any $(\zeta, \boldsymbol{W}) \in \mathcal{W}$, let $\boldsymbol{\kappa}(\boldsymbol{W}) := (\lambda(\boldsymbol{W}_1), \ldots, \lambda(\boldsymbol{W}_r), \sigma(\boldsymbol{W}_{r+1})) \in \Re^m$. Let $\mathcal{C}_2 \subseteq \mathcal{W}$ be the closed subset defined as following if $(t, X) \in \operatorname{bd}\mathcal{K}$,

$$\mathcal{C}_2 := \left\{ (\zeta, \boldsymbol{W}) \in \mathcal{W} \,|\, \sum_{l=1}^{r_0} \operatorname{tr}(\boldsymbol{W}_l) + \|\boldsymbol{\kappa}_\beta(\boldsymbol{W})\|_{(k-k_0)} \leq \zeta \right\}, \tag{3.168a}$$

if $(t,X)\notin \mathrm{bd}\,\mathcal{K}$,

$$\mathcal{C}_2 := \left\{(\zeta,\boldsymbol{W})\in\mathcal{W}\mid \sum_{l=1}^{r_0}\mathrm{tr}(\boldsymbol{W}_l)+\|\boldsymbol{\kappa}_\beta(\boldsymbol{W})\|_{(k-k_0)}\le\zeta,\ \sum_{l=1}^{r_0}\mathrm{tr}(\boldsymbol{W}_l)+\langle u_\beta,\boldsymbol{\kappa}_\beta(\boldsymbol{W})\rangle=\zeta\right\}, \tag{3.168b}$$

where $\|\cdot\|_{(k-k_0)}:\Re^{|\beta|}\to\Re$ is the positive homogeneous convex function defined by

$$\|z\|_{(k-k_0)}=\sum_{i=1}^{k-k_0}|z|_i^{\downarrow},\quad z\in\Re^{|\beta|}.$$

Again, by (3.151), we know that for any $i,j\in\beta$, $u_i=u_j$ if $\sigma_i=\sigma_j$. Therefore, we know that the closed subset $\mathcal{C}_2$ defined by (3.168) is convex. Also, it is easy to see that $\mathcal{C}_2$ is a cone.

Similarly, since the indicator function $\delta_{\mathcal{C}_2}(\cdot)$ is unitarily invariant, we know from Proposition 3.2 that the metric projection operator $\Pi_{\mathcal{C}_2}:\mathcal{W}\to\mathcal{W}$ over the closed convex set $\mathcal{C}_2$ is the spectral operator with respect to the symmetric function $\boldsymbol{\phi}:=(\boldsymbol{\phi}_0,\boldsymbol{\phi}_1,\ldots,\boldsymbol{\phi}_r,\boldsymbol{\phi}_{r+1}):\Re\times\Re^{|a_1|}\times\ldots\times\Re^{|a_r|}\times\Re^{|b|}\to\Re\times\Re^{|a_1|}\times\ldots\times\Re^{|a_r|}\times\Re^{|b|}$, i.e.,

$$\Pi_{\mathcal{C}_2}(\zeta,\boldsymbol{W})=(\boldsymbol{\Phi}_0(\zeta,\boldsymbol{W}),\boldsymbol{\Phi}_1(\zeta,\boldsymbol{W}),\ldots,\boldsymbol{\Phi}_r(\zeta,\boldsymbol{W}),\boldsymbol{\Phi}_{r+1}(\zeta,\boldsymbol{W})) \tag{3.169}$$

with $\boldsymbol{\Phi}_0(\zeta,\boldsymbol{W})=\boldsymbol{\phi}_0(\zeta,\boldsymbol{\kappa}(\boldsymbol{W}))\in\Re$ and

$$\begin{cases}\boldsymbol{\Phi}_l(\zeta,\boldsymbol{W})=R_l\mathrm{diag}\,(\boldsymbol{\phi}_l(\zeta,\boldsymbol{\kappa}(\boldsymbol{W})))R_l^T\in\mathcal{S}^{|a_l|}, & l=1,\ldots,r,\\ \boldsymbol{\Phi}_{r+1}(\zeta,\boldsymbol{W})=E[\mathrm{diag}\,(\boldsymbol{\phi}_{r+1}(\zeta,\boldsymbol{\kappa}(\boldsymbol{W})))\ \ 0]F^T\in\Re^{|b|\times(|b|+n-m)}, & \end{cases}$$

where $R_l\in\mathcal{O}^{|a_l|}(\boldsymbol{W}_l)$, $l=1,\ldots,r$, $(E,F)\in\mathcal{O}^{|b|,|b|+n-m}(\boldsymbol{W}_{r+1})$, and for any $(\zeta,\boldsymbol{\kappa})\in\Re\times\Re^{|\alpha|+|\beta|}$, $\boldsymbol{\phi}(\zeta,\boldsymbol{\kappa})$ is the unique optimal solution of the following convex problem if $(t,X)\in\mathrm{bd}\,\mathcal{K}$,

$$\begin{array}{ll}\min & \frac{1}{2}\big((\eta-\zeta)^2+\|d-\boldsymbol{\kappa}\|^2\big)\\ \text{s.t.} & \langle e_\alpha,d_\alpha\rangle+\|d_\beta\|_{(k-k_0)}\le\eta,\end{array} \tag{3.170a}$$

if $(t,X)\notin\mathrm{bd}\,\mathcal{K}$,

$$\begin{array}{ll}\min & \frac{1}{2}\big((\eta-\zeta)^2+\|d-\boldsymbol{\kappa}\|^2\big)\\ \text{s.t.} & \langle e_\alpha,d_\alpha\rangle+\|d_\beta\|_{(k-k_0)}\le\eta,\\ & \langle e_\alpha,d_\alpha\rangle+\langle u_\beta,d_\beta\rangle=\eta.\end{array} \tag{3.170b}$$

Similarly, define the first divided directional difference $\boldsymbol{g}^{[1]}((t,X);(\tau,H)) \in \Re \times \Re^{m\times n}$ of $\boldsymbol{g}$ at (t,X) along the direction $(\tau,H) \in \Re \times \Re^{m\times n}$ by

$$\boldsymbol{g}^{[1]}((t,X);(\tau,H)) := \left(\boldsymbol{g}_1^{[1]}((t,X);(\tau,H)), \boldsymbol{g}_2^{[1]}((t,X);(\tau,H))\right) \tag{3.171}$$

with

$$\boldsymbol{g}_1^{[1]}((t,X);(\tau,H)) = \boldsymbol{\Phi}_0(\tau, \boldsymbol{D}(\widetilde{H})) \in \Re$$

and

$$\begin{aligned}
&\boldsymbol{g}_2^{[1]}((t,X);(\tau,H)) \\
= \;& \boldsymbol{T}(\widetilde{H}) + \begin{bmatrix} \boldsymbol{\Phi}_1(\tau,\boldsymbol{D}(\widetilde{H})) & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \boldsymbol{\Phi}_r(\tau,\boldsymbol{D}(\widetilde{H})) & 0 \\ 0 & 0 & 0 & \boldsymbol{\Phi}_{r+1}(\tau,\boldsymbol{D}(\widetilde{H})) \end{bmatrix} \in \Re^{m\times n},
\end{aligned}$$

where the linear mapping $\boldsymbol{T}$ is defined by (3.167), $(\tau, \boldsymbol{D}(\widetilde{H})) \in \mathcal{W}$ with

$$\boldsymbol{D}(\widetilde{H}) = \left(S(\widetilde{H}_{a_1a_1}), \ldots, S(\widetilde{H}_{a_ra_r}), [\widetilde{H}_{bb} \;\; \widetilde{H}_{bc}]\right),$$

and $\widetilde{H} = [\overline{U}^T H \overline{V}_1 \;\; \overline{U}^T H \overline{V}_2]$.

Consequently, from Theorem 3.4, we have the following results on the directional differentiability of $\Pi_{\mathcal{K}}$.

Proposition 3.16. *Let $(t,X) \notin \operatorname{int}\mathcal{K} \cup \operatorname{int}\mathcal{K}^\circ$ be given. Suppose X has the singular value decomposition (3.155). Denote $(\bar{t},\overline{X}) = \Pi_{\mathcal{K}}(t,X)$. The metric projection operator $\Pi_{\mathcal{K}}$ is directionally differentiable at (t,X) and the directional derivative at (t,X) along the direction $(\tau,H) \in \Re \times \Re^{m\times n}$ is given by*

$$\Pi'_{\mathcal{K}}((t,X);(\tau,H)) = \left(\boldsymbol{g}_1^{[1]}((t,X);(\tau,H)), \overline{U}\boldsymbol{g}_2^{[1]}((t,X);(\tau,H))\overline{V}^T\right),$$

where the first divided directional difference $\boldsymbol{g}^{[1]}((t,X);(\tau,H)) \in \Re \times \Re^{m\times n}$ is defined by (3.165) if $\sigma_k(\overline{X}) > 0$, and defined by (3.171) if $\sigma_k(\overline{X}) = 0$.

By [113, Theorem 5.2], the following characterization of the F(réchet)-differentiability of $\Pi_{\mathcal{K}}$ follows from Theorem 3.6 directly.

Proposition 3.17. *Let $(t, X) \in \Re \times \Re^{m\times n}$ be given. Denote $(\bar{t}, \overline{X}) = \Pi_{\mathcal{K}}(t, X)$. The metric projection operator $\Pi_{\mathcal{K}}$ is Fréchet differentiable if and only if (t, X) satisfies one of the following conditions:*

(i) $\|X\|_{(k)} < t$;

(ii) $\|X\|_{(k)} > t$, $\sigma_k(\overline{X}) > 0$, $k_1 > k$ *and* $\beta_1 = \emptyset$, $\beta_3 = \emptyset$, *where the index sets* β_1 *and* β_3 *are defined in (3.159);*

(iii) $\|X\|_{(k)} > t$, $\sigma_k(\overline{X}) > 0$, $k_1 = k$;

(iv) $\|X\|_{(k)} > t$, $\sigma_k(\overline{X}) = 0$, $\sum_{i=1}^{m-k_0} u_{k_0+i} < k - k_0$ *and* $\beta_1 = \emptyset$, *where the index set* β_1 *in defined in (3.166).*

Note that (i) of Proposition 3.17 is equivalent with $(t, X) \in \operatorname{int}\mathcal{K}$, and (iv) of Proposition 3.17 includes the case that $(t, X) \in \operatorname{int}\mathcal{K}^\circ$. Moreover, the derivative formula of $\Pi_{\mathcal{K}}$ can be obtained from Theorem 3.6 immediately. For the sake of completeness, we provide the formula as follows.

If $\|X\|_{(k)} < t$, then

$$\Pi'_{\mathcal{K}}(t, X)(\tau, H) = (\tau, H), \quad (\tau, H) \in \Re \times \Re^{m\times n}.$$

If $\|X\|_{(k)} > t$, $\sigma_k(\overline{X}) > 0$, $k_1 > k$ and $\beta_1 = \emptyset$, $\beta_3 = \emptyset$, then for any $(\tau, H) \in \Re\times\Re^{m\times n}$,

$$\begin{aligned}
&\Pi'_{\mathcal{K}}(t, X)(\tau, H) \\
= \;& \Big(\boldsymbol{\Phi}_0(\tau, \boldsymbol{D}(\widetilde{H})), \overline{U}(\boldsymbol{T}(\widetilde{H}) + \begin{bmatrix} \boldsymbol{\Phi}_1(\tau, \boldsymbol{D}(\widetilde{H})) & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \boldsymbol{\Phi}_{r_1}(\tau, \boldsymbol{D}(\widetilde{H})) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix})\overline{V}^T\Big),
\end{aligned} \tag{3.172}$$

where the linear mapping $\boldsymbol{T}$ is defined by (3.160), $\widetilde{H} = [\overline{U}^T H \overline{V}_1 \;\; \overline{U}^T H \overline{V}_2]$, $(\tau, \boldsymbol{D}(\widetilde{H})) \in \mathcal{W}$ with $\boldsymbol{D}(\widetilde{H}) := \Big(S(\widetilde{H}_{a_1 a_1}), \ldots, S(\widetilde{H}_{a_{r_1} a_{r_1}})\Big)$, and $\boldsymbol{\Phi} : \mathcal{W} \to \mathcal{W}$ is defined by (3.163) with respect to the symmetric function $\boldsymbol{\phi} : \Re \times \Re^{|a_1|} \times \ldots \times \Re^{|a_{r_1}|} \to \Re \times \Re^{|a_1|} \times \ldots \times \Re^{|a_{r_1}|}$, i.e., for any $(\zeta, \boldsymbol{\kappa}) \in \Re \times \Re^{|\alpha|+|\beta|}$, $\boldsymbol{\phi}(\zeta, \boldsymbol{\kappa})$ is the unique optimal solution of the following convex problem

$$\begin{array}{ll} \min & \frac{1}{2}\big((\eta - \zeta)^2 + \|d - \boldsymbol{\kappa}\|^2\big) \\ \text{s.t.} & \langle e_\alpha, d_\alpha \rangle + (k - k_0)\omega = \eta \,, \\ & d_i = d_j = \omega, \;\; i, j \in \beta \,. \end{array} \tag{3.173}$$

If $\|X\|_{(k)} > t$, $\sigma_k(\overline{X}) > 0$, $k_1 = k$, then for any $(\tau, H) \in \Re \times \Re^{m \times n}$,

$$\begin{aligned} & \Pi'_{\mathcal{K}}(t, X)(\tau, H) \\ = \;& \Bigg(\boldsymbol{\Phi}_0(\tau, \boldsymbol{D}(\widetilde{H})), \overline{U}(\boldsymbol{T}(\widetilde{H}) + \begin{bmatrix} \boldsymbol{\Phi}_1(\tau, \boldsymbol{D}(\widetilde{H})) & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \boldsymbol{\Phi}_{r_1}(\tau, \boldsymbol{D}(\widetilde{H})) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix})\overline{V}^T\Bigg) \,, \end{aligned} \tag{3.174}$$

where the linear mapping $\boldsymbol{T}$ is defined by (3.160), $\widetilde{H} = [\overline{U}^T H \overline{V}_1 \;\; \overline{U}^T H \overline{V}_2]$, $(\tau, \boldsymbol{D}(\widetilde{H})) \in \mathcal{W}$ with $\boldsymbol{D}(\widetilde{H}) := \Big(S(\widetilde{H}_{a_1 a_1}), \ldots, S(\widetilde{H}_{a_{r_1} a_{r_1}})\Big)$, and $\boldsymbol{\Phi} : \mathcal{W} \to \mathcal{W}$ is defined by (3.163) with respect to the symmetric function $\boldsymbol{\phi} : \Re \times \Re^{|a_1|} \times \ldots \times \Re^{|a_{r_1}|} \to \Re \times \Re^{|a_1|} \times \ldots \times \Re^{|a_{r_1}|}$, i.e., for any $(\zeta, \boldsymbol{\kappa}) \in \Re \times \Re^{|\alpha|+|\beta|}$, $\boldsymbol{\phi}(\zeta, \boldsymbol{\kappa})$ is the unique optimal solution of the following convex problem

$$\begin{array}{ll} \min & \frac{1}{2}\big((\eta - \zeta)^2 + \|d - \boldsymbol{\kappa}\|^2\big) \\ \text{s.t.} & \langle e_\alpha, d_\alpha \rangle + \langle e_\beta, d_\beta \rangle = \eta \,. \end{array} \tag{3.175}$$

If $\|X\|_{(k)} > t$, $\sigma_k(\overline{X}) = 0$, $\sum_{i=1}^{m-k_0} u_{k_0+i} < k - k_0$ and $\beta_1 = \emptyset$, then for any $(\tau, H) \in$

$\Re \times \Re^{m\times n}$,

$$
\begin{aligned}
&\Pi'_{\mathcal{K}}(t,X)(\tau,H)\\
= \;&\left(\boldsymbol{\Phi}_0(\tau,\boldsymbol{D}(\widetilde{H})), \overline{U}(\boldsymbol{T}(\widetilde{H}) + \begin{bmatrix} \boldsymbol{\Phi}_1(\tau,\boldsymbol{D}(\widetilde{H})) & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \boldsymbol{\Phi}_r(\tau,\boldsymbol{D}(\widetilde{H})) & 0 \\ 0 & 0 & 0 & \boldsymbol{\Phi}_{r+1}(\tau,\boldsymbol{D}(\widetilde{H})) \end{bmatrix})\overline{V}^T\right),
\end{aligned} \tag{3.176}
$$

where the linear mapping $\boldsymbol{T}$ is defined by (3.167), $(\tau,\boldsymbol{D}(\widetilde{H})) \in \mathcal{W}$ with

$$
\boldsymbol{D}(\widetilde{H}) := \left(S(\widetilde{H}_{a_1a_1}), \ldots, S(\widetilde{H}_{a_ra_r}), [\widetilde{H}_{bb} \;\; \widetilde{H}_{bc}]\right),
$$

and $\widetilde{H} = [\overline{U}^T H \overline{V}_1 \;\; \overline{U}^T H \overline{V}_2]$, and $\boldsymbol{\Phi} : \mathcal{W} \to \mathcal{W}$ is defined by (3.169) with respect to the symmetric function $\boldsymbol{\phi} : \Re \times \Re^{|a_1|} \times \ldots \times \Re^{|a_r|} \times \Re^{|b|} \to \Re \times \Re^{|a_1|} \times \ldots \times \Re^{|a_r|} \times \Re^{|b|}$, i.e., for any $(\zeta,\boldsymbol{\kappa}) \in \Re \times \Re^{|\alpha|+|\beta|}$, $\boldsymbol{\phi}(\zeta,\boldsymbol{\kappa})$ is the unique optimal solution of the following convex problem

$$
\begin{aligned}
\min \quad & \frac{1}{2}\left((\eta-\zeta)^2 + \|d-\boldsymbol{\kappa}\|^2\right)\\
\text{s.t.} \quad & \langle e_\alpha, d_\alpha\rangle = \eta,\\
& d_\beta = 0.
\end{aligned} \tag{3.177}
$$

Since the symmetric function $\boldsymbol{g}$ defined by (3.144) is piecewise linear, it is well-known that $\boldsymbol{g}$ is strongly semismooth everywhere (see, e.g., [33, Proposition 7.4.7]). Therefore, we know from Theorem 3.12 that the metric projection operator $\Pi_{\mathcal{K}}$ is strongly semismooth everywhere.

We end this section by considering the characterizations of B-subdifferenial $\partial_B\Pi_{\mathcal{K}}$ and Clarke's generalized Jacobian $\partial\Pi_{\mathcal{K}}$ of the metric projector $\Pi_{\mathcal{K}}$. Some useful observations will also be presented. Let $(t,X) \in \Re \times \Re^{m\times n}$ be given. Since the symmetric function $\boldsymbol{g}$ is the metric projection operator over the polyhedral convex set epi $\|\cdot\|_{(k)} \subseteq \Re \times \Re^m$, we know that there exists an open neighborhood $\mathcal{N} \in \Re \times \Re^m$ of zero such that

$$
\boldsymbol{d}(\tau,h) = \boldsymbol{g}((t,\sigma)+(\tau,h)) - \boldsymbol{g}(t,\sigma) - \boldsymbol{g}'((t,\sigma);(\tau,h)) \equiv 0 \quad \forall\, (\tau,h) \in \mathcal{N}.
$$

Therefore, we know from Theorem 3.14 that

$$\partial_B \Pi_{\mathcal{K}}(t,X) = \partial_B \boldsymbol{\Psi}(0,0)\,,$$

where $\boldsymbol{\Psi}(\cdot,\cdot) := \Pi'_{\mathcal{K}}((t,X);(\cdot,\cdot))$ the directional derivative of $\Pi_{\mathcal{K}}$ at (t,X). Meanwhile, by Proposition 3.16, we obtain the following characterizations of $\partial_B\Pi_{\mathcal{K}}$ and $\partial\Pi_{\mathcal{K}}$.

Proposition 3.18. *Let $(t,X) \notin \operatorname{int}\mathcal{K} \cup \operatorname{int}\mathcal{K}^\circ$ be given. Suppose X has the singular value decomposition (3.155). Denote $(\bar{t},\overline{X}) = \Pi_{\mathcal{K}}(t,X)$.*

(i) If $\sigma_k(\overline{X}) > 0$, then $\boldsymbol{V} \in \partial_B\Pi_{\mathcal{K}}(t,X)$ (respectively, $\partial\Pi_{\mathcal{K}}(t,X)$) if and only if there exists $\boldsymbol{K} = (\boldsymbol{K}_0,\boldsymbol{K}_1,\ldots,\boldsymbol{K}_{r_1}) \in \partial_B\Pi_{\mathcal{C}_1}(0,0)$ (respectively, $\partial\Pi_{\mathcal{C}_1}(0,0)$) such that

$$\boldsymbol{V}(\tau,H) = \left(\boldsymbol{V}_0(\tau,H), \boldsymbol{V}_1(\tau,H)\right)\,,$$

where $\widetilde{H} = \overline{U}^T H \overline{V}$, $\boldsymbol{V}_0(\tau,H) = \boldsymbol{K}_0(\tau,\boldsymbol{D}(\widetilde{H}))$,

$$\boldsymbol{V}_1(\tau,H) = \overline{U}\boldsymbol{T}(\widetilde{H})\overline{V}^T + \overline{U}\begin{bmatrix} \boldsymbol{K}_1(\tau,\boldsymbol{D}(\widetilde{H})) & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{K}_{r_1}(\tau,\boldsymbol{D}(\widetilde{H})) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\overline{V}^T\,, \tag{3.178}$$

with $\boldsymbol{D}(\widetilde{H}) = \left(S(\widetilde{H}_{a_1a_1}),\ldots,S(\widetilde{H}_{a_{r_1}a_{r_1}})\right)$, and the linear mapping $\boldsymbol{T}$ is defined by (3.160).

(ii) If $\sigma_k(\overline{X}) = 0$, then $\boldsymbol{V} \in \partial_B\Pi_{\mathcal{K}}(t,X)$ (respectively, $\partial\Pi_{\mathcal{K}}(t,X)$) if and only if there exists $\boldsymbol{K} = (\boldsymbol{K}_0,\boldsymbol{K}_1,\ldots,\boldsymbol{K}_r,\boldsymbol{K}_{r+1}) \in \partial_B\Pi_{\mathcal{C}_2}(0,0)$ (respectively, $\partial\Pi_{\mathcal{C}_2}(0,0)$) such that

$$\boldsymbol{V}(\tau,H) = \left(\boldsymbol{V}_0(\tau,H), \boldsymbol{V}_1(\tau,H)\right)\,,$$

where $\widetilde{H} = \overline{U}^T H \overline{V}$, $\boldsymbol{V}_0(\tau, H) = \boldsymbol{K}_0(\tau, \boldsymbol{D}(\widetilde{H}))$,

$$\boldsymbol{V}_1(\tau, H) = \overline{U}\boldsymbol{T}(\widetilde{H})\overline{V}^T + \overline{U}\begin{bmatrix} \boldsymbol{K}_1(\tau, \boldsymbol{D}(\widetilde{H})) & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \boldsymbol{K}_r(\tau, \boldsymbol{D}(\widetilde{H})) & 0 \\ 0 & 0 & 0 & \boldsymbol{K}_{r+1}(\tau, \boldsymbol{D}(\widetilde{H})) \end{bmatrix}\overline{V}^T, \tag{3.179}$$

with $\boldsymbol{D}(\widetilde{H}) = \Big(S(\widetilde{H}_{a_1a_1}), \ldots, S(\widetilde{H}_{a_ra_r}), [\widetilde{H}_{bb} \;\; \widetilde{H}_{bc}]\Big)$, *and the linear mapping* $\boldsymbol{T}$ *is defined by (3.167).*

The following observation is important to the sensitivity analysis on the linear MCP involving the Ky Fan k-norm in Section 4.2.

Lemma 3.19. *Let* $(t, X) \in \Re \times \Re^{m\times n}$ *be given. Denote* $(\bar{t}, \overline{X}) = \Pi_{\mathcal{K}}(t, X)$. *Suppose that* $\boldsymbol{V} = (\boldsymbol{V}_1, \boldsymbol{V}_2) \in \partial\Pi_{\mathcal{K}}(t, X)$. *Assume that* $(\triangle\zeta, \triangle\Gamma) \in \Re\times\Re^{m\times n}$ *satisfies* $\boldsymbol{V}(\triangle\zeta, \triangle\Gamma) = 0$.

(i) If $\sigma_k(\overline{X}) > 0$, *then*

$$\triangle\Gamma = \overline{U}\begin{bmatrix} -\triangle\zeta I_{|\alpha|} & 0 & 0 & 0 \\ 0 & \triangle\widetilde{\Gamma}_{\beta\beta} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\overline{V}^T, \tag{3.180}$$

where $\triangle\widetilde{\Gamma}_{\beta\beta}$ *is symmetric and*

$$\mathrm{tr}\,(\triangle\widetilde{\Gamma}_{\beta\beta}) + (k - k_0)\triangle\zeta = 0\,, \tag{3.181}$$

where $\triangle\widetilde{\Gamma} = \overline{U}^T\triangle\Gamma\overline{V}$.

(ii) If $\sigma_k(\overline{X}) = 0$, *then*

$$\triangle\Gamma = \overline{U}\begin{bmatrix} -\triangle\zeta I_{|\alpha|} & 0 & 0 \\ 0 & \triangle\widetilde{\Gamma}_{\beta\beta} & \triangle\widetilde{\Gamma}_{\beta c} \end{bmatrix}\overline{V}^T, \tag{3.182}$$

where $\triangle\widetilde{\Gamma} = \overline{U}^T\triangle\Gamma\overline{V}$.

Proof. Without loss of generality, assume that $(t, X) \notin \operatorname{int}\mathcal{K} \cup \operatorname{int}\mathcal{K}^\circ$, since otherwise the results hold trivially.

Case 1. $\sigma_k(\overline{X}) > 0$. Since for any $(\boldsymbol{g}_2)_i(t,\sigma(X)) > (\boldsymbol{g}_2)_j(t,\sigma(X)) > (\boldsymbol{g}_2)_s(t,\sigma(X))$ for any $i \in \alpha$, $j \in \beta$, $s \in \gamma$ and $(\boldsymbol{g}_2)_j(t,\sigma(X)) > 0$ for any $i \in \alpha \cup \beta$, we know from (3.178) that

$$\triangle\widetilde{\Gamma}_{\alpha\alpha} = \begin{bmatrix} \triangle\widetilde{\Gamma}_{a_1a_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \triangle\widetilde{\Gamma}_{a_{r_0}a_{r_0}} \end{bmatrix}, \quad \triangle\widetilde{\Gamma}_{\beta\beta} = \begin{bmatrix} \triangle\widetilde{\Gamma}_{a_{r_0+1}a_{r_0+1}} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \triangle\widetilde{\Gamma}_{a_{r_1}a_{r_1}} \end{bmatrix},$$

$\triangle\widetilde{\Gamma}_{a_la_l} = S(\triangle\widetilde{\Gamma}_{a_la_l}) \in \mathcal{S}^{|a_l|}$, $l = 1, \ldots, r_1$ and

$$\triangle\Gamma = \overline{U} \begin{bmatrix} \triangle\widetilde{\Gamma}_{\alpha\alpha} & 0 & 0 & 0 \\ 0 & \triangle\widetilde{\Gamma}_{\beta\beta} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \overline{V}^T .$$

Therefore, we know that $\triangle\widetilde{\Gamma}_{\beta\beta}$ is symmetric.

For the given (t, X), we first assume that $k < k_1$, i.e., $\beta_3 \neq \emptyset$. Let $\mathcal{W}$ be the Euclid space defined by

$$\mathcal{W} = \mathcal{S}^{|a_1|} \times \ldots \times \mathcal{S}^{|a_{r_1}|} .$$

Since $\boldsymbol{V}(\triangle\zeta, \triangle\Gamma) = 0$, we know from Proposition 3.18 that there exists $\boldsymbol{K} = (\boldsymbol{K}_0, \boldsymbol{K}_1, \ldots, \boldsymbol{K}_{r_1}) \in \partial\Pi_{\mathcal{C}_1}(0,0)$ such that $\boldsymbol{K}_0(\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{\Gamma})) = 0$ and

$$\boldsymbol{K}_l(\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{\Gamma})) = 0, \quad l = 1, \ldots, r_1 ,$$

where $\Pi_{\mathcal{C}_1} : \mathcal{W} \to \mathcal{W}$ is the metric projection operator over the matrix cone $\mathcal{C}_1 \subseteq \mathcal{W}$ (defined in (3.161)), and $\boldsymbol{D}(\triangle\widetilde{\Gamma}) = (\triangle\widetilde{\Gamma}_{a_1a_1}, \ldots, \triangle\widetilde{\Gamma}_{a_{r_1}a_{r_1}}) \in \mathcal{W}$. Denote

$$\Omega := \{(\boldsymbol{W}_1, \ldots, \boldsymbol{W}_{r_1}) \in \mathcal{W} \,|\, \text{for each } l \in \{1, \ldots, r_1\}, \text{ the eigenvalues of } \boldsymbol{W}_l \text{ are distinct}\} .$$

Let $\mathcal{D}_{\Pi_{\mathcal{C}_1}} \subseteq \mathcal{W}$ be the set of points at which $\Pi_{\mathcal{C}_1}$ is differentiable. Since the set $\mathcal{W} \setminus \Omega$ measure zero (in sense of Lebesgue), we know from [109, Theorem 4] that

$$\partial\Pi_{\mathcal{C}_1}(0,0) = \operatorname{conv}\Upsilon ,$$

where $\Upsilon := \left\{ \lim_{(\eta, \boldsymbol{W}) \to (0,0)} \Pi'_{\mathcal{C}_1}(\eta, \boldsymbol{W}) \,|\, (\eta, \boldsymbol{W}) \in \mathcal{D}_{\Pi_{\mathcal{C}_1}} \cap \Omega \right\}$.

Next, we consider the elements of Υ. Suppose that $\Theta \in \Upsilon$. Then there exists a sequence $\{(\eta^{(q)}, \boldsymbol{W}^{(q)})\}$ in $\mathcal{D}_{\Pi_{\mathcal{C}_1}} \cap \Omega$ such that

$$\Theta(\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{\Gamma})) = \lim_{q \to \infty} \Pi'_{\mathcal{C}_1}(\eta^{(q)}, \boldsymbol{W}^{(q)})(\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{\Gamma}))\,.$$

By (3.163), we know that $\Pi_{\mathcal{C}_2}$ is the spectral operator with respect to the symmetric function $\boldsymbol{\phi}$ defined by (3.164). We know from Theorem 3.7 that for each q, $\Pi_{\mathcal{C}_1}$ is differentiable at $(\eta^{(q)}, \boldsymbol{W}^{(q)})$ if and only if $\boldsymbol{\phi}$ is differentiable at $(\eta^{(q)}, \boldsymbol{\lambda}^{(q)})$, where

$$\boldsymbol{\lambda}^{(q)} = \left(\lambda(\boldsymbol{W}_1^{(q)}), \ldots, \lambda(\boldsymbol{W}_{r_1}^{(q)})\right) \in \Re^{|a_1|} \times \ldots \times \Re^{|a_{r_1}|}\,.$$

Correspondingly, for each q, let $R_l^{(q)} \in \mathcal{O}^{|a_l|}(\boldsymbol{W}_l^{(q)})$, $l = 1, \ldots, r_1$. Moreover, we know from [113, Theorem 5.1] that for any $(\eta, \boldsymbol{\lambda})$ sufficiently close to $(0,0)$,

$$\boldsymbol{\phi}(\eta, \boldsymbol{\lambda}) = \boldsymbol{\psi}(t + \eta, \sigma + \boldsymbol{\lambda}) - \boldsymbol{\psi}(t, \sigma)\,,$$

where $\sigma = \sigma(X)_{\alpha\cup\beta}$ and $\boldsymbol{\psi}(t, \sigma) = (\boldsymbol{g}_1(t, \sigma(X)), (\boldsymbol{g}_2(t, \sigma(X)))_{\alpha\cup\beta})$. Therefore, we know that for q sufficiently large, $\boldsymbol{\phi}$ is differentiable at $(\eta^{(q)}, \boldsymbol{\lambda}^{(q)})$ if and only if $\boldsymbol{\psi}$ is differentiable at $(t + \eta^{(q)}, \sigma + \boldsymbol{\lambda}^{(q)})$ and

$$\boldsymbol{\phi}'(\eta^{(q)}, \boldsymbol{\lambda}^{(q)}) = \boldsymbol{\psi}'(t + \eta^{(q)}, \sigma + \boldsymbol{\lambda}^{(q)})\,.$$

For each q, denote

$$\widetilde{\boldsymbol{D}}^{(q)} := \left((R_1^{(q)})^T \triangle\widetilde{\Gamma}_{a_1 a_1} R_1^{(q)}, \ldots, (R_{r_1}^{(q)})^T \triangle\widetilde{\Gamma}_{a_{r_1} a_{r_1}} R_{r_1}^{(q)}\right)$$

and

$$\widetilde{\boldsymbol{d}}^{(q)} = \left(\widetilde{\boldsymbol{d}}_1^{(q)}, \ldots, \widetilde{\boldsymbol{d}}_{r_1}^{(q)}\right) \in \Re^{|a_1|} \times \ldots \times \Re^{|a_{r_1}|}\,,$$

where for each $l \in \{1, \ldots, r_1\}$, $\widetilde{\boldsymbol{d}}_l^{(q)} \in \Re^{|a_l|}$ is the vector whose elements are diagonal elements of $(R_l^{(q)})^T \triangle\widetilde{H}_{a_l a_l} R_l^{(q)}$. For each q, denote

$$(\rho^{(q)}, \boldsymbol{h}^{(q)}) := \boldsymbol{\phi}'(\eta^{(q)}, \boldsymbol{\lambda}^{(q)})(\triangle\zeta, \widetilde{\boldsymbol{d}}^{(q)})$$

Since for $(\eta^{(q)},\boldsymbol{\lambda}^{(q)})$ sufficiently close to $(0,0)$, $k_0'\in\beta_1$ and $k_1'\in\beta_3$ (i.e., $\alpha\subseteq\alpha'$, $\beta_2\subseteq\beta'$ and $k<k_1'$), by considering the KKT condition of the convex problem (3.173), we know that there exists $\theta^{(q)}\geq 0$ such that

$$\begin{aligned}
\rho^{(q)} &= \triangle\zeta+\theta^{(q)}\,, && (3.183)\\
\boldsymbol{h}_i^{(q)} &= \widetilde{\boldsymbol{d}}_i^{(q)}-\theta^{(q)},\quad i=1,\ldots,k_0'\,, && (3.184)\\
\boldsymbol{h}_i^{(q)} &= \boldsymbol{h}_j^{(q)},\quad i,j\in\beta'\,,\\
\sum_{i=k_0'+1}^{k_1'}\boldsymbol{h}_i^{(q)} &= \sum_{i=k_0'+1}^{k_1'}\widetilde{\boldsymbol{d}}_i^{(q)}-(k-k_0')\theta^{(q)}\,. && (3.185)
\end{aligned}$$

Therefore, we know from (3.184) that

$$\frac{\boldsymbol{\psi}_i(\eta^{(q)},\boldsymbol{\lambda}^{(q)})-\boldsymbol{\psi}_j(\eta^{(q)},\boldsymbol{\lambda}^{(q)})}{\boldsymbol{\lambda}_i^{(q)}-\boldsymbol{\lambda}_j^{(q)}}=1\quad\forall\, i\neq j\in\alpha\,. \tag{3.186}$$

For each q, denote

$$(\Delta_0^{(q)},\boldsymbol{\Delta}^{(q)}):=(\Delta_0^{(q)},\boldsymbol{\Delta}_1^{(q)},\ldots,\boldsymbol{\Delta}_{r_1}^{(q)})=\Pi_{\mathcal{C}_1}'(\eta^{(q)},\boldsymbol{W}^{(q)})(\Delta\zeta,\boldsymbol{D}(\Delta\widetilde{\Gamma}))\,.$$

By (3.183), (3.184), (3.186) and (3.185), we know from the derivative formula of spectral operator (3.50) that for each q,

$$\begin{aligned}
\Delta_0^{(q)} &= \triangle\zeta+\theta^{(q)}\,,\\
\boldsymbol{\Delta}_l^{(q)} &= \triangle\widetilde{\Gamma}_{a_la_l}-\theta^{(q)}I_{|a_l|},\quad l=1,\ldots,r_0\,,\\
\sum_{l=r_0+1}^{r_1}\operatorname{tr}(\boldsymbol{\Delta}_l^{(q)}) &= \operatorname{tr}(\triangle\widetilde{\Gamma}_{\beta\beta})-(k-k_0)\theta^{(q)}\,.
\end{aligned}$$

Finally, since $\boldsymbol{K}(\triangle\zeta,\boldsymbol{D}(\triangle\widetilde{\Gamma}))=0$, by taking limits and convex combinations, we know that there exists $\theta\geq 0$ such that

$$\begin{aligned}
0 &= \triangle\zeta+\theta\\
0 &= \triangle\widetilde{\Gamma}_{a_la_l}-\theta I_{|a_l|},\quad l=1,\ldots,r_0\,,\\
0 &= \operatorname{tr}(\triangle\widetilde{\Gamma}_{\beta\beta})-(k-k_0)\theta\,.
\end{aligned}$$

Therefore, we know that (3.180) and (3.181) hold.

For the case that $k = k_1$, then we know from (iii) of Proposition 3.17 that $\Pi_{\mathcal{K}}$ is differentiable. Also, since the singular value function $\sigma(\cdot)$ is globally Lipschitz continuous, we know that when $\{(t^{(q)}, X^q)\}$ sufficiently close to (t, X), we have $k = k_1'$. Therefore, the conclusion (3.180) and (3.181) can be obtained easily by considering the KKT condition of the convex problem (3.175).

Case 2. $\sigma_k(\overline{X}) = 0$. Since for any $\sigma_i(\overline{X}) > 0$ for any $i \in \alpha$, we know from (3.179) that

$$\triangle\widetilde{\Gamma}_{\alpha\alpha} = \begin{bmatrix} \triangle\widetilde{\Gamma}_{a_1a_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \triangle\widetilde{\Gamma}_{a_{r_0}a_{r_0}} \end{bmatrix},$$

$\triangle\widetilde{\Gamma}_{a_la_l} = S(\triangle\widetilde{\Gamma}_{a_la_l}) \in \mathcal{S}^{|a_l|}$, $l = 1, \ldots, r_0$ and

$$\triangle\Gamma = \overline{U} \begin{bmatrix} \triangle\widetilde{\Gamma}_{\alpha\alpha} & 0 & 0 \\ 0 & \triangle\widetilde{\Gamma}_{\beta\beta} & \triangle\widetilde{\Gamma}_{\beta c} \end{bmatrix} \overline{V}^T.$$

Let $\mathcal{W}$ be the Euclid space defined by

$$\mathcal{W} = \mathcal{S}^{|a_1|} \times \ldots \times \mathcal{S}^{|a_r|} \times \Re^{|b|\times(|b|+n-m)}.$$

Since $\boldsymbol{V}(\triangle\zeta, \triangle\Gamma) = 0$, we know from Proposition 3.18 that there exists $\boldsymbol{K} = (\boldsymbol{K}_0, \boldsymbol{K}_1, \ldots, \boldsymbol{K}_{r+1}) \in \partial\Pi_{\mathcal{C}_2}(0,0)$ such that $\boldsymbol{K}_0(\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{\Gamma})) = 0$ and

$$\boldsymbol{K}_l(\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{\Gamma})) = 0, \quad l = 1, \ldots, r+1,$$

where $\Pi_{\mathcal{C}_2} : \mathcal{W} \to \mathcal{W}$ is the metric projection operator over the matrix cone $\mathcal{C}_2 \subseteq \mathcal{W}$ (defined in (3.168)), and $\boldsymbol{D}(\triangle\widetilde{\Gamma}) = (S(\triangle\widetilde{\Gamma}_{a_1a_1}), \ldots, S(\triangle\widetilde{\Gamma}_{a_ra_r}), [\triangle\widetilde{\Gamma}_{bb} \;\; \triangle\widetilde{\Gamma}_{bc}]) \in \mathcal{W}$. Denote

$$\Omega := \{\boldsymbol{W} \in \mathcal{W} \,|\, \text{for each } l \in \{1, \ldots, r+1\}, \text{ the eigenvalues (singular values) of } \boldsymbol{W}_l \text{ are distinct}\}.$$

Let $\mathcal{D}_{\Pi_{\mathcal{C}_2}} \subseteq \mathcal{W}$ be the set of points at which $\Pi_{\mathcal{C}_2}$ is differentiable. Since the set $\mathcal{W} \setminus \Omega$ measure zero (in sense of Lebesgue), we know from [109, Theorem 4] that

$$\partial\Pi_{\mathcal{C}_2}(0,0) = \text{conv}\Upsilon,$$

where $\Upsilon := \left\{ \lim_{(\eta,\boldsymbol{W})\to(0,0)} \Pi'_{\mathcal{C}_2}(\eta,\boldsymbol{W}) \,|\, (\eta,\boldsymbol{W}) \in \mathcal{D}_{\Pi_{\mathcal{C}_2}} \cap \Omega \right\}$.

Consider the elements of Υ. Suppose that $\Theta \in \Upsilon$. Then there exists a sequence $\{(\eta^{(q)},\boldsymbol{W}^{(q)})\}$ in $\mathcal{D}_{\Pi_{\mathcal{C}_2}} \cap \Omega$ such that

$$\Theta(\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{\Gamma})) = \lim_{q\to\infty} \Pi'_{\mathcal{C}_2}(\eta^{(q)},\boldsymbol{W}^{(q)})(\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{\Gamma}))\,.$$

By (3.169), we know that $\Pi_{\mathcal{C}_2}$ is the spectral operator with respect to the symmetric function $\boldsymbol{\phi}$ defined by (3.170). We know from Theorem 3.7 that for each q, $\Pi_{\mathcal{C}_2}$ is differentiable at $(\eta^{(q)},\boldsymbol{W}^{(q)})$ if and only if $\boldsymbol{\phi}$ is differentiable at $(\eta^{(q)},\boldsymbol{\kappa}^{(q)})$, where

$$\boldsymbol{\kappa}^{(q)} = \left(\lambda(\boldsymbol{W}_1^{(q)}),\ldots,\lambda(\boldsymbol{W}_r^{(q)}),\sigma(\boldsymbol{W}_{r+1}^{(q)})\right) \in \Re^m\,.$$

Correspondingly, for each q, let

$$R_l^{(q)} \in \mathcal{O}^{|a_l|}(\boldsymbol{W}_l^{(q)}), \quad l=1,\ldots,r \quad \text{and} \quad (E^{(q)},F^{(q)}) \in \mathcal{O}^{|b|,|b|+n-m}(\boldsymbol{W}_{r+1}^{(q)})\,.$$

Moreover, we know from [113, Theorem 5.1] that for any $(\eta,\boldsymbol{\kappa})$ sufficiently close to $(0,0)$,

$$\boldsymbol{\phi}(\eta,\boldsymbol{\kappa}) = \boldsymbol{\psi}(t+\eta,\sigma+\boldsymbol{\kappa}) - \boldsymbol{\psi}(t,\sigma)\,,$$

where $\sigma = \sigma(X)$ and $\boldsymbol{\psi}(t,\sigma) = \sigma(\overline{X})$. Therefore, we know that for q sufficiently large, $\boldsymbol{\phi}$ is differentiable at $(\eta^{(q)},\boldsymbol{\kappa}^{(q)})$ if and only if $\boldsymbol{\psi}$ is differentiable at $(t+\eta^{(q)},\sigma+\boldsymbol{\kappa}^{(q)})$ and

$$\boldsymbol{\phi}'(\eta^{(q)},\boldsymbol{\kappa}^{(q)}) = \boldsymbol{\psi}'(t+\eta^{(q)},\sigma+\boldsymbol{\kappa}^{(q)})\,.$$

For each q, denote

$$\widetilde{\boldsymbol{D}}^{(q)} = \left((R_1^{(q)})^T\widetilde{\Gamma}_{a_1a_1}R_1^{(q)},\ldots,(R_r^{(q)})^T\widetilde{\Gamma}_{a_ra_r}R_r^{(q)}, E^T[\widetilde{\Gamma}_{bb} \;\; \widetilde{\Gamma}_{bc}]F\right)$$

and

$$\widetilde{\boldsymbol{d}}^{(q)} = \left(\widetilde{\boldsymbol{d}}_1^{(q)},\ldots,\widetilde{\boldsymbol{d}}_r^{(q)},\widetilde{\boldsymbol{d}}_{r+1}^{(q)}\right) \in \Re^{|a_1|}\times\ldots\times\Re^{|a_{r+1}|}\,,$$

where for each $l \in \{1,\ldots,r\}$, $\widetilde{\boldsymbol{d}}_l^{(q)} \in \Re^{|a_l|}$ is the vector whose elements are diagonal elements of $(R_l^{(q)})^T\triangle\widetilde{H}_{a_la_l}R_l^{(q)}$, and $\widetilde{\boldsymbol{d}}_{r+1}^{(q)}$ is the vector whose elements are diagonal elements of $E^T[\widetilde{H}_{bb} \;\; \widetilde{H}_{bc}]F$. For each q, denote

$$(\rho^{(q)},\boldsymbol{h}^{(q)}) := \boldsymbol{\phi}'(\eta^{(q)},\boldsymbol{\lambda}^{(q)})(\triangle\zeta,\widetilde{\boldsymbol{d}}^{(q)})$$

Since for $(\eta^{(q)}, \boldsymbol{\lambda}^{(q)})$ sufficiently close to $(0,0)$, $k_0' \in \beta_1$ (i.e., $\alpha \subseteq \alpha'$), by considering the KKT conditions of the convex problems (3.173), (3.175) and (3.177), we know that there exists $\theta^{(q)} \geq 0$ such that

$$\rho^{(q)} = \triangle\zeta + \theta^{(q)}\,, \tag{3.187}$$

$$\boldsymbol{h}_i^{(q)} = \widetilde{\boldsymbol{d}}_i^{(q)} - \theta^{(q)}, \quad i = 1, \ldots, k_0'\,. \tag{3.188}$$

Therefore, we know from (3.188) that

$$\frac{\boldsymbol{\psi}_i(\eta^{(q)}, \boldsymbol{\kappa}^{(q)}) - \boldsymbol{\psi}_j(\eta^{(q)}, \boldsymbol{\kappa}^{(q)})}{\boldsymbol{\kappa}_i^{(q)} - \boldsymbol{\kappa}_j^{(q)}} = 1 \quad \forall\, i \neq j \in \alpha\,. \tag{3.189}$$

For each q, denote

$$(\Delta_0^{(q)}, \boldsymbol{\Delta}^{(q)}) := (\Delta_0^{(q)}, \boldsymbol{\Delta}_1^{(q)}, \ldots, \boldsymbol{\Delta}_{r+1}^{(q)}) = \Pi'_{\mathcal{C}_2}(\eta^{(q)}, \boldsymbol{W}^{(q)})(\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{\Gamma}))\,.$$

By (3.187), (3.188) and (3.189), we know from the derivative formula of spectral operator that for each q,

$$\Delta_0^{(q)} = \triangle\zeta + \theta^{(q)}\,,$$

$$\boldsymbol{\Delta}_l^{(q)} = \triangle\widetilde{\Gamma}_{a_l a_l} - \theta^{(q)} I_{|a_l|}, \quad l = 1, \ldots, r_0\,.$$

Finally, since $\boldsymbol{K}(\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{\Gamma})) = 0$, by taking limits and convex combinations, we know that there exists $\theta \geq 0$ such that

$$0 = \triangle\zeta + \theta$$

$$0 = \triangle\widetilde{\Gamma}_{a_l a_l} - \theta I_{|a_l|}, \quad l = 1, \ldots, r_0\,.$$

Therefore, we know that (3.182) holds. □

3.8.1 The metric projectors over the epigraphs of the spectral norm and nuclear norm

As we mentioned before, the closed form solutions of the metric projection operators over the epigraphs of the spectral norm and nuclear norm are provided in [30]. On the other hand, for the matrix space $\Re^{m\times n}$, if $k=1$ then the Ky Fan k-norm is the spectral norm of matrices, and if $k=m$ then the Ky Fan k-norm is just the nuclear norm of matrices. Therefore, by considering these two special cases, we list the corresponding results on the metric projection operators over the epigraphs of the spectral norm and nuclear norm. In this subsection, denote the epigraph cone of spectral norm by $\mathcal{K}$, i.e., $\mathcal{K} := \{(t,X)\in\Re\times\Re^{m\times n} \mid \|X\|_2\le t\}$. Since the dual norm of the spectral norm is the nuclear norm $\|\cdot\|_*$, we know from Proposition 1.2 and Proposition 1.1 that the polar of $\mathcal{K}\equiv \mathrm{epi}\|\cdot\|_2$ is $\mathcal{K}^\circ = -\mathrm{epi}\,\|\cdot\|_*$. Moreover, by Moreau decomposition (Theorem 1.4), we have the following simple obversion

$$\Pi_{\mathcal{K}^*}(t,X) = (t,X) + \Pi_{\mathcal{K}}(-t,-X) \quad \forall\,(t,X)\in\Re\times\Re^{m\times n}\,, \tag{3.190}$$

where $\mathcal{K}^*\equiv \mathrm{epi}\,\|\cdot\|_*$ is the epigraph cone of the nuclear norm. Therefore, we will mainly focus on the metric projector over $\mathcal{K}$. The related properties of the metric projector over the epigraph of the nuclear norm can be readily derived by using (3.190).

For any positive constant $\varepsilon>0$, denote the closed convex cone $\mathcal{D}_n^\varepsilon$ by

$$\mathcal{D}_n^\varepsilon := \{(t,x)\in\Re\times\Re^n \mid \varepsilon^{-1}t \ge\ x_i,\ i=1,\ldots,n\}\,.$$

For any $(t,x)\in\Re\times\Re^n$, $\Pi_{\mathcal{D}_n^\varepsilon}(t,x)$ is the unique optimal solution to the following simple quadratic convex optimization problem

$$\begin{array}{ll} \min & \dfrac{1}{2}\big((\tau-t)^2+\|y-x\|^2\big) \\ \text{s.t.} & \varepsilon^{-1}\tau \ge y_i,\ i=1,\ldots,n\,. \end{array} \tag{3.191}$$

Note that the problem (3.191) can be solved at a cost of $O(n)$ operations (see [30] for details). For any positive constant $\varepsilon>0$, define the matrix cone $\mathcal{M}_n^\varepsilon$ in $\mathcal{S}^n$ as the epigraph of the convex function $\varepsilon\lambda_1(\cdot)$, i.e.,

$$\mathcal{M}_n^\varepsilon := \{(t,X)\in\Re\times\mathcal{S}^n \mid \varepsilon^{-1}t\ge\lambda_1(X)\}\,.$$

For $\mathcal{M}_n^\varepsilon$, we have the following result on the metric projection operator $\Pi_{\mathcal{M}_n^\varepsilon}$.

Proposition 3.20. *Let X have the eigenvalue decomposition*

$$X = \overline{P}\mathrm{diag}(\lambda(X))\overline{P}^T\,,$$

where $\overline{P} \in \mathcal{O}^n$. Then,

$$\Pi_{\mathcal{M}_n^\varepsilon}(t, X) = (\bar{t}, \overline{P}\mathrm{diag}(\bar{y})\overline{P}^T) \quad \forall\ (t, X) \in \Re \times \mathcal{S}^n\,,$$

where $(\bar{t}, \bar{y}) = \Pi_{\mathcal{D}_n^\varepsilon}(t, \lambda(X)) \in \Re \times \Re^n$.

Define

$$\mathcal{K}^\varepsilon := \{(t, X) \in \Re \times \Re^{m\times n} \,|\, \varepsilon^{-1}t \geq \|X\|_2\}$$

for $\varepsilon > 0$. We drop ε if it is 1, i.e., $\mathcal{K}$, the epigraph of the operator norm $\|\cdot\|_2$. Consider the metric projector over $\mathcal{K}^\varepsilon$, i.e., the unique optimal solution to the following convex optimization problem

$$\begin{array}{ll} \min & \frac{1}{2}\big((\tau - t)^2 + \|Y - X\|^2\big) \\ \text{s.t.} & \varepsilon^{-1}\tau \geq \|Y\|_2\,. \end{array}$$

Proposition 3.21. *For any $(t, X) \in \Re \times \Re^{m\times n}$, we have*

$$\Pi_{\mathcal{K}^\varepsilon}(t, X) = \Big(\bar{t}, \overline{U}\,[\mathrm{diag}(\bar{y}) \quad 0]\,\overline{V}^T\Big)\,,$$

with

$$(\bar{t}, \bar{y}) = \Pi_{\mathcal{C}_m^\varepsilon}(t, \sigma(X)) \in \Re \times \Re^m\,,$$

where $\Pi_{\mathcal{C}_m^\varepsilon}(t, \sigma(X))$ is the unique optimal solution to the following convex optimization problem

$$\begin{array}{ll} \min & \frac{1}{2}\big((\tau - t)^2 + \|y - \sigma(X)\|^2\big) \\ \text{s.t.} & \varepsilon^{-1}\tau \geq \|y\|_\infty\,. \end{array} \tag{3.192}$$

Note that the simple quadric convex problem (3.192) can be solved in $O(m)$ operations. Moreover, we have the following proposition about the directional differentiability and Fréchet-differentiability of $\Pi_{\mathcal{C}_m^\varepsilon}(t, x)$.

Proposition 3.22. *Assume that $\varepsilon > 0$ and $(t,x) \in \Re \times \Re^n$ are given.*

(i) The continuous mapping $\Pi_{\mathcal{C}^\varepsilon}(\cdot,\cdot)$ is piecewise linear and for any $(\eta,h) \in \Re \times \Re^n$ sufficiently close to $(0,0)$,

$$\Pi_{\mathcal{C}^\varepsilon}(t+\eta, x+h) - \Pi_{\mathcal{C}^\varepsilon}(t,x) = \Pi_{\widehat{\mathcal{C}^\varepsilon}}(\eta,h)\,,$$

where $\widehat{\mathcal{C}^\varepsilon} := T_{\mathcal{C}^\varepsilon}(\bar{t},\bar{x}) \cap ((t,x)-(\bar{t},\bar{x}))^\perp$ is the critical cone of $\mathcal{C}^\varepsilon$ at (t,x) and $T_{\mathcal{C}^\varepsilon}(\bar{t},\bar{x})$ is the tangent cone of $\mathcal{C}^\varepsilon$ at $(\bar{t},\bar{x})$.

(ii) The mapping $\Pi_{\mathcal{C}^\varepsilon}(\cdot,\cdot)$ is differentiable at (t,x) if and only if $t > \varepsilon||x||_\infty$, or $\varepsilon\|x\|_\infty > t > -\varepsilon^{-1}\|x\|_1$ and $|x|^\downarrow_{\bar{k}+1} < (s_k+\varepsilon t)/(\bar{k}+\varepsilon^2)$, or $t < -\varepsilon^{-1}\|x\|_1$.

For convenience, write $\sigma_0(X) = +\infty$ and $\sigma_{n+1}(X) = -\infty$. Let $s_0 = 0$ and $s_k = \sum_{i=1}^k \sigma_i(X)$, $k = 1,\ldots,m$. Let $\bar{k}$ be the smallest integer $k \in \{0,1,\ldots,m\}$ such that

$$\sigma_{k+1}(X) \leq (s_k + \varepsilon t)/(k+\varepsilon^2) < \sigma_k(X)\,. \tag{3.193}$$

Denote

$$\theta(t,\sigma(X)) := (s_{\overline{k}} + \varepsilon t)/(\overline{k}+\varepsilon^2)\,. \tag{3.194}$$

Define three index sets α, β and γ in $\{1,\ldots,n\}$ by

$$\alpha := \{i \,|\, \sigma_i(X) > \theta^\varepsilon(t,\sigma(X))\}, \quad \beta := \{i \,|\, \sigma_i(X) = \theta^\varepsilon(t,\sigma(X))\}$$

and

$$\gamma := \{i \,|\, \sigma_i(X) < \theta^\varepsilon(t,\sigma(X))\}\,.$$

Let $\delta := \sqrt{1+\bar{k}}$. Define a linear operator $\rho : \Re \times \Re^{m\times n} \to \Re$ as follows

$$\rho(\eta,H) := \begin{cases} \delta^{-1}(\eta + \mathrm{Tr}(S(\overline{U}_\alpha^T H \overline{V}_\alpha))) & \text{if } t \geq -\|X\|_*\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Denote

$$\Big(g_0(t,\sigma(X)), g(t,\sigma(X))\Big) := \Pi_{\mathcal{C}_m}(t,\sigma(X))\,.$$

Define $\Omega_1 \in \Re^{m\times m}$, $\Omega_2 \in \Re^{m\times m}$ and $\Omega_3 \in \Re^{m\times(n-m)}$ (depending on X) as follows, for any $i,j \in \{1,\ldots,m\}$,

$$(\Omega_1)_{ij} := \begin{cases} \dfrac{g_i(t,\sigma(X)) - g_j(t,\sigma(X))}{\sigma_i(X)-\sigma_j(X)} & \text{if } \sigma_i(X) \neq \sigma_j(X)\,, \\ 0 & \text{otherwise}\,, \end{cases}$$

$$(\Omega_2)_{ij} := \begin{cases} \dfrac{g_i(t,\sigma(X)) + g_j(t,\sigma(X))}{\sigma_i(X)+\sigma_j(X)} & \text{if } \sigma_i(X) + \sigma_j(X) \neq 0\,, \\ 0 & \text{otherwise} \end{cases}$$

and for any $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n-m\}$

$$(\Omega_3)_{ij} := \begin{cases} \dfrac{g_i(t,\sigma(X))}{\sigma_i(X)} & \text{if } \sigma_i(X) \neq 0\,, \\ 0 & \text{if } \sigma_i(X) = 0\,, \end{cases}$$

The following result can be derived directly from Theorem 3.4. Note that from Part (i) in Proposition 3.22, we have $\Pi_{\mathcal{C}^\varepsilon}$ is Hadamard directionally differentiable at $(t,\sigma(X))$.

Proposition 3.23. *The metric projector over the matrix cone $\mathcal{K}$, $\Pi_{\mathcal{K}}(\cdot,\cdot)$ is directionally differentiable at (t,X). For any given direction $(\eta,H) \in \Re\times\Re^{m\times n}$, let $A := \overline{U}^T H\overline{V}_1$, $B := \overline{U}^T H\overline{V}_2$. Then the directional derivative $\Pi'_{\mathcal{K}}((t,X);(\eta,H))$ can be computed as follows*

(i) if $t > \|X\|_2$, then $\Pi'_{\mathcal{K}}((t,X);(\eta,H)) = (\eta,H)$;

(ii) if $\|X\|_2 \geq t > -\|X\|_*$, then $\Pi'_{\mathcal{K}}((t,X);(\eta,H)) = (\overline{\eta},\overline{H})$ with

$$\begin{aligned} \overline{\eta} &= \delta^{-1}\psi_0^\delta(\eta,H)\,, \\ \overline{H} &= \overline{U}\begin{bmatrix} \overline{\eta}I_{|\alpha|} & 0 & (\Omega_1)_{\alpha\gamma}\circ S(A)_{\alpha\gamma} \\ 0 & \Psi^\delta(\eta,H) & S(A)_{\beta\gamma} \\ (\Omega_1)_{\gamma\alpha}\circ S(A)_{\gamma\alpha} & S(A)_{\gamma\beta} & S(A)_{\gamma\gamma} \end{bmatrix}\overline{V}_1^T \\ &\quad +\overline{U}\begin{bmatrix} (\Omega_2)_{aa}\circ T(A)_{aa} & (\Omega_2)_{ab}\circ T(A)_{ab} \\ (\Omega_2)_{ba}\circ T(A)_{ba} & T(A)_{bb} \end{bmatrix}\overline{V}_1^T + \overline{U}\begin{bmatrix} (\Omega_3)_{ac'}\circ B_{ac'} \\ B_{bc'} \end{bmatrix}\overline{V}_2^T\,, \end{aligned}$$

where $\left(\psi_0^\delta(\eta,H),\Psi^\delta(\eta,H)\right)\in\Re\times\mathcal{S}^{|\beta|}$ *is given by*

$$\left(\psi_0^\delta(\eta,H),\Psi^\delta(\eta,H)\right):=\Pi_{\mathcal{M}^\delta_{|\beta|}}(\rho(\eta,H),S(\overline{U}_\beta^T H\overline{V}_\beta))\,.$$

In particular, if $t=\|X\|_2>0$*, we have that* $\bar{k}=0$*,* $\delta=1$*,* $\alpha=\emptyset$*,* $\rho(\eta,H)=\eta$ *and*

$$\overline{\eta}=\psi_0^\delta(\eta,H),\quad \overline{H}=\overline{U}\begin{bmatrix}\Psi^\delta(\eta,H)+T(A)_{\beta\beta} & A_{\beta\gamma}\\ A_{\gamma\beta} & A_{\gamma\gamma}\end{bmatrix}\overline{V}_1^T+\overline{U}B\overline{V}_2^T\,;$$

(iii) if $t=-\|X\|_*$*, then* $\Pi'_{\mathcal{K}}((t,X);(\eta,H))=(\overline{\eta},\overline{H})$ *with*

$$\begin{aligned}\overline{\eta}&=\delta^{-1}\psi_0^\delta(\eta,H)\,,\\ \overline{H}&=\overline{U}\begin{bmatrix}\overline{\eta}I_{|\alpha|} & 0\\ 0 & \Psi_1^\delta(\eta,H)\end{bmatrix}\overline{V}_1^T+\overline{U}\begin{bmatrix}0\\ \Psi_2^\delta(\eta,H)\end{bmatrix}\overline{V}_2^T\,,\end{aligned}$$

where $\psi_0^\delta(\eta,H)\in\Re$*,* $\Psi_1^\delta(\eta,H)\in\Re^{|\beta|\times|\beta|}$ *and* $\Psi_2^\delta(\eta,H)\in\Re^{|\beta|\times(n-m)}$ *are given by*

$$\begin{aligned}&\left(\psi_0^\delta(\eta,H),\begin{bmatrix}\Psi_1^\delta(\eta,H) & \Psi_2^\delta(\eta,H)\end{bmatrix}\right)\\ :=\;&\Pi_{\mathcal{K}^\delta_{|\beta|,(n-|a|)}}\left(\rho(\eta,H),\begin{bmatrix}\overline{U}_\beta^T H V_\beta & \overline{U}_\beta^T H\overline{V}_2\end{bmatrix}\right).\end{aligned}$$

(iv) if $t<-\|X\|_*$*, then*

$$\Pi'_{\mathcal{K}}((t,X);(\eta,H))=(0,0)\,.$$

The following proposition can be derived directly from Theorem 3.6 and Proposition 3.22.

Proposition 3.24. $\Pi_{\mathcal{K}}(\cdot,\cdot)$ *is* 1*-order B-differentiable everywhere in* $\Re\times\Re^{m\times n}$*.*

By Theorem 3.6 and Proposition 3.22, we obtain the following property on the F-differentiability of $\Pi_{\mathcal{K}}$.

Proposition 3.25. *The metric projector* $\Pi_{\mathcal{K}}(\cdot,\cdot)$ *is differentiable at* $(t,X)\in\Re\times\Re^{m\times n}$ *if and only if* (t,X) *satisfies one of the following three conditions:*

(i) $t>\|X\|_2$*;*

(ii) $\|X\|_2 > t > -\|X\|_*$ *but* $\sigma_{\bar{k}+1}(X) < \theta(t, \sigma(X))$*;*

(iii) $t < -\|X\|_*$.

In this case, for any $(\eta, H) \in \Re \times \Re^{m\times n}$*,* $\Pi'_{\mathcal{K}}(t, X)(\eta, H) = (\bar{\eta}, \overline{H})$*, where under condition (i),* $(\overline{\eta}, \overline{H}) = (\eta, H)$*; under condition (ii),*

$$\overline{\eta} = \delta^{-1}\rho(\eta, H)$$

and

$$\begin{aligned} \overline{H} &= \overline{U}\begin{bmatrix} \delta^{-1}\rho(\eta,H)I_{|\alpha|} & (\Omega_1)_{\alpha\gamma}\circ S(A)_{\alpha\gamma} \\ (\Omega_1)_{\gamma\alpha}\circ S(A)_{\gamma\alpha} & S(A)_{\gamma\gamma} \end{bmatrix}\overline{V}_1^T \\ &+ \overline{U}\begin{bmatrix} (\Omega_2)_{aa}\circ T(A)_{aa} & (\Omega_2)_{ab}\circ T(A)_{ab} \\ (\Omega_2)_{ba}\circ T(A)_{ba} & T(A)_{bb} \end{bmatrix}\overline{V}_1^T + \overline{U}\begin{bmatrix} (\Omega_3)_{ac'}\circ B_{ac'} \\ B_{bc'} \end{bmatrix}\overline{V}_2^T \end{aligned}$$

with $A := \overline{U}^T H\overline{V}_1$*,* $B := \overline{U}^T H\overline{V}_2^T$*; and under condition (iii),* $(\overline{\eta}, \overline{H}) = (0, 0)$*.*

By applying Theorem 3.12 and noting that $\Pi_{\mathcal{C}^\varepsilon}(\cdot,\cdot)$ is globally Lipschitz continuous and piecewise linear, we have the following proposition.

Proposition 3.26. $\Pi_{\mathcal{K}}(\cdot,\cdot)$ *is strongly semismooth everywhere in* $\Re \times \Re^{m\times n}$*.*

Note that for any $(\eta, h) \in \Re \times \Re^n$ sufficiently close to $(0, 0)$,

$$\Pi_{\mathcal{C}^\varepsilon}(t + \eta, x + h) - \Pi_{\mathcal{C}^\varepsilon}(t, x) = \Pi_{\widehat{\mathcal{C}^\varepsilon}}(\eta, h)\,.$$

From Theorem 3.14, we have the following result.

Proposition 3.27. *Let* $(t, X) \in \Re \times \Re^{m\times n}$ *be given. We have*

$$\partial_B \Pi_{\mathcal{K}}(t, X) = \partial_B \Psi(0, 0)\,,$$

where $\Psi(\cdot,\cdot) := \Pi'_{\mathcal{K}}((t, X); (\cdot,\cdot))$*.*

By Proposition 3.16, we obtain the characterizations of $\partial_B\Pi_{\mathcal{K}}$ and $\partial\Pi_{\mathcal{K}}$, which are similar with the results in Proposition 3.18. Finally, from the proof of Lemma 3.19, we can see easily that the corresponding results also hold for the epigraph of the spectral norm.

Chapter 4

Sensitivity analysis of MOPs

In this chapter, we discuss the sensitivity analysis of the matrix optimization problems (MOPs), which is defined in (1.2) or (1.3) in Chapter 1. Instead of considering the general MOP problems, as a starting point, we mainly focus on the sensitivity analysis of the MOP problems with some special structures. For example, the proper closed convex function $f : \mathcal{X} \to (-\infty, \infty]$ in (1.2) is assumed to be a unitarily invariant matrix norm (e.g., the Ky Fan k-norm) or a positively homogenous function (e.g., the sum of k largest eigenvalues of the symmetric matrix). Also, we mainly focus on the simple linear model as the MCP problems (1.48). Certainly, since simplifications , we may lose some kind of generality, which means that some MOP problems are not covered by this work. However, it is worth taking into consideration that the study on the basic models as the linear MCP involving the Ky Fan k-norm cone can serve as a basic tools to study the sensitivity analysis of the more complicated MOP problems. For example, by using the variational properties of the known cones (the second order cone, the SDP cone, and others), it becomes possible to study the sensitivity analysis of the MOP problems involving the second order cone and the SDP cone constraints. Also, the variational results obtained in this chapter on the Ky Fan k-norm cone can be extended to the other matrix cones e.g., the epigraph cone of the sum of k largest eigenvalues of the symmetric matrix. Thus, the corresponding sensitivity results for such MOPs can be obtained similarly by following the derivation of the simple basic model. We will discuss such kind of extensions at the end of this chapter.

As we mentioned, in this chapter, we mainly consider the linear MCP problem involving the Ky Fan k-norm cone ((1.48) in Section 1.3). As two special cases, the linear MCP problems with the Ky Fan k-norm cone include the linear MCP problems which involve the epigraphs of the spectral and nuclear norms. We begin this chapter with a study of the geometrical properties of the Ky Fan k-norm epigraph cone $\mathcal{K} \equiv \text{epi}\,\|\cdot\|_{(k)}$, including the characterizations of tangent cone and the (inner and outer) second order tangent sets of $\mathcal{K}$, the explicit expression of the support function of the second order tangent set, the $\mathcal{C}^2$-cone reducibility of $\mathcal{K}$, the characterization of the critical cone of $\mathcal{K}$. By using these properties, we state the constraint nondegeneracy, the second order necessary condition and the (strong) second order sufficient condition of the linear MCP problem (1.48). Finally, for the linear MCP problem (1.48), we establish the equivalent results among the strong regularity of the KKT point, the strong second order sufficient condition and constraint nondegeneracy, and the non-singularity of both the B-subdifferenitial and Clarke's generalized Jacobian of the nonsmooth system at a KKT point.

Finally, note that the Ky Fan k-norm includes the following two special matrix norms: the spectral norm ($k = 1$) and the nuclear norm ($k = m$). Therefore, all the results obtained in this chapter hold for the linear MCP problems involving the epigraphs of the spectral norm and the nuclear norm, which are two special cases of the linear MCP problem involving the Ky Fan k-norm.

4.1 Variational geometry of the Ky Fan k-norm cone

Consider the epigraph cone $\mathcal{K} \in \Re \times \Re^{m\times n}$ of the Ky Fan k-norm, i.e.,

$$\mathcal{K} = \left\{ (t, X) \in \Re \times \Re^{m\times n} \mid \|X\|_{(k)} \le t \right\} .$$

In this section, we study some important geometric properties of $\mathcal{K}$, including the characterizations of the tangent cone, second order tangent sets and the critical cone of $\mathcal{K}$.

4.1.1 The tangent cone and the second order tangent sets

In this subsection, we first study the tangent cone $\mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X})$ [86, Definition 6.1] of the closed convex cone $\mathcal{K}$ at the given point $(\bar{t},\overline{X})\in\mathcal{K}$, i.e.,

$$\mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X})=\left\{(\tau,H)\in\Re\times\Re^{m\times n}\,|\,\exists\,\rho_n\downarrow 0,\ \operatorname{dist}\big((\bar{t},\overline{X})+\rho_n(\tau,H),\mathcal{K}\big)=o(\rho_n)\right\}\,.$$

For the given $(\bar{t},\overline{X})\in\mathcal{K}$, consider the following three cases.

Case 1. $(\bar{t},\overline{X})\in\operatorname{int}\mathcal{K}$, i.e., $\|\overline{X}\|_{(k)}<\bar{t}$. It is clear that

$$\mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X})=\Re\times\Re^{m\times n}\,.$$

Hence, the lineality space of $\mathcal{T}(\bar{t},\overline{X})$, i.e., the largest linear subspace in $\mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X})$, is given by $\operatorname{lin}\big(\mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X})\big)=\Re\times\Re^{m\times n}$.

Case 2. $(\bar{t},\overline{X})=(0,0)\in\operatorname{bd}\mathcal{K}$. It is easy to see that

$$\mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X})=\mathcal{T}_{\mathcal{K}}(0,0)=\mathcal{K}\,.$$

Then, the lineality space $\operatorname{lin}\big(\mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X})\big)$ coincides with $\{(0,0)\}$.

Case 3. $(\bar{t},\overline{X})\in\operatorname{bd}\mathcal{K}\setminus\{(0,0)\}$, i.e., $\|\overline{X}\|_{(k)}=\bar{t}$ and $\bar{t}>0$. Let $\overline{\sigma}=\sigma(\overline{X})$ and $\overline{\Sigma}=\operatorname{diag}(\overline{\sigma})$. Therefore, there exist two nonnegative integers $0\le k_0<k\le k_1\le m$ such that if $\overline{\sigma}_k>0$,

$$\overline{\sigma}_1\ge\ldots\ge\overline{\sigma}_{k_0}>\overline{\sigma}_{k_0+1}=\ldots=\overline{\sigma}_k=\ldots=\overline{\sigma}_{k_1}>\overline{\sigma}_{k_1+1}\ge\ldots\ge\overline{\sigma}_m\ge 0\,;$$

if $\overline{\sigma}_k=0$,

$$\overline{\sigma}_1\ge\ldots\ge\overline{\sigma}_{k_0}>\overline{\sigma}_{k_0+1}=\ldots=\overline{\sigma}_k=\ldots=\overline{\sigma}_m=0\,.$$

Denote $\alpha=\{1,\ldots,k_0\}$ and $\beta=\{k_0+1,\ldots,k_1\}$. Let $\overline{U}\in\mathcal{O}^m$, $\overline{V}=[\overline{V}_1\ \ \overline{V}_2]\in\mathcal{O}^n$ be such that

$$\overline{X}=\overline{U}[\overline{\Sigma}\ \ 0]\overline{V}^T\,.$$

Since $\|\overline{X}\|_{(k)} = \sum_{i=1}^{k} \overline{\sigma}_i = \bar{t}$, we know from [23, Theorem 2.4.9] that the tangent cone of $\mathcal{K}$ at the point $(\bar{t}, \overline{X})$ can be written as

$$\mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X}) = \left\{ (\tau, H) \in \Re \times \Re^{m\times n} \mid \sum_{i=1}^{k} \sigma_i'(\overline{X}; H) \leq \tau \right\}.$$

Let $a_1, \ldots, a_r$ be the index sets defined by (2.26) for $\overline{X}$. For notational convenience, let $0 \leq r_0 \leq r$ be the nonnegative integer such that $\alpha = \cup_{l=1}^{r_0} a_l$. Therefore, by Proposition 2.15, we know that if $\overline{\sigma}_k > 0$, then

$$\mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X}) = \left\{ (\tau, H) \in \Re \times \Re^{m\times n} \mid \sum_{l=1}^{r_0} \operatorname{tr}(\overline{U}_{a_l}^T H \overline{V}_{a_l}) + \sum_{i=1}^{k-k_0} \lambda_i \left(S(\overline{U}_{\beta}^T H \overline{V}_{\beta}) \right) \leq \tau \right\}; \quad (4.1)$$

if $\overline{\sigma}_k = 0$, then

$$\mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X}) = \left\{ (\tau, H) \in \Re \times \Re^{m\times n} \mid \sum_{l=1}^{r_0} \operatorname{tr}(\overline{U}_{a_l}^T H \overline{V}_{a_l}) + \sum_{i=1}^{k-k_0} \sigma_i \left(\left[\overline{U}_{\beta}^T H \overline{V}_{\beta} \quad \overline{U}_{\beta}^T H \overline{V}_2 \right] \right) \leq \tau \right\}. \quad (4.2)$$

Hence, the lineality space $\operatorname{lin}\left(\mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X})\right)$ takes the following forms: if $\overline{\sigma}_k > 0$,

$$\operatorname{lin}\left(\mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X})\right) = \left\{ (\tau, H) \in \Re \times \Re^{m\times n} \mid S(\overline{U}_{\beta}^T H \overline{V}_{\beta}) = \frac{1}{k-k_0} \left(\tau - \sum_{l=1}^{r_0} \operatorname{tr}(\overline{U}_{a_l}^T H \overline{V}_{a_l}) \right) I_{|\beta|} \right\}; \quad (4.3)$$

if $\overline{\sigma}_k = 0$,

$$\operatorname{lin}\left(\mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X})\right) = \left\{ (\tau, H) \in \Re \times \Re^{m\times n} \mid \sum_{l=1}^{r_0} \operatorname{tr}(\overline{U}_{a_l}^T H \overline{V}_{a_l}) = \tau, \ \left[\overline{U}_{\beta}^T H \overline{V}_{\beta} \quad \overline{U}_{\beta}^T H \overline{V}_2 \right] = 0 \right\}. \quad (4.4)$$

For the polar of $\mathcal{K}$, the tangent cone $\mathcal{T}_{\mathcal{K}^\circ}(\bar{\zeta}, \overline{\Gamma})$ at any given point $(\bar{\zeta}, \overline{\Gamma}) \in \mathcal{K}^\circ$ can be characterized as

$$\mathcal{T}_{\mathcal{K}^\circ}(\bar{\zeta}, \overline{\Gamma}) = \left\{ (\tau, H) \in \Re \times \Re^{m\times n} \mid \Pi_{\mathcal{K}^\circ}'((\bar{\zeta}, \overline{\Gamma}); (\tau, H)) = (\tau, H) \right\}.$$

For the given $(\bar{\zeta}, \overline{\Gamma}) \in \mathcal{K}^\circ$, we know from Theorem 1.4 (the Moreau decomposition) that for any $(\tau, H) \in \Re \times \Re^{m\times n}$,

$$\Pi_{\mathcal{K}^\circ}'((\bar{\zeta}, \overline{\Gamma}); (\tau, H)) = (\tau, H) - \Pi_{\mathcal{K}}'((\bar{\zeta}, \overline{\Gamma}); (\tau, H)),$$

which implies

$$\mathcal{T}_{\mathcal{K}^\circ}(\bar{\zeta}, \overline{\Gamma}) = \left\{(\tau, H) \in \Re \times \Re^{m\times n} \,|\, \Pi'_{\mathcal{K}}((\bar{\zeta}, \overline{\Gamma}); (\tau, H)) = 0\right\} .$$

Thus, the characterization of the tangent cone $\mathcal{T}_{\mathcal{K}^\circ}(\bar{\zeta}, \overline{\Gamma})$ at $(\bar{\zeta}, \overline{\Gamma})$ follows from Proposition 3.16 immediately. Actually, we may consider the singular value decomposition of $\overline{\Gamma}$, i.e.,

$$\overline{\Gamma} = \overline{U}[\Sigma(\overline{\Gamma}) \;\; 0]\overline{V}^T ,$$

where $(\overline{U}, \overline{V}) \in \mathcal{O}^{m,n}(\overline{\Gamma})$. Let $\{a_l\}_{l=1}^r$ and b be the index sets defined by (2.26) with respect to $\overline{\Gamma}$. Assume that $(\bar{\zeta}, \overline{\Gamma}) \in \operatorname{bd}\mathcal{K}^\circ \setminus \{(0,0)\}$. We know that $\Pi_{\mathcal{K}}(\bar{\zeta}, \overline{\Gamma}) = 0$. Denote $\beta = \{1, \ldots, m\}$. Let β_1, β_2 and β_3 be the index sets defined by

$$\beta_1 := \left\{i \in \{1, \ldots, m\} \,|\, \sigma_i(\overline{\Gamma}) = -\bar{\zeta}\right\} , \quad \beta_2 := \left\{i \in \{1, \ldots, m\} \,|\, 0 < \sigma_i(\overline{\Gamma}) < -\bar{\zeta}\right\}$$

and $\beta_3 := \left\{i \in \{1, \ldots, m\} \,|\, \sigma_i(\overline{\Gamma}) = 0\right\}$, respectively. Since $(\bar{\zeta}, \overline{\Gamma}) \in \operatorname{bd}\mathcal{K}^\circ \setminus \{(0,0)\}$, we know that the sets β_1, β_2 and β_3 form a partition of β. For any $(\tau, H) \in \Re \times \Re^{m\times n}$, denote $\widetilde{H} = \overline{U}^T H \overline{V}$ and

$$\boldsymbol{h} = \left(\lambda(\widetilde{H}_{a_r a_r}), \ldots, \lambda(\widetilde{H}_{a_r a_r}), \sigma([\widetilde{H}_{bb} \;\; \widetilde{H}_{bc}])\right) \in \Re^m .$$

Consider the following two cases.

Case 1. $\|\overline{\Gamma}\|_* = -k\bar{\zeta}$, i.e., $\sum_{i=1}^m \sigma_i(\overline{\Gamma}) = -k\bar{\zeta}$. We have

$$\mathcal{T}_{\mathcal{K}^\circ}(\bar{\zeta}, \overline{\Gamma}) = \left\{(\tau, H) \in \Re \times \Re^{m\times n} \,|\, \boldsymbol{h}_i \le -\tau \;\forall\, i \in \beta_1,\; \sum_{i=1}^m \boldsymbol{h}_i \le -k\tau\right\} .$$

Then, the corresponding lineality space $\operatorname{lin}\left(\mathcal{T}_{\mathcal{K}^\circ}(\bar{\zeta}, \overline{\Gamma})\right)$ takes the following form:

$$\operatorname{lin}\left(\mathcal{T}_{\mathcal{K}^\circ}(\bar{\zeta}, \overline{\Gamma})\right) = \left\{(\tau, H) \in \Re \times \Re^{m\times n} \,|\, \widetilde{H}_{\beta_1\beta_1} = -\tau I_{|\beta_1|},\; [\widetilde{H}_{bb} \;\; \widetilde{H}_{bc}] = 0,\; \sum_{l=1}^r \operatorname{tr}(\widetilde{H}_{a_l a_l}) = -k\tau\right\} .$$

Case 2. $\|\overline{\Gamma}\|_* < -k\bar{\zeta}$, i.e., $\sum_{i=1}^m \sigma_i(\overline{\Gamma}) < -k\bar{\zeta}$. We have

$$\mathcal{T}_{\mathcal{K}^\circ}(\bar{\zeta}, \overline{\Gamma}) = \left\{(\tau, H) \in \Re \times \Re^{m\times n} \,|\, \boldsymbol{h}_i \le -\tau \;\forall\, i \in \beta_1\right\} .$$

Hence, the corresponding lineality space $\mathrm{lin}\left(\mathcal{T}_{\mathcal{K}^\circ}(\bar{\zeta},\overline{\Gamma})\right)$ takes the following form:

$$\mathrm{lin}\left(\mathcal{T}_{\mathcal{K}^\circ}(\bar{\zeta},\overline{\Gamma})\right) = \left\{(\tau,H)\in\Re\times\Re^{m\times n} \mid \widetilde{H}_{\beta_1\beta_1} = -\tau I_{|\beta_1|}\right\}.$$

Note that since $(\bar{\zeta},\overline{\Gamma})\in \mathrm{bd}\,\mathcal{K}^\circ\setminus\{(0,0)\}$, we always have $\beta_1\neq\emptyset$. Also, it is obvious that when $(\bar{\zeta},\overline{\Gamma})\in\mathrm{int}\,\mathcal{K}^\circ$, $\mathcal{T}_{\mathcal{K}^\circ}(\bar{\zeta},\overline{\Gamma}) = \Re\times\Re^{m\times n}$.

Next, we study the characterization of the inner and outer second order tangent sets of $\mathcal{K}$. Let $\mathcal{T}^{i,2}_{\mathcal{K}}((\bar{t},\overline{X}),(\bar{\tau},\overline{H}))$ and $\mathcal{T}^{2}_{\mathcal{K}}((\bar{t},\overline{X}),(\bar{\tau},\overline{H}))$ be the inner and outer second order tangent sets [8, Definition 3.28], respectively, to $\mathcal{K}$ at $(\bar{t},\overline{X})\in\mathcal{K}$ along the direction $(\bar{\tau},\overline{H})\in\mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X})$, i.e.,

$$\mathcal{T}^{i,2}_{\mathcal{K}}((\bar{t},\overline{X}),(\bar{\tau},\overline{H})) := \liminf_{\rho\downarrow 0}\frac{\mathcal{K}-(\bar{t},\overline{X})-\rho(\bar{\tau},\overline{H})}{\frac{1}{2}\rho^2}$$

and

$$\mathcal{T}^{2}_{\mathcal{K}}((\bar{t},\overline{X}),(\bar{\tau},\overline{H})) := \limsup_{\rho\downarrow 0}\frac{\mathcal{K}-(\bar{t},\overline{X})-\rho(\bar{\tau},\overline{H})}{\frac{1}{2}\rho^2},$$

where "lim sup" and "lim inf" are Painlevé-Kuratowski outer and inner limit for sets (cf. [86, Definition 4.1]). For the convex set, we have the following result ([8, Proposition 3.34, (3.62) & (3.63)]).

Proposition 4.1. *Let C be a convex set. Then, for any $x\in C$, $h\in\mathcal{T}_C(x)$, the following inclusions hold:*

$$\mathcal{T}^{i,2}_{C}(x,h)+\mathcal{T}_{\mathcal{T}_C(x)}(h)\subseteq\mathcal{T}^{i,2}_{C}(x,h)\subseteq\mathcal{T}_{\mathcal{T}_C(x)}(h),$$

$$\mathcal{T}^{2}_{C}(x,h)+\mathcal{T}_{\mathcal{T}_C(x)}(h)\subseteq\mathcal{T}^{2}_{C}(x,h)\subseteq\mathcal{T}_{\mathcal{T}_C(x)}(h),$$

where $\mathcal{T}_{\mathcal{T}_C(x)}(h)$ is the tangent cone of $\mathcal{T}_C(x)$ at $h\in\mathcal{T}_C(x)$.

Let $(\bar{t},\overline{X})\in\mathcal{K}$ be given. Again, consider the following three cases.

Case 1. $(\bar{t},\overline{X})\in\mathrm{int}\,\mathcal{K}$, i.e., $\|\overline{X}\|<\bar{t}$. Since $\mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X})=\Re\times\Re^{m\times n}$, we know that for any $(\bar{\tau},\overline{H})\in\mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X})$,

$$\mathcal{T}^{i,2}_{\mathcal{K}}((\bar{t},\overline{X}),(\bar{\tau},\overline{H})) = \mathcal{T}^{2}_{\mathcal{K}}((\bar{t},\overline{X}),(\bar{\tau},\overline{H})) = \Re\times\Re^{m\times n} = \mathcal{T}^2. \quad (4.5)$$

Case 2. $(\bar{t}, \overline{X}) = (0,0) \in \mathcal{K}$. Since $\mathcal{R}_{\mathcal{K}}(0,0) = \mathcal{T}_{\mathcal{K}}(0,0) = \mathcal{K}$, where $\mathcal{R}_{\mathcal{K}}(0,0)$ is the *radial cone* of $\mathcal{K}$ at $(0,0)$ (see e.g., [8, Definition 2.54]), we know that for any $(\bar{\tau}, \overline{H}) \in \mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X})$, $(0,0) \in \mathcal{T}^{i,2}_{\mathcal{K}}((\bar{t}, \overline{X}), (\bar{\tau}, \overline{H}))$. Therefore, for any given $(\bar{\tau}, \overline{H}) \in \mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X})$, we have

$$\mathcal{T}^{i,2}_{\mathcal{K}}((\bar{t}, \overline{X}), (\bar{\tau}, \overline{H})) = \mathcal{T}^{2}_{\mathcal{K}}((\bar{t}, \overline{X}), (\bar{\tau}, \overline{H})) = \mathcal{T}_{\mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X})}(\bar{\tau}, \overline{H}) = \mathcal{T}^2 \,. \tag{4.6}$$

Case 3. $(\bar{t}, \overline{X}) \in \operatorname{bd}\mathcal{K} \setminus \{(0,0)\}$, i.e., $\|\overline{X}\|_{(k)} = \bar{t}$ and $\bar{t} > 0$. Let $(\bar{\tau}, \overline{H}) \in \mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X})$ be given. If $\sum_{i=1}^{k} \sigma_i'(\overline{X}; \overline{H}) < \bar{\tau}$, i.e., $(\bar{t}, \overline{H}) \in \operatorname{int}\mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X})$, then it is easy to see that

$$\mathcal{T}^{i,2}_{\mathcal{K}}((\bar{t}, \overline{X}), (\bar{\tau}, \overline{H})) = \mathcal{T}^{2}_{\mathcal{K}}((\bar{t}, \overline{X}), (\bar{\tau}, \overline{H})) = \Re \times \Re^{m\times n} = \mathcal{T}^2 \,. \tag{4.7}$$

If $\sum_{i=1}^{k} \sigma_i'(\overline{X}; \overline{H}) = \bar{\tau}$, then $\mathcal{K}$ can re-written as

$$\mathcal{K} = \left\{(t, X) \in \Re \times \Re^{m\times n} \mid \phi(t, X) \le 0\right\} \,,$$

where $\phi(t, X) := \|X\|_{(k)} - t$ is a closed convex function. Since $\operatorname{int}\mathcal{K} \neq \emptyset$ and the convex and continuous function ϕ is (parabolically) second order directionally differentiable (Definition 2.1) at $(\bar{t}, \overline{X})$, we know from [8, Proposition 3.30] that

$$\mathcal{T}^{i,2}_{\mathcal{K}}((\bar{t}, \overline{X}), (\bar{\tau}, \overline{H})) = \mathcal{T}^{2}_{\mathcal{K}}((\bar{t}, \overline{X}), (\bar{\tau}, \overline{H})) = \mathcal{T}^2$$

with

$$\mathcal{T}^2 := \left\{(\eta, W) \in \Re \times \Re^{m\times n} \mid \sum_{i=1}^{k} \sigma_i''(\overline{X}; \overline{H}, W) \le \eta\right\} , \tag{4.8}$$

where for each $i \in \{1, \ldots, k\}$, $\sigma_i''(\overline{X}; \overline{H}, W)$ is the (parabolic) second order directional derivative of $\sigma_i(\cdot)$ at $\overline{X}$ along $\overline{H}$ and W, which is characterized by Proposition 2.18.

Remark 4.1. *It has been shown that for any given $(\bar{t}, \overline{X}) \in \mathcal{K}$ and $(\bar{\tau}, \overline{H}) \in \mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X})$,*

$$\mathcal{T}^{i,2}_{\mathcal{K}}((\bar{t}, \overline{X}), (\bar{\tau}, \overline{H})) = \mathcal{T}^{2}_{\mathcal{K}}((\bar{t}, \overline{X}), (\bar{\tau}, \overline{H})) = \mathcal{T}^2 \,.$$

Therefore, we denote the convex set $\mathcal{T}^2$ the second order tangent set to $\mathcal{K}$ at $(\bar{t}, \overline{X}) \in \mathcal{K}$ along the direction $(\bar{\tau}, \overline{H}) \in \mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X})$.

In order to study the second order optimality conditions of the linear MCP problem (1.48), we need to consider the support function $\delta^*_{\mathcal{T}^2}(\cdot,\cdot)$ of the second order tangent set $\mathcal{T}^2$ to $\mathcal{K}$ at $(\bar{t},\overline{X}) \in \mathcal{K}$ along $(\bar{\tau},\overline{H}) \in \mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X})$, i.e.,

$$\delta^*_{\mathcal{T}^2}(\zeta,\Gamma) = \sup\left\{\zeta\eta + \langle \Gamma, W\rangle \mid (\eta,W) \in \mathcal{T}^2\right\}, \quad (\zeta,\Gamma) \in \Re \times \Re^{m\times n}.$$

Let $(\bar{t},\overline{X}) \in \mathcal{K}$ and $(\bar{\tau},\overline{H}) \in \mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X})$ be given. From Proposition 4.1, it is easy to see that if $(\zeta,\Gamma) \in \Re \times \Re^{m\times n}$ does not belong the polar of $\mathcal{T}_{\mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X})}(\bar{\tau},\overline{H})$, then $\delta^*_{\mathcal{T}^2}(\zeta,\Gamma) \equiv +\infty$. In fact, since $\mathcal{T}_{\mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X})}(\bar{\tau},\overline{H})$ is nonempty, we may assume that there exists $(\eta^\circ,W^\circ) \in \mathcal{T}_{\mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X})}(\bar{\tau},\overline{H})$ such that

$$\langle(\zeta,\Gamma),(\eta^\circ,W^\circ)\rangle > 0\,.$$

Since $\mathcal{T}^2 \neq \emptyset$, fix any $(\widetilde{\eta},\widetilde{W}) \in \mathcal{T}^2$. Therefore, since for any $\rho > 0$,

$$\rho(\eta^\circ,W^\circ) + (\widetilde{\eta},\widetilde{W}) \in \mathcal{T}_{\mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X})}(\bar{\tau},\overline{H}) + \mathcal{T}^2 \subseteq \mathcal{T}^2\,,$$

we obtain that

$$\rho\langle(\zeta,\Gamma),(\eta^\circ,W^\circ)\rangle + \langle(\zeta,\Gamma),(\widetilde{\eta},\widetilde{W})\rangle \leq \delta^*((\zeta,\Gamma)\mid\mathcal{T}^2)\,,$$

which implies that $\delta^*_{\mathcal{T}^2}(\zeta,\Gamma) = +\infty$.

On the other hand, since $\mathcal{K}$ is a closed convex cone in $\Re \times \Re^{m\times n}$, we have

$$\mathcal{K} \subseteq \mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X}) \subseteq \mathcal{T}_{\mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X})}(\bar{\tau},\overline{H})\,.$$

In particular, we have

$$\pm(\bar{t},\overline{X}) \in \mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X}) \subseteq \mathcal{T}_{\mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X})}(\bar{\tau},\overline{H}) \quad \text{and} \quad \pm\,(\bar{\tau},\overline{H}) \in \mathcal{T}_{\mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X})}(\bar{\tau},\overline{H})\,.$$

Therefore, we know that if $(\zeta,\Gamma) \in (\mathcal{T}_{\mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X})}(\bar{\tau},\overline{H}))^\circ$, then

$$(\zeta,\Gamma) \in \mathcal{K}^\circ, \quad \langle(\zeta,\Gamma),(\bar{t},\overline{X})\rangle = 0 \quad \text{and} \quad \langle(\zeta,\Gamma),(\bar{\tau},\overline{H})\rangle = 0\,. \tag{4.9}$$

Therefore, we know that for any $(\zeta,\Gamma) \in \Re \times \Re^{m\times n}$, $\delta^*_{\mathcal{T}^2}(\zeta,\Gamma) \equiv +\infty$, if (ζ,Γ) does not satisfy the condition (4.9).

For the point $(\zeta,\Gamma)\in\Re\times\Re^{m\times n}$, which satisfies the condition (4.9), consider the following cases.

Case 1. $(\bar t,\overline{X})\in\operatorname{int}\mathcal{K}$. From (4.5), we know that $\delta^*_{\mathcal{T}^2}(\zeta,\Gamma)=0$.

Case 2. $(\bar t,\overline{X})=(0,0)$. For any $(\bar\tau,\overline{H})\in\mathcal{T}_{\mathcal{K}}(0,0)=\mathcal{K}$, we know from (4.6) and (4.9) that $(\zeta,\Gamma)\in(T_{\mathcal{K}}(\bar\tau,\overline{H}))^\circ=(\mathcal{T}^2)^\circ$, which implies $\delta^*_{\mathcal{T}^2}(\zeta,\Gamma)=0$ for any $(\zeta,\Gamma)\in\Re\times\Re^{m\times n}$.

Case 3. $(\bar t,\overline{X})\in\operatorname{bd}\mathcal{K}\setminus\{(0,0)\}$. If $(\bar\tau,\overline{H})\in\operatorname{int}\mathcal{T}_{\mathcal{K}}(\bar t,\overline{X})$, then by (4.7), we know that $\delta^*_{\mathcal{T}^2}(\zeta,\Gamma)=0$. Next, suppose that $(\bar\tau,\overline{H})\in\operatorname{bd}\mathcal{T}_{\mathcal{K}}(\bar t,\overline{X})$ and $(\zeta,\Gamma)\neq(0,0)$. Let $(t,X):=(\bar t,\overline{X})+(\zeta,\Gamma)$. Then, by considering the singular value decomposition of X, we know from the condition (4.9) that

$$(\bar t,\overline{X})=\Pi_{\mathcal{K}}(t,X)\quad\text{and}\quad(\zeta,\Gamma)=\Pi_{\mathcal{K}^\circ}(t,X)$$

with

$$\overline{X}=\overline{U}[\Sigma(\overline{X})\ \ 0]\overline{V}^T\quad\text{and}\quad\Gamma=\overline{U}[\Sigma(\Gamma)\ \ 0]\overline{V}^T,$$

where $(\overline{U},\overline{V})\in\mathcal{O}^{m,n}(X)$. Let a, b, c and a_l, $l=1,\ldots,r$ be the index sets defined by (2.25) and (2.26) for X. Denote $\overline{\sigma}=\sigma(\overline{X})$. Consider the following two sub-cases.

Case 3.1. $\overline{\sigma}_k>0$. Then, $(\bar t,\overline{X})\neq(0,0)$. There exist two integers $0\le k_0\le k-1$ and $k\le k_1\le m$ such that

$$\overline{\sigma}_1\ge\ldots\ge\overline{\sigma}_{k_0}>\overline{\sigma}_{k_0+1}=\ldots=\overline{\sigma}_k=\ldots=\overline{\sigma}_{k_1}>\overline{\sigma}_{k_1+1}\ge\ldots\ge\overline{\sigma}_m\ge0\,.$$

Since $(\bar t,\overline{X})=\Pi_{\mathcal{K}}(t,X)$ and $(\zeta,\Gamma)=(t,X)-(\bar t,\overline{X})$. By the part (i) of Lemma 3.15, we know that there exist $\theta>0$ (since $(t,X)\notin\operatorname{int}\mathcal{K}$) and $u\in\Re^m_+$ such that

$$\zeta=-\theta\quad\text{and}\quad\Gamma=\overline{U}[\operatorname{diag}(\theta u)\ \ 0]\overline{V}^T,\tag{4.10}$$

where $u_i=1$, $i=1,\ldots,k_0$, $u_i=0$, $i=k_1+1,\ldots,m$,

$$1\ge u_{k_0+1}\ge u_{k_0+2}\ge\ldots\ge u_{k_1}\ge0\quad\text{and}\quad\sum_{i=1}^{k_1-k_0}u_{k_0+i}=k-k_0\,.\tag{4.11}$$

Denote $\alpha = \{1, \ldots, k_0\}$, $\beta = \{k_0+1, \ldots, k_1\}$ and $\gamma = \{k_1+1, \ldots, m\}$ and $\overline{\gamma} = \alpha \cup \beta$. Since $\langle (\zeta, \Gamma), (\bar{\tau}, \overline{H}) \rangle = 0$, by Ky Fan's inequality (Lemma 2.3), we know that

$$\begin{aligned} 0 &= \zeta\bar{\tau} + \langle \Gamma, \overline{H} \rangle = \zeta\bar{\tau} + \langle \overline{U}^T \Gamma \overline{V}, \overline{U}^T \overline{H}\, \overline{V} \rangle = \zeta\bar{\tau} + \langle \overline{U}_{\overline{\gamma}}^T \Gamma \overline{V}_{\overline{\gamma}}, \overline{U}_{\overline{\gamma}}^T \overline{H}\, \overline{V}_{\overline{\gamma}} \rangle \\ &= -\theta\bar{\tau} + \langle \theta \mathrm{diag}\,(u_{\overline{\gamma}}), S(\overline{U}_{\overline{\gamma}}^T \overline{H}\, \overline{V}_{\overline{\gamma}}) \rangle \\ &\leq -\theta\bar{\tau} + \theta \sum_{l=1}^{r_0} \mathrm{tr}\,(\overline{U}_{a_l}^T \overline{H}\, \overline{V}_{a_l}) + \theta \sum_{i=1}^{k_1-k_0} u_{k_0+i} \lambda_i \left(S(\overline{U}_{\beta}^T \overline{H}\, \overline{V}_{\beta}) \right) . \end{aligned} \tag{4.12}$$

Since $(\bar{\tau}, \overline{H}) \in \mathrm{bd}\, \mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X})$, we know from (4.1) that

$$\bar{\tau} = \sum_{l=1}^{r_0} \mathrm{tr}(\overline{U}_{a_l}^T \overline{H}\, \overline{V}_{a_l}) + \sum_{i=1}^{k-k_0} \lambda_i \left(S(\overline{U}_{\beta}^T \overline{H}\, \overline{V}_{\beta}) \right) .$$

By substitution, we know from (4.12) and (4.11) that

$$\begin{aligned} 0 &\leq \theta \left(-\sum_{i=1}^{k-k_0} \lambda_i(S(\overline{U}_{\beta}^T \overline{H}\, \overline{V}_{\beta})) + \sum_{i=1}^{k_1-k_0} u_{k_0+i} \lambda_i(S(\overline{U}_{\beta}^T \overline{H}\, \overline{V}_{\beta})) \right) \\ &= \theta \left(\sum_{i=1}^{k-k_0} (u_{k_0+i} - 1) \lambda_i(S(\overline{U}_{\beta}^T \overline{H}\, \overline{V}_{\beta})) + \sum_{i=k-k_0+1}^{k_1-k_0} u_{k_0+i} \lambda_i(S(\overline{U}_{\beta}^T \overline{H}\, \overline{V}_{\beta})) \right) \\ &\leq \theta \lambda_{k-k_0}(S(\overline{U}_{\beta}^T \overline{H}\, \overline{V}_{\beta})) \left(\sum_{i=1}^{k-k_0} (u_{k_0+i} - 1) + \sum_{i=k-k_0+1}^{k_1-k_0} u_{k_0+i} \right) = 0 , \end{aligned}$$

which implies the equality in (4.12) holds and

$$\sum_{i=1}^{k-k_0} \lambda_i(S(\overline{U}_{\beta}^T \overline{H}\, \overline{V}_{\beta})) = \sum_{i=1}^{k_1-k_0} u_{k_0+i} \lambda_i(S(\overline{U}_{\beta}^T \overline{H}\, \overline{V}_{\beta})) . \tag{4.13}$$

Next, consider the eigenvalue decomposition of the symmetric matrix $S(\overline{U}_{\beta}^T \overline{H}\, \overline{V}_{\beta}^T)$. Denote $\widetilde{m} := k_1 - k_0$ and $\widetilde{k} := k - k_0$. Let $\widetilde{\lambda} := \lambda(S(\overline{U}_{\beta}^T \overline{H}\, \overline{V}_{\beta}^T)) \in \Re^{\widetilde{m}}$. Then, we know that there exist two integers $0 \leq \widetilde{k}_0 \leq \widetilde{k} - 1$ and $\widetilde{k} \leq \widetilde{k}_1 \leq \widetilde{m}$ such that

$$\widetilde{\lambda}_1 \geq \ldots \geq \widetilde{\lambda}_{\widetilde{k}_0} > \widetilde{\lambda}_{\widetilde{k}_0+1} = \ldots = \widetilde{\lambda}_{\widetilde{k}} = \ldots = \widetilde{\lambda}_{\widetilde{k}_1} > \widetilde{\lambda}_{\widetilde{k}_1+1} \geq \ldots \geq \widetilde{\lambda}_{\widetilde{m}} .$$

Consider the corresponding index sets $\widetilde{\alpha}_l$, $l = 1, \ldots, \widetilde{r}$ defined by (2.16). Let $\widetilde{r}_0 \in \{1, \ldots, \widetilde{r}\}$ be such that $\widetilde{k} \in \widetilde{\alpha}_{\widetilde{r}_0+1}$. Then, by (4.13), we have

$$\sum_{i=1}^{\widetilde{m}} u_{k_0+i} \widetilde{\lambda}_i = \sup \left\{ \langle y, \lambda \rangle \mid 0 \leq y \leq e,\ \langle e, y \rangle = \widetilde{k},\ y \in \Re^{\widetilde{m}} \right\} ,$$

i.e., $(u_{k_0+1},\ldots,u_{k_1})\in\Re^{\widetilde{m}}$ is the solution of the maximize problem. Therefore, we know from [113, Lemma 2.2] that

$$u_{k_0+i}=1,\quad i=1,\ldots,\widetilde{k}_0,\quad u_{k_0+i}=0,\quad i=\widetilde{k}_1+1,\ldots\widetilde{m} \tag{4.14}$$

and

$$1\geq u_{k_0+\widetilde{k}_0+1}\geq u_{k_0+\widetilde{k}_0+2}\geq\ldots\geq u_{k_0+\widetilde{k}_1}\geq 0\quad\text{and}\quad\sum_{i=1}^{\widetilde{k}_1-\widetilde{k}_0}u_{k_0+\widetilde{k}_0+i}=\widetilde{k}-\widetilde{k}_0\,. \tag{4.15}$$

Since the equality in (4.12) holds, by Lemma 2.3 (Ky Fan's inequality), we know that the symmetric matrices $\operatorname{diag}(u_\beta)$ and $S(\overline{U}_\beta^T\overline{H}\,\overline{V}_\beta^T)$ admit a simultaneous ordered eigenvalue decomposition, i.e., there exists $R\in\mathcal{O}^{\widetilde{m}}$ such that

$$\operatorname{diag}(u_\beta)=R\operatorname{diag}(u_\beta)R^T\quad\text{and}\quad S(\overline{U}_\beta^T\overline{H}\,\overline{V}_\beta^T)=R\Lambda(S(\overline{U}_\beta^T\overline{H}\,\overline{V}_\beta^T))R^T\,.$$

On the other hand, since $(\bar{\tau},\overline{H})\in\operatorname{bd}\mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X})$, we know from (4.8) that $(\eta,W)\in\mathcal{T}^2$ if and only if $\sum_{i=1}^k\sigma_i''(\overline{X};\overline{H},W)\leq\eta$, i.e.,

$$\begin{aligned}&\sum_{l=1}^{r_0}\operatorname{tr}\left(S(\overline{U}_{a_l}^TW\overline{V}_{a_l}^T)\right)-\sum_{l=1}^{r_0}\operatorname{tr}\left(2\overline{P}_{a_l}^T\left[\mathcal{B}(\overline{H})(\mathcal{B}(\overline{X})-\bar{\nu}_lI_{m+n})^\dagger\mathcal{B}(\overline{H})\right]\overline{P}_{a_l}\right)\\&+\sum_{l=1}^{\widetilde{r}_0}\operatorname{tr}\left(R_{\widetilde{\alpha}_l}^T\overline{P}_\beta^T\left[\mathcal{B}(W)-2\mathcal{B}(\overline{H})(\mathcal{B}(\overline{X})-\overline{\sigma}_kI_{m+n})^\dagger\mathcal{B}(\overline{H})\right]\overline{P}_\beta R_{\widetilde{\alpha}_l}\right)\\&+\sum_{i=1}^{\widetilde{k}-\widetilde{k}_0}\lambda_i\left(R_{\widetilde{\alpha}_{\widetilde{r}_0+1}}^T\overline{P}_\beta^T\left[\mathcal{B}(W)-2\mathcal{B}(\overline{H})(\mathcal{B}(\overline{X})-\overline{\sigma}_kI_{m+n})^\dagger\mathcal{B}(\overline{H})\right]\overline{P}_\beta R_{\widetilde{\alpha}_{\widetilde{r}_0+1}}\right)\\&\leq\eta\,.\end{aligned} \tag{4.16}$$

Therefore, for any $(\eta, W) \in \mathcal{T}^2$, by (4.10), we obtain that

$$
\begin{aligned}
& \zeta\eta + \langle \Gamma, W\rangle \\
= \; & \zeta\eta + \left\langle \overline{U}^T \Gamma \overline{V}, \overline{U}^T W \overline{V} \right\rangle \\
= \; & \zeta\eta + \left\langle \theta \mathrm{diag}\,(u_{\overline{\gamma}}), S(\overline{U}_{\overline{\gamma}}^T W \overline{V}_{\overline{\gamma}}) \right\rangle \\
= \; & \zeta\eta + \theta \sum_{l=1}^{r_0} \mathrm{tr}\,(S(\overline{U}_{a_l}^T W \overline{V}_{a_l}^T)) + \left\langle \Sigma_{\beta\beta}(\Gamma), R^T S(\overline{U}_{\beta}^T W \overline{V}_{\beta}^T) R \right\rangle \\
= \; & \Delta(\eta, W) - \zeta \sum_{l=1}^{r_0} \mathrm{tr} \left(2\overline{P}_{a_l}^T \left[\mathcal{B}(\overline{H})(\mathcal{B}(\overline{X}) - \bar{\nu}_l I_{m+n})^\dagger \mathcal{B}(\overline{H}) \right] \overline{P}_{a_l} \right) \\
& + \left\langle \Sigma_{\beta\beta}(\Gamma), 2\overline{P}_{\beta}^T \mathcal{B}(\overline{H})(\mathcal{B}(\overline{X}) - \overline{\sigma}_k I_{m+n})^\dagger \mathcal{B}(\overline{H}) \overline{P}_{\beta} \right\rangle ,
\end{aligned}
$$

where

$$
\begin{aligned}
& \Delta(\eta, W) \\
= \; & \zeta\eta + \theta \left(\sum_{j=1}^{r_0} \mathrm{tr}\,(S(\overline{U}_{a_j}^T W \overline{V}_{a_j}^T)) - \sum_{j=1}^{r_0} \mathrm{tr} \left(2\overline{P}_{a_j}^T \left[\mathcal{B}(\overline{H})(\mathcal{B}(\overline{X}) - \bar{\nu}_j I_{m+n})^\dagger \mathcal{B}(\overline{H}) \right] \overline{P}_{a_j} \right) \right) \\
& + \left\langle \Sigma_{\beta\beta}(\Gamma), R^T \left(S(\overline{U}_{\beta}^T W \overline{V}_{\beta}^T) - 2\overline{P}_{\beta}^T \mathcal{B}(\overline{H})(\mathcal{B}(\overline{X}) - \overline{\sigma}_k I_{m+n})^\dagger \mathcal{B}(\overline{H}) \overline{P}_{\beta} \right) R \right\rangle . \qquad (4.17)
\end{aligned}
$$

Next, we shall show that

$$
\max \left\{ \Delta(\eta, W) \,|\, (\eta, W) \in \mathcal{T}^2 \right\} = 0 \,.
$$

In fact, by (4.14), Lemma 2.3 (Ky Fan's inequality) and (4.15), we have

$$\begin{aligned}
& \left\langle \Sigma_{\beta\beta}(\Gamma), R^T \left(S(\overline{U}_\beta^T W \overline{V}_\beta^T) - 2\overline{P}_\beta^T \mathcal{B}(\overline{H})(\mathcal{B}(\overline{X}) - \overline{\sigma}_k I_{m+n})^\dagger \mathcal{B}(\overline{H})\overline{P}_\beta \right) R \right\rangle \\
\leq \; & \sum_{l=1}^{\widetilde{r}_0} \operatorname{tr}\left(R_{\widetilde{\alpha}_l}^T \overline{P}_\beta^T \left[\mathcal{B}(W) - 2\mathcal{B}(\overline{H})(\mathcal{B}(\overline{X}) - \overline{\sigma}_k I_{m+n})^\dagger \mathcal{B}(\overline{H}) \right] \overline{P}_\beta R_{\widetilde{\alpha}_l} \right) \\
& + \theta \sum_{i=1}^{\widetilde{k}_1 - \widetilde{k}_0} u_{k_0 + \widetilde{k}_0 + i} \lambda_i \left(R_{\widetilde{\alpha}_{\widetilde{r}_0+1}}^T \overline{P}_\beta^T \left[\mathcal{B}(W) - 2\mathcal{B}(\overline{H})(\mathcal{B}(\overline{X}) - \overline{\sigma}_k I_{m+n})^\dagger \mathcal{B}(\overline{H}) \right] \overline{P}_\beta R_{\widetilde{\alpha}_{\widetilde{r}_0+1}} \right) \\
\leq \; & \sum_{l=1}^{\widetilde{r}_0} \operatorname{tr}\left(R_{\widetilde{\alpha}_l}^T \overline{P}_\beta^T \left[\mathcal{B}(W) - 2\mathcal{B}(\overline{H})(\mathcal{B}(\overline{X}) - \overline{\sigma}_k I_{m+n})^\dagger \mathcal{B}(\overline{H}) \right] \overline{P}_\beta R_{\widetilde{\alpha}_l} \right) \\
& + \theta \sum_{i=1}^{\widetilde{k} - \widetilde{k}_0} \lambda_i \left(R_{\widetilde{\alpha}_{\widetilde{r}_0+1}}^T \overline{P}_\beta^T \left[\mathcal{B}(W) - 2\mathcal{B}(\overline{H})(\mathcal{B}(\overline{X}) - \overline{\sigma}_k I_{m+n})^\dagger \mathcal{B}(\overline{H}) \right] \overline{P}_\beta R_{\widetilde{\alpha}_{\widetilde{r}_0+1}} \right) . \qquad (4.18)
\end{aligned}$$

Therefore, we know from (4.16), (4.17) and (4.18) that for any $(\eta, W) \in \mathcal{T}^2$, $\Delta(\eta, W) \leq 0$. Also, it is easy to see that there exists $(\eta^*, W^*) \in \mathcal{T}^2$ such $\Delta(\eta^*, W^*) = 0$. Then, since $\delta^*_{\mathcal{T}^2}(0,0) = 0$, we know that for any (ζ, Γ) satisfying the condition (4.9),

$$\begin{aligned}
\delta^*_{\mathcal{T}^2}(\zeta, \Gamma) \;=\; & -\zeta \sum_{l=1}^{r_0} \operatorname{tr}\left(2\overline{P}_{a_l}^T \left[\mathcal{B}(\overline{H})(\mathcal{B}(\overline{X}) - \bar{\nu}_l I_{m+n})^\dagger \mathcal{B}(\overline{H}) \right] \overline{P}_{a_l} \right) \\
& + \left\langle \Sigma_{\beta\beta}(\Gamma), 2\overline{P}_\beta^T \mathcal{B}(\overline{H})(\mathcal{B}(\overline{X}) - \overline{\sigma}_k I_{m+n})^\dagger \mathcal{B}(\overline{H})\overline{P}_\beta \right\rangle .
\end{aligned}$$

Case 3.2. $\overline{\sigma}_k = 0$. There exists an integer $0 \leq k_0 \leq k-1$ such that

$$\overline{\sigma}_1 \geq \cdots \geq \overline{\sigma}_{k_0} > \overline{\sigma}_{k_0+1} = \ldots = \overline{\sigma}_k = \ldots = \overline{\sigma}_m = 0 .$$

Denote $\alpha = \{1, \ldots, k_0\}$ and $\beta = \{k_0+1, \ldots, m\}$. Since $(\bar{t}, \overline{X}) = \Pi_{\mathcal{K}}(t, X)$ and $(\zeta, \Gamma) = (t, X) - (\bar{t}, \overline{X})$, by the part (ii) of Lemma 3.15, we know that there exist $\theta > 0$ (since $(t, X) \notin \operatorname{int} K$) and $u \in \Re^m_+$ such that

$$\zeta = -\theta \quad \text{and} \quad \Gamma = \overline{U}[\operatorname{diag}(\theta u) \;\; 0]\overline{V}^T , \qquad (4.19)$$

where $u_i = 1$, $i = 1, \ldots, k_0$,

$$1 \geq u_{k_0+1} \geq \ldots \geq u_m \geq 0 \quad \text{and} \quad \sum_{i=1}^{m-k_0} u_{k_0+i} \leq k - k_0 . \qquad (4.20)$$

Let $r_0 \in \{1,\ldots,r\}$ be the integer such that $\alpha = \cup_{l=1}^{r_0} a_l$. Since $\langle(\zeta,\Gamma),(\overline{\tau},\overline{H})\rangle = 0$, we know from von Neumann's trace inequality (Lemma 2.13) that

$$\begin{aligned} 0 &= \zeta\overline{\tau} + \langle\Gamma,\overline{H}\rangle = \zeta\overline{\tau} + \left\langle[\mathrm{diag}\,(\theta u) \ \ 0], \overline{U}^T\overline{H}\,\overline{V}\right\rangle \\ &\leq -\theta\overline{\tau} + \theta\sum_{l=1}^{r_0}\mathrm{tr}\,(\overline{U}_{a_l}^T\overline{H}\,\overline{V}_{a_l}) + \theta\sum_{i=1}^{m-k_0} u_{k_0+i}\sigma_i\left(\left[\overline{U}_\beta^T\overline{H}\,\overline{V}_\beta \ \ \overline{U}_\beta^T\overline{H}\,\overline{V}_2\right]\right). \end{aligned} \tag{4.21}$$

Since $(\overline{\tau},\overline{H}) \in \mathrm{bd}\,\mathcal{T}_{\mathcal{K}}(\overline{t},\overline{X})$, by (4.2), we obtain that

$$\overline{\tau} = \sum_{l=1}^{r_0}\mathrm{tr}(\overline{U}_{a_l}^T\overline{H}\,\overline{V}_{a_l}) + \sum_{i=1}^{k-k_0}\sigma_i\left(\left[\overline{U}_\beta^T\overline{H}\,\overline{V}_\beta \ \ \overline{U}_\beta^T\overline{H}\,\overline{V}_2\right]\right).$$

By substitution, we know from (4.21) and (3.152) that

$$\begin{aligned} 0 &\leq \theta\left(\sum_{i=1}^{m-k_0} u_{k_0+i}\sigma_i\left(\left[\overline{U}_\beta^T\overline{H}\,\overline{V}_\beta \ \ \overline{U}_\beta^T\overline{H}\,\overline{V}_2\right]\right) - \sum_{i=1}^{k-k_0}\sigma_i\left(\left[\overline{U}_\beta^T\overline{H}\,\overline{V}_\beta \ \ \overline{U}_\beta^T\overline{H}\,\overline{V}_2\right]\right)\right) \\ &\leq \theta\sigma_k\left(\left[\overline{U}_\beta^T\overline{H}\,\overline{V}_\beta \ \ \overline{U}_\beta^T\overline{H}\,\overline{V}_2\right]\right)\left(\sum_{i=1}^{k-k_0}(u_{k_0+i}-1) + \sum_{i=k-k_0+1}^{m-k_0} u_{k_0+i}\right) \leq 0\,, \end{aligned}$$

which implies the equality in (4.21) holds and

$$\sum_{i=1}^{m-k_0} u_{k_0+i}\sigma_i\left(\left[\overline{U}_\beta^T\overline{H}\,\overline{V}_\beta \ \ \overline{U}_\beta^T\overline{H}\,\overline{V}_2\right]\right) = \sum_{i=1}^{k-k_0}\sigma_i\left(\left[\overline{U}_\beta^T\overline{H}\,\overline{V}_\beta \ \ \overline{U}_\beta^T\overline{H}\,\overline{V}_2\right]\right). \tag{4.22}$$

Next, consider the singular value decomposition of $\left[\overline{U}_\beta^T\overline{H}\,\overline{V}_\beta \ \ \overline{U}_\beta^T\overline{H}\,\overline{V}_2\right]$. Denote $\widetilde{m} = m - k_0$, $\widetilde{k} = k - k_0$. Let $\widetilde{\sigma} := \sigma\left(\left[\overline{U}_\beta^T\overline{H}\,\overline{V}_\beta \ \ \overline{U}_\beta^T\overline{H}\,\overline{V}_2\right]\right) \in \Re_+^{\widetilde{m}}$ be the corresponding singular values. Let $\widetilde{a}_j$, $j = 1,\ldots,\widetilde{r}$ be the index sets defined by (2.26) and $\widetilde{b}$ be the index set of the zero singular value.

If $\widetilde{\sigma}_{\widetilde{k}} > 0$, then there exist two integers $0 \leq \widetilde{k}_0 \leq \widetilde{k}-1$ and $\widetilde{k} \leq \widetilde{k}_1 \leq \widetilde{m}$ such that

$$\widetilde{\sigma}_1 \geq \ldots \geq \widetilde{\sigma}_{\widetilde{k}_0} > \widetilde{\sigma}_{\widetilde{k}_0+1} = \ldots = \widetilde{\sigma}_{\widetilde{k}} = \ldots = \widetilde{\sigma}_{\widetilde{k}_1} > \widetilde{\sigma}_{\widetilde{k}_1+1} \geq \ldots \geq \widetilde{\sigma}_{\widetilde{m}} \geq 0\,.$$

Let $\widetilde{r}_0 \in \{1,\ldots,\widetilde{r}\}$ be the integer such that $\widetilde{k} \in \widetilde{a}_{\widetilde{r}_0+1}$. Then, from (4.22), we have

$$\sum_{i=1}^{\widetilde{m}} u_{k_0+i}\widetilde{\sigma}_i = \sup\left\{\langle y,\widetilde{\sigma}\rangle \mid y = x - z \in \Re^{\widetilde{m}},\ 0 \leq x, z \leq e,\ \langle e, x+z\rangle = \widetilde{k}\right\},$$

which implies that $(u_{k_0+1}, \ldots, u_m) \in \Re^{\widetilde{m}}$ is the solution of the maximize problem. Therefore, we know from [113, Lemma 2.3] that in this case,

$$u_{k_0+i} = 1, \quad i = 1, \ldots, \widetilde{k}_0, \quad u_{k_0+i} = 0, \quad i = \widetilde{k}_1 + 1, \ldots \widetilde{m} \tag{4.23}$$

and

$$1 \geq u_{k_0+\widetilde{k}_0+1} \geq u_{k_0+\widetilde{k}_0+2} \geq \ldots \geq u_{k_0+\widetilde{k}_1} \geq 0 \quad \text{and} \quad \sum_{i=1}^{\widetilde{k}_1-\widetilde{k}_0} u_{k_0+\widetilde{k}_0+i} = \widetilde{k} - \widetilde{k}_0 \,. \tag{4.24}$$

If $\widetilde{\sigma}_{\widetilde{k}} = 0$, then there exists an integer $0 \leq \widetilde{k}_0 \leq \widetilde{k} - 1$ such that

$$\widetilde{\sigma}_1 \geq \cdots \geq \widetilde{\sigma}_{\widetilde{k}_0} > \widetilde{\sigma}_{\widetilde{k}_0+1} = \ldots = \widetilde{\sigma}_{\widetilde{k}} = \ldots = \widetilde{\sigma}_{\widetilde{m}} = 0 \,.$$

Again, from (4.22) and [113, Lemma 2.3], we know that

$$u_{k_0+i} = 1, \quad i = 1, \ldots, \widetilde{k}_0 \,, \tag{4.25}$$

$$1 \geq u_{k_0+\widetilde{k}_0+1} \geq u_{k_0+\widetilde{k}_0+2} \geq \ldots \geq u_{k_0+\widetilde{k}_1} \geq 0 \quad \text{and} \quad \sum_{i=1}^{\widetilde{k}_1-\widetilde{k}_0} u_{k_0+\widetilde{k}_0+i} \leq \widetilde{k} - \widetilde{k}_0 \,. \tag{4.26}$$

Since the equality in (4.21) holds, by von Neumann's trace inequality, we know that the matrices $[\mathrm{diag}\,(u_\beta) \;\; 0]$ and $\left[\overline{U}_\beta^T \overline{H}\, \overline{V}_\beta \;\; \overline{U}_\beta^T \overline{H}\, \overline{V}_2\right]$ admit a simultaneous ordered singular value decomposition, i.e., there exist two orthogonal matrices $E \in \mathcal{O}^{|\beta|}$, $F \in \mathcal{O}^{|\beta|+n-m}$ such that

$$[\mathrm{diag}\,(u_\beta) \;\; 0] = E[\mathrm{diag}\,(u_\beta) \;\; 0]F^T \quad \text{and} \quad \left[\overline{U}_\beta^T \overline{H}\, \overline{V}_\beta \;\; \overline{U}_\beta^T \overline{H}\, \overline{V}_2\right] = E[\mathrm{diag}\,(\widetilde{\sigma}) \;\; 0]F^T \,.$$

On the other hand, since $(\overline{\tau}, \overline{H}) \in \mathrm{bd}\, \mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X})$, we know from (4.8) that $(\eta, W) \in \mathcal{T}^2$ if and only if $\sum_{i=1}^k \sigma_i''(\overline{X}; \overline{H}, W) \leq \eta$. Therefore, by (ii) and (iii) of Proposition 2.18, we know that if $\widetilde{\sigma}_{\widetilde{k}} > 0$, then

$$\begin{aligned}
&\sum_{j=1}^{r_0} \mathrm{tr}\,(S(\overline{U}_{a_j}^T W \overline{V}_{a_j}^T)) - \sum_{j=1}^{r_0} \mathrm{tr}\left(2\overline{P}_{a_j}^T \left[\mathcal{B}(\overline{H})(\mathcal{B}(\overline{X}) - \bar{\nu}_j I_{m+n})^\dagger \mathcal{B}(\overline{H})\right] \overline{P}_{a_j}\right) \\
&+ \sum_{j=1}^{\widetilde{r}_0} \mathrm{tr}\left(E_{\widetilde{a}_j}^T [\overline{U}_\beta^T W \overline{V}_\beta \;\; \overline{U}_\beta^T W \overline{V}_2] F_{\widetilde{a}_j} - 2E_{\widetilde{a}_j}^T [\overline{U}_\beta^T \overline{H}\,\overline{X}^\dagger \overline{H}\,\overline{V}_\beta \;\; \overline{U}_\beta^T \overline{H}\,\overline{X}^\dagger \overline{H}\,\overline{V}_2] F_{\widetilde{a}_j}\right) \\
&+ \sum_{i=1}^{\widetilde{k}-\widetilde{k}_0} \lambda_i \left(S(E_{\widetilde{a}_{\widetilde{r}_0+1}}^T \left[\overline{U}_\beta^T (W - 2\overline{H}\,\overline{X}^\dagger\,\overline{H})\overline{V}_\beta \;\; \overline{U}_\beta^T (W - 2\overline{H}\,\overline{X}^\dagger\,\overline{H})\overline{V}_2\right] F_{\widetilde{a}_{\widetilde{r}_0+1}})\right) \\
&\leq \eta\,; \qquad (4.27)
\end{aligned}$$

if $\widetilde{\sigma}_{\widetilde{k}} = 0$, then

$$\begin{aligned}
&\sum_{j=1}^{r_0 r} \mathrm{tr}\,(S(\overline{U}_{a_j}^T W \overline{V}_{a_j}^T)) - \sum_{j=1}^{r_0} \mathrm{tr}\left(2\overline{P}_{a_j}^T \left[\mathcal{B}(\overline{H})(\mathcal{B}(\overline{X}) - \bar{\nu}_j I_{m+n})^\dagger \mathcal{B}(\overline{H})\right] \overline{P}_{a_j}\right) \\
&+ \sum_{j=1}^{\widetilde{r}} \mathrm{tr}\left(E_{\widetilde{a}_j}^T A F_{\widetilde{a}_j} - 2E_{\widetilde{a}_j}^T B F_{\widetilde{a}_j}\right) + \sum_{i=1}^{\widetilde{k}-\widetilde{k}_0} \sigma_i \left([E_{\widetilde{b}}^T A F_{\widetilde{b}} \;\; E_{\widetilde{b}}^T A F_2] - 2[E_{\widetilde{b}}^T B F_{\widetilde{b}} \;\; E_{\widetilde{b}}^T B F_2]\right) \\
&\leq \eta\,, \qquad (4.28)
\end{aligned}$$

where $A := [\overline{U}_\beta^T W \overline{V}_\beta \;\; \overline{U}_\beta^T W \overline{V}_2]$ and $B := [\overline{U}_\beta^T \overline{H}\,\overline{X}^\dagger \overline{H}\,\overline{V}_\beta \;\; \overline{U}_\beta^T \overline{H}\,\overline{X}^\dagger \overline{H}\,\overline{V}_2]$. For any $(\eta, W) \in \mathcal{T}^2$, by (4.19), we obtain that

$$\begin{aligned}
& \zeta\eta + \langle \Gamma, W\rangle = \zeta\eta + \langle \overline{U}^T \Gamma \overline{V}, \overline{U}^T W \overline{V}\rangle = \zeta\eta + \langle [\Sigma(\Gamma) \;\; 0], \overline{U}^T W \overline{V}\rangle \\
= \;& \zeta\eta + \theta \sum_{j=1}^{r_0} \mathrm{tr}\,(S(\overline{U}_{a_j}^T W \overline{V}_{a_j}^T)) + \left\langle [\Sigma_{\beta\beta}(\Gamma) \;\; 0], E^T [\overline{U}_\beta^T W \overline{V}_\beta \;\; \overline{U}_\beta^T W \overline{V}_2] F\right\rangle \\
= \;& \Delta(\eta, W) - \zeta \sum_{j=1}^{r_0} \mathrm{tr}\left(2\overline{P}_{a_j}^T \left[\mathcal{B}(\overline{H})(\mathcal{B}(\overline{X}) - \bar{\nu}_j I_{m+n})^\dagger \mathcal{B}(\overline{H})\right] \overline{P}_{a_j}\right) \\
& + \left\langle [\Sigma_{\beta\beta}(\Gamma) \;\; 0], E^T [\overline{U}_\beta^T \overline{H}\,\overline{X}^\dagger \overline{H}\,\overline{V}_\beta \;\; \overline{U}_\beta^T \overline{H}\,\overline{X}^\dagger \overline{H}\,\overline{V}_2] F\right\rangle\,,
\end{aligned}$$

where

$$
\begin{aligned}
&\Delta(\eta, W)\\
= \;& \zeta\eta + \theta\left(\sum_{j=1}^{r_0} \operatorname{tr}\left(S(\overline{U}_{a_j}^T W \overline{V}_{a_j}^T)\right) - \sum_{j=1}^{r_0} \operatorname{tr}\left(2\overline{P}_{a_j}^T\left[\mathcal{B}(\overline{H})(\mathcal{B}(\overline{X}) - \bar{\nu}_j I_{m+n})^{\dagger}\mathcal{B}(\overline{H})\right]\overline{P}_{a_j}\right)\right)\\
& + \left\langle [\Sigma_{\beta\beta}(\Gamma) \;\; 0], E^T[\overline{U}_\beta^T W \overline{V}_\beta \;\; \overline{U}_\beta^T W \overline{V}_2]F - E^T[\overline{U}_\beta^T \overline{H}\,\overline{X}^{\dagger}\overline{H}\,\overline{V}_\beta \;\; \overline{U}_\beta^T \overline{H}\,\overline{X}^{\dagger}\overline{H}\,\overline{V}_2]F\right\rangle .
\end{aligned}
\tag{4.29}
$$

Similarly, we shall show that

$$
\max\left\{\Delta(\eta, W) \,|\, (\eta, W) \in \mathcal{T}^2\right\} = 0\,.
$$

In fact, if $\widetilde{\sigma}_{\widetilde{k}} > 0$, then by Lemma 2.13 (von Neumann's trace inequality), we know from (4.23) and (4.24) that

$$
\begin{aligned}
&\left\langle [\Sigma_{\beta\beta}(\Gamma) \;\; 0], E^T[\overline{U}_\beta^T W \overline{V}_\beta \;\; \overline{U}_\beta^T W \overline{V}_2]F - E^T[\overline{U}_\beta^T \overline{H}\,\overline{X}^{\dagger}\overline{H}\,\overline{V}_\beta \;\; \overline{U}_\beta^T \overline{H}\,\overline{X}^{\dagger}\overline{H}\,\overline{V}_2]F\right\rangle\\
\leq \;& \sum_{j=1}^{\widetilde{r}_0} \operatorname{tr}\left(E_{\widetilde{a}_j}^T[\overline{U}_\beta^T W \overline{V}_\beta \;\; \overline{U}_\beta^T W \overline{V}_2]F_{\widetilde{a}_j} - 2E_{\widetilde{a}_j}^T[\overline{U}_\beta^T \overline{H}\,\overline{X}^{\dagger}\overline{H}\,\overline{V}_\beta \;\; \overline{U}_\beta^T \overline{H}\,\overline{X}^{\dagger}\overline{H}\,\overline{V}_2]F_{\widetilde{a}_j}\right)\\
& + \sum_{i=1}^{\widetilde{k}-\widetilde{k}_0} \lambda_i\left(S(E_{\widetilde{a}_{\widetilde{r}_0+1}}^T[\overline{U}_\beta^T(W - 2\overline{H}\,\overline{X}^{\dagger}\,\overline{H})\overline{V}_\beta \;\; \overline{U}_\beta^T(W - 2\overline{H}\,\overline{X}^{\dagger}\,\overline{H})\overline{V}_2]F_{\widetilde{a}_{\widetilde{r}_0+1}})\right) .
\end{aligned}
\tag{4.30}
$$

Then, by (4.27), (4.29) and (4.30), we know that $\Delta(\eta, W) \leq 0$ for any $(\eta, W) \in \mathcal{T}^2$. Also, it is easy to see that the maximize value can be obtained.

If $\widetilde{\sigma}_{\widetilde{k}} = 0$, then by Lemma 2.13 (von Neumann's trace inequality), we know from (4.25) and (4.26) that

$$
\begin{aligned}
&\left\langle [\Sigma_{\beta\beta}(\Gamma) \;\; 0], E^T[\overline{U}_\beta^T W \overline{V}_b \;\; \overline{U}_\beta^T W \overline{V}_2]F - E^T[\overline{U}_\beta^T \overline{H}\,\overline{X}^{\dagger}\overline{H}\,\overline{V}_\beta \;\; \overline{U}_\beta^T \overline{H}\,\overline{X}^{\dagger}\overline{H}\,\overline{V}_2]F\right\rangle\\
\leq \;& \sum_{j=1}^{\widetilde{r}} \operatorname{tr}\left(E_{\widetilde{a}_j}^T A F_{\widetilde{a}_j} - 2E_{\widetilde{a}_j}^T B F_{\widetilde{a}_j}\right) + \sum_{i=1}^{\widetilde{k}-\widetilde{k}_0} \sigma_i\left([E_{\widetilde{b}}^T A F_{\widetilde{b}} \;\; E_{\widetilde{b}}^T A F_2] - 2[E_{\widetilde{b}}^T B F_{\widetilde{b}} \;\; E_{\widetilde{b}}^T B F_2]\right) .
\end{aligned}
\tag{4.31}
$$

Then, by (4.28), (4.29) and (4.31), we know that $\Delta(\eta, W) \leq 0$ for any $(\eta, W) \in \mathcal{T}^2$. Also, it is easy to see that the maximize value can be obtained. Then, since $\delta^*_{\mathcal{T}^2}(0,0) = 0$, we know that for any (ζ, Γ) satisfying the condition (4.9),

$$\begin{aligned}\delta^*_{\mathcal{T}^2}(\zeta,\Gamma) &= -\zeta \sum_{j=1}^{r_0} \mathrm{tr}\left(2\overline{P}^T_{a_j}\left[\mathcal{B}(\overline{H})(\mathcal{B}(\overline{X})-\bar{\nu}_j I_{m+n})^\dagger \mathcal{B}(\overline{H})\right]\overline{P}_{a_j}\right) \\ &\quad + \left\langle [\Sigma_{\beta\beta}(\Gamma)\ \ 0], [\overline{U}^T_\beta \overline{H}\,\overline{X}^\dagger \overline{H}\,\overline{V}_\beta\ \ \overline{U}^T_\beta \overline{H}\,\overline{X}^\dagger \overline{H}\,\overline{V}_2]\right\rangle .\end{aligned}$$

Next, we summary the result on the support function $\delta^*_{\mathcal{T}^2}$ of the second order tangent set $\mathcal{T}^2$ as follows.

Proposition 4.2. *Let $(\bar{t}, \overline{X}) \in \mathcal{K}$ and $(\bar{\tau}, \overline{H}) \in \mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X})$ be given. Suppose that $(\zeta, \Gamma) \in \Re \times \Re^{m\times n}$ satisfies*

$$(\zeta,\Gamma) \in \mathcal{K}^\circ, \quad \langle (\zeta,\Gamma), (\bar{t},\overline{X})\rangle = 0 \quad \text{and} \quad \langle (\zeta,\Gamma), (\bar{\tau},\overline{H})\rangle = 0\,.$$

(i) *If $(\bar{t}, \overline{X}) \in \mathrm{int}\,\mathcal{K}$, then $\delta^*_{\mathcal{T}^2}(\zeta,\Gamma) = 0$.*

(ii) *If $(\bar{t}, \overline{X}) \in \mathrm{bd}\,\mathcal{K}$ and $(\bar{\tau}, \overline{H}) \in \mathrm{int}\,\mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X})$, then $\delta^*_{\mathcal{T}^2}(\zeta,\Gamma) = 0$.*

(iii) *If $(\bar{t}, \overline{X}) \in \mathrm{bd}\,\mathcal{K}$, $(\bar{\tau}, \overline{H}) \in \mathrm{bd}\,\mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X})$ and $\sigma_k(\overline{X}) > 0$, then*

$$\begin{aligned}\delta^*_{\mathcal{T}^2}(\zeta,\Gamma) &= -\zeta \sum_{j=1}^{r_0} \mathrm{tr}\left(2\overline{P}^T_{a_j}\left[\mathcal{B}(\overline{H})(\mathcal{B}(\overline{X})-\bar{\nu}_j I_{m+n})^\dagger \mathcal{B}(\overline{H})\right]\overline{P}_{a_j}\right) \\ &\quad + \left\langle \Sigma_{\beta\beta}(\Gamma), 2\overline{P}^T_\beta \mathcal{B}(\overline{H})(\mathcal{B}(\overline{X})-\overline{\sigma}_k I_{m+n})^\dagger \mathcal{B}(\overline{H})\overline{P}_\beta\right\rangle .\end{aligned}$$

(iv) *If $(\bar{t}, \overline{X}) \in \mathrm{bd}\,\mathcal{K}$, $(\bar{\tau}, \overline{H}) \in \mathrm{bd}\,\mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X})$ and $\sigma_k(\overline{X}) = 0$, then*

$$\begin{aligned}\delta^*_{\mathcal{T}^2}(\zeta,\Gamma) &= -\zeta \sum_{j=1}^{r_0} \mathrm{tr}\left(2\overline{P}^T_{a_j}\left[\mathcal{B}(\overline{H})(\mathcal{B}(\overline{X})-\bar{\nu}_j I_{m+n})^\dagger \mathcal{B}(\overline{H})\right]\overline{P}_{a_j}\right) \\ &\quad + \left\langle [\Sigma_{\beta\beta}(\Gamma)\ \ 0], [\overline{U}^T_\beta \overline{H}\,\overline{X}^\dagger \overline{H}\,\overline{V}_\beta\ \ \overline{U}^T_\beta \overline{H}\,\overline{X}^\dagger \overline{H}\,\overline{V}_2]\right\rangle .\end{aligned}$$

Definition 4.1. *For any given $(\bar{t}, \overline{X}) \in \mathcal{K}$, define the linear quadratic function $\Upsilon_{(\bar{t},\overline{X})} : \Re \times \Re^{m\times n} \times \Re \times \Re^{m\times n} \to \Re$, which is linear in the first argument and quadratic in the* *second argument, by for any $(\zeta,\Gamma)\in\Re\times\Re^{m\times n}$ and $(\tau,H)\in\Re\times\Re^{m\times n}$, if $\sigma_k(\overline{X})>0$, then*

$$\begin{aligned}\Upsilon_{(\bar t,\overline{X})}((\zeta,\Gamma),(\tau,H)) \;&:=\; -\zeta\sum_{j=1}^{r_0}\operatorname{tr}\left(2\overline{P}_{a_j}^T\left[\mathcal{B}(H)(\mathcal{B}(\overline{X})-\bar\nu_j I_{m+n})^{\dagger}\mathcal{B}(H)\right]\overline{P}_{a_j}\right)\\ &\quad+\left\langle\Sigma_{\beta\beta}(\Gamma),2\overline{P}_{\beta}^T\mathcal{B}(H)(\mathcal{B}(\overline{X})-\bar\nu I_{m+n})^{\dagger}\mathcal{B}(H)\overline{P}_{\beta}\right\rangle\,;\end{aligned}$$

if $\sigma_k(\overline{X})=0$, then

$$\begin{aligned}\Upsilon_{(\bar t,\overline{X})}((\zeta,\Gamma),(\tau,H)) \;&:=\; -\zeta\sum_{j=1}^{r_0}\operatorname{tr}\left(2\overline{P}_{a_j}^T\left[\mathcal{B}(H)(\mathcal{B}(\overline{X})-\bar\nu_j I_{m+n})^{\dagger}\mathcal{B}(H)\right]\overline{P}_{a_j}\right)\\ &\quad+\left\langle[\Sigma_{\beta\beta}(\Gamma)\;\;0],[\overline{U}_{\beta}^T H\,\overline{X}^{\dagger}H\,\overline{V}_{\beta}\;\;\overline{U}_{\beta}^T H\,\overline{X}^{\dagger}H\,\overline{V}_2]\right\rangle\,.\end{aligned}$$

Finally, we will show that the epigraph cone $\mathcal{K}=\mathrm{epi}\|\cdot\|_{(k)}$ of the Ky Fan k-norm is $\mathcal{C}^2$-cone reducible at every point $(\bar t,\overline{X})\in\mathcal{K}$. Hence, $\mathcal{K}$ is second order regular ([8, Definition 3.85]) at every point. We first recall the definition of $\mathcal{C}^2$-cone reducible ([8, Definition 3.135]).

Definition 4.2. *Let $\mathcal{Y}$ and $\mathcal{Z}$ be two finite dimensional Euclidean spaces. Let $K\subseteq\mathcal{Y}$ and $C\subset\mathcal{Z}$ be convex closed sets. We say that the set K is $\mathcal{C}^2$-reducible to the set C, at a point $\bar y\in K$, if there exist a neighborhood $\mathcal{U}$ of y_0 and twice continuously differentiable mapping $\Xi:\mathcal{U}\to\mathcal{Z}$ such that (i) $\Xi'(\bar y):\mathcal{Y}\to\mathcal{Z}$ is onto, and (ii) $K\cap\mathcal{N}=\{y\in\mathcal{U}\,|\,\Xi(\bar y)\in C\}$. We say that the reduction is pointed if the tangent cone $\mathcal{T}_C(\Xi(\bar y))$ is pointed cone. If, in addition, the set $C-\Xi(\bar y)$ is a pointed convex closed cone, we say that K is $\mathcal{C}^2$-cone reducible at $\bar y$. We can assume without loss of generality that $\Xi(\bar y)=0$.*

Proposition 4.3. *The epigraph cone $\mathcal{K}$ of the Ky Fan k-norm is $\mathcal{C}^2$-cone reducible at every point $(\bar t,\overline{X})\in\mathcal{K}$.*

Proof. Since $\mathcal{K}$ is a pointed closed convex cone, we know that $\mathcal{K}$ is $\mathcal{C}^2$-cone reducible at $(\bar t,\overline{X})$ if $(\bar t,\overline{X})\in\operatorname{int}\mathcal{K}$ or $(\bar t,\overline{X})=(0,0)$. Therefore, we only need to consider the case that $(\bar t,\overline{X})\in\operatorname{bd}\mathcal{K}\setminus(0,0)$, i.e., $\|\overline{X}\|_{(k)}=\bar t>0$. Let α and β be the index sets defined by

$$\alpha=\{i\in\{1,\ldots,m\}\,|\,\sigma_i(\overline{X})>\sigma_k(\overline{X})\}\quad\text{and}\quad\beta=\{i\in\{1,\ldots,m\}\,|\,\sigma_i(\overline{X})=\sigma_k(\overline{X})\}\,.$$

Consider the singular value decomposition (3.155) of $\overline{X}$,

$$\overline{X} = \overline{U}[\Sigma(\overline{X}) \;\; 0]\overline{V}^T .$$

Denote $\overline{\sigma} = \sigma(\overline{X})$ and $\overline{\Sigma} = \Sigma(\overline{X})$. Let a_l, $l = 1, \ldots, r$ and $a_{r+1} = b$ be the index sets defined by (2.25) and (2.26) with respect to X_0. Then, we know that there exists $r_0 \in \{1, \ldots, r+1\}$ such that

$$\alpha = \bigcup_{l=1}^{r_0} a_l \quad \text{and} \quad a_{r_0+1} = \beta .$$

For any $Z \in \Re^{m\times n}$ and $W \in \Re^{n\times n}$, recall the definition of the notations $Z_{a_l} \in \Re^{m\times|a_l|}$, $l = 1, \ldots, r+1$ and $W_{a_l} \in \Re^{n\times|a_l|}$, $l = 1, \ldots, r$, i.e., the sub-matrices of Z and W obtained by removing all the columns of Z and W not in a_l, respectively. For simplicity, we also use the notation $W_{a_{r+1}} \in \Re^{n\times(|b|+|c|)}$ to represent the sub-matrix of any matrix $W \in \Re^{n\times n}$ obtained by removing all the columns of W not in $b \cup c$.

Since the single value function $\sigma(\cdot)$ is globally Lipschitz continuous, by using Proposition 2.14, we know that there exists an open neighborhood $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$ of $(\bar{t}, \overline{X})$ such that for each $l \in \{1, \ldots, r+1\}$, the following functions $\mathcal{U}_l : \mathcal{N}_2 \to \Re^{m\times m}$ and $\mathcal{V}_l : \mathcal{N}_2 \to \Re^{n\times n}$ defined by

$$\mathcal{U}_l(X) := \sum_{i\in a_l} u_i(X)u_i(X)^T \quad \text{and} \quad \mathcal{V}_l(X) = \sum_{i\in a_l} v_i(X)v_i(X)^T, \quad X \in \mathcal{N}_2 , \tag{4.32}$$

are well-defined (i.e., for each $l \in \{1, \ldots, r+1\}$ and any $X \in \mathcal{N}_2$, the function values $\mathcal{U}_l(X)$ and $\mathcal{V}_l(X)$ are independent to the choice of the orthogonal pairs $(U(X), V(X)) \in \mathcal{O}^{m,n}(X)$), where $u_i(X) \in \Re^m$ and $v_i(X) \in \Re^n$, $i \in a_l$ are the i-th columns of the orthogonal matrices $U(X) \in \mathcal{O}^m$ and $V(X) \in \mathcal{O}^n$, respectively. By consider the line operator $\mathcal{B} : \Re^{m\times n} \to \mathcal{S}^{m+n}$ defined by (2.28) and the corresponding orthogonal matrix $P \in \mathcal{O}^{m+n}$ defined by (2.43), we have for any $X \in \mathcal{N}_2$,

$$F_l(X) := \mathcal{P}_l(\mathcal{B}(X)) = \sum_{i\in a_l} p_i(\mathcal{B}(X))p_i(\mathcal{B}(X))^T = \frac{1}{2}\begin{bmatrix} \mathcal{U}_l(X) & * \\ * & \mathcal{V}_l(X) \end{bmatrix}, \quad l = 1, \ldots, r$$

and

$$\begin{aligned} F_{r+1}(X) &:= \mathcal{P}_{r+1}(\mathcal{B}(X)) = \sum_{i\in b\cup c\cup b'} p_i(\mathcal{B}(X))p_i(\mathcal{B}(X))^T \\ &= \begin{bmatrix} \mathcal{U}_{r+1}(X) & 0 \\ 0 & \mathcal{V}_{r+1}(X) \end{bmatrix}, \end{aligned}$$

where $\mathcal{V}_{r+1}(X) = \sum_{i\in b} v_i(X)v_i(X)^T + \sum_{i\in c} v_i(X)v_i(X)^T$. We know from Proposition 2.12 that there exists an open neighborhood $\widehat{\mathcal{N}}$ of $\mathcal{B}(\overline{X})$ in $\mathcal{S}^{m+n}$ such that $\mathcal{P}_l(\cdot)$, $l = 1,\ldots,r+1$ are twice continuously differentiable on $\widehat{\mathcal{N}}$. Therefore, by shrinking the neighborhood $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$ if necessary, we know that $F_l(\cdot)$, $l = 1,\ldots,r+1$ are twice continuously differentiable on $\mathcal{N}_2$. Hence, the mappings $\mathcal{U}_l(\cdot)$ and $\mathcal{V}_l(\cdot)$, $l = 1,\ldots,r+1$ are all twice continuously differentiable on $\mathcal{N}_2$.

Next, we first consider the special case that $\overline{X} = \begin{bmatrix}\Sigma & 0\end{bmatrix}$. For any $X \in \mathcal{N}_2$, let $L_l(X)$ and $R_l(X)$, $l = 1,\ldots,r+1$ be the left and right eigenspaces corresponding to the single values $\{\sigma_i(X) : i \in a_l\}$. Actually, for any $X \in \mathcal{N}_2$, the matrices $\mathcal{U}_l(X)$ and $\mathcal{V}_l(X)$, $l = 1,\ldots,r+1$ are the orthogonal projection matrices onto $L_l(X)$ and $R_l(X)$, respectively. For any $X \in \mathcal{N}_2$, denote the columns of $\mathcal{U}_l(X) \in \Re^{m\times m}$ and $\mathcal{V}_l(X) \in \Re^{n\times n}$, $l = 1,\ldots,r+1$ by $\{(\mathcal{U}_l(X))_i\}$ and $\{(\mathcal{V}_l(X))_i\}$. It is obvious that the space spanned by $\{(\mathcal{U}_l(X))_i\}$ and $\{(\mathcal{V}_l(X))_i\}$ coincide with $L_l(X)$ and $R_l(X)$, respectively. Moreover, for each $l \in \{1,\ldots,r+1\}$, we know that for all X sufficiently close to $\overline{X}$, the columns $\{(\mathcal{U}_l(X))_i : i \in a_l\}$ and $\{(\mathcal{V}_l(X))_i : i \in a_l\}$ are linearly independent. In fact, for any $X \in \mathcal{N}_2$ and each $l \in \{1,\ldots,r+1\}$, from the definitions of $\mathcal{U}_l \in \Re^{m\times m}$ and $\mathcal{V}_l \in \Re^{n\times n}$, we know that the j'-th columns of $\mathcal{U}_l(X)$ and $\mathcal{V}_l(X)$ for all $j' \in a_l$ are given by

$$(\mathcal{U}_l(X))_{j'} = \sum_{j\in a_l} U_{j'j}(X) \begin{bmatrix} U_{1j}(X) \\ \vdots \\ U_{nj}(X) \end{bmatrix} \quad \text{and} \quad (\mathcal{V}_l(X))_{j'} = \sum_{j\in a_l} V_{j'j}(X) \begin{bmatrix} V_{1j}(X) \\ \vdots \\ V_{nj}(X) \end{bmatrix}. \tag{4.33}$$

Therefore, for each $l \in \{1,\ldots,r+1\}$, suppose that the real numbers $q_i \in \Re$, $i = 1,\ldots,|a_l|$

such that

$$\sum_{i\in a_l} q_i(\mathcal{U}_l(X))_i = 0\,.$$

Then, since for each $l \in \{1,\ldots,r+1\}$, the columns $\begin{bmatrix} U_{1j}(X) \\ \vdots \\ U_{nj}(X) \end{bmatrix}$, $j \in a_l$ are linearly independent, we obtain that the vector $\begin{bmatrix} q_1 \\ \vdots \\ q_{|a_l|} \end{bmatrix} \in \Re^{|a_l|}$ is the solution of the following linear system

$$U_{a_la_l}(X)\begin{bmatrix} q_1 \\ \vdots \\ q_{|a_l|} \end{bmatrix} = 0\,.$$

From (2.40) in Proposition 2.16, since $\overline{X} = \begin{bmatrix}\Sigma & 0\end{bmatrix}$, for each $l \in \{1,\ldots,r+1\}$, we know that for X sufficiently close to $\overline{X}$, there exists $Q_l \in \mathcal{O}^{|a_l|}$ such that

$$U_{a_la_l}(X) = Q_l + O(\|X-\overline{X}\|)\,.$$

Since the determinant function $\det(\cdot)$ is continuous, for each $l \in \{1,\ldots,r+1\}$, we know that for all X sufficiently close to $\overline{X}$, the matrix $U_{a_la_l}(X)$ is invertible, which implies $q_i = 0$, $i = 1,\ldots,|a_l|$ and the columns $\{(\mathcal{U}_l(X))_i : i \in a_l\}$ are linearly independent. By using the similar arguments, we also have that for X sufficiently close to $\overline{X}$, the columns $\{(\mathcal{U}_l(X))_i : i \in a_l\}$ are also linearly independent. Hence, by shrink $\mathcal{N} = \mathcal{N}_1\times\mathcal{N}_2$ if necessary, we may conclude that for any $X \in \mathcal{N}_2$, $\{(\mathcal{U}_l(X))_i : i \in a_l\}$ and $\{(\mathcal{V}_l(X))_i : i \in a_l\}$, $l = 1,\ldots,r+1$ are the bases of $L_l(X)$ and $R_l(X)$, respectively. Furthermore, for each $l \in \{1,\ldots,r+1\}$, by applying the Gram-Schmit orthonormalization procedure to the columns $\{(\mathcal{U}_l(X))_i : i \in a_l\}$ and $\{(\mathcal{V}_l(X))_i : i \in a_l\}$, for any $X \in \mathcal{N}_2$, we obtain two matrices $M_{a_l}(X) \in \Re^{m\times|a_l|}$ and $N_{a_l}(X) \in \Re^{n\times|a_l|}$ such that the columns of $M_{a_l}(X)$ are the orthogonal bases of the left eigenspace $L_l(X)$ of X and the columns of $N_{a_l}(X)$ are the orthogonal bases of the right eigenspace $R_l(X)$ of X. Moreover, for each $l \in \{1,\ldots,r+1\}$, the mappings $M_{a_l} : \mathcal{N}_2 \to \Re^{m\times|a_l|}$ and $N_{a_l} : \mathcal{N}_2 \to \Re^{m\times|a_l|}$ are twice continuously differentiable on $\mathcal{N}_2$. Therefore, we know that the mappings $M_{a_l}(X)^T X N_{a_l}(X) : \mathcal{N}_2 \to \Re^{|a_l|\times|a_l|}$, $l = 1,\ldots,r$ and $M_{a_{r+1}}(X)^T X N_{a_{r+1}}(X) : \mathcal{N}_2 \to \Re^{b\times(|b|+|c|)}$ are all twice continuously differentiable on $\mathcal{N}_2$, and $M_{a_l}(X)^T X N_{a_l}(X)$, $l = 1,\ldots,r+1$ are diagonal matrices, whose diagonal elements are the singular values $\{\sigma_i(X) : i \in a_l\}$. Since the singular value function $\sigma(\cdot)$ is globally Lipschitz continuous, by further shrinking $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$ if necessary, we have that for any $l, l' \in \{1,\ldots,r+1\}$ and $l < l'$,

$$\sigma_{|a_l|}\left(M_{a_l}(X)^T X N_{a_l}(X)\right) > \sigma_1\left(M_{a_{l'}}(X)^T X N_{a_{l'}}(X)\right) \quad \forall\, X \in \mathcal{N}_2 .$$

In particular, we have

$$M_{a_l}(\overline{X})^T \overline{X} N_{a_l}(\overline{X}) = \overline{\Sigma}_{a_l a_l},\ l = 1,\ldots,r \quad \text{and} \quad M_{a_{r+1}}(\overline{X})^T \overline{X} N_{a_{r+1}}(\overline{X}) = [\overline{\Sigma}_{bb} \ \ 0] .$$

On the other hand, for each $l \in \{1,\ldots,r+1\}$, we know from (2.40) in Proposition 2.16 that for X sufficiently close to $\overline{X}$,

$$U_{ij}(X) = O(\|X-\overline{X}\|) = U_{ji}(X) \quad \text{and} \quad V_{ij}(X) = O(\|X-\overline{X}\|) = V_{ji}(X) \quad \forall\, i \notin a_l \text{ and } j \in a_l .$$

Therefore, we know from (4.33) that for each $l \in \{1,\ldots,r+1\}$, $j' \in a_l$ and any $X \in \mathcal{N}_2$, the j'-th column of $\mathcal{U}_l(X)$ satisfies the following conditions

$$(\mathcal{U}_l(X))_{i'j'} = O(\|X-\overline{X}\|) \quad \forall\, i' \notin a_l ,$$

$$(\mathcal{U}_l(X))_{i'j'} = \sum_{j\in a_l} U_{j'j}(X)U_{i'j}(X) = \sum_{j\notin a_l} U_{j'j}(X)U_{i'j}(X) = O(\|X-\overline{X}\|^2) \quad \forall\, i' \in a_l \text{ but } i' \neq j'$$

and

$$(\mathcal{U}_l(X))_{j'j'} = \sum_{j\in a_l} U_{j'j}(X)^2 = 1 - \sum_{j\notin a_l} U_{j'j}(X)^2 = 1 + O(\|X-\overline{X}\|^2) ,$$

which implies that

$$(\mathcal{U}_l(X))_{a_l} = \begin{bmatrix} O(\|X-\overline{X}\|) \\ I_{|a_l|} + O(\|X-\overline{X}\|^2) \\ O(\|X-\overline{X}\|) \end{bmatrix} .$$

Similarly, we also have for each $l \in \{1, \ldots, r+1\}$ and any $X \in \mathcal{N}_2$,

$$(\mathcal{V}_l(X))_{a_l} = \begin{bmatrix} O(\|X - \overline{X}\|) \\ I_{|a_l|} + O(\|X - \overline{X}\|^2) \\ O(\|X - \overline{X}\|) \end{bmatrix}.$$

Thus, by considering the Gram-Schmit orthonormalization procedure, we obtain that for each $l \in \{1, \ldots, r+1\}$, for any $X \in \mathcal{N}_2$,

$$M_{a_l}(X) = \begin{bmatrix} O(\|X - \overline{X}\|) \\ I_{|a_l|} + O(\|X - \overline{X}\|^2) \\ O(\|X - \overline{X}\|) \end{bmatrix} \quad \text{and} \quad N_{a_l}(X) = \begin{bmatrix} O(\|X - \overline{X}\|) \\ I_{|a_l|} + O(\|X - \overline{X}\|^2) \\ O(\|X - \overline{X}\|) \end{bmatrix}.$$

Denote $H := X - \overline{X}$. Therefore, we obtain that for each $l \in \{1, \ldots, r\}$, for any $X \in \mathcal{N}_2$,

$$M_{a_l}(X)^T X N_{a_l}(X) = M_{a_l}(X)^T \left([\overline{\Sigma} \;\; 0] + H\right) N_{a_l}(X) = \overline{\Sigma}_{a_l a_l} + H_{a_l a_l} + O(\|H\|^2) \quad (4.34)$$

and

$$\begin{aligned} & M_{a_{r+1}}(X)^T X N_{a_{r+1}}(X) = M_{a_{r+1}}(X)^T \left([\overline{\Sigma} \;\; 0] + H\right) N_{a_{r+1}}(X) \\ = \;\; & [\overline{\Sigma}_{bb} \;\; 0] + [H_{bb} \;\; H_{bc}] + O(\|H\|^2) \,. \end{aligned} \quad (4.35)$$

Next, consider the general case that $\overline{X} \neq [\overline{\Sigma} \;\; 0]$. Let $(\overline{U}, \overline{V}) \in \mathcal{O}^{m,n}(\overline{X})$ be fixed. Then, we know that

$$\overline{U}^T X \overline{V} = [\overline{\Sigma} \;\; 0] + \overline{U}^T (X - \overline{X}) \overline{V} \,.$$

Denote $\widetilde{H} = \overline{U}^T (X - \overline{X}) \overline{V}$. It is clear that $\sigma(X) = \sigma(\overline{U}^T X \overline{V})$. Therefore, by replacing X by $\overline{U}^T X \overline{V}$ in the previous arguments, we know that there exists an open neighborhood $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$ of $(\bar{t}, \overline{X})$ such that the mappings

$$\boldsymbol{F}_l(X) = M_{a_l}(\overline{U}^T X \overline{V})^T \overline{U}^T X \overline{V} N_{a_l}(\overline{U}^T X \overline{V}) \in \Re^{|a_l| \times |a_l|}, \quad l = 1, \ldots, r$$

and

$$\boldsymbol{F}_{r+1}(X) = M_{a_{r+1}}(\overline{U}^T X \overline{V})^T \overline{U}^T X \overline{V} N_{a_{r+1}}(\overline{U}^T X \overline{V}) \in \Re^{|b| \times (|b| + |c|)}$$

are twice continuously differentiable on $\mathcal{N}_2$, and for any $X \in \mathcal{N}_2$, the matrices $\boldsymbol{F}_l(X)$, $l = 1, \ldots, r+1$ are diagonal, and the diagonal elements are the singular values $\{\sigma_i(X) : i \in a_l\}$. In particular, we have

$$\boldsymbol{F}_l(\overline{X}) = \overline{\Sigma}_{a_l a_l}, \; l = 1, \ldots, r \quad \text{and} \quad \boldsymbol{F}_{r+1}(\overline{X}) = [\overline{\Sigma}_{bb} \;\; 0] = 0 \,.$$

Thus,

$$\sum_{i \in a_l} \sigma_i(X) = \mathrm{tr}(\boldsymbol{F}_l(X)), \quad l = 1, \ldots, r \,. \tag{4.36}$$

Moreover, we obtain from (4.34) and (4.35) that for any $X \in \mathcal{N}_2$,

$$\boldsymbol{F}_l(X) - \boldsymbol{F}_l(\overline{X}) = \widetilde{H}_{a_l a_l} + O(\|X - \overline{X}\|^2), \quad l = 1, \ldots, r \tag{4.37}$$

and

$$\boldsymbol{F}_{r+1}(X) - \boldsymbol{F}_{r+1}(\overline{X}) = [\widetilde{H}_{bb} \;\; \widetilde{H}_{bc}] + O(\|X - \overline{X}\|^2) \,. \tag{4.38}$$

Finally, in order to show that $\mathcal{K}$ is $\mathcal{C}^2$-cone reducible at $(\bar{t}, \overline{X}) \in \mathrm{bd}\,\mathcal{K} \setminus (0, 0)$, we consider the following two cases.

Case 1. $\sigma_k(\overline{X}) > 0$. Let $0 \leq k_0 \leq k-1$ and $k \leq k_1 \leq m$ be the integers such that

$$\overline{\sigma}_1 \geq \ldots \geq \overline{\sigma}_{k_0} > \overline{\sigma}_{k_0+1} = \ldots = \overline{\sigma}_k = \ldots = \overline{\sigma}_{k_1} > \overline{\sigma}_{k_1+1} \geq \ldots \geq \overline{\sigma}_m \geq 0 \,,$$

which implies that $\alpha = \cup_{l=1}^{r_0} a_l$ and $\beta = a_{r_0+1}$. For each $l \in \{1, \ldots, r_0 + 1\}$, define the linear mapping $\boldsymbol{d}_l : \Re^{|a_l| \times |a_l|} \to \Re^{|a_l|}$ by

$$\boldsymbol{d}_l(Z) = (Z_{11}, Z_{22}, \ldots, Z_{|a_l||a_l|})^T, \quad Z \in \Re^{|a_l| \times |a_l|} \,. \tag{4.39}$$

Therefore, since $\|\overline{X}\|_{(k)} = \bar{t}$, we know from (4.36) that

$$\begin{aligned}
\mathcal{K} \cap \mathcal{N} &= \left\{ (t, X) \in \mathcal{N} \mid \sum_{i=1}^{k} \sigma_i(X) \leq t \right\} \\
&= \left\{ (t, X) \in \mathcal{N} \mid \sum_{i=1}^{k} (\sigma_i(X) - \overline{\sigma}_i) \leq t - \bar{t} \right\} \\
&= \left\{ (t, X) \in \mathcal{N} \mid \sum_{l=1}^{r_0} \left\langle e_{|a_l|}, \widetilde{\boldsymbol{d}}_l \right\rangle + s_{k-k_0}(\widetilde{\boldsymbol{d}}_{r_0+1}) \leq t - \bar{t} \right\} ,
\end{aligned}$$

where $\widetilde{\boldsymbol{d}}_l := \boldsymbol{d}_l(\boldsymbol{F}_l(X) - \boldsymbol{F}_l(\overline{X}))$, $l = 1, \ldots, r_0 + 1$ and $s_{(k-k_0)} : \Re^{|\beta|} \to \Re$ is the positively homogeneous convex function defined by (3.162). Therefore, we may locally define the mapping $\widetilde{\Xi} : \mathcal{N} \to \Re \times \Re^{|\beta|}$ by

$$\widetilde{\Xi}(t, X) = \left(t - \bar{t} - \sum_{l=1}^{r_0} \langle e_{|a_l|}, \widetilde{\boldsymbol{d}}_l \rangle, \widetilde{\boldsymbol{d}}_{r_0+1} \right) \in \Re \times \Re^{|\beta|}, \quad (t, X) \in \mathcal{N} .$$

Thus, we have

$$\mathcal{K} \cap \mathcal{N} = \left\{ (t, X) \in \mathcal{N} \mid \widetilde{\Xi}(t, X) \in \widetilde{\mathcal{C}} \right\} ,$$

where $\widetilde{\mathcal{C}} \subseteq \Re \times \Re^{|\beta|}$ is a closed polyhedral convex cone defined by

$$\widetilde{\mathcal{C}} := \left\{ (s, y) \in \Re \times \Re^{|\beta|} \mid s_{(k-k_0)}(y) \leq s \right\} .$$

Since any polyhedral convex set is $\mathcal{C}^2$-cone reducible, we know that $\widetilde{\mathcal{C}}$ is $\mathcal{C}^2$-cone reducible. Clearly, the mapping $\widetilde{\Xi}$ is twice continuously differentiable on $\mathcal{N}$. Moreover, we know from (4.37) that the derivative $\widetilde{\Xi}'(\bar{t}, \overline{X})$ of $\widetilde{\Xi}$ at $(\bar{t}, \overline{X})$ is given by

$$\widetilde{\Xi}'(\bar{t}, \overline{X})(\tau, H) = \left(\tau - \sum_{l=1}^{r_0} \mathrm{tr}(\widetilde{H}_{a_l a_l}), \boldsymbol{d}_{r_0+1}(\widetilde{H}_{\beta\beta}) \right) \in \Re \times \Re^{|\beta|}, \quad (\tau, H) \in \Re \times \Re^{m \times n} ,$$

where $\widetilde{H} = \overline{U}^T H \overline{V}$, which implies that $\widetilde{\Xi}'(\bar{t}, \overline{X}) : \Re \times \Re^{m \times n} \to \Re \times \Re^{|\beta|}$ is onto. Then, we know from [90, Proposition 3.2] that $\mathcal{K}$ is $\mathcal{C}^2$-cone reducible at $(\bar{t}, \overline{X})$.

Case 2. $\sigma_k(\overline{X}) = 0$. Let $0 \leq k_0 \leq k - 1$ be the integer such that

$$\overline{\sigma}_1 \geq \ldots \geq \overline{\sigma}_{k_0} > \overline{\sigma}_{k_0+1} = \ldots = \overline{\sigma}_k = \ldots = \overline{\sigma}_m = 0 ,$$

which implies that $\alpha = \cup_{l=1}^{r} a_l$ and $\beta = a_{r+1}$. Therefore, we know that

$$\mathcal{K} \cap \mathcal{N} = \left\{ (t, X) \in \mathcal{N} \mid \|\boldsymbol{F}_{r+1}(X)\|_{(k-k_0)} \leq t - \sum_{l=1}^{r} \mathrm{tr}(\boldsymbol{F}_l(X)) \right\} .$$

Define $\Xi : \mathcal{N} \to \Re \times \Re^{|b| \times (|b|+|c|)}$ by

$$\Xi(t, X) := \left(t - \sum_{l=1}^{r} \mathrm{tr}(\boldsymbol{F}_l(X)), \boldsymbol{F}_{r+1}(X) \right) \in \Re \times \Re^{|b| \times (|b|+|c|)}, \quad (t, X) \in \mathcal{N} .$$

Then,

$$\mathcal{K} \cap \mathcal{N} = \left\{ (t, X) \in \mathcal{N} \mid \Xi(t, X) \in \mathrm{epi} \| \cdot \|_{(k-k_0)} \right\}$$

Since $\bar{t} = \sum_{l=1}^{r} \mathrm{tr}(\boldsymbol{F}_l(\overline{X}))$ and $\boldsymbol{F}_{r+1}(\overline{X}) = 0$, we have $\Xi(\bar{t}, \overline{X}) = (0, 0)$. Also, the mapping Ξ is twice continuously differentiable on $\mathcal{N}$. Moreover, by (4.37) and (4.38), we know that the derivative $\Xi'(\bar{t}, \overline{X})$ of Ξ at $(\bar{t}, \overline{X})$ is given by

$$\Xi'(\bar{t}, \overline{X})(\tau, H) = \left(\tau - \sum_{l=1}^{r_0} \mathrm{tr}(\widetilde{H}_{a_l a_l}), [\widetilde{H}_{bb} \quad \widetilde{H}_{bc}] \right) \in \Re \times \Re^{|b| \times (|b|+|c|)}, \quad (\tau, H) \in \Re \times \Re^{m \times n},$$

where $\widetilde{H} = \overline{U}^T H \overline{V}$, which implies that $\Xi'(\bar{t}, \overline{X}) : \Re \times \Re^{m \times n} \to \Re \times \Re^{|b| \times (|b|+|c|)}$ is onto. Since the closed convex cone $\mathrm{epi} \| \cdot \|_{(k-k_0)} \subseteq \Re \times \Re^{|b| \times (|b| \times |c|)}$ is pointed, we obtain from the definition that $\mathcal{K}$ is $\mathcal{C}^2$-cone reducible at $(\bar{t}, \overline{X})$. $\square$

4.1.2 The critical cone

The metric projector $(\bar{t}, \overline{X}) = \Pi_{\mathcal{K}}(t, X)$ of $(t, X) \in \Re \times \Re^{m \times n}$ onto the cone $\mathcal{K}$ satisfies the following complementary condition:

$$\mathcal{K} \ni (\bar{t}, \overline{X}) \perp (t - \bar{t}, X - \overline{X}) \in \mathcal{K}^{\circ}. \tag{4.40}$$

The *critical cone* of $\mathcal{K}$ at $(t, X) \in \Re \times \Re^{m \times n}$, associated with the complementary problem (4.40), is defined as

$$\mathcal{C}_{\mathcal{K}}(t, X) := \mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X}) \cap (t - \bar{t}, X - \overline{X})^{\perp}.$$

Next, for the given $(t, X) \in \Re \times \Re^{m \times n}$, we want to characterize the critical cone $\mathcal{C}_{\mathcal{K}}(t, X)$ of $\mathcal{K}$.

If $(t, X) \in \mathrm{int}\, \mathcal{K}$, then it is clear that

$$\mathcal{C}_{\mathcal{K}}(t, X) = \mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X}) = \Re \times \Re^{m \times n}.$$

If $(t, X) \in \mathrm{bd}\, \mathcal{K}$, then $(t, X) = (\bar{t}, \overline{X})$,

$$\mathcal{C}_{\mathcal{K}}(t, X) = \mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X}),$$

where $\mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X})$, which is completely described by (4.1) and (4.2). Moreover, it is easy to see that the affine hull of $\mathcal{C}_{\mathcal{K}}(t, X)$ is

$$\mathrm{aff}(\mathcal{C}_{\mathcal{K}}(t, X)) = \Re \times \Re^{m \times n}.$$

If $(t, X) \in \operatorname{int} \mathcal{K}^\circ$, then $(\bar{t}, \overline{X}) = (0,0)$ and

$$\mathcal{C}_\mathcal{K}(t,X) = \mathcal{T}_\mathcal{K}(0,0) \cap (t,X)^\perp = \mathcal{K} \cap (t,X)^\perp = \{(0,0)\}\,.$$

Next, we consider the case that $(t,X) \notin \mathcal{K} \cup \operatorname{int}\mathcal{K}^\circ$.

Case 1. $\sigma_k(\overline{X}) > 0$. Then, $(\tau, H) \in \mathcal{C}(t,X)$ if and only if $(\tau,H) \in \Re \times \Re^{m\times n}$ satisfies

$$(\tau,H) \in \mathcal{T}_\mathcal{K}(\bar{t},\overline{X}) \quad \text{and} \quad \langle (\tau,H),(\zeta,\Gamma)\rangle = 0\,,$$

where $(\zeta,\Gamma) = (t-\bar{t}, X-\overline{X})$. Therefore, we know that the equality in (4.12) and (4.13) hold for (τ,H). Thus, we know that (τ,H) satisfies the following conditions.

(i) The symmetric matrix $S(\overline{U}_\beta^T H \overline{V}_\beta) \in \mathcal{S}^{|\beta|}$ has the block-diagonal structure, i.e., for any $l \neq l' \in \{r_0+1,\ldots,r_1\}$, $\left(S(\overline{U}_\beta^T H \overline{V}_\beta)\right)_{a_l a_{l'}} = 0$.

(ii) If $k_1 > k$, for any $i_1 \in \beta_1$, $i_2, i_{2'} \in \beta_2$ and $i_3 \in \beta_3$,

$$\lambda_{i_1}(S(\overline{U}_\beta^T H \overline{V}_\beta)) \geq \lambda_{i_2}(S(\overline{U}_\beta^T H \overline{V}_\beta)) = \ldots = \lambda_{i_{2'}}(S(\overline{U}_\beta^T H \overline{V}_\beta)) \geq \lambda_{i_3}(S(\overline{U}_\beta^T H \overline{V}_\beta))\,.$$

(iii) $\sum_{l=1}^{r_0} \operatorname{tr}(\overline{U}_{a_l}^T H \overline{V}_{a_l}) + \sum_{i=1}^{k-k_0} \lambda_i\left(S(\overline{U}_\beta^T H \overline{V}_\beta)\right) = \tau$.

Moreover, $(\tau,H) \in \operatorname{aff}(\mathcal{C}_\mathcal{K}(t,X))$ if and only if (τ,H) satisfies

(i) The symmetric matrix $S(\overline{U}_\beta^T H \overline{V}_\beta) \in \mathcal{S}^{|\beta|}$ has the block-diagonal structure, i.e., for any $l \neq l' \in \{r_0+1,\ldots,r_1\}$, $\left(S(\overline{U}_\beta^T H \overline{V}_\beta)\right)_{a_l a_{l'}} = 0$;

(ii) if $k_1 > k$, $\lambda_i(S(\overline{U}_\beta^T H \overline{V}_\beta)) = \ldots = \lambda_{i'}(S(\overline{U}_\beta^T H \overline{V}_\beta))$ for any $i, i' \in \beta_2$; if $k = k_1$, $\sum_{l=1}^{r_0} \operatorname{tr}(\overline{U}_{a_l}^T H \overline{V}_{a_l}) + \operatorname{tr}\left(S(\overline{U}_\beta^T H \overline{V}_\beta)\right) = \tau$.

Case 2. $\sigma_k(\overline{X}) = 0$. Then, $(\tau, H) \in \mathcal{C}(t,X)$ if and only if $(\tau,H) \in \Re \times \Re^{m\times n}$ satisfies

$$(\tau,H) \in \mathcal{T}_\mathcal{K}(\bar{t},\overline{X}) \quad \text{and} \quad \langle (\tau,H),(\zeta,\Gamma)\rangle = 0\,,$$

where $(\zeta,\Gamma) = (t-\bar{t}, X-\overline{X})$. Also, we know that the equality in (4.21) and (4.22) hold for (τ,H). Thus, (τ,H) should satisfy the following conditions.

(i) The matrix $[\overline{U}_\beta^T H \overline{V}_\beta \;\; \overline{U}_\beta^T H \overline{V}_2] \in \Re^{|\beta|\times(|\beta|+n-m)}$ has the following block-diagonal structure

$$[\overline{U}_\beta^T H \overline{V}_\beta \;\; \overline{U}_\beta^T H \overline{V}_2] = \begin{bmatrix} \overline{U}_{a_{r_0+1}}^T H \overline{V}_{a_{r_0+1}} & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & \overline{U}_{a_r}^T H \overline{V}_{a_r} & 0 & 0 \\ 0 & 0 & 0 & \overline{U}_b^T H \overline{V}_b & \overline{U}_b^T H \overline{V}_2 \end{bmatrix} \tag{4.41}$$

and the matrices $\overline{U}_{a_l}^T H \overline{V}_{a_l}$, $l = r_0+1, \ldots, r$ are symmetric.

(ii) Denote $\boldsymbol{h} := \left(\lambda(\overline{U}_{a_{r_0+1}}^T H \overline{V}_{a_{r_0+1}}), \ldots, \lambda(\overline{U}_{a_r}^T H \overline{V}_{a_r}), \sigma([\overline{U}_b^T H \overline{V}_b \;\; \overline{U}_b^T H \overline{V}_2])\right) \in \Re^{|\beta|}$. If $\sum_{i\in\beta} u_i = k - k_0$, then for any $i_1 \in \beta_1$, $i_2, i_{2'} \in \beta_2$ and $i_3 \in \beta_3$,

$$\boldsymbol{h}_{i_1} \geq \boldsymbol{h}_{i_2} = \ldots = \boldsymbol{h}_{i_{2'}} \geq \boldsymbol{h}_{i_3} \quad \text{and} \quad \boldsymbol{h}_{i_2} \geq 0\,;$$

if $\sum_{i\in\beta} u_i < k - k_0$, then $\boldsymbol{h}_{i_1} \geq 0$ for any $i_1 \in \beta_1$, $\boldsymbol{h}_{i_2} = 0$ for any $i_2 \in \beta_2 \cup \beta_3$.

(iii) $\sum_{j=1}^{r_0} \operatorname{tr}(\overline{U}_{a_j}^T H \overline{V}_{a_j}) + \sum_{i=1}^{k-k_0} \sigma_i \left([\overline{U}_\beta^T H \overline{V}_\beta \;\; \overline{U}_\beta^T H \overline{V}_2]\right) = \tau$.

Moreover, $(\tau, H) \in \operatorname{aff} \mathcal{C}_\mathcal{K}(t, X)$ if and only if (τ, H) satisfies

(i) The matrix $[\overline{U}_\beta^T H \overline{V}_\beta \;\; \overline{U}_\beta^T H \overline{V}_2] \in \Re^{|\beta|\times(|\beta|+n-m)}$ has the block-diagonal structure (4.41) and the matrices $\overline{U}_{a_l}^T H \overline{V}_{a_l}$, $l = r_0+1, \ldots, r$ are symmetric.

(ii) If $\sum_{i\in\beta} u_i = k - k_0$, then $\boldsymbol{h}_i = \ldots = \boldsymbol{h}_{i'}$ for any $i, i' \in \beta_2$; if $\sum_{i\in\beta} u_i < k - k_0$, then $\boldsymbol{h}_{i_2} = 0$ for any $i_2 \in \beta_2 \cup \beta_3$.

The following observation can be obtained from the characterization of the affine hull of $\mathcal{C}_\mathcal{K}(t, X)$ and the characterization of Clarke's generalized Jacobian of $\Pi_\mathcal{K}$ (Proposition 3.18).

Lemma 4.4. *Let $(t, X) \in \Re \times \Re^{m\times n}$ be given. For any $\boldsymbol{V} = (\boldsymbol{V}_0, \boldsymbol{V}_1) \in \partial\Pi_\mathcal{K}(t, X)$, we have*

$$(\boldsymbol{V}_0(\tau, H), \boldsymbol{V}_1(\tau, H)) \in \operatorname{aff}(\mathcal{C}_\mathcal{K}(t, X)) \quad \forall\, (\tau, H) \in \Re \times \Re^{m\times n}\,.$$

Proof. Without loss of generality, we may assume that $(t, X) \notin \mathcal{K} \cup \operatorname{int} \mathcal{K}^\circ$, since otherwise the result holds trivially (noting that if $(t, X) \in \operatorname{bd} \mathcal{K}$, $\operatorname{aff}(\mathcal{C}_\mathcal{K}(t, X)) = \Re \times \Re^{m \times n}$). On the other hand, since $\partial \Pi_\mathcal{K}(t, X) = \operatorname{conv}\{\partial_B \Pi_\mathcal{K}(t, X)\}$, we only need to show that for any fixed $\boldsymbol{V} = (\boldsymbol{V}_0, \boldsymbol{V}_1) \in \partial_B \Pi_\mathcal{K}(t, X)$ and $(\tau, H) \in \Re \times \Re^{m \times n}$,

$$(\boldsymbol{a}, \boldsymbol{A}) := (\boldsymbol{V}_0(\tau, H), \boldsymbol{V}_1(\tau, H)) \in \operatorname{aff}\left(\mathcal{C}_\mathcal{K}(t, X)\right) . \tag{4.42}$$

Denote $(\bar{t}, \overline{X}) = \Pi_\mathcal{K}(t, X)$ and $\widetilde{\boldsymbol{A}} := \overline{U}^T \boldsymbol{A} \overline{V} = \overline{U}^T \boldsymbol{V}_1(\tau, H) \overline{V}$. Consider the following two cases.

Case 1. $\sigma_k(\overline{X}) > 0$. For the fixed (t, X), let $0 \le k_0 \le k-1$ and $k \le k_1 \le m$ be the integers satisfying the condition (3.149). Let β_1, β_2 and β_3 be the index sets defined by (3.159) for (t, X). From (3.178) and the definition (3.160) of the linear mapping $\boldsymbol{T}$, we know that the symmetric matrix $S(\widetilde{\boldsymbol{A}}_{\beta\beta}) \in \mathcal{S}^{|\beta|}$ has the block-diagonal structure, i.e., for any $l \neq l' \in \{r_0 + 1, \ldots, r_1\}$, $S(\widetilde{\boldsymbol{A}})_{a_l a_{l'}} = 0$.

If $k_1 > k$, since the singular value function $\sigma(\cdot)$ is globally Lipschitz continuous over $\Re^{m \times n}$, we know from the part (i) of Lemma 3.15 (see [113, Lemma 4.2] for details) that if $(t', X') \in \Re \times \Re^{m \times n}$ sufficiently close to the given point (t, X), then $\sigma_k(\overline{X}') > 0$, $k_1' > k$ and

$$k_0' \in \beta_1 \quad (k_0' \equiv k_0 \text{ if } \beta_1 = \emptyset) \quad \text{and} \quad k_1' \in \beta_3 \quad (k_1' \equiv k_1 \text{ if } \beta_3 = \emptyset)\,, \tag{4.43}$$

where $(\bar{t}', \overline{X}') = \Pi_\mathcal{K}(t', X')$, and $0 \le k_0' \le k-1$ and $k \le k_1' \le m$ are two integers defined by (3.149) with respected to $\overline{X}'$. Assume that $(t', X') \in \mathcal{D}_{\Pi_\mathcal{K}}$ converging to (t, X), where $\mathcal{D}_{\Pi_\mathcal{K}}$ is the set of points in $\Re \times \Re^{m \times n}$ where $\Pi_\mathcal{K}$ is differentiable. By the definition of $\partial_B \Pi_\mathcal{K}$, Proposition 3.17 (ii) and (3.172), we know from (3.173) and (4.43) that

$$S(\widetilde{\boldsymbol{A}}_{\beta_2\beta_2}) = cI_{|\beta_2|}\,,$$

for some $c \in \Re$. Therefore, we obtain that

$$\lambda_i(S(\widetilde{\boldsymbol{A}}_{\beta_2\beta_2})) = \lambda_j(S(\widetilde{\boldsymbol{A}}_{\beta_2\beta_2})) \quad \forall\, i, j \in \beta_2\,.$$

If $k_1 = k$, since the singular value function $\sigma(\cdot)$ is globally Lipschitz continuous, we obtain similarly from the part (i) of Lemma 3.15 (see [113, Lemma 4.2] for details) that if (t', X') sufficiently close to the given point (t, X), then $\sigma_k(\overline{X}') > 0$, $k_1' \equiv k$ and

$$k_0' \in \beta_1 \quad (k_0' \equiv k_0 \text{ if } \beta_1 = \emptyset)\,, \tag{4.44}$$

where $(\bar{t}', \overline{X}') := \Pi_{\mathcal{K}}(t', X')$, and $0 \le k_0' \le k-1$ and $k \le k_1' \le m$ are two integers defined by (3.149) with respected to $\overline{X}'$. Assume that $(t', X') \in \mathcal{D}_{\Pi_{\mathcal{K}}}$ converging to (t, X). By the definition of $\partial_B \Pi_{\mathcal{K}}$, Proposition 3.17 (iii) and (3.174), we know from (3.175) and (4.44) that

$$\sum_{l=1}^{r_0} \mathrm{tr}(\widetilde{\boldsymbol{A}}_{a_l a_l}) + \mathrm{tr}(S(\widetilde{\boldsymbol{A}}_{\beta\beta})) = \boldsymbol{a}\,.$$

Therefore, from the obtained characterization of aff$\,(\mathcal{C}_{\mathcal{K}}(t, X))$, we know that (4.42) holds.

Case 2. $\sigma_k(\overline{X}) = 0$. For the fixed (t, X), let $0 \le k_0 \le k-1$ be the integer satisfying the condition (3.153). Let β_1, β_2 and β_3 be the index sets defined by (3.166) for (t, X) and $u \in \Re^m_+$ be the vector satisfying the condition (3.152). From (3.179) and the definition (3.167) of the linear mapping $\boldsymbol{T}$, we know that $[\widetilde{\boldsymbol{A}}_{\beta\beta} \quad \widetilde{\boldsymbol{A}}_{\beta c}]$ has the block-diagonal structure (4.41) and the blocks $\widetilde{\boldsymbol{A}}_{a_l a_l}$, $l = 1, \ldots, r$ are symmetric.

If $\sum_{i \in \beta} u_i < k - k_0$, then since the single value function $\sigma(\cdot)$ is globally Lipschitz continuous, we obtain that for $(t', X') \in \Re \times \Re^{m \times n}$ sufficiently close to the given point (t, X), there exist a positive number $\theta' > 0$ and a integer $k_0' \in \beta_1$ ($k_0' \equiv k_0$ if $\beta_1 = \emptyset$) such that

$$\sigma_{k_0'}(\overline{X}') > \theta' \ge \sigma_{k_0'+1}(\overline{X}') \quad \text{and} \quad \theta' > \frac{1}{k_0'+1} \sum_{i=k_0'+1}^{m} \sigma_i(X')\,,$$

where $\theta' = (\sum_{i=1}^{k_0'} \sigma_i(X') - t')/(k_0'+1) > 0$ (see [113, Lemma 4.1] for details). Thus, we know from [113, Lemma 4.1] that $\sigma_k(\overline{X}') = 0$, where $(\bar{t}', \overline{X}') = \Pi_{\mathcal{K}}(t', X')$ and $0 \le k_0' \le k-1$ is the integer defined by (3.153) with respected to $\overline{X}'$. Assume that $(t', X') \in \mathcal{D}_{\Pi_{\mathcal{K}}}$ converging to (t, X). By the definition of $\partial_B \Pi_{\mathcal{K}}$, Proposition 3.17 (iv) and (3.176), from (3.177) and $k_0' \in \beta_1$, we obtain that $\boldsymbol{h}_{i_2} = 0$ for any $i_2 \in \beta_2 \cup \beta_3$, where $\boldsymbol{h} = \Big(\lambda(\widetilde{\boldsymbol{A}}_{a_1a_1}), \ldots, \lambda(\widetilde{\boldsymbol{A}}_{a_ra_r}), \sigma([\widetilde{\boldsymbol{A}}_{bb} \;\; \widetilde{\boldsymbol{A}}_{bc}])\Big) \in \Re^{|\beta|}$.

If $\sum_{i\in\beta} u_i = k - k_0$, then since $(t, X) \notin \operatorname{int} \mathcal{K}$, we know from [113, Lemma 4.1] that

$$\theta := \frac{1}{k_0+1}(\sum_{i=1}^{k_0} \sigma_i(X) - t) = \frac{1}{k-k_0} \sum_{i\in\beta_1\cup\beta_2} \sigma_i(X) > 0$$

and $\sigma_{k_0}(X) > \theta \geq \sigma_{k_0+1}(X)$,

$$\sigma_{i_1}(X) > \sigma_{i_2}(X), \quad \sigma_i(X) = 0 \quad \forall\, i_1 \in \beta_1,\ i_2 \in \beta_2,\ i_3 \in \beta_3\,,$$

which implies that $\beta_3 = b$. Therefore, by the globally Lipschitz continuity of the single value function $\sigma(\cdot)$, we obtain that for (t', X') sufficiently close to (t, X), if $\sigma_k(\overline{X}') = 0$, then

$$k_0' \in \beta_1 \quad (k_0' \equiv k_0 \text{ if } \beta_1 = \emptyset)\,,$$

where $(\bar{t}', \overline{X}') = \Pi_{\mathcal{K}}(t', X')$, and $0 \leq k_0' \leq k-1$ is the integer defined by (3.153) with respected to $\overline{X}'$; if $\sigma_k(\overline{X}') > 0$, then $k_1' > k$,

$$k_0' \in \beta_1 \quad (k_0' \equiv k_0 \text{ if } \beta_1 = \emptyset) \quad \text{and} \quad k_1' \in \beta_3 \quad (k_1' \equiv m \text{ if } \beta_3 = \emptyset)\,,$$

where $(\bar{t}', \overline{X}') = \Pi_{\mathcal{K}}(t', X')$, and $0 \leq k_0' \leq k-1$ and $k \leq k_1' \leq m$ are two integers defined by (3.149) with respected to $\overline{X}'$. By taking subsequence if necessary, we may assume that for the sequence $\{(t^{(q)}, X^{(q)})\}$ which converges to (t, X), either $\sigma_k(X^{(q)}) = 0$ or $\sigma_k(X^{(q)}) > 0$ for all q. Therefore, if $\sigma_k(X^{(q)}) = 0$ for all q, then by the definition of $\partial_B \Pi_{\mathcal{K}}$, Proposition 3.17 (iv) and (3.176), from (3.177) and $k_0' \in \beta_1$, we obtain that $\boldsymbol{h}_{i_2} = 0$ for any $i_2 \in \beta_2 \cup \beta_3$; if $\sigma_k(X^{(q)}) > 0$ for all q, then by the definition of $\partial_B \Pi_{\mathcal{K}}$, Proposition 3.17 (ii) and (3.172), we know from (3.173) and (4.43) that

$$\boldsymbol{h}_i = \ldots = \boldsymbol{h}_{i'} \quad \forall\, i, i' \in \beta_2\,,$$

where $\boldsymbol{h} = \Big(\lambda(\widetilde{\boldsymbol{A}}_{a_1a_1}), \ldots, \lambda(\widetilde{\boldsymbol{A}}_{a_ra_r}), \sigma([\widetilde{\boldsymbol{A}}_{bb} \;\; \widetilde{\boldsymbol{A}}_{bc}])\Big) \in \Re^{|\beta|}$. Therefore, from the obtained characterization of $\operatorname{aff}(\mathcal{C}_{\mathcal{K}}(t, X))$, we know that (4.42) holds in this case. The proof is completed. □

The following result plays an important role in our subsequent analysis.

Proposition 4.5. *Suppose that* $(\bar{t},\overline{X}) \in \mathcal{K}$ *and* $(\bar{\zeta},\overline{\Gamma}) \in \mathcal{K}^\circ$ *satisfy* $\langle(\bar{t},\overline{X}),(\bar{\zeta},\overline{\Gamma})\rangle = 0$. *Let* $(t,X) = (\bar{t},\overline{X}) + (\bar{\zeta},\overline{\Gamma}) \in \Re\times\Re^{m\times n}$. *Then for any* $\boldsymbol{V} \in \partial\Pi_{\mathcal{K}}(t,X)$ *and* $(\triangle t,\triangle X),(\triangle\zeta,\triangle\Gamma) \in \Re\times\Re^{m\times n}$ *such that* $(\triangle t,\triangle X) = \boldsymbol{V}(\triangle t+\triangle\zeta,\triangle X+\triangle\Gamma)$, *it holds that*

$$\langle(\triangle t,\triangle X),(\triangle\zeta,\triangle\Gamma)\rangle \geq -\Upsilon_{(\bar{t},\overline{X})}\left((\bar{\zeta},\overline{\Gamma}),(\triangle t,\triangle X)\right), \tag{4.45}$$

where the linear quadratic function $\Upsilon_{(\bar{t},\overline{X})}(\cdot,\cdot)$ *is defined in Definition 4.1.*

Proof. By the assumption, we know that $(\bar{t},\overline{X}) = \Pi_{\mathcal{K}}(t,X)$ and $(\bar{\zeta},\overline{\Gamma}) = \Pi_{\mathcal{K}^\circ}(t,X)$. Without loss of generality, assume that $(t,X) \notin \operatorname{int}\mathcal{K}\cup\operatorname{int}\mathcal{K}^\circ$, since otherwise the result holds trivially.

Suppose that $X \in \Re^{m\times n}$ has the singular value decomposition (3.155), i.e., $X = \overline{U}[\Sigma(X)\ \ 0]\overline{V}^T$ with $\overline{U} \in \mathcal{O}^m$ and $\overline{V} \in \mathcal{O}^n$. Let a_l, $l = 1,\ldots,r$ and $a_{r+1} = b$ be the corresponding index sets. Denote $\overline{\sigma} = \sigma(\overline{X})$. Consider the following two cases.

Case 1. $\overline{\sigma}_k > 0$. There exist two integers $0 \leq k_0 \leq k-1$ and $k \leq k_1 \leq m$ such that

$$\overline{\sigma}_1 \geq \ldots \geq \overline{\sigma}_{k_0} > \overline{\sigma}_{k_0+1} = \ldots = \overline{\sigma}_k = \ldots = \overline{\sigma}_{k_1} > \overline{\sigma}_{k_1+1} \geq \ldots \geq \overline{\sigma}_m \geq 0\,.$$

Denote $\alpha = \{1,\ldots,k_0\}$ and $\beta = \{k_0,\ldots,k_1\}$. Let $r_0, r_1 \in \{1,\ldots,r\}$ be the integers such that $\alpha = \cup_{l=1}^{r_0} a_l$ and $\beta = \cup_{l=r_0+1}^{r_1} a_l$. Since $(\bar{t},\overline{X}) = \Pi_{\mathcal{K}}(t,X)$ and $(\bar{\zeta},\overline{\Gamma}) = \Pi_{\mathcal{K}^\circ}(t,X) = (t,X)-(\bar{t},\overline{X})$, we know from the part (i) of Lemma 3.15 that there exist $\theta > 0$ and $\overline{u} \in \Re^m_+$ such that

$$\bar{\zeta} = t-\bar{t} = -\theta \quad \text{and} \quad \overline{\Gamma} = X-\overline{X} = \overline{U}[\operatorname{diag}(\theta\overline{u})\ \ 0]\overline{V}^T$$

with $u_i = 1$, $i = 1,\ldots,k_0$, $u_i = 0$, $i = k_1+1,\ldots,m$,

$$1 \geq u_{k_0+1} \geq u_{k_0+2} \geq \ldots \geq u_{k_1} \geq 0 \quad \text{and} \quad \sum_{i=1}^{k_1-k_0} u_{k_0+i} = k-k_0\,.$$

Therefore, we know that $\overline{\sigma}_i = \overline{\sigma}_j \equiv \overline{\nu}_l$ for any $i,j \in a_l$, $l = 1,\ldots,r+1$ and $\overline{u}_i = \overline{u}_j \equiv \overline{\mu}_l$ for any $i,j \in a_l$, $l = r_0+1,\ldots,r_1$ (noting that $\overline{\nu}_l \equiv \overline{\sigma}_k$ for any $l = r_0 +$

$1,\ldots,r_1$). Denote $\gamma = \{k_1+1,\ldots,m\}$ and $\overline{\gamma} := \{1,\ldots,m\}\setminus\gamma$. Let $\triangle\widetilde{X} = \overline{U}^T\triangle X\overline{V} = [\overline{U}^T\triangle X\overline{V}_1 \;\; \overline{U}^T\triangle X\overline{V}_2] = [\triangle\widetilde{X}_1 \;\; \triangle\widetilde{X}_2]$ and $\triangle\widetilde{\Gamma} = \overline{U}^T\triangle\Gamma\overline{V} = [\overline{U}^T\triangle\Gamma\overline{V}_1 \;\; \overline{U}^T\triangle\Gamma\overline{V}_2] = [\triangle\widetilde{\Gamma}_1 \;\; \triangle\widetilde{\Gamma}_2]$. Since $(\triangle t, \triangle X) = \boldsymbol{V}(\triangle t+\triangle\zeta, \triangle X+\triangle\Gamma)$, we know from Proposition 3.18 that there exists $\boldsymbol{K} = (\boldsymbol{K}_0, \boldsymbol{K}_1, \ldots, \boldsymbol{K}_{r_1}) \in \partial\Pi_{\mathcal{C}_1}(0,0)$ such that $\triangle t = \boldsymbol{K}_0(\triangle t + \triangle\zeta, \boldsymbol{D}(\triangle\widetilde{X}+\triangle\widetilde{\Gamma}))$ and

$$
\begin{aligned}
\triangle\widetilde{X} &= \boldsymbol{T}(\triangle\widetilde{X}+\triangle\widetilde{\Gamma}) \\
&\quad + \begin{bmatrix} \boldsymbol{K}_1(\triangle t+\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{X}+\triangle\widetilde{\Gamma})) & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & 0 & \boldsymbol{K}_{r_1}(\triangle t+\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{X}+\triangle\widetilde{\Gamma})) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},
\end{aligned}
$$

where $\boldsymbol{D}(\triangle\widetilde{X}+\triangle\widetilde{\Gamma}) := \Big(S(\triangle\widetilde{X}_{a_1a_1}+\triangle\widetilde{\Gamma}_{a_1a_1}),\ldots,S(\triangle\widetilde{X}_{a_{r_1}a_{r_1}}+\triangle\widetilde{\Gamma}_{a_{r_1}a_{r_1}})\Big)$, and the linear mapping $\boldsymbol{T}$ is defined by (3.160). Therefore, we have

$$S(\triangle\widetilde{X}_{a_la_l}) = \boldsymbol{K}_l(\triangle t+\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{X}+\triangle\widetilde{\Gamma})), \quad l=1,\ldots,r_1\,, \tag{4.46}$$

$$S(\triangle\widetilde{\Gamma}_{a_la_{l'}}) = 0, \quad l\neq l' \text{ and } l,l'=1,\ldots,r_0\,, \tag{4.47}$$

$$S(\triangle\widetilde{X}_{a_la_{l'}}) = 0, \quad l\neq l' \text{ and } l,l'=r_0+1,\ldots,r_1\,, \tag{4.48}$$

$$S(\triangle\widetilde{X}_{\alpha\beta}) - (\mathcal{E}_1)_{\alpha\beta}\circ S(\triangle\widetilde{X}_{\alpha\beta}) = (\mathcal{E}_1)_{\alpha\beta}\circ S(\triangle\widetilde{\Gamma}_{\alpha\beta})\,, \tag{4.49}$$

$$S(\triangle\widetilde{X}_{\alpha\gamma}) - (\mathcal{E}_1)_{\alpha\gamma}\circ S(\triangle\widetilde{X}_{\alpha\gamma}) = (\mathcal{E}_1)_{\alpha\gamma}\circ S(\triangle\widetilde{\Gamma}_{\alpha\gamma})\,, \tag{4.50}$$

$$S(\triangle\widetilde{X}_{\beta\gamma}) - (\mathcal{E}_1)_{\beta\gamma}\circ S(\triangle\widetilde{X}_{\beta\gamma}) = (\mathcal{E}_1)_{\beta\gamma}\circ S(\triangle\widetilde{\Gamma}_{\beta\gamma})\,, \tag{4.51}$$

$$T(\triangle\widetilde{X}) - \widehat{\mathcal{E}}_2\circ T(\triangle\widetilde{X}) = \widehat{\mathcal{E}}_2\circ T(\triangle\widetilde{\Gamma})\,, \tag{4.52}$$

$$\triangle\widetilde{X}_{\overline{\gamma}c} - \mathcal{F}_{\overline{\gamma}c}\circ\triangle\widetilde{X}_{\overline{\gamma}c} = \mathcal{F}_{\overline{\gamma}c}\circ\triangle\widetilde{\Gamma}_{\overline{\gamma}c}\,, \tag{4.53}$$

$$\begin{bmatrix}\triangle\widetilde{\Gamma}_{\gamma\gamma} & \triangle\widetilde{\Gamma}_{\gamma c}\end{bmatrix} = 0\,, \tag{4.54}$$

where $\widehat{\mathcal{E}}_2 = \begin{bmatrix} (\mathcal{E}_2)_{\alpha\alpha} & (\mathcal{E}_2)_{\alpha\beta} & (\mathcal{E}_2)_{\alpha\gamma} \\ (\mathcal{E}_2)_{\beta\alpha} & (\mathcal{E}_2)_{\beta\beta} & (\mathcal{E}_2)_{\beta\gamma} \\ (\mathcal{E}_2)_{\gamma\alpha} & (\mathcal{E}_2)_{\gamma\beta} & 0 \end{bmatrix}$. By (4.46), we have

$$(\triangle t, \boldsymbol{D}(\triangle \widetilde{X})) = \boldsymbol{K}(\triangle t + \triangle\zeta, \boldsymbol{D}(\triangle \widetilde{X}) + \boldsymbol{D}(\triangle\widetilde{\Gamma}))\,.$$

Therefore, by (c) of [64, Proposition 1], we obtain that

$$\begin{aligned} & \triangle t\triangle\zeta + \sum_{l=1}^{r_1}\Big\langle S(\triangle\widetilde{X}_{a_la_l}), S(\triangle\widetilde{\Gamma}_{a_la_l})\Big\rangle = \triangle t\triangle\zeta + \Big\langle \boldsymbol{D}(\triangle\widetilde{X}), \boldsymbol{D}(\triangle\widetilde{\Gamma})\Big\rangle \\ = \ & \Big\langle \boldsymbol{K}(\triangle t+\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{X}+\triangle\widetilde{\Gamma})), (\triangle t+\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{X}+\triangle\widetilde{\Gamma})) - \boldsymbol{K}(\triangle t+\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{X}+\triangle\widetilde{\Gamma}))\Big\rangle \\ \geq \ & 0 \end{aligned}$$

Therefore, by (4.47), (4.48) and (4.54), we have

$$\begin{aligned} & \triangle t\triangle\zeta + \langle \triangle\widetilde{X}_1, \triangle\widetilde{\Gamma}_1\rangle + \langle \triangle\widetilde{X}_2, \triangle\widetilde{\Gamma}_2\rangle \\ = \ & \triangle t\triangle\zeta + \langle S(\triangle\widetilde{X}_1), S(\triangle\widetilde{\Gamma}_1)\rangle + \langle T(\triangle\widetilde{X}_1), T(\triangle\widetilde{\Gamma}_1)\rangle + \langle \triangle\widetilde{X}_{\overline{\gamma}c}, \triangle\widetilde{\Gamma}_{\overline{\gamma}c}\rangle \\ \geq \ & 2\Big(\langle S(\triangle\widetilde{X}_{\alpha\beta}), S(\triangle\widetilde{\Gamma}_{\alpha\beta})\rangle + \langle S(\triangle\widetilde{X}_{\alpha\gamma}), S(\triangle\widetilde{\Gamma}_{\alpha\gamma})\rangle + \langle S(\triangle\widetilde{X}_{\beta\gamma}), S(\triangle\widetilde{\Gamma}_{\beta\gamma})\rangle\Big) \\ & + \langle T(\triangle\widetilde{X}), T(\triangle\widetilde{\Gamma})\rangle + \langle \triangle\widetilde{X}_{\overline{\gamma}c}, \triangle\widetilde{\Gamma}_{\overline{\gamma}c}\rangle\,. \end{aligned} \tag{4.55}$$

By (4.49), (4.50) and (4.51), we have

$$\begin{aligned} \langle S(\triangle\widetilde{X}_{\alpha\beta}), S(\triangle\widetilde{\Gamma}_{\alpha\beta})\rangle &= \sum_{l=1}^{r_0}\sum_{l'=r_0+1}^{r_1} \frac{\theta}{\overline{\nu}_l - \overline{\nu}_{l'}}\|S(\triangle\widetilde{X}_{a_la_{l'}})\|^2 - \frac{\theta\overline{\mu}_{l'}}{\overline{\nu}_l - \overline{\nu}_{l'}}\|S(\triangle\widetilde{X}_{a_la_{l'}})\|^2\,, \\ \langle S(\triangle\widetilde{X}_{\alpha\gamma}), S(\triangle\widetilde{\Gamma}_{\alpha\gamma})\rangle &= \sum_{l=1}^{r_0}\sum_{l'=r_1+1}^{r+1} \frac{\theta}{\overline{\nu}_l - \overline{\nu}_{l'}}\|S(\triangle\widetilde{X}_{a_la_{l'}})\|^2\,, \\ \langle S(\triangle\widetilde{X}_{\beta\gamma}), S(\triangle\widetilde{\Gamma}_{\beta\gamma})\rangle &= \sum_{l=r_0+1}^{r_1}\sum_{l'=r_1+1}^{r+1} -\frac{\theta\overline{\mu}_l}{\overline{\nu}_{l'} - \overline{\nu}_l}\|S(\triangle\widetilde{X}_{a_la_{l'}})\|^2\,, \end{aligned}$$

which implies

$$
\begin{aligned}
& 2\left(\langle S(\triangle\widetilde{X}_{\alpha\beta}), S(\triangle\widetilde{\Gamma}_{\alpha\beta})\rangle + \langle S(\triangle\widetilde{X}_{\alpha\gamma}), S(\triangle\widetilde{\Gamma}_{\alpha\gamma})\rangle + \langle S(\triangle\widetilde{X}_{\beta\gamma}), S(\triangle\widetilde{\Gamma}_{\beta\gamma})\rangle\right) \\
= & -2\sum_{l=1}^{r_0}\sum_{l'=r_0+1}^{r+1}\frac{\theta}{\overline{\nu}_{l'}-\overline{\nu}_l}\|S(\triangle\widetilde{X}_{a_la_{l'}})\|^2 \\
& -2\sum_{l'=r_0+1}^{r_1}\left(\sum_{l=1}^{r_0}\frac{\theta\overline{\mu}_{l'}}{\overline{\nu}_l-\overline{\nu}_{l'}}\|S(\triangle\widetilde{X}_{a_la_{l'}})\|^2 + \sum_{l=r_1+1}^{r+1}\frac{\theta\overline{\mu}_{l'}}{\overline{\nu}_l-\overline{\nu}_{l'}}\|S(\triangle\widetilde{X}_{a_la_{l'}})\|^2\right).
\end{aligned}
$$

Similarly, by (4.53), we know that

$$
\begin{aligned}
\langle\triangle\widetilde{X}_{\overline{\gamma}c}, \triangle\widetilde{\Gamma}_{\overline{\gamma}c}\rangle &= \langle\triangle\widetilde{X}_{\alpha c}, \triangle\widetilde{\Gamma}_{\alpha c}\rangle + \langle\triangle\widetilde{X}_{\beta c}, \triangle\widetilde{\Gamma}_{\beta c}\rangle \\
&= -\sum_{l=1}^{r_0}\frac{\theta}{-\overline{\nu}_l}\|\triangle\widetilde{X}_{a_lc}\|^2 - \sum_{l'=r_0+1}^{r_1}\frac{\theta\overline{\mu}_{l'}}{-\overline{\nu}}\|\triangle\widetilde{X}_{a_{l'}c}\|^2 .
\end{aligned}
$$

By (4.52), we obtain that

$$
\begin{aligned}
& \langle T(\triangle\widetilde{X}), T(\triangle\widetilde{\Gamma})\rangle \\
= & \langle T(\triangle\widetilde{X}_{\alpha\alpha}), T(\triangle\widetilde{\Gamma}_{\alpha\alpha})\rangle + \langle T(\triangle\widetilde{X}_{\beta\beta}), T(\triangle\widetilde{\Gamma}_{\beta\beta})\rangle \\
& +2\left(\langle T(\triangle\widetilde{X}_{\alpha\beta}), T(\triangle\widetilde{\Gamma}_{\alpha\beta})\rangle + \langle T(\triangle\widetilde{X}_{\alpha\gamma}), T(\triangle\widetilde{\Gamma}_{\alpha\gamma})\rangle + \langle T(\triangle\widetilde{X}_{\beta\gamma}), T(\triangle\widetilde{\Gamma}_{\beta\gamma})\rangle\right) \\
= & -2\sum_{l=1}^{r_0}\sum_{l'=l}^{r_0}\frac{\theta}{-\overline{\nu}_l-\overline{\nu}_{l'}}\|T(\triangle\widetilde{X}_{a_la_{l'}})\|^2 - 2\sum_{l=r_0+1}^{r_1}\sum_{l'=r_0+1}^{r_1}\frac{\theta\overline{\mu}_l}{-2\overline{\nu}}\|T(\triangle\widetilde{X}_{a_la_{l'}})\|^2 \\
& -2\sum_{l=1}^{r_0}\sum_{l'=r_0+1}^{r_1}\frac{\theta+\theta\overline{\mu}_{l'}}{-\overline{\nu}_l-\overline{\nu}}\|T(\triangle\widetilde{X}_{a_la_{l'}})\|^2 - 2\sum_{l=1}^{r_0}\sum_{l'=r_1+1}^{r+1}\frac{\theta}{-\overline{\nu}_l-\overline{\nu}_{l'}}\|T(\triangle\widetilde{X}_{a_la_{l'}})\|^2 \\
& -2\sum_{l=r_0+1}^{r_1}\sum_{l'=r_1+1}^{r+1}\frac{\theta\overline{\mu}_l}{-\overline{\nu}_{l'}-\overline{\nu}}\|T(\triangle\widetilde{X}_{a_la_{l'}})\|^2 \\
= & -2\sum_{l=1}^{r_0}\sum_{l'=l}^{r+1}\frac{\theta}{-\overline{v}_l-\overline{v}_{l'}}\|T(\triangle\widetilde{X}_{a_la_{l'}})\|^2 - 2\sum_{l=r_0+1}^{r_1}\sum_{l'=1}^{r+1}\frac{\theta\overline{\mu}_l}{-\overline{\nu}_{l'}-\overline{\nu}}\|T(\triangle\widetilde{X}_{a_la_{l'}})\|^2 .
\end{aligned}
$$

On the other hand, since $\overline{\zeta} = -\theta$, from the direct calculation, we know that

$$
\begin{aligned}
& -\bar{\zeta}\sum_{j=1}^{r_0} \mathrm{tr}\left(2\overline{P}_{a_j}^T\left[\mathcal{B}(\triangle X)(\mathcal{B}(\overline{X})-\bar{\nu}_j I_{m+n})^{\dagger}\mathcal{B}(\triangle X)\right]\overline{P}_{a_j}\right) \\
= \; & 2\sum_{l=1}^{r_0}\sum_{l'=r_0+1}^{r+1}\frac{\theta}{\overline{\nu}_{l'}-\overline{\nu}_l}\|S(\triangle\widetilde{X}_{a_l a_{l'}})\|^2 + 2\sum_{l=1}^{r_0}\sum_{l'=l}^{r+1}\frac{\theta}{-\overline{v}_l-\overline{v}_{l'}}\|T(\triangle\widetilde{X}_{a_l a_{l'}})\|^2 \\
& +\sum_{l=1}^{r_0}\frac{\theta}{-\overline{\nu}_l}\|\triangle\widetilde{X}_{a_l c}\|^2
\end{aligned}
$$

and

$$
\begin{aligned}
& \left\langle \Sigma_{\beta\beta}(\overline{\Gamma}), 2\overline{P}_\beta^T\mathcal{B}(\triangle X)(\mathcal{B}(\overline{X})-\overline{\nu}I_{m+n})^{\dagger}\mathcal{B}(\triangle X)\overline{P}_\beta\right\rangle \\
= \; & 2\sum_{l'=r_0+1}^{r_1}\left(\sum_{l=1}^{r_0}\frac{\theta\overline{\mu}_{l'}}{\overline{\nu}_l-\overline{\nu}}\|S(\triangle\widetilde{X}_{a_l a_{l'}})\|^2 + \sum_{l=r_1+1}^{r+1}\frac{\theta\overline{\mu}_{l'}}{\overline{\nu}_l-\overline{\nu}}\|S(\triangle\widetilde{X}_{a_l a_{l'}})\|^2\right) \\
& +2\sum_{l=r_0+1}^{r_1}\sum_{l'=1}^{r+1}\frac{\theta\overline{\mu}_l}{-\overline{\nu}_{l'}-\overline{\nu}}\|T(\triangle\widetilde{X}_{a_l a_{l'}})\|^2 + \sum_{l'=r_0+1}^{r_1}\frac{\theta\overline{\mu}_{l'}}{-\overline{\nu}}\|\triangle\widetilde{X}_{a_{l'}c}\|^2 .
\end{aligned}
$$

Finally, by combining with (4.55), we know that the inequality (4.45) holds.

Case 2. $\overline{\sigma}_k = 0$. There exists an integer $0 \le k_0 \le k-1$ such that

$$
\overline{\sigma}_1 \ge \cdots \ge \overline{\sigma}_{k_0} > \overline{\sigma}_{k_0+1} = \ldots = \overline{\sigma}_k = \ldots = \overline{\sigma}_m = 0 .
$$

Again, define $\alpha = \{1,\ldots,k_0\}$ and $\beta = \{k_0,\ldots,m\}$. Since $(\bar{t},\overline{X}) = \Pi_{\mathcal{K}}(t,X)$ and $(\bar{\zeta},\overline{\Gamma}) = \Pi_{\mathcal{K}^\circ}(t,X) = (t,X)-(\bar{t},\overline{X})$, we know from the part (ii) of Lemma 3.15 that there exist $\theta > 0$ and $u \in \Re^m_+$ such that

$$
\overline{\zeta} = t - \bar{t} \quad \text{and} \quad \overline{\Gamma} = X - \overline{X} = \overline{U}[\mathrm{diag}\,(\theta\overline{u}) \;\; 0]\overline{V}^T
$$

with

$$
u_\alpha = e, \quad u_\beta = u_\beta^{\downarrow} \quad \text{and} \quad \sum_{i\in\beta} u_i \le k - k_0 .
$$

Let $r_0 \in \{1,\ldots,r\}$ be the integer such that

$$
\alpha = \bigcup_{l=1}^{r_0} a_l, \quad \beta = \bigcup_{l=r_0+1}^{r+1} a_l \quad (\text{where } a_{r+1} = b) .
$$

Define

$$\beta_1 := \{i \in \beta \,|\, u_i = 1\}, \quad \beta_2 := \{i \in \beta \,|\, 0 < u_i < 1\} \quad \text{and} \quad \beta_3 := \{i \in \beta \,|\, u_i = 0\}\,.$$

Then, we know that $\beta_1 \cup \beta_2 = \bigcup_{l=r_0+1}^{r} a_l$ and $\beta_3 = a_{r+1} = b$. Therefore, we know that $\overline{\sigma}_i = \overline{\sigma}_j \equiv \overline{\nu}_l$ for any $i, j \in a_l$, $l = 1, \ldots, r_0$, $\overline{\sigma}_i = 0$ for any $i \in \beta$, and $\overline{u}_i = \overline{u}_j \equiv \overline{\mu}_l$ for any $i, j \in a_l$, $l = r_0 + 1, \ldots, r + 1$.

Similarly, let $\triangle\widetilde{X} = \overline{U}^T \triangle X \overline{V} = [\overline{U}^T \triangle X \overline{V}_1 \ \ \overline{U}^T \triangle X \overline{V}_2] = [\triangle\widetilde{X}_1 \ \ \triangle\widetilde{X}_2]$ and $\triangle\widetilde{\Gamma} = \overline{U}^T \triangle\Gamma \overline{V} = [\overline{U}^T \triangle\Gamma \overline{V}_1 \ \ \overline{U}^T \triangle\Gamma \overline{V}_2] = [\triangle\widetilde{\Gamma}_1 \ \ \triangle\widetilde{\Gamma}_2]$. Since $(\triangle t, \triangle X) = \boldsymbol{V}(\triangle t + \triangle\zeta, \triangle X + \triangle\Gamma)$, we know from Proposition 3.18 that there exists $\boldsymbol{K} = (\boldsymbol{K}_0, \boldsymbol{K}_1, \ldots, \boldsymbol{K}_{r+1}) \in \partial\Pi_{\mathcal{C}_2}(0,0)$ such that $\triangle t = \boldsymbol{K}_0(\triangle t + \triangle\zeta, \boldsymbol{D}(\triangle\widetilde{X} + \triangle\widetilde{\Gamma}))$ and

$$\begin{aligned}
&\triangle\widetilde{X} = \boldsymbol{T}(\triangle\widetilde{X} + \triangle\widetilde{\Gamma}) \\
&+ \begin{bmatrix} \boldsymbol{K}_1(\triangle t + \triangle\zeta, \boldsymbol{D}(\triangle\widetilde{X} + \triangle\widetilde{\Gamma})) & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \boldsymbol{K}_r(\triangle t + \triangle\zeta, \boldsymbol{D}(\triangle\widetilde{X} + \triangle\widetilde{\Gamma})) & 0 \\ 0 & \cdots & 0 & \boldsymbol{K}_{r+1}(\triangle t + \triangle\zeta, \boldsymbol{D}(\triangle\widetilde{X} + \triangle\widetilde{\Gamma})) \end{bmatrix},
\end{aligned}$$

where

$$\begin{aligned}
&\boldsymbol{D}(\triangle\widetilde{X} + \triangle\widetilde{\Gamma}) \\
:= \ &\Big(S(\triangle\widetilde{X}_{a_1a_1} + \triangle\widetilde{\Gamma}_{a_1a_1}), \ldots, S(\triangle\widetilde{X}_{a_ra_r} + \triangle\widetilde{\Gamma}_{a_ra_r}), \Big[(\triangle\widetilde{X} + \triangle\widetilde{\Gamma})_{bb} \ \ (\triangle\widetilde{X} + \triangle\widetilde{\Gamma})_{bc}\Big]\Big)\,,
\end{aligned}$$

and the linear mapping $\boldsymbol{T}$ is defined by (3.167). Therefore, we have

$$S(\triangle\widetilde{X}_{a_la_l}) = \boldsymbol{K}_l(\triangle t+\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{X}+\triangle\widetilde{\Gamma})), \quad l=1,\ldots,r_0\,, \tag{4.56}$$

$$\triangle\widetilde{X}_{a_la_l} = S(\triangle\widetilde{X}_{a_la_l}) = \boldsymbol{K}_l(\triangle t+\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{X}+\triangle\widetilde{\Gamma})),\ l=r_0+1,\ldots,r\,, \tag{4.57}$$

$$[\triangle\widetilde{X}_{bb}\ \ \triangle\widetilde{X}_{bc}] = \boldsymbol{K}_{r+1}(\triangle t+\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{X}+\triangle\widetilde{\Gamma}))\,, \tag{4.58}$$

$$S(\triangle\widetilde{\Gamma}_{a_la_{l'}}) = 0, \quad l\neq l' \text{ and } l,l'=1,\ldots,r_0\,, \tag{4.59}$$

$$\triangle\widetilde{X}_{a_la_{l'}} = 0, \quad l\neq l' \text{ and } l,l'=r_0+1,\ldots,r+1\,, \tag{4.60}$$

$$\triangle\widetilde{X}_{a_lc} = 0, \quad l=r_0+1,\ldots,r\,, \tag{4.61}$$

$$S(\triangle\widetilde{X}_{\alpha\beta}) - (\mathcal{E}_1)_{\alpha\beta}\circ S(\triangle\widetilde{X}_{\alpha\beta}) = (\mathcal{E}_1)_{\alpha\beta}\circ S(\triangle\widetilde{\Gamma}_{\alpha\beta})\,, \tag{4.62}$$

$$T(\triangle\widetilde{X}_{\alpha\alpha}) - (\mathcal{E}_2)_{\alpha\alpha}\circ T(\triangle\widetilde{X}_{\alpha\alpha}) = (\mathcal{E}_2)_{\alpha\alpha}\circ T(\triangle\widetilde{\Gamma}_{\alpha\alpha})\,, \tag{4.63}$$

$$T(\triangle\widetilde{X}_{\alpha\beta}) - (\mathcal{E}_2)_{\alpha\beta}\circ T(\triangle\widetilde{X}_{\alpha\beta}) = (\mathcal{E}_2)_{\alpha\beta}\circ T(\triangle\widetilde{\Gamma}_{\alpha\beta})\,, \tag{4.64}$$

$$\triangle\widetilde{X}_{\alpha c} - \mathcal{F}_{\alpha c}\circ\triangle\widetilde{X}_{\alpha c} = \mathcal{F}_{\alpha c}\circ\triangle\widetilde{\Gamma}_{\alpha c}\,, \tag{4.65}$$

By (4.56)-(4.58), we know that

$$(\triangle t, \boldsymbol{D}(\triangle\widetilde{X})) = \boldsymbol{K}(\triangle t+\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{X})+\boldsymbol{D}(\triangle\widetilde{\Gamma}))\,.$$

By (c) of [64, Proposition 1], we obtain that

$$\begin{aligned}
& \triangle t\triangle\zeta + \sum_{l=1}^{r}\left\langle S(\triangle\widetilde{X}_{a_la_l}), S(\triangle\widetilde{\Gamma}_{a_la_l})\right\rangle + \left\langle[\triangle\widetilde{X}_{bb}\ \ \triangle\widetilde{X}_{bc}], [\triangle\widetilde{\Gamma}_{bb}\ \ \triangle\widetilde{\Gamma}_{bc}]\right\rangle \\
= \ & \triangle t\triangle\zeta + \left\langle \boldsymbol{D}(\triangle\widetilde{X}), \boldsymbol{D}(\triangle\widetilde{\Gamma})\right\rangle \\
= \ & \left\langle \boldsymbol{K}(\triangle t+\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{X}+\triangle\widetilde{\Gamma})), (\triangle t+\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{X}+\triangle\widetilde{\Gamma})) - \boldsymbol{K}(\triangle t+\triangle\zeta, \boldsymbol{D}(\triangle\widetilde{X}+\triangle\widetilde{\Gamma}))\right\rangle \\
\geq \ & 0
\end{aligned}$$

Therefore, by (4.57) and (4.59)-(4.61), we have

$$\begin{aligned}
& \triangle t\triangle\zeta + \langle \triangle\widetilde{X}_1, \triangle\widetilde{\Gamma}_1\rangle + \langle \triangle\widetilde{X}_2, \triangle\widetilde{\Gamma}_2\rangle \\
= \ & \triangle t\triangle\zeta + \langle S(\triangle\widetilde{X}_1), S(\triangle\widetilde{\Gamma}_1)\rangle + \langle T(\triangle\widetilde{X}_1), T(\triangle\widetilde{\Gamma}_1)\rangle + \langle \triangle\widetilde{X}_{\alpha c}, \triangle\widetilde{\Gamma}_{\alpha c}\rangle \\
\geq \ & 2\langle S(\triangle\widetilde{X}_{\alpha\beta}), S(\triangle\widetilde{\Gamma}_{\alpha\beta})\rangle + \langle T(\triangle\widetilde{X}_{\alpha\alpha}), T(\triangle\widetilde{\Gamma}_{\alpha\alpha})\rangle + 2\langle T(\triangle\widetilde{X}_{\alpha\beta}), T(\triangle\widetilde{\Gamma}_{\alpha\beta})\rangle \\
& + \langle \triangle\widetilde{X}_{\alpha c}, \triangle\widetilde{\Gamma}_{\alpha c}\rangle \, . \qquad (4.66)
\end{aligned}$$

By (4.62), we have

$$\begin{aligned}
& 2\langle S(\triangle\widetilde{X}_{\alpha\beta}), S(\triangle\widetilde{\Gamma}_{\alpha\beta})\rangle \\
= \ & -2\sum_{l=1}^{r_0}\sum_{l'=r_0+1}^{r+1} \frac{\theta}{\overline{\nu}_{l'} - \overline{\nu}_l}\|S(\triangle\widetilde{X}_{a_l a_{l'}})\|^2 - 2\sum_{l'=r_0+1}^{r+1}\sum_{l=1}^{r_0} \frac{\theta\overline{\mu}_{l'}}{\overline{\nu}_l - \overline{\nu}_{l'}}\|S(\triangle\widetilde{X}_{a_l a_{l'}})\|^2 \, .
\end{aligned}$$

From (4.63) and (4.64), we know that

$$\begin{aligned}
& \langle T(\triangle\widetilde{X}_{\alpha\alpha}), T(\triangle\widetilde{\Gamma}_{\alpha\alpha})\rangle + 2\langle T(\triangle\widetilde{X}_{\alpha\beta}), T(\triangle\widetilde{\Gamma}_{\alpha\beta})\rangle \\
= \ & -2\sum_{l=1}^{r_0}\sum_{l'=l}^{r_0} \frac{\theta}{-\overline{\nu}_l - \overline{\nu}_{l'}}\|T(\triangle\widetilde{X}_{a_l a_{l'}})\|^2 - 2\sum_{l=1}^{r_0}\sum_{l'=r_0+1}^{r+1} \frac{\theta + \theta\overline{\mu}_{l'}}{-\overline{\nu}_l - \overline{\nu}_{l'}}\|T(\triangle\widetilde{X}_{a_l a_{l'}})\|^2 \\
= \ & -2\sum_{l=1}^{r_0}\sum_{l'=l}^{r+1} \frac{\theta}{-\overline{\nu}_l - \overline{\nu}_{l'}}\|T(\triangle\widetilde{X}_{a_l a_{l'}})\|^2 - 2\sum_{l=1}^{r_0}\sum_{l'=r_0+1}^{r+1} \frac{\theta\overline{\mu}_{l'}}{-\overline{\nu}_l - \overline{\nu}_{l'}}\|T(\triangle\widetilde{X}_{a_l a_{l'}})\|^2 \, .
\end{aligned}$$

Similarly, by (4.65), we obtain that

$$\langle \triangle\widetilde{X}_{\alpha c}, \triangle\widetilde{\Gamma}_{\alpha c}\rangle = -\sum_{l=1}^{r_0} \frac{\theta}{-\overline{\nu}_l}\|\triangle\widetilde{X}_{a_l c}\|^2 \, .$$

On the other hand, by directly calculating, since $\overline{\zeta} = -\theta$, we know that

$$\begin{aligned}
& -\bar{\zeta}\sum_{j=1}^{r_0} \mathrm{tr}\left(2\overline{P}_{a_j}^T\left[\mathcal{B}(\triangle X)(\mathcal{B}(\overline{X}) - \bar{\nu}_j I_{m+n})^\dagger\mathcal{B}(\triangle X)\right]\overline{P}_{a_j}\right) \\
= \ & 2\sum_{l=1}^{r_0}\sum_{l'=r_0+1}^{r+1} \frac{\theta}{\overline{\nu}_{l'} - \overline{\nu}_l}\|S(\triangle\widetilde{X}_{a_l a_{l'}})\|^2 + 2\sum_{l=1}^{r_0}\sum_{l'=l}^{r+1} \frac{\theta}{-\overline{\nu}_l - \overline{\nu}_{l'}}\|T(\triangle\widetilde{X}_{a_l a_{l'}})\|^2 \\
& + \sum_{l=1}^{r_0} \frac{\theta}{-\overline{\nu}_l}\|\triangle\widetilde{X}_{a_l c}\|^2 \, .
\end{aligned}$$

Meanwhile, since $\left\langle S(\triangle\widetilde{X}_{a_la_{l'}}), T(\triangle\widetilde{X}_{a_la_{l'}})\right\rangle = 0$ for any $l \in \{1,\ldots,r_0\}$ and $l' \in \{r_0 + 1,\ldots,r+1\}$, by directly calculating, we have

$$\begin{aligned}
&\left\langle [\Sigma_{\beta\beta}(\overline{\Gamma}) \ \ 0], [\overline{U}_\beta^T \triangle X\, \overline{X}^\dagger \triangle X\, \overline{V}_\beta \ \ \overline{U}_\beta^T \triangle X\, \overline{X}^\dagger \triangle X\, \overline{V}_2]\right\rangle \\
= \ & 2\sum_{l'=r_0+1}^{r+1}\sum_{l=1}^{r_0} \frac{\theta\overline{\mu}_{l'}}{\overline{\nu}_l - \overline{\nu}_{l'}}\|\triangle\widetilde{X}_{a_la_{l'}}\|^2 \\
= \ & 2\sum_{l'=r_0+1}^{r+1}\sum_{l=1}^{r_0} \frac{\theta\overline{\mu}_{l'}}{\overline{\nu}_l - \overline{\nu}_{l'}}\|S(\triangle\widetilde{X}_{a_la_{l'}})\|^2 + 2\sum_{l'=r_0+1}^{r+1}\sum_{l=1}^{r_0} \frac{\theta\overline{\mu}_{l'}}{-\overline{\nu}_l - \overline{\nu}_{l'}}\|T(\triangle\widetilde{X}_{a_la_{l'}})\|^2 .
\end{aligned}$$

Finally, by combining with (4.66), we know that the inequality (4.45) holds. The proof is completed. □

Let $(t,X) \notin \operatorname{int}\mathcal{K} \cup \operatorname{int}\mathcal{K}^\circ$ be given. We know that both the zero mapping $\boldsymbol{K}^0 \equiv 0$ and the identity mapping $\boldsymbol{K}^{\mathcal{I}} \equiv \mathcal{I}$ from $\mathcal{W} \to \mathcal{W}$ are elements of $\partial_B\Pi_{\mathcal{C}_i}(0,0)$, $i = 1,2$, since both $\mathcal{C}_i$, $i = 1,2$ are closed convex cone in the subspace $\mathcal{W}$. Let $\boldsymbol{V}^0$ and $\boldsymbol{V}^{\mathcal{I}}$ be defined by (3.178) or (3.179) with $\boldsymbol{K}$ being replaced by $\boldsymbol{K}^0$ and $\boldsymbol{K}^{\mathcal{I}}$, respectively. For the given $(t,X) \notin \operatorname{int}\mathcal{K} \cup \operatorname{int}\mathcal{K}^\circ$, define

$$\operatorname{ex}(\partial_B\Pi_{\mathcal{K}}(t,X)) := \{\boldsymbol{V}^0, \boldsymbol{V}^{\mathcal{I}}\}\,. \tag{4.67}$$

4.2 Second order optimality conditions and strong regularity of MCPs

Consider the following linear matrix cone programming (MCP) involving the Ky Fan k-norm

$$\begin{aligned}
\min \quad & \langle (s,C),(t,X)\rangle \\
\text{s.t.} \quad & \mathcal{A}(t,X) = b\,, \\
& (t,X) \in \mathcal{K}\,,
\end{aligned} \tag{4.68}$$

where $\mathcal{K} = \operatorname{epi}\|\cdot\|_{(k)} = \{(t,X) \mid \|X\|_{(k)} \le t\}$, $(s,C) \in \Re \times \Re^{m\times n}$, $b \in \Re^p$ are given, and $\mathcal{A} : \Re \times \Re^{m\times n} \to \Re^p$ is a linear operator. The first oder optimality condition, namely the Karush-Kuhn-Tucker (KKT) condition for (4.68) takes the following form

$$\begin{cases} \mathcal{A}^* y - (\zeta, \Gamma) = (s, C)\,, \\ \mathcal{A}(t, X) = b\,, \\ \mathcal{K} \ni (t, X) \perp (\zeta, \Gamma) \in \mathcal{K}^\circ\,. \end{cases} \tag{4.69}$$

For the given feasible point $(\bar{t}, \overline{X}) \in \Re \times \Re^{m\times n}$, let $\mathcal{M}(\bar{t}, \overline{X})$ be the set of Lagrange multipliers. $(\bar{t}, \overline{X})$ is a stationary point of (4.68) if and only if $\mathcal{M}(\bar{t}, \overline{X}) \neq \emptyset$.

Firstly, we introduce the concept of nondegeneracy for the general constraint, which is first introduced by Robinson [81, 82]. Let $\mathcal{X}$ and $\mathcal{Y}$ be two finite dimensional real vector spaces each equipped with a inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\|\cdot\|$. Let $g : \mathcal{X} \to \mathcal{Y}$ be a continuously differentiable function and K be a nonempty and closed convex set in $\mathcal{Y}$. Consider the following general constraint

$$g(x) \in K, \quad x \in \mathcal{X}\,. \tag{4.70}$$

Assume that $\bar{x} \in \mathcal{X}$ is a feasible solution to (4.70). Let $\mathcal{T}_K(g(\bar{x}))$ be the tangent cone of K at $g(\bar{x})$. Denote the lineality space of $\mathcal{T}_K(g(\bar{x}))$ by $\text{lin}(\mathcal{T}_K(g(\bar{x})))$. Then, we define the constraint nondegeneracy condition for (4.70) as follows.

Definition 4.3. *A feasible point $\bar{x}$ to the problem (4.70) is constraint nondegenerate if*

$$g'(\bar{x})\mathcal{X} + \text{lin}(\mathcal{T}_K(g(\bar{x}))) = \mathcal{Y}\,. \tag{4.71}$$

For the MCP problem (4.68), the Euclidean spaces $\mathcal{X} = \mathcal{Y} = \Re \times \Re^{m\times n}$, $g = (\mathcal{A}, \mathcal{I})$, where $\mathcal{I}$ is the identical mapping in $\Re \times \Re^{m\times n}$, and the convex set $K \equiv \{0\} \times \mathcal{K}$. Then, for a feasible point $(\bar{t}, \overline{X}) \in \Re \times \Re^{m\times n}$, the constraint nondegeneracy can be specified as follows.

Definition 4.4. *We say that the constraint nondegeneracy holds at a feasible point $(\bar{t}, \overline{X}) \in \Re \times \Re^{m\times n}$ to the MCP problem (4.68) if*

$$\begin{bmatrix} \mathcal{A} \\ \mathcal{I} \end{bmatrix} \Re \times \Re^{m\times n} + \begin{bmatrix} \{0\} \\ \text{lin}(\mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X})) \end{bmatrix} = \begin{bmatrix} \Re^p \\ \Re \times \Re^{m\times n} \end{bmatrix}. \tag{4.72}$$

Let $\overline{\boldsymbol{Z}} := \big((\bar{t},\overline{X}),\bar{y},(\bar{\zeta},\overline{\Gamma})\big) \in \Re\times\Re^{m\times n}\times\Re^p\times\Re\times\Re^{m\times n}$ be a KKT point satisfying the KKT conditions (4.69). Then, since $\mathcal{K}$ is a closed convex cone, we know from [32] that

$$\begin{aligned} & \mathcal{K} \ni (t,X) \perp (\zeta,\Gamma) \in \mathcal{K}^\circ \\ \Longleftrightarrow \quad & (t,X) - \Pi_{\mathcal{K}}(t+\zeta, X+\Gamma) = (\zeta,\Gamma) - \Pi_{\mathcal{K}^\circ}(t+\zeta, X+\Gamma) = 0\,. \end{aligned}$$

Therefore, $\overline{\boldsymbol{Z}} = \big((\bar{t},\overline{X}),\bar{y},(\bar{\zeta},\overline{\Gamma})\big)$ satisfies the KKT condition (4.69) if and only if $\overline{\boldsymbol{Z}}$ is a solution to the following non-smooth equation

$$F\left((t,X),y,(\zeta,\Gamma)\right) := \begin{bmatrix} (s,C) - \mathcal{A}^*y + (\zeta,\Gamma) \\ \mathcal{A}(t,X) - b \\ (t,X) - \Pi_{\mathcal{K}}(t+\zeta, X+\Gamma) \end{bmatrix} = 0\,, \tag{4.73}$$

where $((t,X),y,(\zeta,\Gamma)) \in \Re\times\Re^{m\times n}\times\Re^p\times\Re\times\Re^{m\times n}$. It is well-known that both (4.69) and (4.73) are equivalent to the following generalized equation

$$0 \in \begin{bmatrix} (s,C) - \mathcal{A}^*y + (\zeta,\Gamma) \\ \mathcal{A}(t,X) - b \\ -(t,X) \end{bmatrix} + \begin{bmatrix} \mathcal{N}_{\Re\times\Re^{m\times n}}(t,X) \\ \mathcal{N}_{\Re^p}(y) \\ \mathcal{N}_{\mathcal{K}^\circ}(\zeta,\Gamma) \end{bmatrix}. \tag{4.74}$$

Robinson [80] introduced an important concept called strong regularity for a solution of generalized equations. We define the strong regularity for (4.74) as follows.

Definition 4.5. *Let $\mathcal{Z} \equiv \Re\times\Re^{m\times n}\times\Re^p\times\Re\times\Re^{m\times n}$. We say that a KKT point $\overline{\boldsymbol{Z}} = \big((\bar{t},\overline{X}),\bar{y},(\bar{\zeta},\overline{\Gamma})\big) \in \mathcal{Z}$ is a strongly regular solution of the generalized equation (4.74) if there exist neighborhoods $\mathcal{U}$ of the origin $0\in\mathcal{Z}$ and $\mathcal{V}$ of $\overline{Z}$ such that for every $\boldsymbol{\delta}\in\mathcal{U}$, the following generalized equation*

$$\boldsymbol{\delta} \in \begin{bmatrix} (s,C) - \mathcal{A}^*y + (\zeta,\Gamma) \\ \mathcal{A}(t,X) - b \\ -(t,X) \end{bmatrix} + \begin{bmatrix} \mathcal{N}_{\Re\times\Re^{m\times n}}(t,X) \\ \mathcal{N}_{\Re^p}(y) \\ \mathcal{N}_{\mathcal{K}^\circ}(\zeta,\Gamma) \end{bmatrix} \tag{4.75}$$

has a unique solution in $\mathcal{V}$, denoted by $\boldsymbol{Z}_{\mathcal{V}}(\boldsymbol{\delta})$, and the mapping $\boldsymbol{Z}_{\mathcal{V}} : \mathcal{U} \to \mathcal{V}$ is Lipschitz continuous.

The following result on the relationship between the strong regularity of (4.74) and the locally Lipschitz homeomorphism of F defined in (4.73) can be proved in the similar way to that of in [17, Lemma 11]. We omit the proof here.

Lemma 4.6. *Let $\mathcal{Z} \equiv \Re \times \Re^{m\times n} \times \Re^p \times \Re \times \Re^{m\times n}$. Let $F : \mathcal{Z} \to \mathcal{Z}$ be defined by (4.73) and $\overline{\boldsymbol{Z}} = \big((\bar{t}, \overline{X}), \bar{y}, (\bar{\zeta}, \overline{\Gamma})\big)$ be a KKT point of the MCP problem. Then, F is locally Lipschitz homeomorphism near $\overline{\boldsymbol{Z}}$ if and only if $\overline{\boldsymbol{Z}}$ is a strong regular solution of the generalized equation (4.74).*

Let $(\bar{t}, \overline{X})$ be a feasible solution to the MCP problem (4.68). The critical cone $\mathcal{C}(\bar{t}, \overline{X})$ of (4.68) at $(\bar{t}, \overline{X})$ is defined by

$$\mathcal{C}(\bar{t}, \overline{X}) := \left\{(\tau, H) \in \Re \times \Re^{m\times n} \,|\, \mathcal{A}(\tau, H) = 0,\ (\tau, H) \in \mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X}),\ s\tau + \langle C, H\rangle \leq 0\right\} . \tag{4.76}$$

If $(\bar{t}, \overline{X})$ is a stationary point of MCP, i.e., $\mathcal{M}(\bar{t}, \overline{X})$ is nonempty, then

$$\mathcal{C}(\bar{t}, \overline{X}) = \left\{(\tau, H) \in \Re \times \Re^{m\times n} \,|\, \mathcal{A}(\tau, H) = 0,\ (\tau, H) \in \mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X}),\ s\tau + \langle C, H\rangle = 0\right\} .$$

Let $(\bar{y}, (\bar{\zeta}, \overline{\Gamma})) \in \mathcal{M}(\bar{t}, \overline{X})$. Denote $(t, X) = (\bar{t} + \bar{\zeta}, \overline{X} + \overline{\Gamma})$. For such $(\bar{y}, (\bar{\zeta}, \overline{\Gamma})) \in \mathcal{M}(\bar{t}, \overline{X})$, we know from the KKT condition (4.69) that

$$\mathcal{C}(\bar{t}, \overline{X}) = \left\{(\tau, H) \in \Re \times \Re^{m\times n} \,|\, \mathcal{A}(\tau, H) = 0,\ (\tau, H) \in \mathcal{C}_{\mathcal{K}}(t, X)\right\} , \tag{4.77}$$

where $\mathcal{C}_{\mathcal{K}}(t, X)$ is the critical cone of $\mathcal{K}$ at (t, X), which is completely characterized in Section 4.1.2.

For the MCP problem (4.68), Robinson's constraint qualification (CQ) (Robinson [79]) can be equivalently written as

$$\begin{bmatrix} \mathcal{A} \\ \mathcal{I} \end{bmatrix} \Re \times \Re^{m\times n} + \begin{bmatrix} \{0\} \\ \mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X}) \end{bmatrix} = \begin{bmatrix} \Re^p \\ \Re \times \Re^{m\times n} \end{bmatrix} . \tag{4.78}$$

The following result on the uniqueness of Lagrange multiplier of the MCP problem (4.68 can be obtained from [8, Proposition 4.50], directly.

Proposition 4.7. *Let $(\bar{t},\overline{X})$ be a feasible solution to the MCP problem (4.68) and $(\bar{y},(\bar{\zeta},\overline{\Gamma}))\in\mathcal{M}(\bar{t},\overline{X})$. Suppose that $(\bar{y},(\bar{\zeta},\overline{\Gamma}))$ satisfies the following strict constraint qualification:*

$$\begin{bmatrix} \mathcal{A} \\ \mathcal{I} \end{bmatrix} \Re\times\Re^{m\times n} + \begin{bmatrix} \{0\} \\ \mathcal{T}_{\mathcal{K}}(\bar{t},\overline{X})\cap(\bar{y},(\bar{\zeta},\overline{\Gamma}))^{\perp} \end{bmatrix} = \begin{bmatrix} \Re^{p} \\ \Re\times\Re^{m\times n} \end{bmatrix}. \tag{4.79}$$

Then $\mathcal{M}(\bar{t},\overline{X})$ is a singleton.

Let $G:\Re\times\Re^{m\times n}\to\Re^{p}\times\Re^{m\times n}$ be defined by

$$G(t,X):=\begin{bmatrix} \mathcal{A}(t,X)-b \\ (t,X) \end{bmatrix} \qquad (t,X)\in\Re\times\Re^{m\times n}.$$

Then, for any $(\bar{y},(\bar{\zeta},\overline{\Gamma}))\in\mathcal{M}(\bar{t},\overline{X})$ and $(\tau,H)\in\mathcal{C}(\bar{t},\overline{X})$, the second order tangent set $\mathcal{T}^{2}_{\{0\}\times\mathcal{K}}\big(G(\bar{t},\overline{X}),G'(\bar{t},\overline{X})(\tau,H)\big)$ to $\{0\}\times\mathcal{K}$ at $G(\bar{t},\overline{X})$ along the direction $G'(\bar{t},\overline{X})(\tau,H)$ is given by

$$\begin{aligned} \mathcal{T}^{2}_{\{0\}\times\mathcal{K}}\big(G(\bar{t},\overline{X}),G'(\bar{t},\overline{X})(\tau,H)\big) &= \mathcal{T}^{2}_{\{0\}}\big(\mathcal{A}(\bar{t},\overline{X})-b,\mathcal{A}(\tau,H)\big)\times\mathcal{T}^{2}_{\mathcal{K}}\big((\bar{t},\overline{X}),(\tau,H)\big) \\ &= \mathcal{T}^{2}_{\{0\}}\times\mathcal{T}^{2}_{\mathcal{K}}. \end{aligned}$$

Since the support function value $\delta^{*}_{\mathcal{T}^{2}_{\{0\}}}(\bar{y})=0$, we know that

$$\delta^{*}_{\mathcal{T}^{2}_{\{0\}\times\mathcal{K}}}(\bar{y},(\bar{\zeta},\overline{\Gamma}))=\delta^{*}_{\mathcal{T}^{2}_{\mathcal{K}}}(\bar{\zeta},\overline{\Gamma}).$$

Let $(\bar{t},\overline{X})\in\mathcal{K}$ be an optimal solution to the MCP problem (4.68). By Proposition 4.2, we have the following proposition.

Proposition 4.8. *Let $(\bar{t},\overline{X})$ be a feasible solution to the MCP problem (4.68) such that $\mathcal{M}(\bar{t},\overline{X})$ is nonempty. Then for any $(\bar{y},(\bar{\zeta},\overline{\Gamma}))\in\mathcal{M}(\bar{t},\overline{X})$, one has*

$$\delta^{*}_{\mathcal{T}^{2}_{\mathcal{K}}}(\bar{\zeta},\overline{\Gamma})=\Upsilon_{(\bar{t},\overline{X})}\big((\bar{\zeta},\overline{\Gamma}),(\tau,H)\big)\quad\forall\,(\tau,H)\in\mathcal{C}(\bar{t},\overline{X}),$$

where the linear quadratic function $\Upsilon_{(\bar{t},\overline{X})}(\cdot,\cdot)$ is defined in Definition 4.1.

Recall that $\mathcal{K}$ is $\mathcal{C}^2$-cone reducible (Proposition 4.3). Note that $\{0\}$ is also $\mathcal{C}^2$-cone reducible, and the Cartesian product of $\mathcal{C}^2$-cone reducible sets is again $\mathcal{C}^2$-cone reducible. Then, by combining Theorem 3.45, Proposition 3.136 and Theorem 3.137 in Bonnans and Shapiro [8], we can state in the following theorem on the second order necessary condition and the second order sufficient condition for the MCP problem (4.68).

Theorem 4.9. *Suppose that $(\bar{t}, \overline{X})$ is a locally optimal solution to the linear MCP (4.68) and Robinson's CQ holds at $(\bar{t}, \overline{X})$. Then, the following inequality holds:*

$$\sup_{(\bar{y},(\bar{\zeta},\bar{\Gamma}))\in\mathcal{M}(\bar{t},\overline{X})} \left\{-\Upsilon_{(\bar{t},\overline{X})}\left((\bar{\zeta},\bar{\Gamma}),(\tau,H)\right)\right\} \geq 0 \quad \forall\,(\tau,H)\in\mathcal{C}(\bar{t},\overline{X})\,. \tag{4.80}$$

Conversely, let $(\bar{t}, \overline{X})$ be a feasible solution to MCP such that $\mathcal{M}(\bar{t}, \overline{X})$ is nonempty. Suppose that Robinson's CQ holds at $(\bar{t}, \overline{X})$. Then the following condition

$$\sup_{(\bar{y},(\bar{\zeta},\bar{\Gamma}))\in\mathcal{M}(\bar{t},\overline{X})} \left\{-\Upsilon_{(\bar{t},\overline{X})}\left((\bar{\zeta},\bar{\Gamma}),(\tau,H)\right)\right\} > 0 \quad \forall\,(\tau,H)\in\mathcal{C}(\bar{t},\overline{X})\setminus\{(0,0)\} \tag{4.81}$$

is necessary and sufficient for the quadratic growth condition at $(\bar{t}, \overline{X})$, i.e., $\forall\,(t,X)\in N$ such that (t,X) is feasible,

$$\langle (s,C),(t,X)\rangle \geq \langle (s,C),(\bar{t},\overline{X})\rangle + c\|(\bar{t},\overline{X})-(t,X)\|^2\,, \tag{4.82}$$

for some constant $c>0$ and a neighborhood N of $(\bar{t}, \overline{X})$ is $\Re\times\Re^{m\times n}$.

For the stationary point $(\bar{t}, \overline{X})$, in order to introduce the strong second order sufficient condition for the MCP problem (4.68), we define the following outer approximation set to the affine hull of $\mathcal{C}(\bar{t}, \overline{X})$ with respect to $(\bar{y},(\bar{\zeta},\bar{\Gamma}))\in\mathcal{M}(\bar{t}, \overline{X})$ by

$$\operatorname{app}(\bar{y},(\bar{\zeta},\bar{\Gamma})) := \left\{(\tau,H)\in\Re\times\Re^{m\times n}\,|\,\mathcal{A}(\tau,H)=0,\ (\tau,H)\in\operatorname{aff}(\mathcal{C}_{\mathcal{K}}(t,X))\right\}\,. \tag{4.83}$$

Therefore, the strong second order sufficient condition for the MCP problem (4.68) is defined as follows.

Definition 4.6. *Let $(\bar{t}, \overline{X})$ be an optimal solution to (4.68) such that $\mathcal{M}(\bar{t}, \overline{X})$ is nonempty. We say that the strong second order sufficient condition holds at $(\bar{t}, \overline{X})$ if*

$$\sup_{(\bar{y},(\bar{\zeta},\bar{\Gamma}))\in\mathcal{M}(\bar{t},\overline{X})} \left\{-\Upsilon_{(\bar{t},\overline{X})}\left((\bar{\zeta},\bar{\Gamma}),(\tau,H)\right)\right\} > 0 \quad \forall\,(\tau,H)\in\widehat{\mathcal{C}}(\bar{t},\overline{X})\setminus\{(0,0)\}\,, \tag{4.84}$$

where for any $(\bar{y}, (\bar{\zeta}, \overline{\Gamma})) \in \mathcal{M}(\bar{t}, \overline{X})$, $\bar{y} \in \Re^p$, $(\bar{\zeta}, \overline{\Gamma}) \in \Re \times \Re^{m \times n}$ *and*

$$\widehat{\mathcal{C}}(\bar{t}, \overline{X}) := \bigcap_{(\bar{y}, (\bar{\zeta}, \overline{\Gamma})) \in \mathcal{M}(\bar{t}, \overline{X})} \mathrm{app}(\bar{y}, (\bar{\zeta}, \overline{\Gamma}))\,.$$

Let $(\bar{y}, (\bar{\zeta}, \overline{\Gamma})) \in \mathcal{M}(\bar{t}, \overline{X})$. Denote $(t, X) \equiv (\bar{t} + \bar{\zeta}, \overline{X} + \overline{\Gamma})$. Without loss of generality, from now on, we always assume that $(t, X) \notin \mathrm{int}\,\mathcal{K} \cup \mathrm{int}\,\mathcal{K}^\circ$. By [17, Lemma 1], it is clear that $\boldsymbol{U} \in \partial_B F((\bar{t}, \overline{X}), \bar{y}, (\bar{\zeta}, \overline{\Gamma}))$ if and only if there exists a $\boldsymbol{V} \in \partial_B \Pi_{\mathcal{K}}(t, X)$ such that

$$\boldsymbol{U}\left((\triangle t, \triangle X), \triangle y, (\triangle \zeta, \triangle \Gamma)\right) = \begin{bmatrix} -\mathcal{A}^*(\triangle y) + (\triangle \zeta, \triangle \Gamma) \\ \mathcal{A}(\triangle t, \triangle X) \\ (\triangle t, \triangle X) - \boldsymbol{V}(\triangle t + \triangle \zeta, \triangle X + \triangle \Gamma) \end{bmatrix} \tag{4.85}$$

for all $((\triangle t, \triangle X), \triangle y, (\triangle \zeta, \triangle \Gamma)) \in \mathcal{Z}$. Let $\mathrm{ex}(\partial_B \Pi_{\mathcal{K}}(t, X))$ be defined by (4.67). For $\boldsymbol{V}^0, \boldsymbol{V}^{\mathcal{I}} \in \mathrm{ex}(\partial_B \Pi_{\mathcal{K}}(t, X))$, let $\boldsymbol{U}^0$ and $\boldsymbol{U}^{\mathcal{I}}$ be defined by (4.85), respectively. Denote

$$\mathrm{ex}\left(\partial_B F((\bar{t}, \overline{X}), \bar{y}, (\bar{\zeta}, \overline{\Gamma}))\right) := \left\{\boldsymbol{U}^0, \boldsymbol{U}^{\mathcal{I}}\right\}\,.$$

Proposition 4.10. *Let* $\left((\bar{t}, \overline{X}), \bar{y}, (\bar{\zeta}, \overline{\Gamma})\right)$ *be a KKT point of the MCP problem (4.68). If* $\boldsymbol{U}^0 \in \mathrm{ex}\left(\partial_B F((\bar{t}, \overline{X}), \bar{y}, (\bar{\zeta}, \overline{\Gamma}))\right)$ *is nonsingular, then the constraint nondegenerate condition (4.72) holds at* $(\bar{t}, \overline{X})$.

Proof. Assume on the contrary that (4.72) does not hold. Then, we have

$$\left\{\begin{bmatrix} \mathcal{A} \\ \mathcal{I} \end{bmatrix} \Re \times \Re^{m \times n}\right\}^\perp \bigcap \begin{bmatrix} 0 \\ \mathrm{lin}(\mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X})) \end{bmatrix}^\perp \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in \begin{bmatrix} \Re^p \\ \Re \times \Re^{m \times n} \end{bmatrix},$$

which implies that there exists

$$0 \neq (\triangle y, (\triangle \zeta, \triangle \Gamma)) \in \left\{\begin{bmatrix} \mathcal{A} \\ \mathcal{I} \end{bmatrix} \Re \times \Re^{m \times n}\right\}^\perp \bigcap \begin{bmatrix} 0 \\ \mathrm{lin}(\mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X})) \end{bmatrix}^\perp.$$

From $(\triangle y, (\triangle \zeta, \triangle \Gamma)) \in \left\{\begin{bmatrix} \mathcal{A} \\ \mathcal{I} \end{bmatrix} \Re \times \Re^{m \times n}\right\}^\perp$, we know that

$$\langle (\triangle y, (\triangle \zeta, \triangle \Gamma)), (\mathcal{A}(\tau, H), (\tau, H)) \rangle = 0 \quad \forall\, (\tau, H) \in \Re \times \Re^{m \times n}\,,$$

which implies

$$\mathcal{A}^*(\triangle y) + (\triangle\zeta, \triangle\Gamma) = -\mathcal{A}^*(\triangle y) + (-\triangle\zeta, -\triangle\Gamma) = 0\,.$$

Meanwhile, from $(\triangle y, (\triangle\zeta, \triangle\Gamma)) \in \begin{bmatrix} 0 \\ \mathrm{lin}(\mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X})) \end{bmatrix}^{\perp}$, we obtain that

$$-\tau\triangle\zeta - \langle H, \triangle\Gamma\rangle = 0 \quad \forall\, (\tau, H) \in \mathrm{lin}(\mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X}))\,.$$

Therefore, we know from (4.3) and (4.4) that

$$\boldsymbol{T}(\overline{U}^T \triangle\Gamma \overline{V}) = 0\,,$$

where the linear operator $\boldsymbol{T} : \Re^{m\times n} \to \Re^{m\times n}$ is defined by (3.160) if $\overline{\sigma}_k > 0$, and (3.167) if $\overline{\sigma}_k = 0$. By Proposition 3.18, we know that $\boldsymbol{V}_1^0(-\triangle\zeta, -\triangle\Gamma) = 0 \in \Re^{m\times n}$. Therefore, since $\boldsymbol{V}_0^0(\triangle t - \triangle\zeta, \triangle X - \triangle\Gamma) \equiv 0 \in \Re$, for $(\triangle t, \triangle X) \equiv (0, 0)$, we have

$$\begin{bmatrix} \triangle t \\ \triangle X \end{bmatrix} - \begin{bmatrix} \boldsymbol{V}_0^0(\triangle t - \triangle\zeta, \triangle X - \triangle\Gamma) \\ \boldsymbol{V}_1^0(\triangle t - \triangle\zeta, \triangle X - \triangle\Gamma) \end{bmatrix} = 0\,,$$

which implies that

$$\boldsymbol{U}^0((\triangle t, \triangle X), \triangle y, (-\triangle\zeta, -\triangle\Gamma)) = \begin{bmatrix} -\mathcal{A}^*(\triangle y) + (-\triangle\zeta, -\triangle\Gamma) \\ \mathcal{A}(\triangle t, \triangle X) \\ (\triangle t, \triangle X) - \boldsymbol{V}^0(\triangle t - \triangle\zeta, \triangle X - \triangle\Gamma) \end{bmatrix} = 0\,.$$

Since $0 \neq (\triangle y, (\triangle\zeta, \triangle\Gamma))$, we know that $\boldsymbol{U}^0$ is singular. This contradiction shows that the constraint nondegenerate condition (4.72) holds at $(\bar{t}, \overline{X})$. □

When $\mathcal{M}(\bar{t}, \overline{X})$ is a singleton, we have the following result on the strong second order sufficient condition (4.84).

Proposition 4.11. *Let $(\bar{t}, \overline{X})$ be a feasible point of the MCP problem (4.68). Assume that $\mathcal{M}(\bar{t}, \overline{X}) = \{\bar{y}, (\bar{\zeta}, \overline{\Gamma})\}$. If $\boldsymbol{U}^{\mathcal{I}} \in \mathrm{ex}(\partial_B F((\bar{t}, \overline{X}), \bar{y}, (\bar{\zeta}, \overline{\Gamma})))$ is nonsingular, then the strong second order sufficient condition (4.84) holds at $(\bar{t}, \overline{X})$.*

Proof. Since $\mathcal{M}(\bar{t},\overline{X}) = \big\{\bar{y},(\bar{\zeta},\overline{\Gamma})\big\}$, the strong second order sufficient condition (4.84) can be written as

$$-\Upsilon_{(\bar{t},\overline{X})}\big((\bar{\zeta},\overline{\Gamma}),(\tau,H)\big) > 0 \quad \forall\,(\tau,H)\in \mathrm{app}(\bar{y},(\bar{\zeta},\overline{\Gamma}))\setminus\{(0,0)\}\,. \tag{4.86}$$

Suppose that the condition (4.86) does not hold at $(\bar{t},\overline{X})$. By noting that for any $(\tau,H)\in \mathrm{app}(\bar{y},(\bar{\zeta},\overline{\Gamma}))$, $-\Upsilon_{(\bar{t},\overline{X})}\big((\bar{\zeta},\overline{\Gamma}),(\tau,H)\big)\geq 0$, we know that there exists $0\neq (\tau,H)\in \mathrm{app}(\bar{y},(\bar{\zeta},\overline{\Gamma}))$ such that

$$\mathcal{A}(\tau,H)=0 \quad \text{and} \quad -\Upsilon_{(\bar{t},\overline{X})}\big((\bar{\zeta},\overline{\Gamma}),(\tau,H)\big)=0\,.$$

Therefore, by the definition (Definition 4.1) of $\Upsilon_{(\bar{t},\overline{X})}\big((\bar{\zeta},\overline{\Gamma}),(\tau,H)\big)$ and the proof of Proposition 4.5, we know that if $\sigma_k(\overline{X})>0$,

$$\begin{cases} \widetilde{H}_{\alpha\alpha}\in\mathcal{S}^{|\alpha|}, \quad \begin{bmatrix} \widetilde{H}_{\beta_1\beta_1} & \widetilde{H}_{\beta_1\beta_2} \\ \widetilde{H}_{\beta_2\beta_1} & \widetilde{H}_{\beta_2\beta_2}\end{bmatrix}\in\mathcal{S}^{|\beta_1|+|\beta_2|}\,, \\ \widetilde{H}_{\beta_1\beta_3}=(\widetilde{H}_{\beta_3\beta_1})^T, \quad \widetilde{H}_{\beta_2\beta_3}=(\widetilde{H}_{\beta_3\beta_2})^T \\ \widetilde{H}_{\alpha\beta_2}=(\widetilde{H}_{\beta_2\alpha})^T=0, \quad \widetilde{H}_{\alpha\beta_3}=(\widetilde{H}_{\beta_3\alpha})^T=0\,, \\ \widetilde{H}_{\alpha\gamma}=(\widetilde{H}_{\gamma\alpha})^T=0\,, \\ \widetilde{H}_{\beta_1\gamma}=(\widetilde{H}_{\gamma\beta_1})^T=0, \quad \widetilde{H}_{\beta_2\gamma}=(\widetilde{H}_{\gamma\beta_2})^T=0\,, \\ \widetilde{H}_{\alpha c}=0, \quad \widetilde{H}_{\beta_1 c}=0, \quad \widetilde{H}_{\beta_2 c}=0\,, \end{cases} \tag{4.87}$$

where $\widetilde{H}=\overline{U}^T H\overline{V}$, and the index sets α, β, γ, and β_i, $i=1,2,3$ are defined by (3.150) and (3.159), respectively; if $\sigma_k(\overline{X})=0$,

$$\begin{cases} \widetilde{H}_{\alpha\alpha}\in\mathcal{S}^{|\alpha|}\,, \\ \widetilde{H}_{\alpha\beta_2}=(\widetilde{H}_{\beta_2\alpha})^T=0, \quad \widetilde{H}_{\alpha\beta_3}=(\widetilde{H}_{\beta_3\alpha})^T=0\,, \\ \widetilde{H}_{\alpha c}=0\,, \end{cases} \tag{4.88}$$

where $\widetilde{H}=\overline{U}^T H\overline{V}$, and the index sets α, β, and β_i, $i=1,2,3$ are defined by (3.154) and (3.166), respectively. By Proposition 3.18, we know from (4.87) and (4.88) that

$$(\tau,H)=\boldsymbol{V}^{\mathcal{I}}(\tau,H)\,.$$

Finally, by (4.85), we have for $(\triangle y, (\triangle\zeta, \triangle\Gamma)) = 0 \in \Re^p \times \Re \times \Re^{m\times n}$ that

$$\boldsymbol{U}^{\mathcal{I}}((\tau, H), \triangle y, (\triangle\zeta, \triangle\Gamma)) = \begin{bmatrix} -\mathcal{A}^*(\triangle y) + (\triangle\zeta, \triangle\Gamma) \\ \\ \mathcal{A}(\tau, H) \\ \\ (\tau, H) - \boldsymbol{V}^{\mathcal{I}}(\tau + \triangle\zeta, H + \triangle\Gamma) \end{bmatrix} = 0\,,$$

which, implies that $\boldsymbol{U}^{\mathcal{I}}$ is singular. This contradiction shows that the strong second order sufficient condition (4.86) holds at $(\bar{t}, \overline{X})$. □

The following proposition relates the strong second order sufficient condition and constraint nondegeneracy to the nonsingularity of Clarke's Jacobian of the mapping F and the strong regularity of a solution to the generalized equation (4.74).

Proposition 4.12. *Let $(\bar{t}, \overline{X})$ be a feasible solution of the MCP problem (4.68). Let $\bar{y} \in \Re^p$, $(\bar{\zeta}, \overline{\Gamma}) \in \Re \times \Re^{m\times n}$ be such that $(\bar{y}, (\bar{\zeta}, \overline{\Gamma})) \in \mathcal{M}(\bar{t}, \overline{X})$. Consider the following three statements:*

(a) The strong second order sufficient condition (4.84) holds at $(\bar{t}, \overline{X})$ and $(\bar{t}, \overline{X})$ is constraint nondegenerate.

(b) Any element in $\partial F((\bar{t}, \overline{X}), \bar{y}, (\bar{\zeta}, \overline{\Gamma}))$ is nonsingular.

(c) The KKT point $\big((\bar{t}, \overline{X}), \bar{y}, (\bar{\zeta}, \overline{\Gamma})\big)$ is a strong regular solution of the generalized equation (4.74).

It holds that (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c).

Proof. "(a) $\Longrightarrow$ (b)" Since the constraint nondegeneracy condition (4.72) holds at $(\bar{t}, \overline{X})$, $(\bar{y}, (\bar{\zeta}, \overline{\Gamma}))$ satisfies the strict constraint qualification (4.79). Thus, we know from Proposition 4.7 that $\mathcal{M}(\bar{t}, \overline{X}) = \{(\bar{t}, \overline{X}),\ (\bar{y}, (\bar{\zeta}, \overline{\Gamma}))\}$. The strong second order sufficient condition (4.84) then takes the following form

$$-\,\Upsilon_{(\bar{t},\overline{X})}\big((\bar{\zeta}, \overline{\Gamma}), (\tau, H)\big) > 0 \quad \forall\, (\tau, H) \in \operatorname{app}(\bar{y}, (\bar{\zeta}, \overline{\Gamma})) \setminus \{(0,0)\}\,. \tag{4.89}$$

Let $(t,X)=(\bar t+\bar\zeta,\overline{X}+\overline{\Gamma})$.

Let $\boldsymbol{U}$ be an arbitrary element in $\partial F((\bar t,\overline{X}),\bar y,(\bar\zeta,\overline{\Gamma}))$. We will show that $\boldsymbol{U}$ is nonsingular. Let $((\triangle t,\triangle X),\triangle y,(-\triangle\zeta,-\triangle\Gamma))\in\Re\times\Re^{m\times n}\times\Re^p\times\Re\times\Re^{m\times n}$ be such that

$$\boldsymbol{U}\,((\triangle t,\triangle X),\triangle y,(-\triangle\zeta,-\triangle\Gamma))=0\,.$$

Then, we know that there exists a $\boldsymbol{V}\in\partial\Pi_{\mathcal{K}}(t,X)$ such that

$$\boldsymbol{U}\,((\triangle t,\triangle X),\triangle y,(\triangle\zeta,\triangle\Gamma))=\begin{bmatrix}-\mathcal{A}^*(\triangle y)+(\triangle\zeta,\triangle\Gamma)\\ \mathcal{A}(\triangle t,\triangle X)\\ (\triangle t,\triangle X)-\boldsymbol{V}(\triangle t+\triangle\zeta,\triangle X+\triangle\Gamma)\end{bmatrix}=0\,. \quad (4.90)$$

From the third equation of (4.90), we know that $(\triangle t,\triangle X)=\boldsymbol{V}(\triangle t+\triangle\zeta,\triangle X+\triangle\Gamma)$. By Lemma 4.4 and the second equation of (4.90), we obtain that

$$(\triangle t,\triangle X)\in\operatorname{app}(\bar y,(\bar\zeta,\overline{\Gamma}))\,.$$

From the first and second equations of (4.90), we know that

$$0=-\langle\mathcal{A}(\triangle t,\triangle X),\triangle y\rangle+\langle(\triangle t,\triangle X),(\triangle\zeta,\triangle\Gamma)\rangle=\langle(\triangle t,\triangle X),(\triangle\zeta,\triangle\Gamma)\rangle\,,$$

which, together with the third equation of (4.90) and Proposition 4.5, implies that

$$0\geq-\Upsilon_{(\bar t,\overline{X})}\left((\bar\zeta,\overline{\Gamma}),(\triangle t,\triangle X)\right)\,.$$

Therefore, by (4.89), we have

$$(\triangle t,\triangle X)=0\,.$$

Thus, (4.89) reduces to

$$\begin{bmatrix}-\mathcal{A}^*(\triangle y)+(\triangle\zeta,\triangle\Gamma)\\ \boldsymbol{V}(\triangle\zeta,\triangle\Gamma)\end{bmatrix}=0 \quad (4.91)$$

By the constraint nondegeneracy condition (4.72), we know that there exist $(a,A)\in\Re\times\Re^{m\times n}$ and $(\tau,H)\in\operatorname{lin}(\mathcal{T}_{\mathcal{K}}(\bar t,\overline{X}))$ such that

$$\mathcal{A}(a,A)=-\triangle y\quad\text{and}\quad(a+\tau,A+H)=(\triangle\zeta,\triangle\Gamma)\,. \quad (4.92)$$

By (4.92) and the first equation of (4.91), we know that

$$\begin{aligned} & \langle \triangle y, \triangle y\rangle + \langle (\triangle\zeta, \triangle\Gamma), (\triangle\zeta, \triangle\Gamma)\rangle \\ = \ & \langle -\mathcal{A}(a, A), \triangle y\rangle + \langle (a+\tau, A+H), (\triangle\zeta, \triangle\Gamma)\rangle \\ = \ & \langle (a, A), -\mathcal{A}^*(\triangle y) + (\triangle\zeta, \triangle\Gamma)\rangle + \langle (\tau, H), (\triangle\zeta, \triangle\Gamma)\rangle \\ = \ & \tau\triangle\zeta + \langle H, \triangle\Gamma\rangle = \tau\triangle\zeta + \langle \widetilde{H}, \triangle\widetilde{\Gamma}\rangle\,, \end{aligned} \tag{4.93}$$

where $\widetilde{H} = \overline{U}^T H \overline{V}$ and $\triangle\widetilde{\Gamma} = \overline{U}^T \triangle\Gamma \overline{V}$. Next, consider the following two cases.

Case 1. $\sigma_k(\overline{X}) > 0$. Since $(\tau, H) \in \mathrm{lin}(\mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X}))$, by (4.3), we know that

$$S(\widetilde{H}_{\beta\beta}) = \frac{1}{k-k_0}\left(\tau - \sum_{l=1}^{r_0} \mathrm{tr}(\widetilde{H}_{a_l a_l})\right) I_{|\beta|}\,.$$

Hence, from the part (i) of Lemma 3.19, we know that

$$\begin{aligned} \tau\triangle\zeta + \langle \widetilde{H}, \triangle\widetilde{\Gamma}\rangle &= \triangle\zeta\tau + \left\langle \widetilde{H}, \begin{bmatrix} -\triangle\zeta I_{|\alpha|} & 0 & 0 & 0 \\ 0 & \triangle\widetilde{\Gamma}_{\beta\beta} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \right\rangle \\ = \ & \triangle\zeta\tau - \triangle\zeta \sum_{l=1}^{r_0} \mathrm{tr}(\widetilde{H}_{a_l a_l}) + \langle S(\widetilde{H}_{\beta\beta}), \triangle\widetilde{\Gamma}_{\beta\beta}\rangle \quad (\text{since } \triangle\widetilde{\Gamma}_{\beta\beta} \text{ is symmetric}) \\ = \ & \triangle\zeta\tau - \triangle\zeta \sum_{l=1}^{r_0} \mathrm{tr}(\widetilde{H}_{a_l a_l}) + \frac{1}{k-k_0}\left(\tau - \sum_{l=1}^{r_0} \mathrm{tr}(\widetilde{H}_{a_l a_l})\right) \mathrm{tr}(\triangle\widetilde{\Gamma}_{\beta\beta}) \\ = \ & -\triangle\zeta\left(-\tau + \sum_{l=1}^{r_0} \mathrm{tr}(\widetilde{H}_{a_l a_l})\right) - \triangle\zeta\left(\tau - \sum_{l=1}^{r_0} \mathrm{tr}(\widetilde{H}_{a_l a_l})\right) = 0\,. \end{aligned}$$

Case 2. $\sigma_k(\overline{X}) = 0$. Since $(\tau, H) \in \mathrm{lin}(\mathcal{T}_{\mathcal{K}}(\bar{t}, \overline{X}))$, by (4.4), we know that

$$\sum_{l=1}^{r_0} \mathrm{tr}(\widetilde{H}_{a_l a_l}) = \tau \quad \text{and} \quad \begin{bmatrix} \widetilde{H}_{\beta\beta} & \widetilde{H}_{\beta c} \end{bmatrix} = 0\,.$$

From the part (ii) of Lemma 3.19, we know that

$$\begin{aligned} \tau\triangle\zeta + \langle \widetilde{H}, \triangle\widetilde{\Gamma}\rangle &= \triangle\zeta\tau + \left\langle \widetilde{H}, \begin{bmatrix} -\triangle\zeta I_{|\alpha|} & 0 & 0 \\ 0 & \triangle\widetilde{\Gamma}_{\beta\beta} & \triangle\widetilde{\Gamma}_{\beta c} \end{bmatrix} \right\rangle \\ = \ & \triangle\zeta\tau - \triangle\zeta \sum_{l=1}^{r_0} \mathrm{tr}(\widetilde{H}_{a_l a_l}) = 0\,. \end{aligned}$$

Thus, from (4.93), we obtain that

$$\triangle y = 0 \quad \text{and} \quad (\triangle\zeta, \triangle\Gamma) = 0\,.$$

This, together with $(\triangle t, \triangle X) = 0$, shows that $\boldsymbol{U}$ is nonsingular.

"(b) $\Longrightarrow$ (c)" By Clarke's inverse function theorem [22, 23], we know that F is a locally Lipschitz homeomorphism near $\big((\bar{t}, \overline{X}), \bar{y}, (\bar{\zeta}, \overline{\Gamma})\big)$. Thus, from Lemma 4.6, $\big((\bar{t}, \overline{X}), \bar{y}, (\bar{\zeta}, \overline{\Gamma})\big)$ is a strong regular solution of the generalized equation (4.74). □

Now, we are ready to state our main results of this chapter.

Theorem 4.13. *Let $\big((\bar{t}, \overline{X}), \bar{y}, (\bar{\zeta}, \overline{\Gamma})\big)$ be a KKT point satisfying the KKT condition (4.69) and F be defined by (4.73). Then, the following statements are all equivalent:*

(i) The KKT point $\big((\bar{t}, \overline{X}), \bar{y}, (\bar{\zeta}, \overline{\Gamma})\big)$ is a strongly regular solution of the generalized equation (4.74).

(ii) The function F is locally Lipschitz homeomorphism near $\big((\bar{t}, \overline{X}), \bar{y}, (\bar{\zeta}, \overline{\Gamma})\big)$.

(iii) The strong second order sufficient condition (4.84) holds at $(\bar{t}, \overline{X})$ and $(\bar{t}, \overline{X})$ is constraint nondegenerate.

(iv) Every element in $\partial F((\bar{t}, \overline{X}), \bar{y}, (\bar{\zeta}, \overline{\Gamma}))$ is nonsingular.

(v) Every element in $\partial_B F((\bar{t}, \overline{X}), \bar{y}, (\bar{\zeta}, \overline{\Gamma}))$ is nonsingular.

(vi) The two elements in $\operatorname{ex}\big(\partial_B F((\bar{t}, \overline{X}), \bar{y}, (\bar{\zeta}, \overline{\Gamma})\big)$ are nonsingular.

Proof. The relation $(i) \Longleftrightarrow (ii)$ follows from Lemma 4.6. We know from Proposition 4.12, Proposition 4.10 and Proposition 4.11 that $(iii) \Longleftrightarrow (iv) \Longleftrightarrow (v) \Longleftrightarrow (vi) \Longrightarrow (i)$. Finally, we know from [50] that $(ii) \Longrightarrow (v)$. Thus, the proof is completed. □

4.3 Extensions to other MOPs

In pervious sections, we have studied the variational analysis of the Ky Fan k-norm cone and the sensitivity analysis of the linear MCP problem involving the Ky Fan k-norm cone. In this section, we consider the extensions of the corresponding sensitivity results to other MOP problems.

The first kind of MOPs considering in this section is the linear MCP problem involving the epigraph cone $\mathcal{M}$ of the sum of k largest eigenvalues of the symmetric matrix ((1.49) in Section 1.3), which comes from the applications such as eigenvalue optimization [69, 70, 71, 55]. Note that the epigraph cone $\mathcal{M}$ can be regarded as the symmetric counterpart of the Ky Fan k-norm cone $\mathcal{K}$. By using the properties of the eigenvalue function $\lambda(\cdot)$ of the symmetric matrix (see e.g., Section 2.1), the corresponding variational properties of $\mathcal{M}$ such as the characterizations of tangent cone and the second order tangent sets of $\mathcal{M}$, the explicit expression of the support function of the second order tangent set of $\mathcal{M}$, the $\mathcal{C}^2$-cone reducibility of $\mathcal{M}$ and the characterization of the critical cone of $\mathcal{M}$, can be obtained in the similar but simple way to those of the Ky Fan k-norm cone $\mathcal{K}$. Similarly, we can state the constraint nondegeneracy, the second order necessary condition and the (strong) second order sufficient condition of the linear matrix cone programming (MCP) problem (1.49). Also, by using the properties of the spectral operator (the metric projection operator over the epigraph cone $\mathcal{M}$), for the considering linear matrix cone programming (MCP) problem (1.49), we can consider the relationships among the strong regularity of the KKT point, the strong second order sufficient condition and constraint nondegeneracy, and the nonsingularity of both the B-subdifferenitial and Clarke's generalized Jacobian of the nonsmooth system at a KKT point.

The second kind of MOPs considering in this section is the nonlinear MCP problems with the Ky Fan k-norm cone $\mathcal{K}$, where the smooth objective function and constraints in (4.68) are not necessary linear. For example, the problem (1.10), (1.12) and (1.14) can be reformulated as the nonlinear MCP problems with the Ky Fan k-norm cone $\mathcal{K}$. Since the epigraph cone $\mathcal{K}$ is $\mathcal{C}^2$-cone reducible, by combining the variational properties of $\mathcal{K}$ which we obtained in this thesis and the sensitivity results for the general conic programming in literature [5, 7, 8], we can establish the constraint nondegeneracy, the second order necessary condition and the (strong) second order sufficient condition for the nonlinear MCP problem involving $\mathcal{K}$ directly. Furthermore, as the nonlinear SDP problem [94], we can consider the various characterizations for the strong regularity for a local solution of the nonlinear MCP with the Ky Fan k-norm cone $\mathcal{K}$. Actually, the results in Proposition 4.12 for the linear MCP problem (4.68) can be extended easily to the nonlinear MCP problem involving the Ky Fan k-norm cone $\mathcal{K}$. Finally, as the nonlinear SDP problem [94], for a local solution of the considering nonlinear MCP problem, we are able to consider the relationships among the strong second-order sufficient condition and constraint nondegeneracy, the non-singularity of Clarke's Jacobian of the Karush-Kuhn-Tucker (KKT) system and the strong regularity of the KKT point, under the Robinson's CQ.

The third kind of MOPs considering in this section is the linear MCP problem (1.4) where the matrix cone $\mathcal{K}$ is the Cartesian product of the Ky Fan k-norm cone and some well understood symmetric cones (e.g., nonnegative orthant, the second order cone and the SDP cone). For example, the problem (1.17), (1.18) and others can be reformulated as this separable cone constraints MCP problem. Since the variational properties of such symmetric cones are well studied in literature [33, 86, 35, 97] and all the cones considering right now are $\mathcal{C}^2$-cone reducible, by combining the variational properties of the Ky Fan k-norm cone which we obtained before, we can derive the corresponding sensitivity results for the linear MCP problem with the separable cone constraints. Therefore, the sensitivity analysis results obtained in this chapter can be extended immediately to such linear MCP problems.

Finally, as we mentioned before, the work done on the sensitivity analysis of MOPs is far from comprehensive. It can be seen that some MOP problems may not be covered by this work due to the inseparable structure. For example, in order to study the sensitivity results of the MOP problem defined in (1.46), we must first study the variational properties of the epigraph cone $\mathcal{Q}$ of the positively homogenous convex function $f \equiv \max\{\lambda(\cdot), \|\cdot\|_2\} : \mathcal{S}^n \times \Re^{m\times n} \to (-\infty, \infty]$ such as the characterizations of tangent cone and the (inner and outer) second order tangent sets of $\mathcal{Q}$, the explicit expression of the support function of the second order tangent set of $\mathcal{Q}$, the $\mathcal{C}^2$-cone reducibility of $\mathcal{M}$ and the characterization of the critical cone of $\mathcal{Q}$. Certainly, the properties of spectral operators (the metric projection operator over the convex cone $\mathcal{Q}$) will play an important role in this study. Also, this is our future research direction.

Chapter 5

Conclusions

In this thesis, we study a class of optimization problems, which involve minimizing the sum of a linear function and a proper closed convex function subject to an affine constraint in the matrix space. Such optimization problems are said to be matrix optimization problems (MOPs). Many important optimization problems in diverse applications arising from a wide range of fields can be cast in the form of MOPs. In order to solve the defined MOP by the proximal point algorithms (PPAs), as an initial step, we do a systematic study on spectral operators. Several fundamental properties of spectral operators are studied, including the well-definiteness, the directional differentiability, the Fréchet-differentiability, the locally Lipschitz continuity, the ρ-order B(ouligand)-differentiability, the ρ-order G-semismooth and the characterization of Clarke's generalized Jacobian. This systematical study of spectral operators is of crucial importance in terms of the study of MOPs, since it provides the powerful tools to study both the efficient algorithms and the optimal theory of MOPs.

In the second part of this thesis, we discuss the sensitivity analysis of some MOP problems. We mainly focus on the linear MCP problems involving the Ky Fan k-norm epigraph cone $\mathcal{K}$. Firstly, we study some important variational properties of the Ky Fan k-norm epigraph cone $\mathcal{K}$, including the characterizations of tangent cone and the (inner and outer) second order tangent sets of $\mathcal{K}$, the explicit expression of the support function of the second order tangent set, the $\mathcal{C}^2$-cone reducibility of $\mathcal{K}$, the characterization of the critical cone of $\mathcal{K}$. By using these properties, we state the constraint nondegeneracy, the second order necessary condition and the (strong) second order sufficient condition of the linear matrix cone programming (MCP) problem involving the Ky Fan k-norm. For such linear MCP problems, we establish the equivalent links among the strong regularity of the KKT point, the strong second order sufficient condition and constraint nondegeneracy, and the non-singularity of both the B-subdifferenitial and Clarke's generalized Jacobian of the nonsmooth system at a KKT point. The extensions to other MOP problems are also discussed.

The work done in this thesis is far from comprehensive. There are many interesting topics for our future research. Firstly, the general framework of the classical PPAs for MOPs discussed in this thesis is heuristics. For applications, a careful study on the numerical implementation is an important issue. There is a great demand for efficient and robust solvers for solving MOPs, especially for problems that are large scale. On the other hand, our idea for solving MOPs is built on the classical PPA method. One may use other methods to solve MOPs. For example, in order to design the efficient and robust interior point method to MCPs, more insightful research on the geometry of the non-symmetric matrix cones as the Ky Fan k-norm cone is needed. In this thesis, we only study the sensitivity analysis of some MOP problems with special structures, such as the linear MCP problems involving the Ky Fan k-norm epigraph cone $\mathcal{K}$ and others. Another important research topic is the sensitivity analysis of the general MOP problems such as the nonlinear MCP problems and the MOP problems (1.2) and (1.3) with the general convex functions.

Bibliography

[1] F. Alizadeh, *Interior point methods in semidefinite programming with applications to combinatorial optimization*, SIAM Journal on Optimization, 5 (1995), pp. 13–51.

[2] A. Ben-Tal and A. Nemirovski, *Lectures on Modern Convex Optimization: analysis, algorithms, and engineering applications*, vol. 2, Society for Industrial Mathematics, 2001.

[3] R. Bhatia, *Matrix Analysis*, Springer Verlag, 1997.

[4] J. Bolte, A. Daniilidis, and A. Lewis, *Tame functions are semismooth*, Mathematical Programming, 117 (2009), pp. 5–19.

[5] J. Bonnans, R. Cominetti, and A. Shapiro, *Sensitivity analysis of optimization problems under second order regular constraints*, Mathematics of Operations Research, 23 (1998), pp. 806–831.

[6] ——, *Second order optimality conditions based on parabolic second order tangent sets*, SIAM Journal on Optimization, 9 (1999), pp. 466–492.

[7] J. Bonnans and A. Shapiro, *Optimization problems with perturbations: A guided tour*, SIAM review, 40 (1998), pp. 228–264.

[8] ——, *Perturbation Analysis of Optimization Problems*, Springer Verlag, 2000.

[9] S. Boyd, P. Diaconis, P. Parrilo, and L. Xiao, *Fastest mixing Markov chain on graphs with symmetries*, SIAM Journal on Optimization, 20 (2009), pp. 792–819.

[10] S. Boyd, P. Diaconis, and L. Xiao, *Fastest mixing Markov chain on a graph*, SIAM review, 46 (2004), pp. 667–689.

[11] S. Burer and R. Monteiro, *A nonlinear programming algorithm for solving semidefinite programs via low-rank factorization*, Mathematical Programming, 95 (2003), pp. 329–357.

[12] ——, *Local minima and convergence in low-rank semidefinite programming*, Mathematical Programming, 103 (2005), pp. 427–444.

[13] J. Cai, E. Candès, and Z. Shen, *A singular value thresholding algorithm for matrix completion*, SIAM Journal on Optimization, 20 (2010), pp. 1956–1982.

[14] E. Candès, X. Li, Y. Ma, and J. Wright, *Robust principal component analysis?*, Journal of the ACM (JACM), 58 (2011), p. 11.

[15] E. Candès and B. Recht, *Exact matrix completion via convex optimization*, Foundations of Computational Mathematics, 9 (2009), pp. 717–772.

[16] E. Candès and T. Tao, *The power of convex relaxation: Near-optimal matrix completion*, Information Theory, IEEE Transactions on, 56 (2010), pp. 2053–2080.

[17] Z. Chan and D. Sun, *Constraint nondegeneracy, strong regularity, and nonsingularity in semidefinite programming*, SIAM Journal on Optimization, 19 (2008), pp. 370–396.

[18] V. Chandrasekaran, S. Sanghavi, P. Parrilo, and A. Willsky, *Rank-sparsity incoherence for matrix decomposition*, SIAM Journal on Optimization, 21 (2011), pp. 572–596.

[19] X. Chen, H. Qi, and P. Tseng, *Analysis of nonsmooth symmetric-matrix-valued functions with applications to semidefinite complementarity problems*, SIAM Journal on Optimization, 13 (2003), pp. 960–985.

[20] X. Chen and P. Tseng, *Non-interior continuation methods for solving semidefinite complementarity problems*, Mathematical Programming, 95 (2003), pp. 431–474.

[21] M. Chu, R. Funderlic, and R. Plemmons, *Structured low rank approximation*, Linear algebra and its applications, 366 (2003), pp. 157–172.

[22] F. Clarke, *On the inverse function theorem*, Pacific J. Math, 64 (1976), pp. 97–102.

[23] ——, *Optimization and Nonsmooth Analysis.*, JOHN WILEY & SONS, NEW YORK, 1983.

[24] M. Coste, *An Introduction to o-minimal Geometry*, RAAG Notes, Institut de Recherche Mathématiques de Rennes, 1999.

[25] C. Davis, *All convex invariant functions of hermitian matrices*, Archiv der Mathematik, 8 (1957), pp. 276–278.

[26] B. De Moor, M. Moonen, L. Vandenberghe, and J. Vandewalle, *A geometrical approach for the identification of state space models with singular value decomposition*, in Acoustics, Speech, and Signal Processing, 1988. ICASSP-88., 1988 International Conference on, IEEE, 1988, pp. 2244–2247.

[27] V. Demyanov and A. Rubinov, *On quasidifferentiable mappings*, Optimization, 14 (1983), pp. 3–21.

[28] C. Ding, D. Sun, and J. Jane, *First order optimality conditions for mathematical programs with semidefinite cone complementarity constraints*, Preprint available at http://www.optimization-online.org/DB_FILE/2010/11/2820.pdf, (2010).

[29] C. Ding, D. Sun, J. Sun, and K. Toh, *Spectral operator of matrices*, Manuscripts in Preparation, National University of Singapore, (2012).

[30] C. Ding, D. Sun, and K. Toh, *An introduction to a class of matrix cone programming*, http://www.math.nus.edu.sg/~matsundf/IntroductionMCP_Sep_15.pdf, (2010).

[31] W. Donoghue, *Monotone Matrix Functions and Analytic Continuation, Grundlehren der mathematichen Wissenschaften 207*, Springer Verlag, 1974.

[32] B. Eaves, *On the basic theorem of complementarity*, Mathematical Programming, 1 (1971), pp. 68–75.

[33] F. Facchinei and J. Pang, *Finite-dimensional Variational Inequalities and Complementarity Problems*, vol. 1, Springer Verlag, 2003.

[34] K. Fan, *On a theorem of weyl concerning eigenvalues of linear transformations I*, Proceedings of the National Academy of Sciences of the United States of America, 35 (1949), pp. 652–655.

[35] F. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Clarendon Press, Oxford, 1994.

[36] T. Flett, *Differential Analysis: differentiation, differential equations, and differential inequalities*, Cambridge University Press, Cambridge, England, 1980.

[37] Y. Gao and D. Sun, *A majorized penalty approach for calibrating rank constrained correlation matrix problems*, Preprint available at http://www.math.nus.edu.sg/~matsundf/MajorPen.pdf, (2010).

[38] A. Greenbaum and L. Trefethen, *GMRES/CR and Arnoldi/Lanczos as matrix approximation problems*, SIAM Journal on Scientific Computing, 15 (1994), pp. 359–359.

[39] D. Gross, *Recovering low-rank matrices from few coefficients in any basis*, Information Theory, IEEE Transactions on, 57 (2011), pp. 1548–1566.

[40] N. Higham, *Computing a nearest symmetric positive semidefinite matrix*, Linear algebra and its applications, 103 (1988), pp. 103–118.

[41] J. Hiriart-Urruty and C. Lemaréchal, *Convex Analysis and Mminimization Algorithms, Vols. 1 and 2*, Springer-Verlag, 1993.

[42] R. Horn and C. Johnson, *Matrix Analysis*, Cambridge University Press, 1985.

[43] ——, *Topics in Matrix Analysis*, Cambridge University Press, 1991.

[44] A. Ioffe, *An invitation to tame optimization*, SIAM Journal on Optimization, 19 (2009), pp. 1894–1917.

[45] K. Jiang, D. Sun, and K. Toh, *A proximal point method for matrix least squares problem with nuclear norm regularization*, Technique report, National University of Singapore, (2010).

[46] ——, *A partial proximal point algorithm for nuclear norm regularized matrix least squares problems*, Technique report, National University of Singapore, (2012).

[47] R. Keshavan, A. Montanari, and S. Oh, *Matrix completion from a few entries*, Information Theory, IEEE Transactions on, 56 (2010), pp. 2980–2998.

[48] D. Klatte and B. Kummer, *Nonsmooth Equations in Optimization: regularity, calculus, methods, and applications*, Kluwer Academic Publishers, 2002.

[49] A. Korányi, *Monotone functions on formally real Jordan algebras*, Mathematische Annalen, 269 (1984), pp. 73–76.

[50] B. Kummer, *Lipschitzian inverse functions, directional derivatives, and applications in $C^{1,1}$-optimization*, Journal of Optimization Theory and Applications, 70 (1991), pp. 561–582.

[51] P. Lancaster, *On eigenvalues of matrices dependent on a parameter*, Numerische Mathematik, 6 (1964), pp. 377–387.

[52] W. Larimore, *Canonical variate analysis in identification, filtering, and adaptive control*, in Decision and Control, 1990., Proceedings of the 29th IEEE Conference on, Ieee, 1990, pp. 596–604.

[53] C. Lemaréchal and C. Sagastizábal, *Practical aspects of the Moreau-Yosida regularization: theoretical preliminaries*, SIAM Journal on Optimization, 7 (1997), pp. 367–385.

[54] A. Lewis, *Derivatives of spectral functions*, Mathematics of Operations Research, 21 (1996), pp. 576–588.

[55] A. Lewis and M. Overton, *Eigenvalue optimization*, Acta numerica, 5 (1996), pp. 149–190.

[56] A. Lewis and H. Sendov, *Twice differentiable spectral functions*, SIAM Journal on Matrix Analysis and Applications, 23 (2001), pp. 368–386.

[57] ——, *Nonsmooth analysis of singular values. Part II: Applications*, Set-Valued Analysis, 13 (2005), pp. 243–264.

[58] X. Lin and S. Boyd, *Fast linear iterations for distributed averaging*, Systems & Control Letters, 53 (2004), pp. 65–78.

[59] Z. Liu and L. Vandenberghe, *Interior-point method for nuclear norm approximation with application to system identification*, SIAM Journal on Matrix Analysis and Applications, 31 (2009), pp. 1235–1256.

[60] Z. Liu and L. Vandenberghe, *Semidefinite programming methods for system realization and identification*, in Decision and Control, 2009 held jointly with the 2009 28th Chinese Control Conference. CDC/CCC 2009. Proceedings of the 48th IEEE Conference on, IEEE, 2009, pp. 4676–4681.

[61] K. Löwner, *Über monotone matrixfunktionen*, Mathematische Zeitschrift, 38 (1934), pp. 177–216.

[62] N. A. Lynch, *Distributed algorithms*, Morgan Kaufmann, 1996.

[63] J. Malick, J. Povh, F. Rendl, and A. Wiegele, *Regularization methods for semidefinite programming*, SIAM Journal on Optimization, 20 (2009), pp. 336–356.

[64] F. Meng, D. Sun, and G. Zhao, *Semismoothness of solutions to generalized equations and the moreau-yosida regularization*, Mathematical programming, 104 (2005), pp. 561–581.

[65] B. Mordukhovich, *Generalized differential calculus for nonsmooth and set-valued mappings*, Journal of Mathematical Analysis and Applications, 183 (1994), pp. 250–288.

[66] J. Moreau, *Proximité et dualité dans un espace hilbertien*, Bull. Soc. Math. France, 93 (1965), pp. 273–299.

[67] M. Nashed, *Differentiability and related properties of nonlinear operators: Some aspects of the role of differentials in nonlinear functional analysis*, in Nonlinear Functional Analysis and Applications, L. Rall, ed., Academic Press, New York, 1971.

[68] Y. Nesterov and A. Nemirovsky, *Interior Point Polynomial Methods in Convex Programming*, SIAM Studies in Applied Mathematics, 1994.

[69] M. Overton, *On minimizing the maximum eigenvalue of a symmetric matrix*, SIAM Journal on Matrix Analysis and Applications, 9 (1988), pp. 256–268.

[70] M. Overton and R. Womersley, *On the sum of the largest eigenvalues of a symmetric matrix*, SIAM Journal Matrix Analysis and Applications, 13 (1992), pp. 41–45.

[71] ——, *Optimality conditions and duality theory for minimizing sums of the largest eigenvalues of symmetric matrices*, Mathematical Programming, 62 (1993), pp. 321–357.

[72] J. Pang, D. Sun, and J. Sun, *Semismooth homeomorphisms and strong stability of semidefinite and lorentz complementarity problems*, Mathematics of Operations Research, 28 (2003), pp. 39–63.

[73] G. Pataki, *On the rank of extreme matrices in semidefinite programs and the multiplicity of optimal eigenvalues*, Mathematics of Operations Research, 23 (1998), pp. 339–358.

[74] J. Povh, F. Rendl, and A. Wiegele, *A boundary point method to solve semidefinite programs*, Computing, 78 (2006), pp. 277–286.

[75] H. Qi and X. Yang, *Semismoothness of spectral functions*, SIAM Journal on Matrix Analysis and Applications, 25 (2004), pp. 784–803.

[76] L. Qi, *Convergence analysis of some algorithms for solving nonsmooth equations*, Mathematics of Operations Research, 18 (1993), pp. 227–244.

[77] B. Recht, *A simpler approach to matrix completion*, Preprint available at `http://pages.cs.wisc.edu/~brecht/publications.html`, (2009).

[78] B. Recht, M. Fazel, and P. Parrilo, *Guaranteed minimum-rank solutions of linear matrix equations via nuclear norm minimization*, SIAM Review, 52 (2010), pp. 471–501.

[79] S. Robinson, *First order conditions for general nonlinear optimization*, SIAM Journal on Applied Mathematics, 30 (1976), pp. 597–607.

[80] ——, *Strongly regular generalized equations*, Mathematics of Operations Research, 5 (1980), pp. 43–62.

[81] ——, *Local structure of feasible sets in nonlinear programming. ii: Nondegeneracy*, Mathematical programming study, 22 (1984), pp. 217–230.

[82] ——, *Local structure of feasible sets in nonlinear programming. iii: Stability and sensitivity*, Mathematical programming study, 30 (1987), pp. 45–66.

[83] R. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.

[84] ——, *Augmented Lagrangians and applications of the proximal point algorithm in convex programming*, Mathematics of operations research, 1 (1976), pp. 97–116.

[85] ——, *Monotone operators and the proximal point algorithm*, SIAM Journal on Control and Optimization, 14 (1976), pp. 877–898.

[86] R. Rockafellar and R.-B. Wets, *Variational Analysis*, Springer Verlag, 1998.

[87] S. Scholtes, *Introduction to Piecewise Differentiable Equations*, PhD thesis, Inst. für Statistik und Math. Wirtschaftstheorie, 1994.

[88] N. Schwertman and D. Allen, *Smoothing an indefinite variance-covariance matrix*, Journal of Statistical Computation and Simulation, 9 (1979), pp. 183–194.

[89] A. Shapiro, *On differentiability of symmetric matrix valued functions*, Preprint available at `http://www.optimization-online.org/DB_FILE/2002/07/499.pdf`, (2002).

[90] ——, *Sensitivity analysis of generalized equations*, Journal of Mathematical Sciences, 115 (2003), pp. 2554–2565.

[91] G. Stewart and J. Sun, *Matrix Perturbation Theory*, Academic press, 1990.

[92] J. Sturm, *Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones*, Optimization methods and software, 11 (1999), pp. 625–653.

[93] D. Sun, *Algorithms and Convergence Analysis for Nonsmooth Optimization and Nonsmooth Equations*, PhD thesis, Institute of Applied Mathematics, Chinese Academy of Sciences, China, 1994.

[94] ——, *The strong second-order sufficient condition and constraint nondegeneracy in nonlinear semidefinite programming and their implications*, Mathematics of Operations Research, 31 (2006), pp. 761–776.

[95] D. Sun and J. Sun, *Semismooth matrix-valued functions*, Mathematics of Operations Research, 27 (2002), pp. 150–169.

[96] ——, *Strong semismoothness of eigenvalues of symmetric matrices and its application to inverse eigenvalue problems*, SIAM Journal on Numerical Analysis, 40 (2003), pp. 2352–2367.

[97] ——, *Löwner's operator and spectral functions in euclidean jordan algebras*, Mathematics of Operations Research, 33 (2008), pp. 421–445.

[98] R. Tibshirani, *The LASSO method for variable selection in the cox model*, Statistics in medicine, 16 (1997), pp. 385–395.

[99] K. Toh, *GMRES vs. ideal GMRES*, SIAM Journal on Matrix Analysis and Applications, 18 (1997), pp. 30–36.

[100] K. Toh and L. Trefethen, *The chebyshev polynomials of a matrix*, SIAM Journal on Matrix Analysis and Applications, 20 (1998), pp. 400–419.

[101] M. Torki, *Second-order directional derivatives of all eigenvalues of a symmetric matrix*, Nonlinear analysis, 46 (2001), pp. 1133–1150.

[102] P. Tseng, *Merit functions for semi-definite complemetarity problems*, Mathematical Programming, 83 (1998), pp. 159–185.

[103] R. Tütüncü, K. Toh, and M. Todd, *Solving semidefinite-quadratic-linear programs using SDPT3*, Mathematical programming, 95 (2003), pp. 189–217.

[104] P. Van Overschee and B. De Moor, *N4SID: Subspace algorithms for the identification of combined deterministic-stochastic systems*, Automatica, 30 (1994), pp. 75–93.

[105] L. Vandenberghe and S. Boyd, *Semidefinite programming*, SIAM review, 38 (1996), pp. 49–95.

[106] M. Verhaegen, *Identification of the deterministic part of mimo state space models given in innovations form from input-output data*, Automatica, 30 (1994), pp. 61–74.

[107] M. Viberg, *Subspace-based methods for the identification of linear time-invariant systems*, Automatica, 31 (1995), pp. 1835–1851.

[108] J. von Neumann, *Some matrix inequalities and metrization of matric space*, Tomsk University Review, 1 (1937), pp. 286–300.

[109] J. Warga, *Fat homeomorphisms and unbounded derivate containers*, Journal of Mathematical Analysis and Applications, 81 (1981), pp. 545–560.

[110] G. Watson, *On matrix approximation problems with Ky Fan k norms*, Numerical Algorithms, 5 (1993), pp. 263–272.

[111] Z. Wen, D. Goldfarb, and W. Yin, *Alternating direction augmented lagrangian methods for semidefinite programming*, Mathematical Programming Computation, 2 (2010), pp. 1–28.

[112] J. Wright, A. Ganesh, S. Rao, and Y. Ma, *Robust principal component analysis: Exact recovery of corrupted low-rank matrices via convex optimization*, submitted to Journal of the ACM, (2009).

[113] B. Wu, C. Ding, D. Sun, and K. Toh, *On the Moreau-Yosida regularization of the vector k-norm related functions*, Preprint available at `http://www.optimization-online.org/DB_FILE/2011/03/2978.pdf`, (2011).

[114] Z. Yang, *A study on nonsymmetric matrix-valued functions*, Master's thesis, Department of Mathematics, National University of Singapore, 2009.

[115] L. Zhang, N. Zhang, and X. Xiao, *The second order directional derivative of symmetric matrix-valued functions*, Preprint available at `www.optimization-online.org/DB_FILE/2011/04/3010.pdf`, (2011).

[116] X. Zhao, *A semismooth Newton-CG augmented Lagrangian method for large scale linear and convex quadratic SDPs*, PhD thesis, National University of Singapore, 2009.

[117] X. Zhao, D. Sun, and K. Toh, *A Newton-CG augmented Lagrangian method for semidefinite programming*, SIAM Journal on Optimization, 20 (2010), pp. 1737–1765.

Index