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FULL LENGTH PAPER

A multi-stage convex relaxation approach to noisy structured low-rank matrix recovery

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Abstract

This paper concerns with a noisy structured low-rank matrix recovery problem which can be modeled as a structured rank minimization problem. We reformulate this problem as a mathematical program with a generalized complementarity constraint (MPGCC), and show that its penalty version, yielded by moving the generalized complementarity constraint to the objective, has the same global optimal solution set as the MPGCC does whenever the penalty parameter is over a certain threshold. Then, by solving the exact penalty problem in an alternating way, we obtain a multi-stage convex relaxation approach. We provide theoretical guarantees for our approach under a mild restricted eigenvalue condition, by quantifying the reduction of the error and approximate rank bounds of the first stage convex relaxation in the subsequent stages and establishing the geometric convergence of the error sequence in a statistical sense. Numerical experiments are conducted for some structured low-rank matrix recovery examples to confirm our theoretical findings. Our code can be achieved from https://doi.org/10.5281/zenodo.3600639.

Keywords Structured rank minimization · MPGCC · Exact penalty · Convex relaxation

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1 Introduction

The task of noisy structured low-rank matrix recovery is to seek a low-rank matrix with a certain structure consistent with some noisy linear measurements. Let $\overline{X}$ be the target matrix to be recovered and $b = \mathcal{A}\overline{X} + \xi$ be the noisy measurement vector, where $\mathcal{A} : \mathbb{R}^{n_1\times n_2} \to \mathbb{R}^m$ is the sampling operator and $\xi \in \mathbb{R}^m$ is the noisy vector with $\|\xi\| \le \delta$ for some $\delta > 0$. The noisy structured low-rank matrix recovery problem can be modeled as the rank minimization problem

$$\min_{X\in\mathbb{R}^{n_1\times n_2}} \Big\{\mathrm{rank}(X) \ \text{ s.t. } \|\mathcal{A}X - b\| \le \delta,\ X \in \Omega\Big\}, \tag{1}$$

where $\Omega \subseteq \mathbb{R}^{n_1\times n_2}$ is a compact convex set describing the structure of $\overline{X}$. Throughout this paper, we assume that $\overline{X}$ is a global optimal solution of (1) with $\mathrm{rank}(\overline{X}) = r$, and that the sampling operator $\mathcal{A}$ is defined by $\mathcal{A}X := (\langle A_1, X\rangle, \ldots, \langle A_m, X\rangle)^{\mathbb{T}}$ for $X \in \mathbb{R}^{n_1\times n_2}$, where $A_1, \ldots, A_m$ are the given matrices in $\mathbb{R}^{n_1\times n_2}$. Such a structured rank minimization problem has wide applications in system identification and control [11,13], signal and image processing [7,17], machine learning [36], multi-dimensional scaling in statistics [31], finance [30], and quantum tomography [16]. For instance, one is often led to seek a low-rank Hankel matrix in system identification and control, a low-rank correlation matrix in finance and a low-rank density matrix in quantum tomography.

Due to the combinatorial property of the rank function, the problem (1) is generally NP-hard. One popular way to deal with NP-hard problems is to use the convex relaxation technique, which typically yields a desirable local optimal solution via a single or a sequence of numerically tractable convex optimization problems. Fazel [11] initiated the research for the nuclear norm relaxation method, motivated by the fact that the nuclear norm is the convex envelope of the rank function in the unit ball on the spectral norm. In the past decade or so, this relaxation method has received much attention from many fields such as information, computer science, statistics, optimization, and so on (see, e.g., [5,16,20,21,28,33,38]), and it has been shown that a single nuclear norm minimization problem can recover the target matrix $\overline{X}$ under a certain restricted isometry property (RIP) of $\mathcal{A}$ when $\delta = 0$ and $\Omega = \mathbb{R}^{n_1\times n_2}$ [33] or yield a solution satisfying a certain error bound when $\delta > 0$ and $\Omega = \mathbb{R}^{n_1\times n_2}$ [4]. For its recoverability and error bounds under other conditions, the reader may refer to [10,28,34] and references therein.

Most of the existing low-rank matrix optimization models target the case $\Omega = \mathbb{R}^{n_1\times n_2}$. When the structure on the target matrix is known, it is reasonable to consider the rank minimization problem (1) with Ω indicating the available information. However, the (hard) constraint $X \in \Omega$ often contradicts the role of the nuclear norm in promoting a low-rank solution. For example, when Ω consists of the set of correlation matrices, the nuclear norm relaxation method for (1) may fail in generating a low-rank solution since the nuclear norm becomes a constant in the set Ω. In addition, although

some error bounds have been established for the nuclear norm relaxation method in the noisy setting [4,28,29], they are minimax-optimal up to a logarithmic factor of the dimension [29], instead of a constant factor like the l_1-norm relaxation method for sparse regression [32]. These two considerations motivate us to seek more efficient convex relaxations.

1.1 Main contribution

The main contribution of this work is the introduction of a multi-stage convex relaxation approach via an equivalent Lipschitz optimization reformulation. This approach can efficiently reduce the error bounds obtained from the nuclear norm convex relaxation. More specifically, we reformulate (1) as an equivalent MPGCC by using a variational characterization of the rank function and verify that its penalized version, yielded by moving the generalized complementarity constraint to the objective, has the same global optimal solution set as the MPGCC does once the penalty parameter is over a certain threshold. This exact penalty problem not only has a convex feasible set but also possesses a Lipschitz objective function with a bilinear structure, which offers a favorable Lipschitz reformulation for (1). To the best of our knowledge, this is the first equivalent Lipschitz characterization for low-rank optimization problems (although the nuclear norm relaxation is a Lipschitz characterization for low-rank optimization problems, it generally does not have the same global optimal solution set as the rank optimization problem does). With this reformulation, we propose a multi-stage convex relaxation approach by solving the exact penalty problem in an alternating way. Under a restricted eigenvalue condition weaker than the RIP condition used in [4,25], we quantify the reduction of the error and approximate rank bounds of the first stage nuclear norm convex relaxation in the subsequent stages, and establish the geometric convergence of the error sequence in a statistical sense. Among others, the latter entails an upper estimation for the stage number of the convex relaxations to make the estimation error to reach the statistical error level. The analysis shows that the error and approximate rank bounds of the nuclear norm relaxation are reduced most in the second stage and the reduction rate is at least 40% for the problems with a relatively worse restricted eigenvalue property, and the reduction becomes less as the number of stages increases and can be ignored after the fifth stage.

1.2 Related works

The idea of using the multi-stage convex relaxation for low-rank optimization problems is not new. In order to improve the solution quality of the nuclear norm relaxation method, some researchers pay their attention to nonconvex surrogates of low-rank optimization problems. Since seeking a global optimal solution of a nonconvex surrogate problem is almost as difficult as solving a low-rank optimization problem itself, they relax nonconvex surrogates into a sequence of simple matrix optimization problems, and develop the reweighted minimization methods (see [12,22,26]). In contrast to our multi-stage convex relaxation approach, such sequential convex relaxation methods are designed by solving a sequence of convex relaxation problems of nonconvex sur-

rogates instead of the equivalent reformulation. We also notice that the theoretical analysis in [25] for the reweighted trace norm minimization method [12] depends on the special property of the log-determinant function, which is not applicable to general low-rank optimization problems, and the theoretical guarantees in [22] were established only for the noiseless recovery problem.

Additionally, some researchers have reformulated low-rank optimization problems as smooth nonconvex problems with the help of low-rank decomposition of matrices in the attempt to achieve a desirable solution by solving the smooth nonconvex problems in an alternating way (actually by solving a sequence of simple convex matrix optimization problems); see, e.g., [19,35]. This class of convex relaxation methods has a theoretical guarantee, but is not applicable to those problems with hard constraints such as the problem (1).

Finally, it is worthwhile to point out that our multi-stage convex relaxation approach is highly relevant to the one proposed by Zhang [43] for sparse regularization problems and the rank-corrected procedure for the matrix completion problem with fixed coefficients [24]. The former is designed via solving a sequence of convex relaxation problems for the nonconvex surrogates of the zero-norm regularization problem. Since the singular values vectors are involved in low-rank matrix recovery, the analysis technique in [43] is not applicable to our multi-stage convex approach to problem (1). In particular, for low-rank matrix recovery, it is not clear whether the error sequence yielded by the multi-stage convex relaxation approach shrinks geometrically or not in a statistical sense, and if it does, under what conditions. We will answer these questions affirmatively in Sect. 4. The rank-corrected procedure [24] is actually a two-stage convex relaxation approach in which the first-stage is to find a good initial estimator and the second-stage is to solve the rank-corrected problem. This procedure has already been applied to nonlinear dimensionality reduction problems [8] and tensor completion problems [1]. However, when the rank of the true matrix is unknown, the rank-corrected problem in [24] needs to be constructed heuristically with the knowledge of the initial estimator, while each subproblem in our multi-stage convex relaxation approach stems from the global exact penalty of the equivalent MPGCC. In addition, the analysis in [24] is more reliant on concentration inequalities in probability analysis, whereas our analysis is deterministic and relies on the restricted eigenvalue property of $\mathcal{A}$.

1.3 Notation

We stipulate $n_1 \leq n_2$. Let $\mathbb{R}^{n_1\times n_2}$ be the vector space of all $n_1 \times n_2$ real matrices endowed with the trace inner product $\langle\cdot,\cdot\rangle$ and its induced norm $\|\cdot\|_F$. Let $\mathbb{O}^{n\times\kappa}$ be the set in $\mathbb{R}^{n\times\kappa}$ consisting of all matrices whose columns are of unit length and are mutually orthogonal to each other, and denote $\mathbb{O}^{n\times n}$ by $\mathbb{O}^n$. For a given matrix $X \in \mathbb{R}^{n_1\times n_2}$, we denote by $\|X\|_*$ and $\|X\|$ the nuclear norm and the spectral norm of X, respectively, and by $\sigma(X) \in \mathbb{R}^{n_1}$ the singular value vector of X with entries arranged in a non-increasing order, and write $\mathbb{O}^{n_1,n_2}(X) := \{(U, V) \in \mathbb{O}^{n_1} \times \mathbb{O}^{n_2} \mid X = U[\mathrm{Diag}(\sigma(X)) \;\; 0]V^{\mathbb{T}}\}$. Let e and I be the vector of all ones and the identity matrix whose dimensions are known from the context.

Let Φ be the family of closed proper convex functions $\phi: \mathbb{R} \to (-\infty, +\infty]$ satisfying

$$\mathrm{int}(\mathrm{dom}\,\phi) \supseteq [0,1], \quad 1 > t^* := \arg\min_{0\le t\le 1} \phi(t), \quad \phi(t^*) = 0 \quad \text{and} \quad \phi'_-(1) < +\infty. \tag{2}$$

For each $\phi \in \Phi$, let $\psi : \mathbb{R} \to (-\infty, +\infty]$ be the associated closed proper convex function:

$$\psi(t) := \begin{cases} \phi(t) & \text{if } t \in [0,1], \\ +\infty & \text{otherwise.} \end{cases} \tag{3}$$

Then from convex analysis [37] we know that the conjugate ψ^* of ψ has the properties:

$$\begin{cases} \partial\psi^*(t) = \left[(\psi^*)'_-(t), (\psi^*)'_+(t)\right] \subseteq [0,1] \quad \forall t \in \mathbb{R}, & \text{(4a)} \\ (\psi^*)'_+(t_1) \le (\psi^*)'_-(t) \le (\psi^*)'_+(t) \le (\psi^*)'_-(t_2) \quad \forall t_1 < t < t_2. & \text{(4b)} \end{cases}$$

In addition, we also need the eigenvalues of $\mathcal{A}^*\mathcal{A}$ restricted to a set of low-rank matrices, where $\mathcal{A}^*$ denotes the adjoint of $\mathcal{A}$. To this end, for a given positive integer k, we define

$$\vartheta_+(k) := \sup_{0<\mathrm{rank}(X)\le k} \frac{\langle X, \mathcal{A}^*\mathcal{A}(X)\rangle}{\|X\|_F^2} \quad \text{and} \quad \vartheta_-(k) := \inf_{0<\mathrm{rank}(X)\le k} \frac{\langle X, \mathcal{A}^*\mathcal{A}(X)\rangle}{\|X\|_F^2}. \tag{5}$$

2 Exact penalty for an equivalent reformulation

First of all, we shall provide an equivalent reformulation of the rank minimization problem (1) with the help of the following variational characterization of the rank function.

Lemma 1 *Let $\phi \in \Phi$. Then, for any given $X \in \mathbb{R}^{n_1\times n_2}$, it holds that*

$$\phi(1)\mathrm{rank}(X) = \min_{W\in\mathbb{R}^{n_1\times n_2}} \left\{ \sum_{i=1}^{n_1} \phi(\sigma_i(W)) : \|X\|_* - \langle W, X\rangle = 0,\ \|W\| \le 1 \right\}. \tag{6}$$

Proof We first argue that $\phi(1)\mathrm{rank}(X)$ is a lower bound for the optimal value of (6). Indeed, let W be an arbitrary feasible point of (6). From [18, Equation (3.3.25)],

$$\|X\|_* = \langle W, X\rangle \le \langle \sigma(W), \sigma(X)\rangle \le \|\sigma(X)\|_1 = \|X\|_*,$$

which implies that $\sum_{i=1}^{n_1}(1-\sigma_i(W))\sigma_i(X)=0$. Along with $\sigma_i(W)\in[0,1]$ for $i=1,\ldots,n_1$, we obtain $\sigma_i(W)=1$ if $\sigma_i(X)\neq 0$, and consequently $\sum_{i=1}^{n_1}\phi(\sigma_i(W))\geq \phi(1)\mathrm{rank}(X)$, i.e., $\phi(1)\mathrm{rank}(X)$ is a lower bound for the optimal value of (6). Now consider the matrix

$$W^*=U_1V_1^{\mathbb{T}}+t^*U_2[\mathrm{Diag}(e)\ \ 0]V_2^{\mathbb{T}}\quad\text{with}\quad([U_1\ \ U_2],[V_1\ \ V_2])\in\mathbb{O}^{n_1,n_2}(X),$$

where $U_1\in\mathbb{O}^{n_1\times\kappa}$ and $V_1\in\mathbb{O}^{n_2\times\kappa}$ for $\kappa=\mathrm{rank}(X)$, and t^* is defined in (2). It is immediate to check that W^* is feasible to (6) and $\sum_{i=1}^{n_1}\phi(\sigma_i(W^*))=\phi(1)\mathrm{rank}(X)$. This shows that the optimal value of (6) is equal to $\phi(1)\mathrm{rank}(X)$. The proof is completed. □

Recall that $\phi(1)>0$ for each $\phi\in\Phi$. By Lemma 1, we readily have the following result.

Proposition 1 *Let $\phi\in\Phi$. Then, the rank minimization problem (1) is equivalent to*

$$\begin{aligned}\min_{X,W\in\mathbb{R}^{n_1\times n_2}}&\sum_{i=1}^{n_1}\phi(\sigma_i(W))\\ \text{s.t. }&\|\mathcal{A}(X)-b\|\leq\delta,\ X\in\Omega,\\ &\|X\|_*-\langle W,X\rangle=0,\ \|W\|\leq 1\end{aligned}\tag{7}$$

in the sense that if X^* is globally optimal to (1), then $(X^*,U_1^*(V_1^*)^{\mathbb{T}}+t^*U_2^*[\mathrm{Diag}(e)\ 0](V_2^*)^{\mathbb{T}})$ is a global optimal solution of the problem (7) where $([U_1^*\ \ U_2^*],[V_1^*\ \ V_2^*])\in\mathbb{O}^{n_1,n_2}(X^*)$ with $U_1^*\in\mathbb{O}^{n_1\times r}$ and $V_1^*\in\mathbb{O}^{n_2\times r}$ for $r=\mathrm{rank}(X^*)$; and conversely, if (X^*,W^*) is a global optimal solution to (7), then X^* is globally optimal to (1).

The constraints $\|X\|_*-\langle W,X\rangle=0$ and $\|W\|\leq 1$ involve a complementarity relation which, for the positive semidefinite (PSD) rank minimization problem, is exactly the PSD cone complementarity relation. In view of this, we call the problem (7) an MPGCC. Due to the presence of the nonconvex constraint $\|X\|_*-\langle W,X\rangle=0$, the MPGCC (7) is as difficult as the original problem (1). Nevertheless, it provides us a new view to tackle the difficult rank minimization problem (1). Since numerically it is usually more convenient to handle nonconvex objective functions than to handle nonconvex constraints, we are motivated to investigate the following penalization of (7):

$$\begin{aligned}\min_{X,W\in\mathbb{R}^{n_1\times n_2}}&\sum_{i=1}^{n_1}\phi(\sigma_i(W))+\rho(\|X\|_*-\langle W,X\rangle)\\ \text{s.t.}\quad&\|\mathcal{A}(X)-b\|\leq\delta,\ X\in\Omega,\ \|W\|\leq 1.\end{aligned}\tag{8}$$

Next we shall verify that (8) is an exact penalty version for (7) in the sense that there exists a constant $\overline{\rho}>0$ such that the global optimal solution set of (8) associated to any $\rho>\overline{\rho}$ coincides with that of (7). To the best of our knowledge, there are only a few works devoted to mathematical programs with matrix cone complementarity

constraints [9,41], which mainly focus on the optimality conditions, but not the exact penalty conditions.

Theorem 1 *Let $\phi \in \Phi$ and denote by $\mathcal{F}$ the feasible set of the problem* (1)*. Then, there exists a constant $\alpha > 0$ such that for all $X \in \mathcal{F}$, $\sigma_r(X) \geq \alpha$, and the global optimal solution set of* (8) *associated to any $\rho > \phi'_-(1)/\alpha$ is the same as that of* (7)*.*

Proof We prove the first part of the conclusions by contradiction. Suppose that there exists a sequence $\{X^k\} \subset \mathcal{F}$ such that $\sigma_r(X^k) \to 0$. Notice that $\{X^k\}$ is bounded since $\mathcal{F}$ is bounded. Let $\widehat{X}$ be an accumulation point of $\{X^k\}$. By the closedness of $\mathcal{F}$ and the continuity of $\sigma_r(\cdot)$, we have $\widehat{X} \in \mathcal{F}$ and $\sigma_r(\widehat{X}) = 0$. This implies $\mathrm{rank}(\widehat{X}) \leq r-1$, contradicting the fact that the optimal value of (1) is equal to r. So, such α exists.

Fix an arbitrary $\rho > \phi'_-(1)/\alpha$. Then, for any $X \in \mathcal{F}$ and each $i \in \{1, 2, \ldots, r\}$,

$$\{1\} = \arg\min_{t\in[0,1]} \big\{\phi(t) + \rho\sigma_i(X)(1-t)\big\}. \tag{9}$$

Let $\mathcal{S}_\rho$ and $\mathcal{S}_\rho^*$ be the feasible set and the global optimal solution set of the penalty problem (8) associated to ρ, respectively, and denote by $\mathcal{S}$ and $\mathcal{S}^*$ the feasible set and the global optimal solution set of (7), respectively. We first establish the inclusion $\mathcal{S}_\rho^* \subseteq \mathcal{S}^*$. To achieve this goal, we need to verify that each $(X^*, W^*) \in \mathcal{S}_\rho^*$ satisfies

$$\|X^*\|_* - \langle W^*, X^*\rangle = 0 \quad \text{and} \quad \mathrm{rank}(X^*) = r. \tag{10}$$

Since $\mathcal{S}^* \subset \mathcal{S} \subset \mathcal{S}_\rho$ and $r\phi(1)$ is the optimal value of the problem (7), it holds that

$$r\phi(1) \geq \sum_{i=1}^{n_1} \phi(\sigma_i(W^*)) + \rho(\|X^*\|_* - \langle W^*, X^*\rangle). \tag{11}$$

In addition, from [18, Equation (3.3.25)], it follows that

$$\begin{aligned}
&\sum_{i=1}^{n_1} \phi(\sigma_i(W^*)) + \rho(\|X^*\|_* - \langle W^*, X^*\rangle) \\
&\quad \geq \sum_{i=1}^{n_1} \big[\phi(\sigma_i(W^*)) + \rho\sigma_i(X^*)(1-\sigma_i(W^*))\big] \\
&\quad \geq \sum_{i=1}^{r} \big[\phi(\sigma_i(W^*)) + \rho\sigma_i(X^*)(1-\sigma_i(W^*))\big] \\
&\quad \geq \sum_{i=1}^{r} \min_{t\in[0,1]} \big[\phi(t) + \rho\sigma_i(X^*)(1-t)\big] = r\phi(1),
\end{aligned}$$

where the second inequality is by the nonnegativity of $\phi(\sigma_i(W^*))$ and $\sigma_i(X^*)(1-\sigma_i(W^*))$ for all i, and the last one is due to (9). Together with (11), we obtain that

$$\begin{aligned}&\sum_{i=1}^{n_1}\phi(\sigma_i(W^*))+\rho(\|X^*\|_*-\langle W^*,X^*\rangle)\\&\quad=\sum_{i=1}^{r}\big[\phi(\sigma_i(W^*))+\rho\sigma_i(X^*)(1-\sigma_i(W^*))\big]\\&\quad=\sum_{i=1}^{r}\min_{t\in[0,1]}\big[\phi(t)+\rho\sigma_i(X^*)(1-t)\big]=r\phi(1).\end{aligned}$$

This, along with (9), implies that $\sigma_i(W^*)=1$ for $i=1,\ldots,r$. Substituting $\sigma_i(W^*)=1$ for $i=1,\ldots,r$ into the last equation and using the nonnegativity of ϕ in $[0,1]$, we deduce that $\sum_{i=r+1}^{n_1}\phi(\sigma_i(W^*))=0$ and $\|X^*\|_*=\langle W^*,X^*\rangle=\langle\sigma(X^*),\sigma(W^*)\rangle$. This means that $\sigma_i(W^*)=t^*$ for $i=r+1,\ldots,n_1$ and $\mathrm{rank}(X^*)=r$. Thus, the claimed equalities in (10) hold. Hence, $\mathcal{S}^*_\rho\subset\mathcal{S}$ and $\sum_{i=1}^{n_1}\phi(\sigma_i(W^*))=r\phi(1)$ for each $(X^*,W^*)\in\mathcal{S}^*_\rho$. Since the optimal value of (7) is $r\phi(1)$, we have $\mathcal{S}^*_\rho\subseteq\mathcal{S}^*$. For the reverse inclusion, let (X^*,W^*) be an arbitrary point from $\mathcal{S}^*$. Then $(X^*,W^*)\in\mathcal{S}_\rho$ and $\sum_{i=1}^{n_1}\phi(\sigma_i(W^*))=r\phi(1)$. While the last equation implies that the optimal value of (8) is exactly $r\phi(1)$. Thus, $S^*\subseteq\mathcal{S}^*_\rho$. The proof is then completed. □

Theorem 1 extends the exact penalty result of [2, Theorem 3.3] for the zero-norm minimization to the matrix setting, and further develops the exact penalty result of the rank-constrained minimization problems in [3, Theorem 3.1]. Observe that the objective function of (8) is globally Lipschitzian over its feasible set. Combining Theorem 1 with Proposition 1, we conclude that the rank minimization problem (1) is equivalent to the Lipschitzian optimization problem (8).

3 A multi-stage convex relaxation approach

The penalty problem (8) is equivalent to the problem (1), but it depends on the lower bound α for the rth largest singular value of all $X\in\mathcal{F}$, which may be difficult to estimate. This means that a sequence of penalty problems of the form (8) with non-decreasing ρ should be solved so as to target achieving a global optimal solution of (1). The problem (8) associated to a given $\rho>0$ is not globally solvable due to the nonconvexity of the objective function, but it becomes a nuclear semi-norm minimization with respect to X if the variable W is fixed and has a closed form solution of W (as will be shown later) if the variable X is fixed. This motivates us to propose a multi-stage convex relaxation approach to (1) by solving (8) in an alternating way.

Algorithm 1 (A multi-stage convex relaxation approach)

Initialization: Choose a function $\phi \in \Phi$. Take $W^0 = 0$ and set $k := 1$.

while the stopping conditions are not satisfied **do**

1. Solve the following nuclear semi-norm minimization problem

$$X^k \in \mathop{\arg\min}_{X\in\mathbb{R}^{n_1\times n_2}} \big\{\|X\|_* - \langle W^{k-1}, X\rangle : \ \|\mathcal{A}(X) - b\| \le \delta, \ X \in \Omega\big\}. \tag{12}$$

   If $k = 1$, select a suitable $\rho_1 > 0$ and go to Step (S.3); else go to Step (S.2).
2. Select a suitable ratio factor $\mu_k \ge 1$ and set $\rho_k := \mu_k \rho_{k-1}$.
3. Solve the following minimization problem

$$W^k \in \mathop{\arg\min}_{W\in\mathbb{R}^{n_1\times n_2}} \big\{\sum_{i=1}^{n_1} \phi(\sigma_i(W)) - \rho_k\langle W, X^k\rangle : \ \|W\| \le 1\big\}. \tag{13}$$

4. Let $k \leftarrow k + 1$, and then go to Step (S.1).

end while

The subproblem (12) corresponds to the penalty problem (8) associated to ρ_{k-1} with the variable W fixed to W^{k-1}. Since the set Ω is assumed to be compact, its solution X^k is well defined. Let X^k have the SVD as $U^k[\mathrm{Diag}(\sigma(X^k)) \ \ 0](V^k)^{\mathbb{T}}$. By [18, Eq.(3.3.25)], it is easy to check that $Z^* = U^k[\mathrm{Diag}(z^*) \ \ 0](V^k)^{\mathbb{T}}$ is globally optimal to (13) where

$$z^* \in \mathop{\arg\min}_{z\in\mathbb{R}^{n_1}} \left\{\sum_{i=1}^{n_1} \psi(z_i) - \rho\langle z, \sigma(X^k)\rangle\right\}; \tag{14}$$

and conversely, if W^* is globally optimal to (13), then $\sigma(W^*)$ is optimal to (14). Write

$$W^k := U^k[\mathrm{Diag}\big(w_1^k, \ldots, w_{n_1}^k\big) \ \ 0](V^k)^{\mathbb{T}} \ \text{ with } \ w_i^k \in \partial\psi^*(\rho_k\sigma_i(X^k)). \tag{15}$$

Together with [37, Theorem 23.5], it follows that such W^k is an optimal solution of the subproblem (13). This means that the main computational work of Algorithm 1 consists of solving a sequence of subproblems (12). Unless otherwise stated, in the sequel we choose $w_i^k = w_j^k$ whenever $\sigma_i(X^k) = \sigma_j(X^k)$, which ensures that $1 \ge w_1^k \ge \cdots \ge w_{n_1}^k \ge 0$.

Since $\|W^{k-1}\| \le 1$, the function $\|\cdot\|_* - \langle W^{k-1}, \cdot\rangle$ is a semi-norm over $\mathbb{R}^{n_1\times n_2}$. So, the subproblem (12) is a nuclear semi-norm minimization problem. When $k = 1$, it reduces to the nuclear norm minimization problem, i.e., the first stage of Algorithm 1 is exactly the nuclear norm convex relaxation. It should be emphasized that Algorithm 1 is different from the reweighted trace norm minimization method [12,25] and the iterative reweighted algorithm [22]. The former is proposed from the primal and dual viewpoint by solving an equivalent Lipschitz reformulation in an alternating way, whereas the latter is proposed from the primal viewpoint by relaxing a smooth nonconvex surrogate of (1).

To close this section, we illustrate the choice of w_i^k in (15) with two specific $\phi \in \Phi$.

Example 1 Let $\phi_1(t) = t$ for $t \in \mathbb{R}$. Clearly, $\phi_1 \in \Phi$ with $t^* = 0$. Moreover, for the function ψ_1 defined by (3) with ϕ_1, an elementary calculation yields that

$$\psi_1^*(s) = \begin{cases} s-1 & \text{if } s > 1; \\ 0 & \text{if } s \le 1 \end{cases} \quad \text{and} \quad \partial\psi_1^*(s) = \begin{cases} \{1\} & \text{if } s > 1; \\ [0,1] & \text{if } s = 1; \\ \{0\} & \text{if } s < 1. \end{cases} \tag{16}$$

Thus, one may choose $w_i^k = \begin{cases} 1 \text{ if } \sigma_i(X^k) \ge \frac{1}{\rho_k}; \\ 0 \text{ otherwise} \end{cases}$ for the matrix W^k in formula (15).

Example 2 Let $\phi_2(t) = -t - \frac{q-1}{q}(1-t+\epsilon)^{\frac{q}{q-1}} + \epsilon + \frac{q-1}{q}$ for $t \in (-\infty, 1+\epsilon)$ with $0 < q < 1$, where $\epsilon \in (0,1)$ is a constant. One can check that $\phi_2 \in \Phi$ with $t^* = \epsilon$. For the function ψ_2 defined by the Eq. (3) with ϕ_2, an elementary calculation yields that

$$\partial\psi_2^*(s) = \begin{cases} \{1\} & \text{if } \ s \ge \epsilon^{\frac{1}{q-1}} - 1; \\ \{1+\epsilon-(s+1)^{q-1}\} & \text{if } (1+\epsilon)^{\frac{1}{q-1}} - 1 < s < \epsilon^{\frac{1}{q-1}} - 1; \\ \{0\} & \text{if } s \le (1+\epsilon)^{\frac{1}{q-1}} - 1. \end{cases}$$

Hence, one may take $w_i^k = \min\left[1+\epsilon-(\rho_k\sigma_i(X^k)+1)^{q-1}, 1\right]$ for the matrix W^k in (15).

Remark 1 A constant $\epsilon \in (0,1)$ is introduced in ϕ_2 so as to ensure that $(\phi_2)'_-(1) < +\infty$, and then the problem (8) is a global exact penalization of (1). Thus, once $(\widehat{X}, \widehat{W})$ yielded by Algorithm 1 satisfies $\|X\|_* - \langle X, W\rangle = 0$, $\widehat{X}$ is at least a local minimum of the problem (1) since each feasible solution of (1) is locally optimal.

4 Theoretical guarantees of Algorithm 1

In this section, we shall provide the theoretical guarantees of Algorithm 1 under a mild condition for the restricted eigenvalues of $\mathcal{A}^*\mathcal{A}$, which is stated as follows.

Assumption 1 There exist a constant $c \in [0, \sqrt{2})$ and an integer $s \in [1, \frac{n_1-2r}{2}]$ such that $\frac{\vartheta_+(s)}{\vartheta_-(2r+2s)} \le 1 + \frac{2c^2 s}{r}$, where $\vartheta_+(\cdot)$ and $\vartheta_-(\cdot)$ are the functions defined by (5).

Assumption 1 requires the restricted eigenvalue ratio of $\mathcal{A}^*\mathcal{A}$ to grow sublinearly in s. This condition, extending the sparse eigenvalue condition used for the analysis of sparse regularization (see [42,43]), is weaker than the RIP condition $\delta_{4r} < \sqrt{2}-1$ used in [4] for $n_1 \ge 4r$, where δ_{kr} is the kr-restricted isometry constant of $\mathcal{A}$ defined as in [4]. Indeed, from the definitions of $\vartheta_+(\cdot)$ and $\vartheta_-(\cdot)$, it is immediate to have that

$$\frac{\vartheta_+(r)}{\vartheta_-(2r+2r)} \le \frac{1+\delta_{4r}}{1-\delta_{4r}} < 1 + \frac{2\sqrt{2}-2}{2-\sqrt{2}} < 1 + 2 \times 0.843^2.$$

This shows that $c = 0.843$ is such that $\frac{\vartheta_+(s)}{\vartheta_-(2r+2s)} \le 1 + \frac{2c^2 s}{r}$ for $s = r$. In addition, this condition is also weaker than the RIP condition $\delta_{3r} < 2\sqrt{5} - 4$ used in [25] for $n_1 \ge 3r$, where r is an arbitrary even number or r is an odd number greater than 11. To see this, let r be an arbitrary even number or be an odd number greater than 11. Then,

$$\max\left(\frac{\vartheta_+(r/2)}{\vartheta_-(2r+r)}, \frac{\vartheta_+((r-1)/2)}{\vartheta_-(2r+r-1)}\right) \le \frac{1+\delta_{r/2}}{1-\delta_{3r}} \le \frac{1+\delta_{3r}}{1-\delta_{3r}} < 1 + \frac{4\sqrt{5}-8}{5-2\sqrt{5}}. \tag{17}$$

So, $c = 1.34$ and 1.403 are respectively such that $\frac{\vartheta_+(s)}{\vartheta_-(2r+2s)} \le 1 + \frac{2c^2 s}{r}$ for $s = \frac{r}{2}$ and $\frac{r-1}{2}$.

Recall that $\overline{X}$ is assumed to be a global optimal solution of (1) with $\mathrm{rank}(\overline{X}) = r$. In the sequel, we let $\overline{X}$ have the SVD as $\overline{U}[\mathrm{Diag}(\sigma(\overline{X}))\ 0]\overline{V}^{\mathbb{T}}$, where $\overline{U} = [\overline{U}_1\ \overline{U}_2] \in \mathbb{O}^{n_1}$ and $\overline{V} = [\overline{V}_1\ \overline{V}_2] \in \mathbb{O}^{n_2}$ with $\overline{U}_1 \in \mathbb{O}^{n_1 \times r}$ and $\overline{V}_1 \in \mathbb{O}^{n_2 \times r}$, and write $\mathcal{T} := \mathcal{T}(\overline{X})$ where $\mathcal{T}(\overline{X})$ is the tangent space at $\overline{X}$ associated to the constraint $\mathrm{rank}(X) \le r$ (see (32)). Let

$$\gamma_{k-1} := \frac{\|\mathcal{P}_{\mathcal{T}}(W^{k-1} - \overline{U}_1\overline{V}_1^{\mathbb{T}})\|_F}{\sqrt{2r}(1 - \|\mathcal{P}_{\mathcal{T}^\perp}(W^{k-1})\|)} \quad \text{for } k = 1, 2, \ldots. \tag{18}$$

The proofs of all the results in the subsequent subsections are given in "Appendix 3".

4.1 Error and approximate rank bounds

Under Assumption 1, when $\gamma_{k-1} \in [0, 1/c)$ for some $k \ge 1$, we can establish the following error bound and approximate rank bound for the solution X^k of the kth subproblem.

Proposition 2 *Suppose that Assumption 1 holds and* $0 \le \gamma_{k-1} < 1/c$ *for some* $k \ge 1$. *Then*

$$\left\|X^k - \overline{X}\right\|_F \le \Xi(\gamma_{k-1}) \quad \text{and} \quad \left\|\mathcal{P}_{\mathcal{T}^\perp}(X^k)\right\|_* \le \Gamma(\gamma_{k-1}), \tag{19}$$

where $\Xi : [0, 1/c) \to \mathbb{R}_+$ *and* $\Gamma : [0, 1/c) \to \mathbb{R}_+$ *are the increasing functions defined by*

$$\Xi(t) := \frac{2\delta\sqrt{\vartheta_+(2r+s)}}{\vartheta_-(2r+s)} \cdot \frac{1}{1-ct}\sqrt{1 + \frac{rt^2}{2s}} \quad \text{and} \quad \Gamma(t) := \frac{2\delta\sqrt{\vartheta_+(2r+s)}}{\vartheta_-(2r+s)} \cdot \frac{\sqrt{2r}t}{1-ct}. \tag{20}$$

Remark 2 **(a)** Since $\|\mathcal{P}_{\mathcal{T}^\perp}(X^k)\|_* = 0$ implies that $\mathrm{rank}(X^k) \le 2r$, it is reasonable to view $\|\mathcal{P}_{\mathcal{T}^\perp}(X^k)\|_*$ as a measure for the approximate rank of X^k. So, the second inequality in (19) provides an approximate rank bound for X^k. The error and approximate rank bounds in (19) consist of two parts: one part is the statistical error

$\Xi(0) = \frac{2\delta\sqrt{\vartheta_+(2r+s)}}{\vartheta_-(2r+s)}$ from the noise and the operator $\mathcal{A}$, and the other part is the estimation error from γ_{k-1}.

(b) Since $W^0 = 0$, we have $\gamma_0 = \frac{1}{\sqrt{2r}}\|\overline{U}_1\overline{V}_1^{\mathbb{T}}\|_F = \frac{1}{\sqrt{2}} < \frac{1}{c}$. Hence, under Assumption 1, the error and approximate rank bounds of the nuclear norm convex relaxation are

$$\left\|X^1 - \overline{X}\right\|_F \le \Xi(\gamma_0) = \Xi(1/\sqrt{2}) \quad \text{and} \quad \left\|\mathcal{P}_{\mathcal{T}^\perp}(X^1)\right\|_* \le \Gamma(\gamma_0) = \Gamma(1/\sqrt{2}). \tag{21}$$

Moreover, if Assumption 1 is satisfied with $s = r/2$ and $c < \sqrt{2} - \frac{2(1-\delta_{3r}(1+\sqrt{5}/2))}{\sqrt{3}(1-\delta_{3r})}$ for $\delta_{3r} < 2\sqrt{5}-4$, then the error bound $\Xi(\gamma_0)$ is tighter than the bound $\frac{3\delta\sqrt{1+\delta_{3r}}}{1-\delta_{3r}(1+\sqrt{5}/2)}$ given by [25, Theorem III.1] with $C_{1,1} = 1$ for the nuclear norm relaxation because

$$\Xi(\gamma_0) = \frac{\sqrt{\vartheta_+(2.5r)}\sqrt{6}\delta}{\big(1-c/\sqrt{2}\big)\vartheta_-(2.5r)} \le \frac{\sqrt{1+\delta_{3r}}\sqrt{6}\delta}{\big(1-c/\sqrt{2}\big)(1-\delta_{3r})} < \frac{3\delta\sqrt{1+\delta_{3r}}}{1-\delta_{3r}(1+\sqrt{5}/2)}.$$

Remark 2 (b) says that under Assumption 1 the solution X^1 of the first stage convex relaxation has the error and approximate rank bounds as in (21). However, it is not clear whether X^k ($k \ge 2$) has such error and approximate rank bounds or not. The following theorem states that if in addition $\sigma_r(\overline{X}) > 2\Xi(\gamma_0)$ and ρ_1 and μ_k are appropriately chosen, all X^k ($k \ge 2$) have the bounds as in (19), and more importantly, their error and approximate rank bounds are, respectively, smaller than those of X^1. To achieve this result, we need the sequence $\{\widetilde{\gamma}_k\}_{k\ge1}$, which is defined recursively with $\widetilde{\gamma}_0 = \gamma_0$ as

$$\begin{cases} \widetilde{\gamma}_k := \dfrac{\sqrt{r}(1-\widetilde{b}_k) + (\sqrt{2}\widetilde{a}_k + 1)\widetilde{\beta}_k}{\sqrt{2r}(1-\widetilde{a}_k)(1-\widetilde{\beta}_k^2)} \quad \text{with} \quad \widetilde{a}_k = (\psi^*)'_+\big[\rho_k\Xi(\widetilde{\gamma}_{k-1})\big], & \text{(22a)} \\ \widetilde{b}_k = (\psi^*)'_-\big[\rho_k(\sigma_r(\overline{X}) - \Xi(\widetilde{\gamma}_{k-1}))\big], \quad \widetilde{\beta}_k = -\dfrac{1}{\sqrt{2}}\ln\Big[1 - \dfrac{\sqrt{2}\Xi(\widetilde{\gamma}_{k-1})}{\sigma_r(\overline{X})}\Big]. & \text{(22b)} \end{cases}$$

Theorem 2 *Suppose that Assumption 1 holds and $\sigma_r(\overline{X}) > 2\Xi(\gamma_0)$. If the parameters ρ_1 and μ_k are chosen such that $\widetilde{a}_1 < \frac{(\widetilde{b}_1-\widetilde{\beta}_1^2)\sqrt{r}-\widetilde{\beta}_1}{(1-\widetilde{\beta}_1^2)\sqrt{r}+\sqrt{2}\widetilde{\beta}_1}$ and $\mu_k \in \big[1, \frac{\Xi(\widetilde{\gamma}_{k-2})}{\Xi(\widetilde{\gamma}_{k-1})}\big]$, respectively, then all X^k ($k \ge 1$) satisfy the inequalities in (19), and for $k = 2, 3, \ldots$ it holds that*

$$\begin{aligned} &\left\|X^k - \overline{X}\right\|_F \le \Xi(\gamma_{k-1}) \le \Xi(\widetilde{\gamma}_{k-1}) < \Xi(\widetilde{\gamma}_{k-2}) < \cdots < \Xi(\widetilde{\gamma}_0) = \Xi(\gamma_0), \\ &\|\mathcal{P}_{\mathcal{T}^\perp}(X^k)\|_* \le \Gamma(\gamma_{k-1}) \le \Gamma(\widetilde{\gamma}_{k-1}) < \Gamma(\widetilde{\gamma}_{k-2}) < \cdots < \Gamma(\widetilde{\gamma}_0) = \Gamma(\gamma_0). \end{aligned}$$

Table 1 Reduction rate of the error bounds of the first stage in the 2nd–5th stage

	ρ_1	c	0	0.1	0.3	0.5	0.7	0.9
ϕ_1	$\left[\frac{0.29\alpha}{\sigma_r(\overline{X})}, \frac{\alpha}{\sigma_r(\overline{X})}\right)$	$\Xi(\widetilde{\gamma}_1)/\Xi(\gamma_0)$	0.819	0.766	0.658	0.547	0.433	0.316
		$\Xi(\widetilde{\gamma}_2)/\Xi(\gamma_0)$	0.818	0.763	0.652	0.537	0.420	0.302
		$\Xi(\widetilde{\gamma}_3)/\Xi(\gamma_0)$	0.818	0.763	0.651	0.536	0.420	0.302
		$\Xi(\widetilde{\gamma}_4)/\Xi(\gamma_0)$	0.818	0.763	0.651	0.536	0.420	0.302
ϕ_2	$\left[\frac{0.24\alpha}{\sigma_r(\overline{X})}, \frac{4.42\alpha}{\sigma_r(\overline{X})}\right]$	$\Xi(\widetilde{\gamma}_1)/\Xi(\gamma_0)$	0.975	0.969	0.955	0.934	0.905	0.856
		$\Xi(\widetilde{\gamma}_2)/\Xi(\gamma_0)$	0.967	0.958	0.931	0.888	0.816	0.689
		$\Xi(\widetilde{\gamma}_3)/\Xi(\gamma_0)$	0.965	0.954	0.920	0.760	0.752	0.572
		$\Xi(\widetilde{\gamma}_4)/\Xi(\gamma_0)$	0.965	0.953	0.915	0.744	0.714	0.516
		$\Xi(0)/\Xi(\gamma_0)$	0.817	0.759	0.644	0.528	0.413	0.297

Remark 3 **(a)** Theorem 2 shows that under Assumption 1 and $\sigma_r(\overline{X}) > 2\Xi(\gamma_0)$, if ρ_1 and μ_k are chosen appropriately, then the error and approximate rank bounds of X^k ($k \geq 2$) improve those of X^1 at least by $1 - \frac{\Xi(\widetilde{\gamma}_{k-1})}{\Xi(\gamma_0)}$ and $1 - \frac{\Gamma(\widetilde{\gamma}_{k-1})}{\Gamma(\gamma_0)}$, respectively. **(b)** The choice of ρ_1 depends on $\Xi(\gamma_0)$. Take the function ϕ_1 in Example 1 for instance. If $\sigma_r(\overline{X}) = \alpha\Xi(\gamma_0)$ for $\alpha \geq 2.5$, by virtue of the definitions of $\widetilde{a}_1, \widetilde{b}_1$ and $\widetilde{\beta}_1$ and (16), it is easy to check that $\widetilde{a}_1 = 0, \widetilde{b}_1 = 1$ and $\widetilde{\beta}_1 \in [0, 0.6)$, and hence $(\widetilde{b}_1 - \widetilde{\beta}_1^2)\sqrt{r} - \widetilde{\beta}_1 > 0$. This means that $\left(\frac{1}{(\alpha-1)\Xi(\gamma_0)}, \frac{1}{\Xi(\gamma_0)}\right)$ is the range of choice for ρ_1. For numerical computations, one may estimate r and $\sigma_r(\overline{X})$ with the help of $\sigma(X^1)$.

To close this subsection, we illustrate the ratios $\frac{\Xi(\widetilde{\gamma}_{k-1})}{\Xi(\gamma_0)}$ and $\frac{\Gamma(\widetilde{\gamma}_{k-1})}{\Gamma(\gamma_0)}$ by using ϕ_1 and ϕ_2 with $q = 1/2$ and $\epsilon = 10^{-3}$. To this end, we suppose that Assumption 1 holds with $r = 10, s = r/2$ and $\sigma_r(\overline{X}) = \alpha\Xi(\gamma_0)$ for $\alpha \geq 4.5$. Then, for those c in the first row of Table 1, one may compute the ratios $\frac{\Xi(\widetilde{\gamma}_{k-1})}{\Xi(\gamma_0)}$ and $\frac{\Gamma(\widetilde{\gamma}_{k-1})}{\Gamma(\gamma_0)}$ as those in the last six columns of Table 1 with ρ_1 chosen as the middle point of the interval and $\mu_k \equiv 1$. We see that the error bound of the first stage is reduced most in the second stage, and as the number of stages increases, the reduction becomes less. For Algorithm 1 with ϕ_1, the reduction is close to the limit $\frac{\Xi(0)}{\Xi(\widetilde{\gamma}_0)}$ when $k = 5$, but for Algorithm 1 with ϕ_2, there is a little room for the reduction especially for those $\mathcal{A}^*\mathcal{A}$ with $c \geq 0.5$.

4.2 Geometric convergence

Generally speaking, because of the presence of the noise, it is impossible for the error sequence $\{\|X^k - \overline{X}\|_F\}_{k\geq 1}$ to decrease and then converge geometrically. However, one may achieve its geometric convergence in a statistical sense as in the following theorem.

Theorem 3 *Suppose that Assumption 1 holds and* $\sigma_r(\overline{X}) > \max(2, \sqrt{2} + \alpha)\Xi(\gamma_0)$ *holds with* $\alpha = \frac{1+\sqrt{2}\widetilde{a}_1}{(1-\widetilde{a}_1)(1-\widetilde{\beta}_1^2)\sqrt{r+4s}}$. *If* ρ_1 *and* μ_k *are chosen as in Theorem 2, then for* $k = 1, 2, \ldots,$

$$\|X^k - \overline{X}\|_F \le \frac{\Xi(0)}{1-c\widetilde{\gamma}_1}\Big[1+\frac{(1-\widetilde{b}_1)\sqrt{r}}{2(1-\widetilde{a}_1)(1-\widetilde{\beta}_1^2)\sqrt{s}}\Big] + \Big[\frac{\alpha\,\Xi(\gamma_0)}{\sigma_r(\overline{X})-\sqrt{2}\Xi(\gamma_0)}\Big]^{k-1}\|X^1-\overline{X}\|_F. \tag{23}$$

Remark 4 **(a)** The requirement $\sigma_r(\overline{X}) > \max(2,\sqrt{2}+\alpha)\Xi(\gamma_0)$ in Theorem 3 is bit stronger than $\sigma_r(\overline{X}) > 2\Xi(\gamma_0)$. Take ϕ_1 for example. When $\sigma_r(\overline{X}) \ge 2.4\Xi(\gamma_0)$, this requirement is automatically satisfied. Also, now we have that $\varrho := \frac{\alpha\,\Xi(\gamma_0)}{\sigma_r(\overline{X})-\sqrt{2}\Xi(\gamma_0)} \le 0.76$.

(b) The first term of the sum on the right hand side of (23) represents the statistical error arising from the noise and the sampling operator $\mathcal{A}$, and the second term is the estimation error related to the multi-stage convex relaxation. Clearly, the statistical error is of a certain order of $\Xi(0)$. Thus, to guarantee that the second term is less than the statistical error, at most $\overline{k}$ stage convex relaxations are required, where

$$\overline{k} = \frac{\log(\Xi(0)) - \log(\|X^1-\overline{X}\|_F)}{\log\varrho} + 1 \le \frac{\log(\Xi(0)/\Xi(\gamma_0))}{\log\varrho} + 1.$$

Take $\varrho = 0.7$ for example. When $s = r$, one can calculate that $\overline{k} \le 2$ if $c = 0.3$, and $\overline{k} \le 4$ if $c = 0.7$. This means that, for those $\mathcal{A}^*\mathcal{A}$ with a worse restricted eigenvalue condition, more than two stage convex relaxations are needed to yield a satisfactory solution.

For the analysis in the previous two subsections, the condition $\sigma_r(\overline{X}) \ge \alpha\Xi(\gamma_0)$ for a certain $\alpha > 2$ is required for the decreasing of the error and approximate rank bounds of the first stage convex relaxation and the contraction of the error sequence. Such a condition is necessary for the low-rank recovery since, when the smallest nonzero singular is mistaken as a zero, the additional singular vectors will yield a large error. In fact, in the geometric convergence analysis of sparse vector optimization (see [43]), the error bound of the first stage was implicitly assumed not to be too large. In addition, we observe that the structure information of $\overline{X}$ does not lend any help to the low-rank matrix recovery in terms of convergence rates. However, when the true matrix has a certain structure, it is necessary to incorporate such structure information into model (1). Otherwise, the solution X^k yielded by the multi-stage convex relaxation may not satisfy the structure constraint, and then it is impossible to control the error of X^k to the true matrix $\overline{X}$.

Finally, we point out that when the components $\xi_1, \xi_2, \ldots, \xi_m$ of the noisy vector ξ are independent (but not necessarily identically distributed) sub-Gaussians, i.e., there exists a constant $\sigma \ge 0$ such that $\mathbb{E}[e^{t\xi_i}] \le e^{\sigma^2t^2/2}$ holds for all i and any $t \in \mathbb{R}$, by Lemma 8 in "Appendix 3", the conclusions of Theorems 2 and 3 hold with $\delta = \sqrt{m}\sigma$ with probability at least $1 - \exp(1 - \frac{c_1 m}{4})$ for an absolute constant $c_1 > 0$. For the random $\mathcal{A}$, the following result is immediate by [4, Theorem 2.3] and the first inequality in (17).

Theorem 4 *Fix $\overline{\delta} \in (0, 1/2)$ and let $\mathcal{A}$ be a random measurement ensemble obeying the following conditions: for any given $X \in \mathbb{R}^{n_1 \times n_2}$ and any fixed $0 < t < 1$,*

$$\mathbb{P}\left\{ \left| \|\mathcal{A}(X)\|^2 - \|X\|_F^2 \right| > t\|X\|_F^2 \right\} \leq C \exp(-c_2 m) \tag{24}$$

for fixed constants $C, c_2 > 0$ (which may depend on t). If $m \geq 3C(n_1 + n_2 + 1)r$ with $C > \frac{\log(36\sqrt{2}/\overline{\delta})}{c_2}$, then Assumption 1 holds for $s = r/2$ and $c = \sqrt{\frac{2\overline{\delta}}{1-\overline{\delta}}}$ with probability exceeding $1-2\exp(-dm)$ where $d = c_2 - \frac{\log(36\sqrt{2}/\overline{\delta})}{C}$. Consequently, when $0 \leq \gamma_{k-1} < 1/c$, the bounds in (19) holds with probability at least $1 - 2\exp(-dm)$ for such random measurements.

As remarked after [4, Theorem 2.3], the condition in (24) holds when $\mathcal{A}$ is a Gaussian random measurement ensemble (i.e., $A_1, \ldots, A_m$ are independent from each other and each A_i contains i.i.d. entries $\mathcal{N}(0, 1/m)$); or when each entry of each A_i has i.i.d. entries that are equally likely to take $\frac{1}{\sqrt{m}}$ or $-\frac{1}{\sqrt{m}}$; or when $\mathcal{A}$ is a random projection (see [33]).

5 Numerical experiments

In this section, we shall test the theoretical results of Sect. 4 with Algorithm 1 solving low-rank matrix recovery problems, including matrix sensing and matrix completion problems. During the testing, we choose ϕ_2 with $q = 1/2$ and $\epsilon = 10^{-3}$ for the function ϕ in Algorithm 1. Although Table 1 shows that Algorithm 1 with ϕ_1 reduces the error faster than Algorithm 1 with ϕ_2 does, our preliminary testing indicates that the latter has a little better performance in reducing the relative error. In addition, we choose $\rho_1 = 10/\|X^1\|$ and $\mu_k = 5/4$ ($k \geq 2$) for Algorithm 1. All the results were run on the Windows system with an Intel(R) Core(TM) i7-7700 CPU 2.80GHz.

5.1 Low-rank matrix sensing problems

We test the performance of Algorithm 1 with some matrix sensing problems in which some entries are known exactly. Specifically, there are **5** entries of the true $\overline{X} \in \mathbb{R}^{n_1 \times n_2}$ assumed to be known exactly. We generate the true $\overline{X}$ of rank r in the following command

```
XR = randn(n1,r);  XL = randn(n2,r); Xbar = XR*XL'.
```

For these problems, $\Omega = \left\{X \in \mathbb{R}^{n_1 \times n_2} \mid \mathcal{B}(X) = d, \|X\| \leq R\right\}$ for a constant $R > 0$ with

$$\mathcal{B}(X) := (\langle E_{ij}, X\rangle : (i, j) \in \Upsilon_{\text{fix}})^{\mathbb{T}} \text{ and } d := (\langle E_{ij}, \overline{X}\rangle : (i, j) \in \Upsilon_{\text{fix}})^{\mathbb{T}}, \tag{25}$$

where E_{ij} is an $n_1 \times n_2$ matrix with the (i, j)th entry being 1 and other entries being 0, and Υ_{fix} is the set consisting of the indices of known entries. We successively generate the matrices $A_1, \ldots, A_m \in \mathbb{R}^{n_1 \times n_2}$ with i.i.d. standard normal entries to formulate the

sampling operator $\mathcal{A}$. Such $\mathcal{A}$ satisfies the RIP property with a high probability by [33], which means that the restricted eigenvalues of $\mathcal{A}^*\mathcal{A}$ can satisfy Assumption 1 with a high probability from the discussions after Assumption 1. Then, we successively generate the standard Gaussian noises $\varepsilon_1, \ldots, \varepsilon_m$ to formulate the observation vector b by

$$b = \mathcal{A}(\overline{X}) + 0.1(\|\mathcal{A}(\overline{X})\|/\|\varepsilon\|)\varepsilon \quad \text{with} \quad \varepsilon = (\varepsilon_1, \ldots, \varepsilon_m)^{\mathbb{T}}. \tag{26}$$

For the testing in the rest of this subsection, we choose $R = 10^3\|\overline{X}\|$ and $\delta = 0.1\|b\|$.

Let $\delta_S(\cdot)$ denote the indicator function of a set S, i.e., $\delta_S(x) = 0$ if $x \in S$ and otherwise $\delta_S(x) = +\infty$, and let $C = W^{k-1}$. Then the subproblem (12) can be equivalently written as

$$\begin{aligned} \min_{X,Z\in\mathbb{R}^{n_1\times n_2},z\in\mathbb{R}^m} & \|X\|_* - \langle C, X\rangle + \delta_{\mathcal{R}}(z) + \delta_{\Lambda}(Z) \\ \text{s.t. } & \mathcal{A}(X) - z - b = 0,\ \mathcal{B}(X) - d = 0,\ X - Z = 0 \end{aligned} \tag{27}$$

where $\mathcal{R} := \{z \in \mathbb{R}^m \mid \|z\| \le \delta\}$ and $\Lambda := \{Z \in \mathbb{R}^{n_1\times n_2} \mid \|Z\| \le R\}$. After an elementary calculation, one may obtain the dual problem of (27) as follows

$$\begin{aligned} \min_{Y,\Gamma\in\mathbb{R}^{n_1\times n_2},\xi,u\in\mathbb{R}^m,\zeta\in\mathbb{R}^{|\gamma_{\rm fix}|}} & \langle b,\xi\rangle + \langle d,\zeta\rangle + \delta\|u\| + R\|Y\|_* \\ \text{s.t. } & C - \mathcal{A}^*(\xi) - \mathcal{B}^*(\zeta) - Y - \Gamma = 0,\ \xi - u = 0,\ \|\Gamma\| \le 1. \end{aligned} \tag{28}$$

Based on the optimality condition of (27), we measure the accuracy of an approximate optimal solution $(X, Z, z, Y, \Gamma, \xi, u, \zeta)$ for the problem (27) and its dual (28) via

$$\eta = \max\left\{\eta_P, \eta_z, \eta_Z, \eta_D, \eta_\Gamma\right\} \quad \text{and} \quad \eta_{\rm gap} := \frac{|{\rm obj}_P + {\rm obj}_D|}{1 + |{\rm obj}_P| + |{\rm obj}_D|}$$

where

$$\begin{aligned} \eta_P &:= \frac{\sqrt{\|\mathcal{A}(X) - z - b\|^2 + \|\mathcal{B}(X) - d\|^2 + \|X - Z\|_F^2}}{1 + \|b\|}, \\ \eta_z &= \frac{\max(\|z\| - \delta, 0)}{1 + \|z\|}, \quad \eta_Z = \frac{\max(\|Z\| - R, 0)}{1 + \|Z\|}, \quad \eta_\Gamma = \frac{\max(\|\Gamma\| - 1, 0)}{1 + \|\Gamma\|}, \\ \eta_D &:= \frac{\sqrt{\|C - \mathcal{A}^*(\xi) - \mathcal{B}^*(\zeta) - Y - \Gamma\|_F^2 + \|\xi - u\|^2}}{1 + \|C\|_F}. \end{aligned}$$

We solve the problem (27) with the powerful Schur-complement based semi-proximal ADMM (alternating direction method of multipliers) [23] for its dual (28). We terminate the semi-proximal ADMM when $\max(\eta, \eta_{\rm gap}) \le 10^{-6}$. In the sequel, if X^k is the output of Algorithm 1 in a certain stage, its relative error is defined by $\|X^k - \overline{X}\|_F/\|\overline{X}\|_F$.

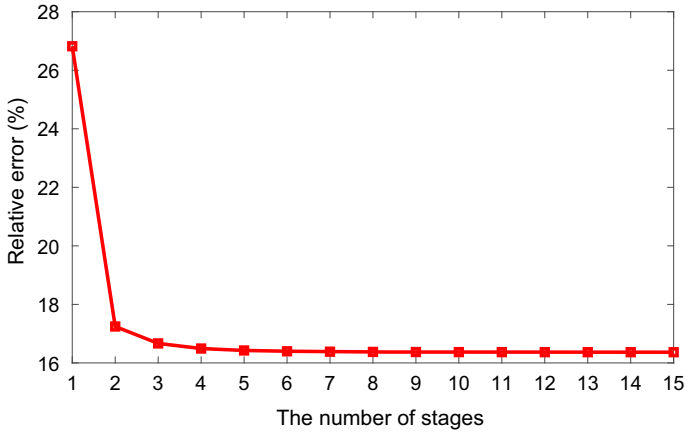

Fig. 1 Performance of Algorithm 1 in the first fifteen stages

5.1.1 Performance of Algorithm 1 in different stages

We generate randomly a matrix sensing problem with 5 known entries as above with $n_1 = n_2 = 100$, $r = 6$ and $m = 2328$ to test the performance of Algorithm 1 in different stages. Figure 1 plots the relative error of Algorithm 1 in the first fifteen stages. We see that Algorithm 1 reduces the relative error of the nuclear norm relaxation method most in the second stage, and after the third stage the reduction becomes insignificant. This performance coincides with the analysis results shown as in Table 1.

5.1.2 Performance of Algorithm 1 with different samples

We generate randomly a matrix sensing problem with 5 known entries as above with $n_1 = n_2 = 100$ and $r = 5$ to test the performance of Algorithm 1 with the number of samples $m = \alpha r(2n_1 - r)$ for $\alpha \in \{1.0, 1.1, \ldots, 3.0\}$. Figure 2 plots the relative error and rank curves of the first stage convex relaxation and the first five stages convex relaxation, respectively. We see that the relative errors of the first stage convex relaxation and the first five stages convex relaxation decrease as the number of samples increases, but the relative error of the latter is always smaller than that of the former. Moreover, the first five stages convex relaxation reduces those of the first stage convex relaxation at least 25% for $\alpha \in [1.0, 3.0]$, and the reduction becomes less as the number of samples increases. In particular, the rank of X^1 is higher than that of $\overline{X}$ even for $\alpha = 3$, i.e., the number of samples $m = 3r(2n_1 - r)$, but the rank of X^5 equals that of $\overline{X}$ for $m = 1.2r(2n_1 - r)$.

5.2 Comparison with the nuclear norm relaxation method

In this subsection, we shall compare the performance of Algorithm 1 with that of the popular nuclear norm relaxation method (NNRM) by taking the low-rank PSD matrix completion problem for example. Though the sampling operators for such problems

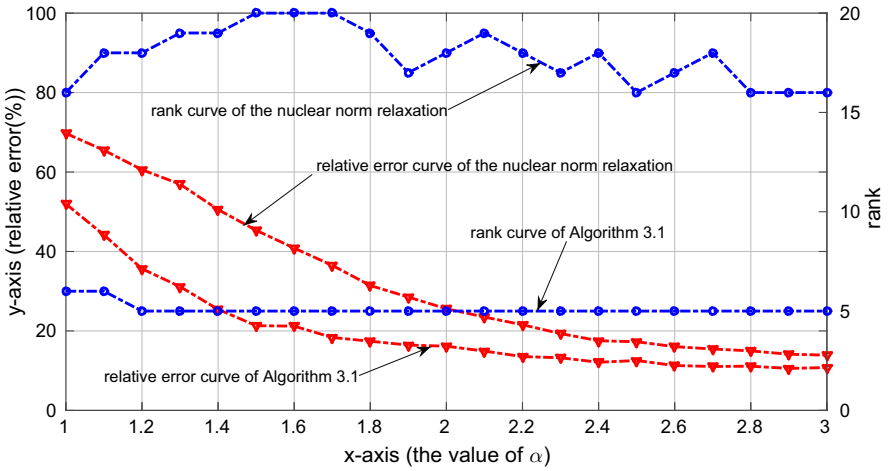

Fig. 2 Performance of the first stage and the first five stages convex relaxation

do not satisfy the RIP property, it is possible for the restricted eigenvalues of $\mathcal{A}^*\mathcal{A}$ to satisfy Assumption 1. For these problems, the sampling operator $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ is defined by

$$\mathcal{A}(X) := (\langle E_{ij}, X\rangle : (i, j) \in \Upsilon_{\text{sample}})^{\mathbb{T}} \tag{29}$$

where E_{ij} is same as the one in Eq. (25), and Υ_{sample} is the set consisting of the indices of sampled entries; the observation vector b is generated randomly in the same way as in (26); and $\Omega = \left\{X \in \mathbb{S}^n_+ \mid \mathcal{E}_1(X) = d,\ \mathcal{E}_2(X) \le g\right\}$ where $\mathcal{E}_1 : \mathbb{S}^n \to \mathbb{R}^{l_1}$ and $\mathcal{E}_2 : \mathbb{S}^n \to \mathbb{R}^{l_2}$ are the linear operators, and $d \in \mathbb{R}^{l_1}$ and $g \in \mathbb{R}^{l_2}$ are the given vectors.

Since $\Omega \subseteq \mathbb{S}^n_+$, the objective function $\|X\|_* - \langle W^{k-1}, X\rangle$ of (12) over the feasible set becomes $\langle I - W^{k-1}, X\rangle$. Write $C = I - W^{k-1}$. Then, the subproblem (12) takes the following form

$$\begin{aligned} \min_{X\in\mathbb{S}^n, z\in\mathbb{R}^m, y\in\mathbb{R}^{l_2}} & \langle C, X\rangle + \delta_{\mathbb{S}^n_+}(X) + \delta_{\mathcal{R}}(z) + \delta_{\mathbb{R}^{l_2}_+}(y) \\ \text{s.t. } & \mathcal{A}X - z - b = 0,\ \begin{pmatrix}\mathcal{E}_1\\ \mathcal{E}_2\end{pmatrix} X - \begin{pmatrix} d\\ g\end{pmatrix} + \begin{pmatrix} 0\\ y\end{pmatrix} = 0. \end{aligned} \tag{30}$$

After an elementary calculation, the dual problem of (30) has the following form

$$\begin{aligned} \min_{\Gamma\in\mathbb{S}^n, \xi\in\mathbb{R}^m, \zeta\in\mathbb{R}^{l_1}, s,u\in\mathbb{R}^{l_2}} & \langle b, \xi\rangle + \langle d, \zeta\rangle + \langle g, s\rangle + \delta\|\xi\| + \delta_{\mathbb{S}^n_+}(\Gamma) + \delta_{\mathbb{R}^{l_2}_+}(u) \\ \text{s.t. } & C + \mathcal{A}^*(\xi) + \mathcal{E}_1^*(\zeta) + \mathcal{E}_2^*(s) - \Gamma = 0,\ s - u = 0. \end{aligned} \tag{31}$$

Notice that the NNRM for the problem (1) is solving the problem (30) with $C \equiv 0$. For Algorithm 1 and the NNRM, we solve the subproblem of the form (30) with the Schur-complement based semi-proximal ADMM [23] for its dual (31). Based on

the optimality condition of (30), we measure the accuracy of an approximate optimal solution $(X, z, y, \Gamma, \xi, \zeta, s, u)$ for the problem (30) and its dual problem (31) in terms of

$$\eta = \max\left\{\eta_P, \eta_X, \eta_y, \eta_D, \eta_\Gamma, \eta_u\right\} \quad \text{and} \quad \eta_{\rm gap} := \frac{|{\rm obj}_P + {\rm obj}_D|}{1 + |{\rm obj}_P| + |{\rm obj}_D|},$$

where η_P and η_D are defined as in Sect. 5.1, and $\eta_X, \eta_y, \eta_\Gamma, \eta_u$ are defined by

$$\eta_X = \frac{\|X - \Pi_{\mathbb{S}_+}(X)\|_F}{1 + \|X\|_F}, \quad \eta_y = \frac{\|y - \Pi_{\mathbb{R}^{l_2}_+}(y)\|}{1 + \|y\|}, \quad \eta_\Gamma = \frac{\|\Gamma - \Pi_{\mathbb{S}_+}(\Gamma)\|_F}{1 + \|\Gamma\|_F},$$
$$\eta_u = \frac{\|u - \Pi_{\mathbb{R}^{l_2}_+}(u)\|}{1 + \|u\|}.$$

Here, for a given closed convex set S, $\Pi_S(\cdot)$ denotes the projection mapping onto S. During the testing, we terminate the semi-proximal ADMM once $\max(\eta, \eta_{\rm gap}) \le 10^{-6}$, and terminate Algorithm 1 at the kth iterate when ${\rm rank}(X^{k-1}) = {\rm rank}(X^k)$, where ${\rm rank}(X^k)$ is the number of nonzero singular values of X^k less than $10^{-8}\|X^k\|$. In the rest of this subsection, the sampling ratio is defined by $\frac{2m}{n(n+1)-2n_{\rm fix}}$ where m is the number of samples and $n_{\rm fix}$ is the number of known entries, and the relative error is defined by $\frac{\|X^f - \overline{X}\|_F}{\|\overline{X}\|_F}$ where X^f is the output of solvers.

5.2.1 Low-rank correlation matrix completion problems

A correlation matrix is a real symmetric PSD matrix with all diagonals being 1. We generate the true correlation matrix $\overline{X} \in \mathbb{S}^n_+$ of rank r in the following command:

```
L = randn(n,r);  W = weight*L(:,1:1);  L(:,1:1) = W; G = L*L';
M = diag(1./sqrt(diag(G)))*G*diag(1./sqrt(diag(G))); Xbar = (M+M')/2.
```

In this way, one can control the ratio of the largest eigenvalue and the smallest nonzero eigenvalue of $\overline{X}$ by `weight`. We assume that some off-diagonal entries of $\overline{X}$ are known. Thus, $\mathcal{E}_1(X) = \begin{pmatrix} {\rm diag}(X) \\ \mathcal{B}(X) \end{pmatrix}$ for $X \in \mathbb{S}^n$, $g_1 = \begin{pmatrix} e \\ d \end{pmatrix}$, $\mathcal{E}_2 \equiv 0$ and $g_2 = 0$, where the operator $\mathcal{B} : \mathbb{S}^n \to \mathbb{R}^{|\Upsilon_{\rm fix}|}$ and the vector $d \in \mathbb{R}^{|\Upsilon_{\rm fix}|}$ are defined as in Sect. 5.1. The noise vector ξ and the observation vector b are generated in the same way as in (26).

Table 2 reports the numerical results of NNRM and Algorithm 1 for some examples generated randomly with $n = 1000$. The information of $\overline{X}$ is reported in the first three columns, where the second column lists the number of known off-diagonal entries for $\overline{X}$, and the third column gives the ratio of the largest eigenvalue of $\overline{X}$ to the smallest nonzero eigenvalue of $\overline{X}$. For each test example, we sample partial unknown off-diagonal entries uniformly at random to formulate the operator $\mathcal{A}$, where the sample ratio is **1.92**% for ${\rm rank}(\overline{X}) = 5$ and **4.32**% for ${\rm rank}(\overline{X}) = 10$. The fourth and the fifth columns report the results of the NNRM and Algorithm 1, respectively, where **relerr** and **rank** mean the relative error and the rank of solutions, **iter** and **time** are the total

Table 2 Performance of NNRM and Algorithm 1 for low-rank correlation completion

r	off-diag	eigr	NNRM			Algorithm 1			
			relerr(rank)	iter	time	relerr(rank)	iter	time	n_s
5	0	1.19	5.06e-1(1000)	95	25.7	1.58e-1(5)	2279	579.7	4
	0	2.86	3.86e-1(1000)	101	25.1	1.51e-1(5)	906	228.9	3
	0	4.36	2.68e-1(1000)	105	27.1	1.51e-1(5)	883	226.5	3
	100	1.17	4.92e-1(1000)	94	23.5	1.53e-1(5)	1941	491.1	4
	100	2.79	3.40e-1(1000)	114	28.6	1.47e-1(5)	916	231.3	3
	100	4.23	2.73e-1(1000)	105	27.8	1.48e-1(5)	932	240.3	3
10	0	1.36	3.29e-1(1000)	73	19.0	1.44e-1(10)	849	217.9	3
	0	3.52	2.59e-1(1000)	76	19.7	1.39e-1(10)	626	163.2	3
	0	6.39	1.80e-1(1000)	84	21.5	1.33e-1(10)	520	132.2	3
	100	1.42	3.08e-1(1000)	73	18.2	1.42e-1(10)	949	241.8	3
	100	3.31	2.50e-1(1000)	76	19.8	1.38e-1(10)	623	159.3	3
	100	6.35	1.86e-1(1000)	85	22.4	1.37e-1(10)	551	141.5	3

number of iterations and the total computing time in second for the semi-proximal ADMM, and n_s is the number of stages required by Algorithm 1.

We see that the solutions yielded by the NNRM have high relative errors as well as full ranks, while those given by Algorithm 1 not only have much lower relative error but also achieve the rank of the true matrix. Among others, the relative error of the NNRM is reduced at least 25% by that of Algorithm 1, and for those problems with **1.92**% sample ratio, the reduction is close to 45%. The last column of Table 2 shows that Algorithm 1 yields the desirable results for almost all problems within 3 stages.

5.2.2 Low-rank covariance matrix completion problems

We generate the true covariance matrix $\overline{X} \in \mathbb{S}^n_+$ of rank r in the following command:

```
L = randn(n,r)/sqrt(sqrt(n));  W = weight*L(:,1:1);
L(:,1:1) = W;  G = L*L'; Xbar = (G+G')/2.
```

In this case, $\mathcal{E}_1 = \mathcal{B}$ and $g_1 = d$ where $\mathcal{B} : \mathbb{S}^n \to \mathbb{R}^{|\Upsilon_{\rm fix}|}$ and $d \in \mathbb{R}^{|\Upsilon_{\rm fix}|}$ are defined as in Sect. 5.1, $\mathcal{E}_2(X) := \langle E_{ii}, X\rangle$ for $(i,i) \in \Upsilon_{\rm diag}$ where E_{ii} is an $n \times n$ matrix with the (i,i)th entry being 1 and other entries being 0, and $\Upsilon_{\rm diag}$ is the index set of unknown diagonal entries of $\overline{X}$, and $g_2 \in \mathbb{R}^{|\Upsilon_{\rm diag}|}$ is the vector consisting of the upper bounds for unknown diagonal entries of $\overline{X}$. We set $g_2 = (1 + 0.01\mathbf{rand}(1,1))\|\overline{X}\|_\infty \mathbf{ones}(|\Upsilon_{\rm diag}|, 1)$.

Table 3 reports the numerical results of NNRM and Algorithm 1 for some problems generated randomly with $n = 1000$. The information of the true covariance matrix $\overline{X}$ is reported in the first two columns, where the second column lists the number of known diagonal and off-diagonal entries of $\overline{X}$, and the third one reports the ratio of the largest eigenvalue of $\overline{X}$ to the smallest nonzero eigenvalue of $\overline{X}$. For each test example, we sample the upper triangular entries uniformly at random to formulate the

Table 3 Performance of NNRM and Algorithm 1 for low-rank covariance completion

r	(diag, offdiag)	eigr	NNRM			Algorithm 1			
			relerr(rank)	iter	time	relerr(rank)	iter	time	n_s
5	(200, 0)	1.18	4.80e-1(36)	950	244.6	2.25e-1(5)	2443	631.1	4
	(200, 0)	4.59	3.19e-1(32)	777	205.6	1.82e-1(5)	2292	597.4	4
	(0, 200)	1.20	4.86e-1(36)	350	88.1	2.21e-1(5)	1240	316.2	4
	(0, 200)	4.26	3.24e-1(33)	272	69.9	1.90e-1(5)	827	213.1	4
	(100, 100)	1.21	4.74e-1(36)	1409	366.5	2.24e-1(5)	3696	960.2	4
	(100, 100)	4.07	3.33e-1(33)	715	181.1	1.80e-1(5)	2206	563.4	4
13	(200, 0)	5.33	2.20e-1(53)	290	75.2	1.54e-1(13)	638	164.2	3
	(200, 0)	7.72	1.90e-1(48)	309	79.1	1.45e-1(13)	619	159.8	3
	(0, 200)	5.17	2.20e-1(53)	147	37.6	1.55e-1(13)	386	101.3	3
	(0, 200)	9.01	1.78e-1(45)	142	36.7	1.43e-1(13)	385	99.2	3
	(100, 100)	4.58	2.30e-1(54)	286	74.8	1.56e-1(13)	599	155.0	3
	(100, 100)	8.11	1.85e-1(48)	248	64.4	1.46e-1(13)	481	124.2	3

sampling operator $\mathcal{A}$, where the sample ratio is $\mathbf{1.91}\%$ for $\text{rank}(\overline{X}) = 5$ and $\mathbf{5.72}\%$ for $\text{rank}(\overline{X}) = 13$. The fourth and the fifth columns report the results of NNRM and Algorithm 1, respectively.

We see that the solutions yielded by the NNRM have high relative errors and ranks, while the solutions given by Algorithm 1 have the desirable relative errors as well as the same rank as the true matrix does. The relative error of the NNRM is reduced at least 20% by Algorithm 1, and for those problems with $\mathbf{1.92}\%$ sample ratio, the reduction is at least 40%. Comparing with the time columns in Table 2, we see that for the low-rank matrix covariance completion, the time gap between Algorithm 1 and NNRM becomes much smaller, and the time of the former is only about twice that of the latter. In addition, along with the results in Table 2, Algorithm 1 has no direct relation with the ratio of the largest eigenvalue and the smallest nonzero eigenvalue of $\overline{X}$.

5.3 Applications to real data

Let M be an estimated $n \times n$ correlation matrix. In this part, we shall seek a low-rank correlation matrix under a given noise level $\delta > 0$ by applying Algorithm 1 to the problem (1) with $b = \mathcal{A}(M)$, where $\mathcal{A}$ is the sampling operator formulated as in Sect. 5.2.

Example 3 The M is a 500×500 correlation matrix extracted from the correlation matrix, which is based on a $10,000$ gene micro-array data set obtained from 256 drugs treated rat livers; see Natsoulis et al. [27] for details.

Example 4 The M is an estimated 943×943 correlation matrix based on $100,000$ ratings for 1682 movies by 943 users. Due to missing data, the generated matrix M

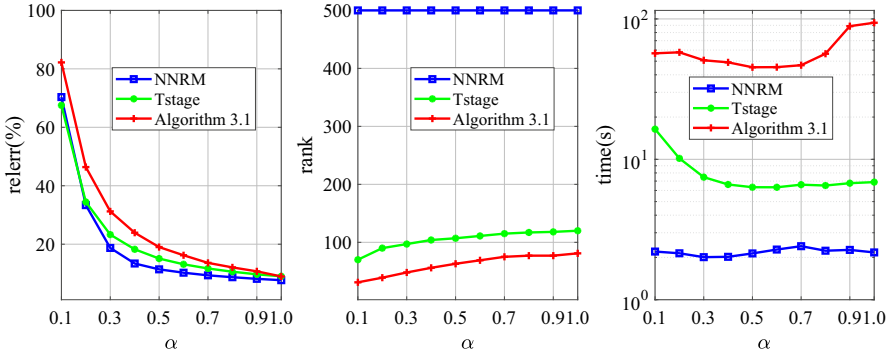

Fig. 3 Performance of NNRM, Tstage and Algorithm 1 for low-rank correlation estimation

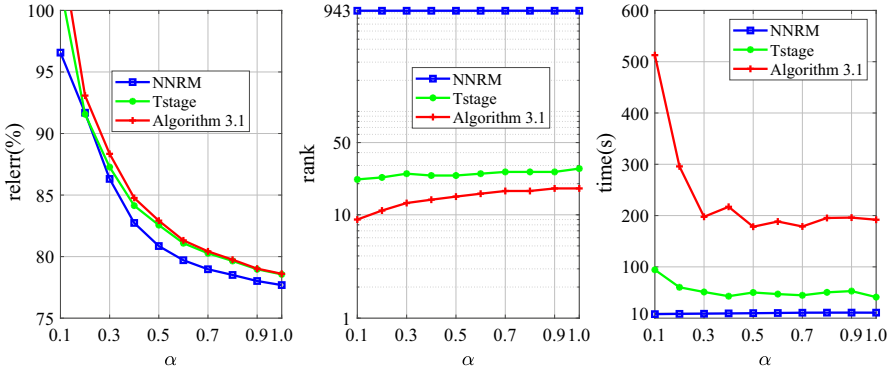

Fig. 4 Performance of NNRM, Tstage and Algorithm 1 for low-rank correlation estimation

is not positive semi-definite [15]. This rating data set can be downloaded from http://www.grouplens.org/node/73.

We apply NNRM, Tstage (Algorithm 1 with the first two stages) and Algorithm 1 to solving the problem (1) with $b = \mathcal{A}(M)$. We adopt the stopping criterion as described in Sect. 5.2 for Algorithm 1. Figures 3 and 4 below plot the relative error, rank and time curves of the three solvers with M from Examples 3 and 4 , respectively, under the number of samples $m = \alpha(\frac{n^2+n}{2} - n_{\text{fix}})$ where $n_{\text{fix}} = \text{nfix\_diag} = n$ for $\alpha \in \{0.1, 0.2, \ldots, 1.0\}$. Here, the relative error is defined by $\frac{\|X^f - M\|_F}{\|M\|_F}$ where X^f denotes the output of three solvers. Since the matrix M in Example 4 is highly polluted, we take $\delta = 0.75$ for it instead of $\delta = 0.1$ as for M from Example 3.

From Figs. 3 and 4 , we see that NNRM gives the outputs with the lowest relative error but full rank within the least time, Tstage yields the outputs with much lower rank and a little higher relative error than those of NNRM by using about 5 times computing time of NNRM, while Algorithm 1 gives the outputs with the lowest rank and a little higher relative error than those of NNRM and Tstage though it requires more computing time. We find that when the number of samples is over $0.7(\frac{n^2+n}{2} - n_{\text{fix}})$, for the M in Example 3, Tstage and Algorithm 1 yield a solution with the rank lower than

120 and **81**, respectively, and the relative error is less than **0.12** and **0.14**, respectively; while for the M from Example 4, they give a solution with the rank lower than **28** and **18** respectively, and the relative error is less than **0.8027** and **0.8043**, respectively. That is, the relative error of the solution to the M is close to the given noise level. Notice that the matrix M from these two examples has a high rank; for example, for the M from Example 3, its rank (i.e., the number of singular values greater than $10^{-5}\|M\|$) is **300**. So, if one wants to seek the lowest rank estimation, Algorithm 1 is an ideal choice; and if one only wants to seek a low rank estimation, then Algorithm 1 with the first two stages is a desirable choice.

6 Conclusions

We have proposed a multi-stage convex relaxation approach to the structured rank minimization problem (1) by solving the exact penalty problem of its equivalent MPGCC in an alternating way. It turned out that this approach not only has favorable theoretical guarantees but also reduces effectively the rank and error of the nuclear norm relaxation method. There are several topics worthwhile to pursue, such as to develop more effective algorithms for seeking the solution of subproblems, to establish the theoretical guarantee for the case where the subproblems are solved inexactly, and to apply this approach to other classes of low-rank optimization problems, say, low-rank plus sparse problems.

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Compliance with ethical standards

Conflict of interest The authors declare that they have no conflict of interest.

7 Appendix

7.1 Technical lemmas

Let $M \in \mathbb{R}^{n_1\times n_2}$ be a matrix of rank $\kappa > 0$, and let $\mathcal{T}(M)$ be the tangent space at M associated to the rank constraint $\mathrm{rank}(X) \le \kappa$ (see [6, Section 2.3]). Then, the subspace $\mathcal{T}(M)$ and its orthogonal complementarity $\mathcal{T}(M)^\perp$ in $\mathbb{R}^{n_1\times n_2}$ take the form of

$$\begin{aligned}\mathcal{T}(M) &= \big\{X \in \mathbb{R}^{n_1\times n_2} \mid X = U_1U_1^{\mathbb{T}}X + XV_1V_1^{\mathbb{T}} - U_1U_1^{\mathbb{T}}XV_1V_1^{\mathbb{T}}\big\},\\ \mathcal{T}(M)^\perp &= \big\{X \in \mathbb{R}^{n_1\times n_2} \mid X = U_2U_2^{\mathbb{T}}XV_2V_2^{\mathbb{T}}\big\},\end{aligned} \tag{32}$$

where $([U_1\ \ U_2], [V_1\ \ V_2]) \in \mathbb{O}^{n_1,n_2}(M)$ with $U_1 \in \mathbb{O}^{n_1\times\kappa}$ and $V_1 \in \mathbb{O}^{n_2\times\kappa}$. In this part, letting $\widetilde{X} \in \mathbb{R}^{n_1\times n_2}$ be a matrix of rank $\kappa > 0$, and letting $([\widetilde{U}_1\ \ \widetilde{U}_2], [\widetilde{V}_1\ \ \widetilde{V}_2]) \in \mathbb{O}^{n_1,n_2}(\widetilde{X})$ with $\widetilde{U}_1 \in \mathbb{O}^{n_1\times\kappa}$ and $\widetilde{V}_1 \in \mathbb{O}^{n_2\times\kappa}$, we shall derive an upper bound for the

projection of the perturbed $\widetilde{U}_1\widetilde{V}_1^{\mathbb{T}}$ by a matrix $W \in \mathbb{R}^{n_1\times n_2}$ onto $\mathcal{T}(\widetilde{X})$ and $\mathcal{T}(\widetilde{X})^{\perp}$, respectively.

Lemma 2 *For a given $W \in \mathbb{R}^{n_1\times n_2}$, by letting $([U_1\ \ U_2], [V_1\ \ V_2]) \in \mathbb{O}^{n_1,n_2}(W)$ with $U_1 \in \mathbb{O}^{n_1\times\kappa}$ and $V_1 \in \mathbb{O}^{n_2\times\kappa}$, and writing $w = \sigma(W)$, the following inequalities hold:*

$$\|\mathcal{P}_{\mathcal{T}(\widetilde{X})^{\perp}}(W)\| \le w_{\kappa+1} + (w_1 - w_{\kappa+1})\|\widetilde{U}_1\widetilde{V}_1^{\mathbb{T}} - U_1V_1^{\mathbb{T}}\|^2, \tag{33a}$$

$$\|\mathcal{P}_{\mathcal{T}(\widetilde{X})}(\widetilde{U}_1\widetilde{V}_1^{\mathbb{T}} - W)\|_F \le (1+\sqrt{2}w_{\kappa+1})\|U_1V_1^{\mathbb{T}} - \widetilde{U}_1\widetilde{V}_1^{\mathbb{T}}\|_F + \sqrt{\kappa}\max\big(|1-w_1|, |1-w_\kappa|\big). \tag{33b}$$

Proof Let $\Sigma_1 := \mathrm{Diag}(w_1, \ldots, w_\kappa)$ and $\Sigma_2 := \mathrm{Diag}(w_{\kappa+1}, \ldots, w_{n_1})$. Then, we have

$$\begin{aligned}
&\big\|\mathcal{P}_{\mathcal{T}(\widetilde{X})^{\perp}}(W)\big\| \\
&= \big\|\widetilde{U}_2\widetilde{U}_2^{\mathbb{T}}\big[U_1(\Sigma_1 - w_{\kappa+1}I)V_1^{\mathbb{T}} + (w_{\kappa+1}U_1V_1^{\mathbb{T}} + U_2[\Sigma_2\ \ 0]V_2^{\mathbb{T}})\big]\widetilde{V}_2\widetilde{V}_2^{\mathbb{T}}\big\| \\
&\le \big\|\widetilde{U}_2\widetilde{U}_2^{\mathbb{T}}U_1\big\|\big\|\Sigma_1 - w_{\kappa+1}I\big\|\big\|\widetilde{V}_2\widetilde{V}_2^{\mathbb{T}}V_1\big\| + w_{\kappa+1} \\
&= (w_1 - w_{\kappa+1})\big\|\widetilde{U}_2\widetilde{U}_2^{\mathbb{T}}U_1V_1^{\mathbb{T}}\big\|\big\|\widetilde{V}_2\widetilde{V}_2^{\mathbb{T}}V_1U_1^{\mathbb{T}}\big\| + w_{\kappa+1} \\
&= (w_1 - w_{\kappa+1})\big\|\widetilde{U}_2\widetilde{U}_2^{\mathbb{T}}(U_1V_1^{\mathbb{T}} - \widetilde{U}_1\widetilde{V}_1^{\mathbb{T}})\big\|\big\|\widetilde{V}_2\widetilde{V}_2^{\mathbb{T}}(V_1U_1^{\mathbb{T}} - \widetilde{V}_1\widetilde{U}_1^{\mathbb{T}})\big\| + w_{\kappa+1} \\
&\le (w_1 - w_{\kappa+1})\big\|U_1V_1^{\mathbb{T}} - \widetilde{U}_1\widetilde{V}_1^{\mathbb{T}}\big\|^2 + w_{\kappa+1},
\end{aligned}$$

where the first inequality is using $\|w_{\kappa+1}U_1V_1^{\mathbb{T}} + U_2[\Sigma_2\ \ 0]V_2^{\mathbb{T}}\| \le w_{\kappa+1}$, and the second equality is due to $\|Z\| = \|ZQ^{\mathbb{T}}\|$ for any Z and Q with $Q^{\mathbb{T}}Q = I$. So, the inequality (33a) holds. In order to establish (33b), we first notice that for any $Z \in \mathbb{R}^{(n_1-\kappa)\times(n_2-\kappa)}$,

$$\begin{aligned}
\big\|\mathcal{P}_{\mathcal{T}(\widetilde{X})}(U_2ZV_2^{\mathbb{T}})\big\|_F &= \sqrt{\big\|\widetilde{U}_1\widetilde{U}_1^{\mathbb{T}}U_2ZV_2^{\mathbb{T}}\big\|_F^2 + \big\|\widetilde{U}_2\widetilde{U}_2^{\mathbb{T}}U_2ZV_2^{\mathbb{T}}\widetilde{V}_1\widetilde{V}_1^{\mathbb{T}}\big\|_F^2} \\
&\le \sqrt{\|Z\|^2\big\|\widetilde{U}_1^{\mathbb{T}}U_2\big\|_F^2 + \|Z\|^2\big\|V_2^{\mathbb{T}}\widetilde{V}_1\big\|_F^2} \\
&= \|Z\|\sqrt{\big\|(\widetilde{V}_1\widetilde{U}_1^{\mathbb{T}} - V_1U_1^{\mathbb{T}})U_2\big\|_F^2 + \big\|V_2^{\mathbb{T}}(\widetilde{V}_1\widetilde{U}_1^{\mathbb{T}} - V_1U_1^{\mathbb{T}})\big\|_F^2} \\
&\le \sqrt{2}\|Z\|\|\widetilde{V}_1\widetilde{U}_1^{\mathbb{T}} - V_1U_1^{\mathbb{T}}\|_F,
\end{aligned}$$

where the first equality is by the expression of $\mathcal{P}_{\mathcal{T}(\widetilde{X})}(\cdot)$. Then, it holds that

$$\begin{aligned}
&\big\|\mathcal{P}_{\mathcal{T}(\widetilde{X})}(W - \widetilde{U}_1\widetilde{V}_1^{\mathbb{T}})\big\|_F \\
&\le \big\|\mathcal{P}_{\mathcal{T}(\widetilde{X})}(\widetilde{U}_1\widetilde{V}_1^{\mathbb{T}} - U_1\Sigma_1V_1^{\mathbb{T}})\big\|_F + \big\|\mathcal{P}_{\mathcal{T}(\widetilde{X})}(U_2[\Sigma_2\ \ 0]V_2^{\mathbb{T}})\big\|_F \\
&\le \big\|\widetilde{U}_1\widetilde{V}_1^{\mathbb{T}} - U_1\Sigma_1V_1^{\mathbb{T}}\big\|_F + \sqrt{2}\|[\Sigma_2\ \ 0]\|\|\widetilde{V}_1\widetilde{U}_1^{\mathbb{T}} - V_1U_1^{\mathbb{T}}\|_F \\
&\le (1+\sqrt{2}w_{\kappa+1})\big\|U_1V_1^{\mathbb{T}} - \widetilde{U}_1\widetilde{V}_1^{\mathbb{T}}\big\|_F + \big\|U_1(I-\Sigma_1)V_1^{\mathbb{T}}\big\|_F \\
&\le (1+\sqrt{2}w_{\kappa+1})\|U_1V_1^{\mathbb{T}} - \widetilde{U}_1\widetilde{V}_1^{\mathbb{T}}\|_F + \sqrt{\kappa}\max\big(|1-w_1|, |1-w_\kappa|\big).
\end{aligned}$$

This shows that the inequality (33b) holds. Thus, we complete the proof. □

When the matrix W in Lemma 2 and a matrix X close to $\widetilde{X}$ have a simultaneous ordered SVD, the term $\|\widetilde{U}_1\widetilde{V}_1^{\mathbb{T}} - U_1V_1^{\mathbb{T}}\|$ in (33a)–(33b) can be upper bounded as follows.

Lemma 3 ([24, Theorem 3]) *Let $X \in \mathbb{R}^{n_1\times n_2}$ be an arbitrary matrix of rank $\kappa > 0$. For any given $\omega > 2$, if $\|X - \widetilde{X}\|_F \le \eta$ for some $\eta \in \big(0, \sigma_\kappa(\widetilde{X})/\omega\big]$, then it holds that*

$$\big\|U_1V_1^{\mathbb{T}} - \widetilde{U}_1\widetilde{V}_1^{\mathbb{T}}\big\|_F \le \frac{1}{\sqrt{2}}\ln\Big(\frac{\omega}{\omega-\sqrt{2}}\Big),$$

where $([U_1\ \ U_2], [V_1\ \ V_2]) \in \mathbb{O}^{n_1,n_2}(X)$ with $U_1 \in \mathbb{O}^{n_1\times\kappa}$ and $V_1 \in \mathbb{O}^{n_2\times\kappa}$.

7.2 Properties of restricted eigenvalues

This part includes two results on the restricted eigenvalues of $\mathcal{A}^*\mathcal{A}$. The first gives a relation among $\vartheta_+(\cdot)$, $\vartheta_-(\cdot)$ and $\pi(\cdot,\cdot)$ where for given positive integers k, l with $k + l \le n_1$,

$$\pi(k,l) := \sup_{\substack{0<\mathrm{rank}(X)\le k,\\ 0<\mathrm{rank}(Y)\le l, \langle X,Y\rangle=0}} \frac{\langle X, \mathcal{A}^*\mathcal{A}(Y)\rangle}{\|\mathcal{A}(X)\|^2\|Y\|}\|X\|_F. \tag{34}$$

Lemma 4 *For any given positive integer k, l with $k + l \le n_1$, $\pi(k,l) \le \frac{\sqrt{l}}{2}\sqrt{\frac{\vartheta_+(l)}{\vartheta_-(k+l)} - 1}$.*

Since the proof of Lemma 4 is similar to that of [42, Proposition 3.1], we omit it. The second one is an extension of [42, Lemma 10.1] in the matrix setting, stated as follows.

Lemma 5 *Let $G \in \mathbb{R}^{n_1\times n_2}, U_J \in \mathbb{O}^{n_1\times|J|}$ with $J \subseteq \{1,\ldots,n_1\}$ and $V_{J'} \in \mathbb{O}^{n_2\times|J'|}$ with $J' \subseteq \{1,\ldots,n_2\}$ be given matrices. Let $([P_1\ \ P_2], [Q_1\ \ Q_2]) \in \mathbb{O}^{|J|,|J'|}(U_J^{\mathbb{T}}GV_{J'})$ with $P_1 \in \mathbb{O}^{|J|\times s}$ and $Q_1 \in \mathbb{O}^{|J'|\times s}$ for an integer $1 \le s \le \min(|J|,|J'|)$. Define $\mathcal{G} := \mathcal{L}^\perp \oplus \mathcal{J}_1$ where $\mathcal{L} = \big\{U_JZV_{J'}^{\mathbb{T}} \mid Z \in \mathbb{R}^{|J|\times|J'|}\big\}$ and $\mathcal{J}_1 = \big\{U_JP_1Z(V_{J'}Q_1)^{\mathbb{T}} \mid Z \in \mathbb{R}^{s\times s}\big\}$. Then, for any $H \in \mathcal{G}$, the following inequality holds with $l = \max_{Z\in\mathcal{L}^\perp}\mathrm{rank}(Z)$:*

$$\begin{aligned}\max\big(0, \langle H, \mathcal{A}^*\mathcal{A}(G)\rangle\big) \ge{}& \vartheta_-(l+s)\big(\|H\|_F - s^{-1}\pi(l+s,s)\big\|\mathcal{P}_{\mathcal{L}}(G)\big\|_*\big)\|H\|_F\\ &- \vartheta_+(l+s)\|H\|_F\|\mathcal{P}_{\mathcal{G}}(G-H)\|_F.\end{aligned}$$

Proof Let H be an arbitrary matrix from $\mathcal{G}$. If $\|H\|_F \le \frac{\pi(l+s,s)}{s}\big\|\mathcal{P}_{\mathcal{L}}(G)\big\|_*$, the conclusion is clear. So, we assume that $\|H\|_F > \frac{\pi(l+s,s)}{s}\big\|\mathcal{P}_{\mathcal{L}}(G)\big\|_*$. By the definition of $\vartheta_+(l+s)$, $\|\mathcal{A}\mathcal{P}_{\mathcal{G}}(H-G)\|^2 \le \vartheta_+(l+s)\|\mathcal{P}_{\mathcal{G}}(H-G)\|_F^2$ and $\|\mathcal{A}(H)\|^2 \le \vartheta_+(l+s)\|H\|_F^2$. Then,

$$\begin{aligned}&\big\langle H, \mathcal{A}^*\mathcal{A}\mathcal{P}_{\mathcal{G}}(G-H)\big\rangle\\ &\quad\geq -\|\mathcal{A}(H)\|\|\mathcal{A}\mathcal{P}_{\mathcal{G}}(H-G)\| \geq -\vartheta_+(l+s)\|H\|_F\|\mathcal{P}_{\mathcal{G}}(H-G)\|_F. \quad (35)\end{aligned}$$

We proceed the arguments by considering the following two cases.
Case 1 $\mathrm{rank}(U_J^{\mathbb{T}}GV_{J'}) \leq s \leq \min(|J|, |J'|)$. Now, by the expression of $\mathcal{P}_{\mathcal{J}_1}$, we have

$$\begin{aligned}\mathcal{P}_{\mathcal{J}_1}(G) &= U_J P_1 P_1^{\mathbb{T}} U_J^{\mathbb{T}} G V_{J'} Q_1 Q_1^{\mathbb{T}} V_{J'}^{\mathbb{T}} = U_J P_1\big[\mathrm{Diag}(\sigma(U_J^{\mathbb{T}}GV_{J'}))\ \ 0\big]Q_1^{\mathbb{T}}\\ &\quad V_{J'}^{\mathbb{T}} = U_J U_J^{\mathbb{T}} G V_{J'} V_{J'}^{\mathbb{T}},\end{aligned}$$

where the last two equalities are due to $U_J^{\mathbb{T}}GV_{J'} = P_1[\mathrm{Diag}\big(\sigma(U_J^{\mathbb{T}}GV_{J'})\big)\ \ 0]Q_1^{\mathbb{T}}$. Note that $\mathcal{P}_{\mathcal{L}}(G) = U_J U_J^{\mathbb{T}} G V_{J'} V_{J'}^{\mathbb{T}}$ by the definition of $\mathcal{L}$. So, $\mathcal{P}_{\mathcal{L}}(G) = \mathcal{P}_{\mathcal{J}_1}(G)$, i.e., $G \in \mathcal{G}$. Then,

$$\begin{aligned}\langle \mathcal{A}(H), \mathcal{A}(G)\rangle &= \langle \mathcal{A}(H), \mathcal{A}(H)\rangle + \langle \mathcal{A}(H), \mathcal{A}\mathcal{P}_{\mathcal{G}}(G-H)\rangle\\ &\geq \vartheta_-(l+s)\|H\|_F^2 - \vartheta_+(l+s)\|H\|_F\|\mathcal{P}_{\mathcal{G}}(H-G)\|_F.\end{aligned}$$

This inequality implies the desired result. Thus, we complete the proof for this case.
Case 2 $1 \leq s < \mathrm{rank}(U_J^{\mathbb{T}}GV_{J'})$. Let k be the smallest positive integer such that $sk \geq \min(|J|, |J'|)$. Clearly, $k \geq 2$. Let l_i and $\widetilde{l}_i$ for $i = 1, 2, \ldots, k$ be such that

$$l_1 = \cdots = l_{k-1} = s,\ l_k = |J| - s(k-1),\quad \widetilde{l}_1 = \cdots = \widetilde{l}_{k-1} = s,\quad \widetilde{l}_k = |J'| - s(k-1).$$

For each $2 \leq i \leq k$, define the subspace $\mathcal{J}_i := \big\{U_J \widetilde{P}_i Z (V_{J'}\widetilde{Q}_i)^{\mathbb{T}} \mid Z \in \mathbb{R}^{l_i \times \widetilde{l}_i}\big\}$, where $\widetilde{P}_i \in \mathbb{O}^{|J|\times l_i}$ is the matrix consisting of the $(\sum_{j=1}^{i-1} l_j + 1)$th column to the $(\sum_{j=1}^{i} l_j)$th column of P; and $\widetilde{Q}_i \in \mathbb{O}^{|J'|\times \widetilde{l}_i}$ is the matrix consisting of the $(\sum_{j=1}^{i-1} \widetilde{l}_j + 1)$th column to the $(\sum_{j=1}^{i} \widetilde{l}_j)$th column of Q. Clearly, $\mathcal{J}_1 \perp \mathcal{J}_i$ for $i \geq 2$. From the definition of $\mathcal{G}$, we have $\mathcal{G} \perp \mathcal{J}_i$ for $i \neq 1$. For each $i \geq 1$, it is easy to calculate that

$$\mathcal{P}_{\mathcal{J}_i}(Z) = U_J \widetilde{P}_i (U_J \widetilde{P}_i)^{\mathbb{T}} Z V_{J'} \widetilde{Q}_i (V_{J'}\widetilde{Q}_i)^{\mathbb{T}} \quad \forall Z \in \mathbb{R}^{n_1\times n_2}.$$

This, together with $\mathcal{P}_{\mathcal{L}}(G) = U_J U_J^{\mathbb{T}} G V_{J'} V_{J'}^{\mathbb{T}}$, implies that $\mathcal{P}_{\mathcal{L}}(G) = \sum_{i=1}^{k} \mathcal{P}_{J_i}(G)$. Then, $\langle H, \mathcal{A}^*\mathcal{A}(G)\rangle = \langle H, \mathcal{A}^*\mathcal{A}\mathcal{P}_{\mathcal{G}}(G)\rangle + \sum_{i>1}\big\langle H, \mathcal{A}^*\mathcal{A}\mathcal{P}_{J_i}(G)\big\rangle$. Consequently, we have

$$\begin{aligned}&\big\langle H, \mathcal{A}^*\mathcal{A}(G)\big\rangle - \big\langle H, \mathcal{A}^*\mathcal{A}\mathcal{P}_{\mathcal{G}}(G-H)\big\rangle\\ &\quad= \langle H, \mathcal{A}^*\mathcal{A}(H)\rangle + \sum_{i>1}\big\langle \mathcal{P}_{\mathcal{G}}(H), \mathcal{A}^*\mathcal{A}\mathcal{P}_{\mathcal{J}_i}(G)\big\rangle\\ &\quad= \langle H, \mathcal{A}^*\mathcal{A}(H)\rangle\left[1 + \sum_{i>1}\frac{\langle H, \mathcal{A}^*\mathcal{A}\mathcal{P}_{\mathcal{J}_i}(G)\rangle\|H\|_F}{\|\mathcal{A}(H)\|^2\|\mathcal{P}_{\mathcal{J}_i}(G)\|}\frac{\|\mathcal{P}_{\mathcal{J}_i}(G)\|}{\|H\|_F}\right]\\ &\quad\geq \langle H, \mathcal{A}^*\mathcal{A}(H)\rangle\Big[1 - \pi(l+s,s)\frac{\sum_{i>1}\|\mathcal{P}_{\mathcal{J}_i}(G)\|}{\|H\|_F}\Big]\end{aligned}$$

$$\geq \langle H, \mathcal{A}^*\mathcal{A}(H)\rangle\Big[1 - \frac{\pi(l+s,s)\|\mathcal{P}_{\mathcal{L}}(G)\|_*}{s\|H\|_F}\Big]$$
$$\geq \vartheta_-(l+s)\|H\|_F\Big[\|H\|_F - s^{-1}\pi(l+s,s)\|\mathcal{P}_{\mathcal{L}}(G)\|_*\Big], \tag{36}$$

where the first inequality is using the definition of π by the fact that $H \in \mathcal{G}, \mathcal{P}_{\mathcal{J}_i}(G) \in \mathcal{J}_i$ and $\mathrm{rank}(\mathcal{P}_{\mathcal{J}_i}(G)) \leq s, \mathcal{G} \perp \mathcal{J}_i$ for $i > 1$, and the second inequality is due to

$$\sum_{i>1}\|\mathcal{P}_{\mathcal{J}_i}(G)\| \leq s^{-1}\sum_{i=1}\big\|\mathcal{P}_{\mathcal{J}_i}(G)\big\|_* = s^{-1}\|\mathcal{P}_{\mathcal{L}}(G)\|_*$$

implied by $\|\mathcal{P}_{\mathcal{J}_{i+1}}(G)\| \leq s^{-1}\|\mathcal{P}_{\mathcal{J}_i}(G)\|_*$. Combining (36) with (35), we get the result. □

7.3 Proofs of the results in Sect. 4

This part includes the proofs of all the results in Sect. 4. For convenience, in this part we write $\Delta^k := X^k - \overline{X}$ for $k = 1, 2, \ldots$. We first establish two preliminary lemmas.

Lemma 6 *If $\|\mathcal{P}_{\mathcal{T}^\perp}(W^{k-1})\| < 1$ for some $k \geq 1$, then with γ_{k-1} defined by (18) we have*

$$\|\mathcal{P}_{\mathcal{T}^\perp}(\Delta^k)\|_* \leq \gamma_{k-1}\sqrt{2r}\|\mathcal{P}_{\mathcal{T}}(\Delta^k)\|_F.$$

Proof By the optimality of X^k and the feasibility of $\overline{X}$ to the subproblem (12),

$$\|X^k\|_* - \langle W^{k-1}, X^k\rangle \leq \|\overline{X}\|_* - \langle W^{k-1}, \overline{X}\rangle.$$

Recall from [40] that $\partial\|\overline{X}\|_* = \{\overline{U}_1\overline{V}_1^{\mathbb{T}} + W \mid W \in \mathbb{R}^{(n_1-r)\times(n_2-r)} \text{ with } \|W\| \leq 1\}$. Then,

$$\|X^k\|_* - \|\overline{X}\|_* \geq \langle \overline{U}_1\overline{V}_1^{\mathbb{T}}, X^k - \overline{X}\rangle + \|\mathcal{P}_{\mathcal{T}^\perp}(X^k - \overline{X})\|_*.$$

The last two equations imply that $\langle \overline{U}_1\overline{V}_1^{\mathbb{T}}, \Delta^k\rangle + \|\mathcal{P}_{\mathcal{T}^\perp}(\Delta^k)\|_* \leq \langle W^{k-1}, \Delta^k\rangle$. Hence,

$$\langle \overline{U}_1\overline{V}_1^{\mathbb{T}}, \mathcal{P}_{\mathcal{T}}(\Delta^k)\rangle + \|\mathcal{P}_{\mathcal{T}^\perp}(\Delta^k)\|_* \leq \langle W^{k-1}, \Delta^k\rangle.$$

This, along with $\langle W^{k-1}, \Delta^k\rangle = \langle \mathcal{P}_{\mathcal{T}^\perp}(W^{k-1}), \mathcal{P}_{\mathcal{T}^\perp}(\Delta^k)\rangle + \langle W^{k-1}, \mathcal{P}_{\mathcal{T}}(\Delta^k)\rangle$, yields that

$$\|\mathcal{P}_{\mathcal{T}^\perp}(\Delta^k)\|_* - \langle \mathcal{P}_{\mathcal{T}^\perp}(W^{k-1}), \mathcal{P}_{\mathcal{T}^\perp}(\Delta^k)\rangle \leq \langle \mathcal{P}_{\mathcal{T}}(W^{k-1} - \overline{U}_1\overline{V}_1^{\mathbb{T}}), \mathcal{P}_{\mathcal{T}}(\Delta^k)\rangle.$$

Using the relation $|\langle Y, Z\rangle| \le \|Y\|\|Z\|_*$ for any $Y, Z \in \mathbb{R}^{n_1\times n_2}$, we obtain that

$$\big(1 - \|\mathcal{P}_{\mathcal{T}^\perp}(W^{k-1})\|\big)\|\mathcal{P}_{\mathcal{T}^\perp}(\Delta^k)\|_* \le \|\mathcal{P}_{\mathcal{T}}(W^{k-1} - \overline{U}_1\overline{V}_1^{\mathbb{T}})\|_F\|\mathcal{P}_{\mathcal{T}}(\Delta^k)\|_F.$$

From this inequality and the definition of γ_{k-1}, we obtain the desired result. □

Lemma 7 *Suppose that* $\|\mathcal{P}_{\mathcal{T}^\perp}(W^{k-1})\| < 1$ *for some* $k \ge 1$. *Let* $([P_1^k\ \ P_2^k], [Q_1^k\ \ Q_2^k]) \in \mathbb{O}^{n_1-r,n_2-r}(\overline{U}_2^{\mathbb{T}}\Delta^k\overline{V}_2)$ *with* $P_1^k \in \mathbb{O}^{(n_1-r)\times s}$ *and* $Q_1^k \in \mathbb{O}^{(n_2-r)\times s}$ *for an integer* $s \in [1, n_1-r]$, *and define* $\mathcal{M}^k := \mathcal{T}\oplus\mathcal{H}^k$ *with* $\mathcal{H}^k = \big\{\overline{U}_2P_1^kY(\overline{V}_2Q_1^k)^{\mathbb{T}} \mid Y \in \mathbb{R}^{s\times s}\big\}$. *Then,*

$$\big\|\Delta^k\big\|_F \le \sqrt{1 + r\gamma_{k-1}^2/(2s)}\,\big\|\mathcal{P}_{\mathcal{M}^k}(\Delta^k)\big\|_F.$$

Proof By the definitions of the subspaces $\mathcal{T}^\perp$ and $\mathcal{H}^k$, for any $Z \in \mathbb{R}^{n_1\times n_2}$ we have

$$\mathcal{P}_{\mathcal{T}^\perp}(Z) = \overline{U}_2\overline{U}_2^{\mathbb{T}}Z\overline{V}_2\overline{V}_2^{\mathbb{T}} \quad\text{and}\quad \mathcal{P}_{\mathcal{H}^k}(Z) = \overline{U}_2P_1^k(\overline{U}_2P_1^k)^{\mathbb{T}}Z\overline{V}_2Q_1^k(\overline{V}_2Q_1^k)^{\mathbb{T}}.$$

By this, it is easy to check that $\mathcal{P}_{\mathcal{H}^k}(\Delta^k) = \mathcal{P}_{\mathcal{H}^k}(\mathcal{P}_{\mathcal{T}^\perp}(\Delta^k))$. By the SVD of $\overline{U}_2^{\mathbb{T}}\Delta^k\overline{V}_2$,

$$\mathcal{P}_{\mathcal{T}^\perp}(\Delta^k) = \overline{U}_2\big(\overline{U}_2^{\mathbb{T}}\Delta^k\overline{V}_2\big)\overline{V}_2^{\mathbb{T}} = \overline{U}_2P^k\big[\mathrm{Diag}(\sigma(\overline{U}_2^{\mathbb{T}}\Delta^k\overline{V}_2))\ \ 0\big](Q^k)^{\mathbb{T}}\overline{V}_2^{\mathbb{T}} \quad (37)$$

where $P^k = [P_1^k\ \ P_2^k]$ and $Q^k = [Q_1^k\ \ Q_2^k]$. Together with the expression of $\mathcal{P}_{\mathcal{H}^k}(\mathcal{P}_{\mathcal{T}^\perp}(\Delta^k))$,

$$\mathcal{P}_{\mathcal{H}^k}(\Delta^k) = \mathcal{P}_{\mathcal{H}^k}(\mathcal{P}_{\mathcal{T}^\perp}(\Delta^k)) = \overline{U}_2P_1^k\big[\mathrm{Diag}\big(\sigma^{s,\downarrow}(\overline{U}_2^{\mathbb{T}}\Delta^k\overline{V}_2)\big)\ \ 0\big](Q_1^k)^{\mathbb{T}}\overline{V}_2^{\mathbb{T}}, \quad (38)$$

where $\sigma^{s,\downarrow}(\overline{U}_2^{\mathbb{T}}\Delta^k\overline{V}_2)$ is the vector consisting of the first s components of $\sigma(\overline{U}_2^{\mathbb{T}}\Delta^k\overline{V}_2)$. Notice that $\mathcal{P}_{\mathcal{M}^k}(\Delta^k) = \mathcal{P}_{\mathcal{T}}(\Delta^k) + \mathcal{P}_{\mathcal{H}^k}(\Delta^k)$ since the subspaces $\mathcal{T}$ and $\mathcal{H}^k$ are orthogonal. By combining this with equalities (37) and (38), it follows that

$$\begin{aligned}
&\|\Delta^k - \mathcal{P}_{\mathcal{M}^k}(\Delta^k)\| = \|\mathcal{P}_{\mathcal{T}^\perp}(\Delta^k) - \mathcal{P}_{\mathcal{H}^k}(\Delta^k)\| \le s^{-1}\|\mathcal{P}_{\mathcal{H}^k}(\Delta^k)\|_*,\\
&\|\Delta^k - \mathcal{P}_{\mathcal{M}^k}(\Delta^k)\|_* = \|\mathcal{P}_{\mathcal{T}^\perp}(\Delta^k) - \mathcal{P}_{\mathcal{H}^k}(\Delta^k)\|_* = \|\mathcal{P}_{\mathcal{T}^\perp}(\Delta^k)\|_* - \|\mathcal{P}_{\mathcal{H}^k}(\Delta^k)\|_*.
\end{aligned}$$

Together with $\|\mathcal{P}_{(\mathcal{M}^k)^\perp}(\Delta^k)\|_F^2 \le \|\mathcal{P}_{(\mathcal{M}^k)^\perp}(\Delta^k)\|\|\mathcal{P}_{(\mathcal{M}^k)^\perp}(\Delta^k)\|_*$ and Lemma 6,

$$\begin{aligned}
\|\mathcal{P}_{(\mathcal{M}^k)^\perp}(\Delta^k)\|_F &\le \big(\|\Delta^k - \mathcal{P}_{\mathcal{M}^k}(\Delta^k)\|\|\Delta^k - \mathcal{P}_{\mathcal{M}^k}(\Delta^k)\|_*\big)^{1/2} \le \frac{1}{2\sqrt{s}}\|\mathcal{P}_{\mathcal{T}^\perp}(\Delta^k)\|_*\\
&\le \frac{\gamma_{k-1}\sqrt{2r}}{2\sqrt{s}}\big\|\mathcal{P}_{\mathcal{T}}(\Delta^k)\big\|_F\\
&\le \frac{\gamma_{k-1}\sqrt{2r}}{2\sqrt{s}}\big\|\mathcal{P}_{\mathcal{M}^k}(\Delta^k)\big\|_F,
\end{aligned}$$

where the second inequality is using the fact that $ab \le (a+b)^2/4$ for $a, b \in \mathbb{R}$. The result then follows by noting that $\|\Delta^k\|_F^2 = \|\mathcal{P}_{\mathcal{M}^k}(\Delta^k)\|_F^2 + \|\mathcal{P}_{(\mathcal{M}^k)^\perp}(\Delta^k)\|_F^2$. □

Proof of Proposition 2 Since $\gamma_{k-1} \in [0, 1/c)$, by the definition of γ_{k-1}, it is clear that $\|\mathcal{P}_{\mathcal{T}^\perp}(W^{k-1})\| < 1$. From Assumption 1 and Lemma 4 of "Appendix B",

$$\frac{\pi(2r+s,s)\gamma_{k-1}}{s} \le \frac{c_{k-1}}{\sqrt{2r}} \quad \text{with} \quad c_{k-1} = c\gamma_{k-1} < 1. \tag{39}$$

Let $\mathcal{H}^k$ and $\mathcal{M}^k$ be defined as in Lemma 7. Using Lemma 5 of "Appendix B" with $\mathcal{L} = \mathcal{T}^\perp, \mathcal{J}_1 = \mathcal{H}^k, \mathcal{G} = \mathcal{M}^k, H = \mathcal{P}_{\mathcal{M}^k}(\Delta^k)$ and $G = \Delta^k$ and the fact $\mathcal{P}_{\mathcal{M}^k}(G-H) = 0$, we have

$$\begin{aligned}
&\max\left(0, \langle \mathcal{P}_{\mathcal{M}^k}(\Delta^k), \mathcal{A}^*\mathcal{A}(\Delta^k)\rangle\right) \\
&\ge \vartheta_-(2r+s)\left(\left\|\mathcal{P}_{\mathcal{M}^k}(\Delta^k)\right\|_F - \frac{\pi(2r+s,s)}{s}\left\|\mathcal{P}_{\mathcal{T}^\perp}(\Delta^k)\right\|_*\right)\left\|\mathcal{P}_{\mathcal{M}^k}(\Delta^k)\right\|_F \\
&\ge \vartheta_-(2r+s)\left(\|\mathcal{P}_{\mathcal{M}^k}(\Delta^k)\|_F - c_{k-1}\left\|\mathcal{P}_{\mathcal{T}}(\Delta^k)\right\|_F\right)\|\mathcal{P}_{\mathcal{M}^k}(\Delta^k)\|_F \\
&\ge \vartheta_-(2r+s)(1-c_{k-1})\|\mathcal{P}_{\mathcal{M}^k}(\Delta^k)\|_F^2 \ge 0,
\end{aligned} \tag{40}$$

where the second inequality is due to Lemma 6 and Eq. (39), and the last one is due to $\|\mathcal{P}_{\mathcal{T}}(\Delta^k)\|_F \le \|\mathcal{P}_{\mathcal{M}^k}(\Delta^k)\|_F$. In addition, by the definition of $\vartheta_+(\cdot)$, it holds that

$$\max\left(0, \langle \mathcal{P}_{\mathcal{M}^k}(\Delta^k), \mathcal{A}^*\mathcal{A}(\Delta^k)\rangle\right) \le \|\mathcal{A}(\mathcal{P}_{\mathcal{M}^k}(\Delta^k))\|\|\mathcal{A}(\Delta^k)\| \le 2\delta\sqrt{\vartheta_+(2r+s)}\|\mathcal{P}_{\mathcal{M}^k}(\Delta^k)\|_F$$

where the second inequality is using $\|\mathcal{A}(\Delta^k)\| \le \|\mathcal{A}(X^k) - b\| + \|\mathcal{A}(\overline{X}) - b\| \le 2\delta$. Together with (40), we obtain $\|\mathcal{P}_{\mathcal{M}^k}(\Delta^k)\|_F \le \frac{2\delta\sqrt{\vartheta_+(2r+s)}}{(1-c_{k-1})\vartheta_-(2r+s)}$. The first inequality in (19) then follows by Lemma 7. For the second inequality in (19), from Lemma 6 it follows that

$$\begin{aligned}
\left\|\mathcal{P}_{\mathcal{T}^\perp}(X^k)\right\|_* &= \left\|\mathcal{P}_{\mathcal{T}^\perp}(\Delta^k)\right\|_* \le \sqrt{2r}\gamma_{k-1}\left\|\mathcal{P}_{\mathcal{T}}(\Delta^k)\right\|_F \le \sqrt{2r}\gamma_{k-1}\left\|\mathcal{P}_{\mathcal{M}^k}(\Delta^k)\right\|_F \\
&\le \frac{2\delta\sqrt{2r}\gamma_{k-1}\sqrt{\vartheta_+(2r+s)}}{(1-c_{k-1})\vartheta_-(2r+s)}
\end{aligned}$$

where the first equality is due to $\overline{X} \in \mathcal{T}$, and the second inequality is since $\mathcal{M}^k = \mathcal{T} \oplus \mathcal{H}^k$. This shows that the second inequality in (19) holds. We complete the proof. □

Proof of Theorem 2 By the strict increasing of $\Xi(\cdot)$ in (20), it suffices to prove that

$$0 \le \gamma_k \le \widetilde{\gamma}_k < \widetilde{\gamma}_{k-1} < \cdots < \widetilde{\gamma}_1 < \widetilde{\gamma}_0 = 1/\sqrt{2}. \tag{41}$$

To establish the relations in (41), by the definition of $\widetilde{\gamma}_k$ in (22a), we need to prove that

$$\begin{cases} 0 \le \widetilde{a}_k \le \widetilde{a}_{k-1} \le \cdots \le \widetilde{a}_1 \le \widetilde{b}_1 \le \cdots \le \widetilde{b}_{k-1} \le \widetilde{b}_k \le 1, & (42a) \\ 0 \le \widetilde{\beta}_k < \widetilde{\beta}_{k-1} < \cdots < \widetilde{\beta}_1 < 1. & (42b) \end{cases}$$

By the definitions of $\widetilde{a}_k$ and $\widetilde{b}_k$ and Eq. (4a), $\{\widetilde{a}_k\}_{k\ge1} \subseteq [0,1]$ and $\{\widetilde{b}_k\}_{k\ge1} \subseteq [0,1]$. We next establish the monotone relations in (42a)–(42b) and (41) by induction on k. Let $(U^1, V^1) \in \mathbb{O}^{n_1,n_2}(X^1)$ where $U^1 = [U_1^1 \;\; U_2^1]$ with $U_1^1 \in \mathbb{O}^{n_1\times r}$ and $V^1 = [V_1^1 \;\; V_2^1]$ with $V_1^1 \in \mathbb{O}^{n_2\times r}$. By (15), $W^1 = U^1\big[\mathrm{Diag}(w_1^1,\ldots,w_{n_1}^1)\;\; 0\big](V^1)^{\mathbb{T}}$ with $1 \ge w_1^1 \ge \cdots \ge w_{n_1}^1 \ge 0$. Since $\gamma_0 = 1/\sqrt{2}$, by Proposition 2 we have $\|X^1 - \overline{X}\|_F \le \Xi(\gamma_0) = \Xi(\widetilde{\gamma}_0)$. From [18, Theorem 3.3.16], $\sigma_i(X^1) \ge \sigma_r(\overline{X}) - \Xi(\widetilde{\gamma}_0)$ for $i = 1,\ldots,r$ and $\sigma_i(X^1) \le \Xi(\widetilde{\gamma}_0)$ for $i = r+1,\ldots,n_1$. From the given assumption $\sigma_r(\overline{X}) > 2\Xi(\gamma_0) = 2\Xi(\widetilde{\gamma}_0)$, clearly, $\rho_1\Xi(\widetilde{\gamma}_0) < \rho_1(\sigma_r(\overline{X}) - \Xi(\widetilde{\gamma}_0))$. Together with (22a)–(22b) and (4b), we obtain $\widetilde{a}_1 \le \widetilde{b}_1$. In addition, by recalling that $w_i^1 \in \partial\psi^*(\rho_1\sigma_i(X^1))$ for each i, from (22a)–(22b) and (4b),

$$w_i^1 \ge \widetilde{b}_1,\; i = 1,2,\ldots,r \quad \text{and} \quad 0 \le w_i^1 \le \widetilde{a}_1,\; i = r+1,\ldots,n_1. \tag{43}$$

Now using Lemma 2 with $\widetilde{X} = \overline{X}$ and $W = W^1$ and the relations in (43) yields that

$$\begin{aligned} &\|\mathcal{P}_{\mathcal{T}^\perp}(W^1)\| \le w_{r+1}^1 + (1 - w_{r+1}^1)\big\|U_1^1(V_1^1)^{\mathbb{T}} - \overline{U}_1\overline{V}_1^{\mathbb{T}}\big\|^2, \\ &\|\mathcal{P}_{\mathcal{T}}(W^1 - \overline{U}_1\overline{V}_1^{\mathbb{T}})\|_F \le \sqrt{r}(1-\widetilde{b}_1) + (\sqrt{2}\widetilde{a}_1 + 1)\big\|U_1^1(V_1^1)^{\mathbb{T}} - \overline{U}_1\overline{V}_1^{\mathbb{T}}\big\|. \end{aligned}$$

Since $\|X^1 - \overline{X}\|_F \le \Xi(\widetilde{\gamma}_0)$, applying Lemma 3 with $\omega = \sigma_r(\overline{X})/\Xi(\widetilde{\gamma}_0)$, $\widetilde{X} = \overline{X}$, $X = X^1$ and $\eta = \Xi(\widetilde{\gamma}_0)$ we obtain $\|U_1^1(V_1^1)^{\mathbb{T}} - \overline{U}_1\overline{V}_1^{\mathbb{T}}\| \le \widetilde{\beta}_1 < 1$. Thus, it holds that

$$\begin{cases} 1 - \|\mathcal{P}_{\mathcal{T}^\perp}(W^1)\| \ge (1-\widetilde{a}_1)\big(1-\widetilde{\beta}_1^2\big), & (44a) \\ \|\mathcal{P}_{\mathcal{T}}(W^1 - \overline{U}_1\overline{V}_1^{\mathbb{T}})\|_F \le \sqrt{r}(1-\widetilde{b}_1) + (\sqrt{2}\widetilde{a}_1 + 1)\widetilde{\beta}_1. & (44b) \end{cases}$$

Along with $\widetilde{a}_1 < \frac{\sqrt{r}(\widetilde{b}_1 - \widetilde{\beta}_1^2) - \widetilde{\beta}_1}{\sqrt{r}(1-\widetilde{\beta}_1^2) + \sqrt{2}\widetilde{\beta}_1} < 1$ and the definitions of γ_1 and $\widetilde{\gamma}_1$, we have $0 \le \gamma_1 \le \widetilde{\gamma}_1$. Also, $\widetilde{\gamma}_1 < \frac{1}{\sqrt{2}}$ is implied by $\widetilde{a}_1 < \frac{\sqrt{r}(\widetilde{b}_1 - \widetilde{\beta}_1^2) - \widetilde{\beta}_1}{\sqrt{r}(1-\widetilde{\beta}_1^2) + \sqrt{2}\widetilde{\beta}_1}$. The desired (41) holds for $k = 1$.

Now assume that the conclusion holds for $k \le l-1$ with $l \ge 2$. We shall show that it holds for $k = l$. Since the conclusion holds for $k = l-1$, we have $\gamma_{l-1} \le \widetilde{\gamma}_{l-1} < 1/\sqrt{2}$. This means that the assumption of Proposition 2 holds for $k = l$. Consequently,

$$\|X^l - \overline{X}\|_F \le \Xi(\gamma_{l-1}) \le \Xi(\widetilde{\gamma}_{l-1}).$$

By [18, Theorem 3.3.16], $\sigma_i(X^l) \ge \sigma_r(\overline{X}) - \Xi(\widetilde{\gamma}_{l-1})$ for $i = 1,\ldots,r$ and $\sigma_i(X^l) \le \Xi(\widetilde{\gamma}_{l-1})$ for $i = r+1,\ldots,n_1$. Let $(U^l, V^l) \in \mathbb{O}^{n_1,n_2}(X^l)$ where $U^l = [U_1^l \;\; U_2^l]$ with $U_1^l \in \mathbb{O}^{n_1\times r}$ and $V^l = [V_1^l \;\; V_2^l]$ with $V_1^l \in \mathbb{O}^{n_2\times r}$. From Eq. (15), $W^l =$

$U^l\big[\mathrm{Diag}(w_1^l,\ldots,w_{n_1}^l)\ \ 0\big](V^l)^{\mathbb{T}}$ with $1\geq w_1^l\geq\cdots\geq w_{n_1}^l\geq 0$. Along with the definitions of $\widetilde{a}_l$ and $\widetilde{b}_l$ and Eq. (4b),

$$w_i^l\geq\widetilde{b}_l,\ i=1,2,\ldots,r\quad\text{and}\quad 0\leq w_i^l\leq\widetilde{a}_l,\ i=r+1,\ldots,n_1. \tag{45}$$

Since the conclusion holds for $k=l-1$, we have $\Xi(\widetilde{\gamma}_{l-1})<\Xi(\widetilde{\gamma}_0)$, and then $\frac{\sigma_r(\overline{X})}{\Xi(\widetilde{\gamma}_{l-1})}>2$. Using Lemma 2 with $\widetilde{X}=\overline{X}$ and $W=W^l$ and Lemma 3 with $\omega=\frac{\sigma_r(\overline{X})}{\Xi(\widetilde{\gamma}_{l-1})}$, $\widetilde{X}=\overline{X}$, $X=X^l$ and $\eta=\Xi(\widetilde{\gamma}_{l-1})$ and following the same arguments as those for $k=1$, we have

$$\begin{cases} 1-\|\mathcal{P}_{\mathcal{T}^\perp}(W^l)\|\geq(1-\widetilde{a}_l)(1-\widetilde{\beta}_l^2), & \text{(46a)}\\ \|\mathcal{P}_{\mathcal{T}}(W^l-\overline{U}_1\overline{V}_1^{\mathbb{T}})\|_F\leq\sqrt{r}(1-\widetilde{b}_l)+(\sqrt{2}\widetilde{a}_l+1)\widetilde{\beta}_l. & \text{(46b)}\end{cases}$$

Notice that $1\leq\mu_l\leq\frac{\Xi(\widetilde{\gamma}_{l-2})}{\Xi(\widetilde{\gamma}_{l-1})}$. So, $\rho_{l-1}\leq\rho_l\leq\frac{\rho_{l-1}\Xi(\widetilde{\gamma}_{l-2})}{\Xi(\widetilde{\gamma}_{l-1})}$. By the definitions of $\widetilde{a}_l$ and $\widetilde{b}_l$ and Eq. (4b), $\widetilde{a}_l\leq\widetilde{a}_{l-1}$ and $\widetilde{b}_l\geq\widetilde{b}_{l-1}$. In addition, since $\widetilde{\gamma}_{l-1}<\widetilde{\gamma}_{l-2}$, we have $\Xi(\widetilde{\gamma}_{l-1})<\Xi(\widetilde{\gamma}_{l-2})$, and then $\widetilde{\beta}_l<\widetilde{\beta}_{l-1}$. Equations (46a) and (46b) and the definitions of γ_l and $\widetilde{\gamma}_l$ imply that $0\leq\gamma_l\leq\widetilde{\gamma}_l<\widetilde{\gamma}_{l-1}$. Thus, the conclusion holds for $k=l$. □

Proof of Theorem 3 Notice that the assumption of Theorem 2 is satisfied. The monotone relations in (42a)–(42b) hold for all $k\geq 2$. For $k=1$, clearly, (23) holds. Now fix $k\geq 2$. Let $(U^{k-1},V^{k-1})\in\mathbb{O}^{n_1,n_2}(X^{k-1})$ where $U^{k-1}=[U_1^{k-1}\ \ U_2^{k-1}]$ with $U_1^{k-1}\in\mathbb{O}^{n_1\times r}$ and $V^{k-1}=[V_1^{k-1}\ \ V_2^{k-1}]$ with $V_1^{k-1}\in\mathbb{O}^{n_2\times r}$. Then $W^{k-1}=U^{k-1}[\mathrm{Diag}\big(w_1^{k-1},\ldots,w_{n_1}^{k-1}\big)\ \ 0](V^{k-1})^{\mathbb{T}}$ with $1\geq w_1^{k-1}\geq\cdots\geq w_{n_1}^{k-1}\geq 0$. By following the same arguments as those for Theorem 2, we have

$$\begin{aligned}&1-\|\mathcal{P}_{\mathcal{T}^\perp}(W^{k-1})\|\geq(1-\widetilde{a}_{k-1})(1-\widetilde{\beta}_{k-1}^2),\\ &\big\|\mathcal{P}_{\mathcal{T}}(W^{k-1}-\overline{U}_1\overline{V}_1^{\mathbb{T}})\big\|_F\leq\sqrt{r}(1-\widetilde{b}_{k-1})+(1+\sqrt{2}\widetilde{a}_{k-1})\big\|U_1^{k-1}(V_1^{k-1})^{\mathbb{T}}-\overline{U}_1\overline{V}_1^{\mathbb{T}}\big\|.\end{aligned}$$

Also, from [24, Equation(49)-(51)], $\big\|U_1^{k-1}(V_1^{k-1})^{\mathbb{T}}-\overline{U}_1\overline{V}_1^{\mathbb{T}}\big\|_F\leq\frac{\|X^{k-1}-\overline{X}\|_F}{\sigma_r(\overline{X})-\sqrt{2}\Xi(\gamma_0)}$. Thus, together with the definition of γ_{k-1}, it immediately follows that

$$\gamma_{k-1}\leq\frac{1-\widetilde{b}_1}{\sqrt{2}(1-\widetilde{a}_1)(1-\widetilde{\beta}_1^2)}+\frac{1+\sqrt{2}\widetilde{a}_1}{\sqrt{2r}(1-\widetilde{a}_1)(1-\widetilde{\beta}_1^2)}\cdot\frac{\|X^{k-1}-\overline{X}\|_F}{\sigma_r(\overline{X})-\sqrt{2}\Xi(\gamma_0)}.$$

From the first part of Theorem 2 and the first inequality of (19), it follows that

$$\begin{aligned}\|X^k-\overline{X}\|_F&\leq\frac{2\delta\sqrt{\vartheta_+(2r+s)}}{(1-c\gamma_{k-1})\vartheta_-(2r+s)}\Big(1+\sqrt{\frac{r}{2s}}\gamma_{k-1}\Big)=\frac{\Xi(0)}{(1-c\gamma_{k-1})}\Big(1+\sqrt{\frac{r}{2s}}\gamma_{k-1}\Big)\\&\leq\frac{\Xi(0)}{1-c\widetilde{\gamma}_1}\Big[1+\frac{(1-\widetilde{b}_1)\sqrt{r}}{2(1-\widetilde{a}_1)(1-\widetilde{\beta}_1^2)\sqrt{s}}\Big]+\Big[\frac{\alpha\,\Xi(\gamma_0)}{\sigma_r(\overline{X})-\sqrt{2}\Xi(\gamma_0)}\Big]\|X^{k-1}-\overline{X}\|_F\end{aligned} \tag{47}$$

where the second inequality is using $\Xi(\gamma_0) = \frac{\Xi(0)}{1-c\widetilde{\gamma}_0}\sqrt{\frac{4s+r}{4s}}$. Since $\sigma_r(\overline{X}) > (\sqrt{2}+\alpha)\Xi(\gamma_0)$ implies $0 \le \frac{\alpha\,\Xi(\gamma_0)}{\sigma_r(\overline{X})-\sqrt{2}\Xi(\gamma_0)} < 1$, the desired inequality follows by the recursion (47). □

Lemma 8 *If the components $\xi_1, \xi_2, \ldots, \xi_m$ of ξ are independent sub-Gaussians, then $\|\xi\| \le \sqrt{m}\sigma$ with probability at least $1 - \exp(1 - \frac{c_1 m}{4})$ for an absolute constant $c_1 > 0$.*

Proof Notice that $\|\xi\| = \sup_{u\in\mathcal{S}^{m-1}}\langle u, \xi\rangle$, where $\mathcal{S}^{m-1}$ denotes the unit sphere in $\mathbb{R}^m$. Let $\mathcal{U} := \{u^1, \ldots, u^m\}$ denote $1/2$ covering of $\mathcal{S}^{m-1}$. Then, for any $u \in \mathcal{S}^{m-1}$, there exists $\overline{u} \in \mathcal{U}$ such that $u = \overline{u} + \Delta u$ with $\|\Delta u\| \le 1/2$. Consequently, $\langle u, \xi\rangle = \langle \overline{u}, \xi\rangle + \langle \Delta u, \xi\rangle \le \langle \overline{u}, \xi\rangle + \frac{1}{2}\|\xi\|$. This, by $\|\xi\| = \sup_{u\in\mathcal{S}^{m-1}}\langle u, \xi\rangle$, implies that $\|\xi\| \le 2\langle \overline{u}, \xi\rangle = 2\sum_{i=1}^m \overline{u}_i\xi_i$. By the Hoeffding-type inequality (see [39]), there exists an absolute constant $c_1 > 0$ such that for every $t > 0$,

$$\mathbb{P}\{\|\xi\| \ge t\} \le \mathbb{P}\left\{\Big|\sum_{i=1}^m \overline{u}_i\xi_i\Big| \ge t/2\right\} \le \exp\big(1 - c_1 t^2/(4\sigma^2)\big).$$

Taking $t = \sqrt{m}\sigma$, we obtain the desired result from the last equation. □

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