

THE AUBIN PROPERTY FOR GENERALIZED EQUATIONS OVER C^2 -CONE REDUCIBLE SETS

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ABSTRACT. This paper establishes the equivalence of the Aubin property and the strong regularity for generalized equations over C^2 -cone reducible sets. This result resolves a long-standing question in variational analysis and extends the well-known equivalence theorem for polyhedral sets to a significantly broader class of non-polyhedral cases. Our proof strategy departs from traditional variational techniques, integrating insights from convex geometry with powerful tools from algebraic topology. A cornerstone of our analysis is a new fundamental lemma concerning the local structure of the normal cone map for arbitrary closed convex sets, which reveals how the dimension of normal cones varies in the neighborhood of a boundary point. This geometric insight is the key to applying degree theory, allowing us to prove that a crucial function associated with the problem has a topological index of ± 1 . This, via a homological version of the inverse mapping theorem, implies that the function is a local homeomorphism, which in turn yields the strong regularity of the original solution map. This result unifies and extends several existing stability results for problems such as conventional nonlinear programming, nonlinear second-order cone programming, and nonlinear semidefinite programming under a single general framework.

Keywords. Aubin property, Strong regularity, C^2 -cone reducible set, Generalized equation, Convex geometry, Degree theory

1. INTRODUCTION

In this paper, we aim to prove that the Aubin property and the strong regularity are equivalent for canonically perturbed generalized equations over C^2 -cone reducible sets. Specifically, we consider the following two types of generalized equations:

$$(1) \quad y \in \varphi(x) + N_S(x) \quad \text{and} \quad y \in \varphi(x) + N_S^{-1}(x), \quad x \in \mathbb{R}^n,$$

where $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuously differentiable function, $S \subset \mathbb{R}^n$ is a nonempty closed convex set, $y \in \mathbb{R}^n$ is the parameter vector and $N_S : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is the normal cone map of S . Many problems in optimization and variational analysis can be written in the form of (1), such as the Karush-Kuhn-Tucker (KKT) system and the variational inequality. For convenience, let $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ denote the right-hand side of the generalized equation in (1), i.e.,

$$(2) \quad \Phi(x) = \varphi(x) + N_S(x) \quad \text{or} \quad \Phi(x) = \varphi(x) + N_S^{-1}(x), \quad x \in \mathbb{R}^n.$$

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In a seminal paper [12], Dontchev and Rockafellar proved the equivalence between the Aubin property and the strong regularity for Φ^{-1} when S is a polyhedral set (they proved it for the first type of Φ , and in fact one can transform the second type to the first type when S is polyhedral). The approach in [12] highly depends on the structure of the polyhedral set and the associated piecewise affine normal map proposed by Robinson [17]. Whether one can use a similar approach for the general non-polyhedral case is still open. Recently, Chen et al. proved the equivalence between the Aubin property and the strong regularity for the KKT solution mapping of canonically perturbed nonlinear second-order cone programming [10] and nonlinear semidefinite programming [9] at a locally optimal solution, both of which can be written in the form of Φ^{-1} over a non-polyhedral set S with a special structure.

So far, for all known related results, the set S is C^2 -cone reducible at each $x \in S$ in the sense of [6, Definition 3.135], or C^2 -cone reducible for short. A natural question is whether the Aubin property and the strong regularity for Φ^{-1} are equivalent whenever S is C^2 -cone reducible. In this paper, we will provide an affirmative answer to this question.

For a set-valued map $\Psi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$, the inverse of Ψ is defined as $\Psi^{-1}(y) := \{x \in \mathbb{R}^n \mid y \in \Psi(x)\}$. The graph of Ψ is $\text{gph } \Psi := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid y \in \Psi(x)\}$. We are concerned with the following two localized Lipschitzian properties (the Aubin property and the strong regularity) for Ψ around $(x_0, y_0) \in \text{gph } \Psi$:

- (L1) Ψ has the Aubin property [2] around (x_0, y_0) . That is, there exist open neighborhoods U of x_0 and V of y_0 and a positive constant $\lambda > 0$ such that

$$\Psi(x) \cap V \subset \Psi(x') + \lambda \|x - x'\| \mathbb{B}, \quad \forall x, x' \in U,$$

where $\mathbb{B}$ is the closed unit ball in $\mathbb{R}^n$.

- (L2) Ψ is locally single-valued and Lipschitz continuous around (x_0, y_0) . That is, there exist open neighborhoods U of x_0 and V of y_0 such that the map $x \mapsto \Psi(x) \cap V$ is single-valued and Lipschitz continuous on U .

Let $(x_0, y_0) \in \text{gph } \Phi$. In a landmark paper [16], Robinson found that generalized equations $y \in \Phi(x)$ are closely related to the corresponding linearized generalized equations $y \in \widehat{\Phi}(x)$, where

$$\widehat{\Phi}(x) = \varphi(x_0) + \varphi'(x_0)(x - x_0) + N_S(x) \quad \text{or} \quad \widehat{\Phi}(x) = \varphi(x_0) + \varphi'(x_0)(x - x_0) + N_S^{-1}(x).$$

The property (L2) for $\widehat{\Phi}^{-1}$ around (y_0, x_0) is called the strong regularity for Φ^{-1} around (y_0, x_0) ; see [16] and [6, Definition 5.12]. Actually, according to [6, Theorem 5.13], the property (L2) for Φ^{-1} around (y_0, x_0) and the property (L2) for $\widehat{\Phi}^{-1}$ around (y_0, x_0) are equivalent. So, the strong regularity and the property (L2) for Φ^{-1} are the same.

Clearly, for any set-valued map, (L2) implies (L1), but the converse does not necessarily hold in general. In this paper, we prove the equivalence between the properties (L1) and (L2) for Φ^{-1} around $(y_0, x_0) \in \text{gph } \Phi^{-1}$ when S is C^2 -cone reducible. From now on, we always assume that S in (1) is C^2 -cone reducible. By the C^2 -cone reduction and other operations, we find that Φ is closely related to a function $N : \mathbb{R}^n \rightarrow \mathbb{R}^n$ in the following form:

$$(3) \quad N(z) = A(z - \Pi_K(z)) + \Pi_K(z), \quad z \in \mathbb{R}^n,$$

where $K \subset \mathbb{R}^n$ is a nonempty closed convex set, A is an n by n matrix and $\Pi_K : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is the metric projection onto K given by $\Pi_K(z) := \arg \min_{z' \in K} \|z - z'\|$, $z \in \mathbb{R}^n$. Note that N is similar to the normal map $z \mapsto \varphi(\Pi_S(z)) + (z - \Pi_S(z))$ induced by the generalized equation $y \in \varphi(x) + N_S(x)$ and proposed by Robinson [17], which can be transformed into the form of N without affecting our problem. In fact, with appropriately chosen K and A , the property of Φ around some point is related to the property of N around some point $z_0 \in K$. The condition that z_0 lies in the set K is crucial, as our topological approach relies on analyzing the local behavior of N . This is the reason why we need the C^2 -cone reducible property.

Since the set K in (3) is an arbitrary closed convex set without other specific structures, we shall use a topological approach combined with convex geometry. Our proof relies on two key tools: the dimension of $N_K(z)$ and (topological) degree theory [14, 15]. Degree theory is a powerful tool for analyzing the existence, multiplicity, and qualitative properties of solutions to equations involving continuous functions, even in the absence of differentiability assumptions. So, it is suitable for the study of the continuous function N and the normal map.

The general idea of our proof goes as follows: Given Φ and $(x_0, y_0) \in \text{gph } \Phi$, suppose that Φ^{-1} has the Aubin property around (y_0, x_0) . There is a continuous function (similar to the normal map) $\hat{N} : \mathbb{R}^n \rightarrow \mathbb{R}^n$ induced by Φ such that $\hat{N}^{-1}$ also has the Aubin property around $(\hat{N}(\hat{z}_0), \hat{z}_0)$ where $\hat{z}_0 \in \mathbb{R}^n$ is determined by Φ and (x_0, y_0) . The property of $\hat{N}$ around $\hat{z}_0$ is closely related to another function N in the form of (3) around a point $z_0 \in K$. The most critical step is to use the geometric structure of N_K around z_0 to prove that the (topological) index of N at z_0 is ± 1 as long as N satisfies some conditions at z_0 , which are guaranteed by the Aubin property of $\hat{N}^{-1}$ around $(\hat{N}(\hat{z}_0), \hat{z}_0)$. This in turn implies that the index of $\hat{N}$ is ± 1 at $\hat{z}_0$. According to the homological version of the inverse mapping theorem [3], $\hat{N}$ is a local homeomorphism at $\hat{z}_0$, which yields the strong regularity of Φ^{-1} around (y_0, x_0) . The detailed proofs are presented in the following sections.

The remaining parts of this paper are organized as follows. Section 2 introduces the notation and some preliminary results used in this paper. In Section 3, we study the index of N . The equivalence between the Aubin property and the strong regularity for Φ^{-1} over C^2 -cone reducible set S is proved in Section 4. We conclude the paper in Section 5.

2. NOTATION AND PRELIMINARIES

We use the standard inner product $\langle \cdot, \cdot \rangle$ and the Euclidean norm $\|\cdot\|$ in $\mathbb{R}^n$. Let $\Omega \subset \mathbb{R}^n$. We use $\text{int } \Omega$ and $\bar{\Omega}$ to denote the interior and closure of Ω , respectively. The boundary of Ω is denoted by $\partial\Omega := \bar{\Omega} \setminus \text{int } \Omega$. A point $x \in \Omega$ is an isolated point of Ω if there exists a neighborhood U of x such that $U \cap \Omega = \{x\}$. The distance from $x \in \mathbb{R}^n$ to Ω is defined by $\text{dist}(x, \Omega) := \inf\{\|x - x'\| \mid x' \in \Omega\}$. Let $L \subset \mathbb{R}^n$ be a linear subspace. The orthogonal complement of L is denoted by $L^\perp$. The orthogonal projection matrix P onto L is an n by n matrix such that $Px = x$ for all $x \in L$ and $Px = 0$ for all $x \in L^\perp$.

For a nonempty closed convex cone $C \subset \mathbb{R}^n$, the polar cone of C is given by $C^\circ := \{u \in \mathbb{R}^n \mid \langle x, u \rangle \leq 0, \forall x \in C\}$. Let $K \subset \mathbb{R}^n$ be a nonempty closed convex

set. The normal cone map $N_K : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is defined by

$$N_K(x) := \begin{cases} \{v \in \mathbb{R}^n \mid \langle v, x' - x \rangle \leq 0, \forall x' \in K\} & \text{if } x \in K, \\ \emptyset & \text{if } x \notin K, \end{cases}$$

and the tangent cone map $T_K : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ of K is given by

$$T_K(x) := \begin{cases} N_K^\circ(x) & \text{if } x \in K, \\ \emptyset & \text{if } x \notin K. \end{cases}$$

Note that for any $x \in K$, $v \in N_K(x)$ if and only if $x = \Pi_K(x + v)$. The dimension of K , i.e., the dimension of the affine hull of K , is denoted by $\dim K$. The relative interior and relative boundary of K is denoted by $\text{ri } K$ and $\text{rb } K$, respectively. A face of K is a convex subset $F \subset K$ such that $\lambda x + (1 - \lambda)x' \in F$ with $x, x' \in K$ and $0 < \lambda < 1$ implies $x, x' \in F$. A supporting half-space to K is a closed half-space containing K and a boundary point of K , which can be represented in the form $\{x \in \mathbb{R}^n \mid \langle x, u \rangle \leq \alpha\}$ where $u \neq 0$ is called an (outer) normal vector of K . A supporting hyperplane to K is the boundary of a supporting half-space to K . For each supporting hyperplane H to K , the set $K \cap H$ is a face of K called an exposed face of K .

A continuously differentiable function is called the C^1 function, and a twice continuously differentiable function is called the C^2 function. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a function and $x_0 \in \mathbb{R}^n$. We say that f is a local homeomorphism at x_0 if there exists an open neighborhood U of x_0 such that $f(U)$ is open and the restriction $f|_U : U \rightarrow f(U)$ is a homeomorphism. If f is differentiable at x_0 , the Jacobian matrix is denoted by $f'(x_0)$, whose transpose is denoted by $\nabla f(x_0) = (f'(x_0))^T$. We say that f is open around x_0 , if there exists an open neighborhood U of x_0 such that $f|_U$ (the restriction of f to U) is an open map, that is, for any open set $V \subset U$, $f(V)$ is open in $\mathbb{R}^m$. The function f is said to be discrete at x_0 if x_0 is an isolated point of $f^{-1}(f(x_0))$, i.e., there exists an open neighborhood W of x_0 such that $f^{-1}(f(x_0)) \cap W = \{x_0\}$. If there exists an open neighborhood U of x_0 such that f is discrete at each $x \in U$, then we say that f is discrete around x_0 .

Given a set-valued map $\Psi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ and $x_0 \in \mathbb{R}^n$. Ψ is outer semicontinuous at x_0 if

$$\limsup_{x \rightarrow x_0} \Psi(x) \subset \Psi(x_0),$$

and Ψ is inner semicontinuous at x_0 if

$$\liminf_{x \rightarrow x_0} \Psi(x) \supset \Psi(x_0),$$

where “ $\limsup$ ” and “ $\liminf$ ” are the outer limit and the inner limit in Painlevé-Kuratowski convergence for subsets, respectively. Ψ is called outer (inner) semicontinuous if Ψ is outer (inner) semicontinuous at each $x \in \mathbb{R}^n$. According to [18, Corollary 6.29], for a nonempty closed convex set $K \subset \mathbb{R}^n$, the normal cone map $N_K : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is outer semicontinuous, and the tangent cone map $T_K : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is inner semicontinuous. We say Ψ is inner semicontinuous around $(x_0, y_0) \in \text{gph } \Psi$ if there exist open neighborhoods U of x_0 and V of y_0 such that the map $x \mapsto \Psi(x) \cap V$ is inner semicontinuous at each $x \in U$. We have the following lemma.

Lemma 1. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a continuous function and $x_0 \in \mathbb{R}^n$. If f^{-1} is inner semicontinuous around $(f(x_0), x_0)$, then f is open around x_0 .*

Proof. Assume that f^{-1} is inner semicontinuous around $(f(x_0), x_0)$. Then there exist open neighborhoods U of x_0 and V of $f(x_0)$ such that the following set-valued map $\Psi : \mathbb{R}^m \rightrightarrows \mathbb{R}^n$ is inner semicontinuous:

$$\Psi(y) := \begin{cases} \{x \in U \mid f(x) = y\} & \text{if } y \in V, \\ \emptyset & \text{if } y \notin V. \end{cases}$$

So $\Psi^{-1}(W)$ is open for every open set $W \subset \mathbb{R}^n$ [18, Theorem 5.7, Part (c)]. Let $W \subset U \cap f^{-1}(V)$ be an open set. Then the set $\Psi^{-1}(W) = \{y \in \mathbb{R}^m \mid \Psi(y) \cap W \neq \emptyset\} = \{y \in V \mid \{x \in W \mid f(x) = y\} \neq \emptyset\} = f(W)$ is open. Therefore, f is open around x_0 , since $U \cap f^{-1}(V)$ is an open set containing x_0 . $\square$

We know, by definition, that the Aubin property of Ψ around $(x_0, y_0) \in \text{gph } \Psi$ implies that Ψ is inner semicontinuous around (x_0, y_0) . So, by Lemma 1, for a continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $x_0 \in \mathbb{R}^n$, if f^{-1} has the Aubin property around $(f(x_0), x_0)$, then f is open around x_0 .

2.1. Classification of normal vectors. Consider the function N defined in (3). We have

$$N(x + u) = A(x + u - \Pi_K(x + u)) + \Pi_K(x + u) = Au + x, \quad \forall (x, u) \in \text{gph } N_K.$$

So in a sense, the function N acts as a “linear transform” on the normal vectors of K . Therefore, it is important to understand the structure of the normal vectors of K around a point $x \in K$.

We need the classification of normal vectors of a compact convex set $K \subset \mathbb{R}^n$ described in [19, Section 2.2], which was first studied by Bonnesen and Fenchel [7]. Let $u \in \mathbb{R}^n \setminus \{0\}$. Then u is an outer normal vector of K at each point in the exposed face $N_K^{-1}(u)$ of K . For each $x \in \text{ri } N_K^{-1}(u)$, the normal cone of K at x is the same, that is, $N_K(x) = N_K(\text{ri } N_K^{-1}(u))$. The touching cone of K at u , denoted by $T(K, u)$, is the smallest face of $N_K(\text{ri } N_K^{-1}(u))$ that contains u . Moreover, for each $x \in N_K^{-1}(u)$, $T(K, u)$ is the smallest face of $N_K(x)$ that contains u ; see [19, Note 6 for Section 2.2]. The vector u is called an r -extreme normal vector of K if $\dim T(K, u) \leq r + 1$. And u is called an r -exposed normal vector of K if $\dim N_K(\text{ri } N_K^{-1}(u)) \leq r + 1$. An important relationship between these two concepts is established in the following lemma, which is a dual version of the classical result of Asplund [1].

Lemma 2 ([19, Theorem 2.2.9]). *Let $K \subset \mathbb{R}^n$ be a nonempty compact convex set and $r \in \{0, 1, \dots, n - 1\}$. Then each r -extreme normal vector of K is a limit of r -exposed normal vectors of K .*

2.2. Degree theory and inverse mapping theorem. For any bounded open set $D \subset \mathbb{R}^n$, continuous function $f : \overline{D} \rightarrow \mathbb{R}^n$, and $y \in \mathbb{R}^n \setminus f(\partial D)$, the degree of f on D at y is an integer denoted by $\deg(f, D, y)$. Here we list some properties of the degree that we will use. For more details, the reader may refer to [14, 15].

Proposition 1. *Let $D \subset \mathbb{R}^n$ be a bounded open set and $f : \overline{D} \rightarrow \mathbb{R}^n$ be a continuous function.*

- *Let $y \in \mathbb{R}^n \setminus f(\partial D)$. If f is differentiable on an open set containing $f^{-1}(y)$ and the Jacobian matrix $f'(x)$ is nonsingular for all $x \in f^{-1}(y)$, then $\deg(f, D, y) = \sum_{x \in f^{-1}(y)} \text{sgn } \det f'(x)$.*

- (Local constancy) The degree $\deg(f, D, \cdot)$ is constant on every connected component of $\mathbb{R}^n \setminus f(\partial D)$.
- (Homotopy invariance) If $H : [0, 1] \times \overline{D} \rightarrow \mathbb{R}^n$ is continuous, $\gamma : [0, 1] \rightarrow \mathbb{R}^n$ is continuous, and $\gamma(t) \notin H(t, \partial D)$ for all $t \in [0, 1]$, then $\deg(H(t, \cdot), D, \gamma(t))$ does not depend on t .
- (Multiplication formula) Let E be a bounded open set and $g : \overline{E} \rightarrow \mathbb{R}^n$ be a continuous function. If E is connected, $f(\overline{D}) \subset \overline{E}$ and $f(\partial D) \subset \partial E$, then for any $z \in \mathbb{R}^n \setminus g(\partial E)$ and $y \in f(D)$, we have

$$\deg(g \circ f, D, z) = \deg(g, E, z) \deg(f, D, y).$$

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous function. Suppose that $x \in \mathbb{R}^n$ is an isolated point of $f^{-1}(f(x))$. Take any bounded open set D such that $f^{-1}(f(x)) \cap D = \{x\}$. Then the degree $\deg(f|_{\overline{D}}, D, f(x))$ is independent of the choice of D , which is called the index (local degree) of f at x and denoted by $\text{ind}(f, x)$. By the homotopy invariance and the multiplication formula of the degree, it is easy to check that for any $x', y' \in \mathbb{R}^n$, we have $\text{ind}(f, x) = \text{ind}(f(\cdot + x') + y', x - x')$.

With the assumption on the index, we have the following homological version of the inverse mapping theorem [3], which follows from the classical result of Černavskii [21, 22] about discrete open maps on manifolds; see [20] and [4].

Lemma 3 ([3, Theorem 1.2]). *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous function. If f is open and discrete around $x_0 \in \mathbb{R}^n$ and $|\text{ind}(f, x_0)| = 1$, then f is a local homeomorphism at x_0 .*

2.3. Strictly stationary property. A set-valued map $\Psi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is called locally closed-valued around $(x_0, y_0) \in \text{gph } \Psi$, if there exist neighborhoods U of x_0 and V of y_0 such that $\Psi(x) \cap \overline{V}$ is closed for each $x \in U$. Consider the following property for Ψ around $(x_0, y_0) \in \text{gph } \Psi$:

(L3) Ψ is locally closed-valued and has the Aubin property around (x_0, y_0) .

Note that for a continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, the properties **(L1)** and **(L3)** are the same for f^{-1} since $f^{-1}(y)$ is closed for any $y \in \mathbb{R}^m$. A function $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is called strictly stationary [11] at $x_0 \in \mathbb{R}^n$ if, for any $\varepsilon > 0$, there exists an open neighborhood U of x_0 , such that

$$\|\psi(x) - \psi(x')\| < \varepsilon \|x - x'\|, \quad \forall x, x' \in U.$$

Lemma 4 ([11]). *Let $\Psi : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ be a set-valued map and $(x_0, y_0) \in \text{gph } \Psi$. Let $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a function which is strictly stationary at x_0 . Then Ψ^{-1} has the property **(L2)** (respectively, **(L3)**) around (y_0, x_0) if and only if $(\Psi + \psi)^{-1}$ has the property **(L2)** (respectively, **(L3)**) around $(y_0 + \psi(x_0), x_0)$.*

According to Lemma 4, strictly stationary property allows us to transfer the problem from one set-valued map to another. So, it is important to find out what kind of functions are strictly stationary. Note that $\psi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is strictly stationary at x_0 if and only if each component $\psi_k : \mathbb{R}^n \rightarrow \mathbb{R}$ is strictly stationary at x_0 . We have the following lemma. Its proof is direct, and thus omitted.

Lemma 5. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$, $g : \mathbb{R}^n \rightarrow \mathbb{R}$ and $x_0 \in \mathbb{R}^n$. Consider the following cases:*

- f, g are strictly stationary at x_0 . Let $\psi(x) = f(x) + g(x)$.
- f is continuously differentiable around x_0 . Let $\psi(x) = f'(x_0)x - f(x)$.

- f is strictly stationary at x_0 , $f(x_0) = 0$ and g is Lipschitz around x_0 . Let $\psi(x) = f(x)g(x)$.
- f, g are Lipschitz around x_0 and $f(x_0) = g(x_0) = 0$. Let $\psi(x) = f(x)g(x)$.

Then ψ is strictly stationary at x_0 in each case.

The following lemma allows us to keep the Aubin property while keeping the index.

Lemma 6. *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a continuous function such that f is discrete at $x_0 \in \mathbb{R}^n$ and f^{-1} has the Aubin property around $(f(x_0), x_0)$. Let $g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a function which is strictly stationary at x_0 . Then $f + g$ is discrete at x_0 and $\text{ind}(f, x_0) = \text{ind}(f + g, x_0)$.*

Proof. Proof Since f^{-1} has the Aubin property around $(f(x_0), x_0)$ and $\text{gph } f$ is closed, we have that f is metrically regular at $(x_0, f(x_0))$ by [8], that is, there exist a constant $\kappa > 0$ and bounded open neighborhoods U of x_0 and V of $f(x_0)$ such that for each $x \in U$ and $y \in V$, we have

$$\text{dist}(x, f^{-1}(y)) \leq \kappa \text{dist}(f(x), y).$$

Since f is discrete at x_0 , by shrinking U if necessary, we can assume that $f^{-1}(f(x_0)) \cap U = \{x_0\}$ and for each $x \in \overline{U}$,

$$\text{dist}(x, f^{-1}(f(x_0))) = \|x - x_0\| \leq \kappa \|f(x) - f(x_0)\|.$$

Since g is strictly stationary at x_0 , by shrinking U if necessary, we can assume that for each $x \in \overline{U}$,

$$\|g(x) - g(x_0)\| \leq (2\kappa)^{-1} \|x - x_0\| \leq \|f(x) - f(x_0)\|/2.$$

Define the homotopy $H : [0, 1] \times \overline{U} \rightarrow \mathbb{R}^n$ by

$$H(t, x) := f(x) + tg(x).$$

Then $H(t, x_0) \notin H(t, \overline{U} \setminus \{x_0\})$ for all $t \in [0, 1]$. By the homotopy invariance of the degree, we have

$$\text{ind}(f, x_0) = \deg(f|_{\overline{U}}, U, f(x_0)) = \deg((f + g)|_{\overline{U}}, U, f(x_0) + g(x_0)) = \text{ind}(f + g, x_0),$$

which completes the proof. $\square$

Lemma 6 is crucial as it allows us to add strictly stationary terms to a function without altering its index, which is a key step for simplifying the operator in our main arguments in Section 4.

2.4. C^2 -cone reduction. Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set. We say that K is C^2 -cone reducible at $x_0 \in K$ [6, Definition 3.135], if there exist a pointed closed convex cone $C \subset \mathbb{R}^m$, a neighborhood U of x_0 and a C^2 function $\Xi : \mathbb{R}^n \rightarrow \mathbb{R}^m$ such that $\Xi'(x_0) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is onto, $\Xi(x_0) = 0$, and $K \cap U = \{x \in U \mid \Xi(x) \in C\}$. Note that $m \leq n$. If $m = n$, then Ξ is a C^2 local homeomorphism around x_0 as the Jacobian matrix $\Xi'(x_0)$ is nonsingular. And if $m < n$, the reduction changes the dimension of space and Ξ is not a local homeomorphism around x_0 . However, we show that Ξ can always be extended to a C^2 local homeomorphism in the following lemma.

Lemma 7. *Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set. If K is C^2 -cone reducible at $\hat{x} \in K$, then there exist a closed convex cone $C \subset \mathbb{R}^n$, open neighborhoods U of $\hat{x}$ and V of 0 and a C^2 homeomorphism $h : U \rightarrow V$ such that $h(\hat{x}) = 0$ and $h(K \cap U) = C \cap V$.*

Proof. Proof By the definition of C^2 -cone reduction, there exist a closed convex cone $\hat{C} \subset \mathbb{R}^m$, an open neighborhood $\hat{U}$ of $\hat{x}$ and a C^2 function $\hat{h} : \hat{U} \rightarrow \mathbb{R}^m$ such that $\hat{h}'(\hat{x}) : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is onto, $\hat{h}(\hat{x}) = 0$ and $K \cap \hat{U} = \{x \in \hat{U} \mid \hat{h}(x) \in \hat{C}\}$. Note that $m \leq n$. Without loss of generality, assume that the $m \times m$ submatrix consisting of the first m columns of $\hat{h}'(\hat{x})$ is nonsingular. Define the function $h : \hat{U} \rightarrow \mathbb{R}^n$

$$h(x) := (\hat{h}_1(x), \dots, \hat{h}_m(x), x_{m+1} - \hat{x}_{m+1}, \dots, x_n - \hat{x}_n), \quad x \in \hat{U}.$$

So $h'(\hat{x})$ is nonsingular. Since h is C^2 , there exist open neighborhoods $U \subset \hat{U}$ of $\hat{x}$ and V of $h(\hat{x}) = 0$ such that $h : U \rightarrow V$ is a C^2 homeomorphism. Define the closed convex cone $C \subset \mathbb{R}^n$ as:

$$C := \{y \in \mathbb{R}^n \mid (y_1, \dots, y_m) \in \hat{C}\}.$$

We now show that $h(K \cap U) = C \cap V$. Note that for a point $x \in U \subset \hat{U}$, we have $x \in K$ if and only if $\hat{h}(x) \in \hat{C}$ since $K \cap \hat{U} = \{x \in \hat{U} \mid \hat{h}(x) \in \hat{C}\}$. So

$$y \in h(K \cap U) \Leftrightarrow y \in V \wedge h^{-1}(y) \in K \Leftrightarrow y \in V \wedge \hat{h}(h^{-1}(y)) \in \hat{C} \Leftrightarrow y \in C \cap V.$$

This completes the proof. $\square$

We say that $S \subset \mathbb{R}^n$ is C^2 -cone reducible if S is a nonempty closed convex set that is C^2 -cone reducible at each $x \in S$. According to [5, Theorem 7.2] and [6, Proposition 3.136], the metric projection onto a C^2 -cone reducible set is directionally differentiable, which can be combined with the following lemma in our subsequent analysis.

Lemma 8 ([13, Theorem 3.3]). *Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a Lipschitz continuous and directionally differentiable function, such that f^{-1} has the Aubin property around $(f(x_0), x_0)$. Then f is discrete around x_0 .*

Note that with Lemmas 1 and 8, the condition that f is open and discrete around x_0 in Lemma 3 is easily satisfied, and the most difficult part is $|\text{ind}(f, x_0)| = 1$, which is highly related to the structure of f around x_0 .

3. INDEX OF N

Consider the function N defined in (3). We want to prove that the index of N at a point $x_0 \in K$ is ± 1 under the assumption that N is open and discrete around x_0 . When $x_0 \in \text{ri } K$, N is linear around x_0 , so we easily get $|\text{ind}(N, x_0)| = 1$. However, it is difficult to compute $\text{ind}(N, x_0)$ directly when $x_0 \in \text{rb } K$. The key point is to prove that there exists a bounded open neighborhood U of x_0 such that $N^{-1}(N(x)) = \{x\}$ for each $x \in \text{ri } K \cap U$, and then we can find a sequence $\{x_i\} \subset \text{ri } K \cap U$ such that $x_i \rightarrow x_0$ and $|\deg(N|_{\overline{D}}, D, N(x_i))| = 1$. By the local constancy of the degree, we can prove $|\text{ind}(N, x_0)| = 1$.

We need the following lemma on the local structure of the normal cone map for closed convex sets, which plays a fundamental role in our following proofs and reveals how the dimension of normal cones varies in the neighborhood of a boundary point.

Lemma 9. *Let $K \subset \mathbb{R}^n$ be a closed convex set with nonempty interior. Then for any $x_0 \in \partial K$ and $u_0 \in \text{rb } N_K(x_0)$, there exist sequences $\{x_i\}$ and $\{u_i\}$ converging to x_0 and u_0 , respectively, such that $u_i \in N_K(x_i)$ and $\dim N_K(x_i) < \dim N_K(x_0)$ for each i .*

Proof. Proof By taking $K := K \cap \{x \in \mathbb{R}^n \mid \|x - x_0\| \leq r\}$ for some $r > 0$ if necessary, we can assume that K is compact. Let $d := \dim N_K(x_0)$. So $d \geq 1$ since $x_0 \in \partial K$. If $u_0 = 0$, then we can prove the lemma by choosing $x_i \in \text{int } K$ and $u_i = 0$ and we have $\dim N_K(x_i) = 0 < d$.

Assume $u_0 \neq 0$. Then we have $d \geq 2$. The touching cone $T(K, u_0)$ is the smallest face of $N_K(x_0)$ that contains u_0 . Thus, $\dim T(K, u_0) < d$, since $u_0 \in \text{rb } N_K(x_0)$, which implies that u_0 is a $(d-2)$ -extreme normal vector of K . By Lemma 2, there is a sequence of $(d-2)$ -exposed normal vectors $\{u_k\}$ converging to u_0 . For each k , choose any $\hat{x}_k \in \text{ri } N_K^{-1}(u_k)$. So $\dim N_K(\hat{x}_k) = \dim N_K(\text{ri } N_K^{-1}(u_k)) < d$ by the definition of $(d-2)$ -exposed normal vectors. Since K is compact and $\hat{x}_k \in K$, there is a convergent subsequence $\{\hat{x}_{k_i}\}$ such that $\hat{x}_{k_i} \rightarrow \hat{x} \in K$. The outer semicontinuity of N_K implies $u_0 \in N_K(\hat{x})$. If $\dim N_K^{-1}(u_0) = 0$, then $\hat{x} = x_0$, which implies that the sequences $\{\hat{x}_{k_i}\}$ and $\{u_{k_i}\}$ satisfy the conditions required in this lemma.

Next, we consider the case that $\dim N_K^{-1}(u_0) > 0$. Let L be the smallest linear subspace containing $N_K^{-1}(u_0) - x_0$, and let $L^\perp$ be the orthogonal complement of L . So $u_0 \in N_K(\text{ri } N_K^{-1}(u_0)) \subset L^\perp$. For any $\delta > 0$, define the closed convex set

$$D^\delta := \{x \in x_0 + L \mid \|x - x_0\| \leq \delta\} + L^\perp.$$

So $K \cap D^\delta$ is a compact convex set with nonempty interior and $N_{K \cap D^\delta}(x) = N_K(x) + N_{D^\delta}(x)$ for each $x \in \mathbb{R}^n$. Note that $u_0 \in \text{rb } N_K(x_0) = \text{rb } N_{K \cap D^\delta}(x_0)$. Using the above approach to x_0, u_0 and $K \cap D^\delta$, there exist $\{x_k^\delta\}$ and $\{u_k^\delta\}$ such that $x_k^\delta \rightarrow x^\delta, u_k^\delta \rightarrow u_0, u_k^\delta \in N_{K \cap D^\delta}(x_k^\delta)$ and $\dim N_{K \cap D^\delta}(x_k^\delta) < d$. So $\|x^\delta - x_0\| \leq \delta$, since $x^\delta \in N_{K \cap D^\delta}^{-1}(u_0) = N_K^{-1}(u_0) \cap D^\delta \subset x_0 + L$.

By utilizing $\{x_k^\delta\}$ and $\{u_k^\delta\}$, we now construct sequences $\{\hat{x}_k^\delta\}$ and $\{\hat{u}_k^\delta\}$ that converge to x^δ and u_0 respectively, while satisfying the conditions $\hat{u}_k^\delta \in N_K(\hat{x}_k^\delta)$ and $\dim N_K(\hat{x}_k^\delta) < d$. If $\|x^\delta - x_0\| < \delta$, then $x^\delta \in \text{int } D^\delta$, which implies that we can find a subsequence $\{x_{k_i}^\delta\} \subset \text{int } D^\delta$. So $N_{K \cap D^\delta}(x_{k_i}^\delta) = N_K(x_{k_i}^\delta)$. Thus, $u_{k_i}^\delta \in N_K(x_{k_i}^\delta)$ and $\dim N_K(x_{k_i}^\delta) < d$.

Suppose $\|x^\delta - x_0\| = \delta$. For each k , there exists $v_k \in N_{D^\delta}(x_k^\delta)$ such that $\hat{u}_k^\delta := u_k^\delta - v_k \in N_K(x_k^\delta)$. We claim that $v_k \rightarrow 0$. Suppose for the sake of contradiction that it is not true. Then there are two cases. One is that there exists a subsequence $\{v_{k_i}\}$ such that $\|v_{k_i}\| \rightarrow \infty$. So $\|\hat{u}_{k_i}^\delta\| \rightarrow \infty$. Without loss of generality, we can assume that $v_{k_i} \neq 0$ and $\hat{u}_{k_i}^\delta \neq 0$ for each i and the sequence $\{\hat{u}_{k_i}^\delta / \|\hat{u}_{k_i}^\delta\|\}$ converges to e . Since $\hat{u}_{k_i}^\delta \in N_K(x_{k_i}^\delta)$, we have $\langle \hat{u}_{k_i}^\delta, x_{k_i}^\delta - x_0 \rangle \geq 0$ which implies $\langle e, x^\delta - x_0 \rangle \geq 0$. The outer semicontinuity of N_{D^δ} implies $v_{k_i} / \|v_{k_i}\| \rightarrow (x^\delta - x_0) / \delta$. So $\langle \hat{u}_{k_i}^\delta / \|\hat{u}_{k_i}^\delta\|, v_{k_i} / \|v_{k_i}\| \rangle \rightarrow \langle e, (x^\delta - x_0) / \delta \rangle \geq 0$. Therefore, $\|u_{k_i}^\delta\|^2 = \|\hat{u}_{k_i}^\delta + v_{k_i}\|^2 \rightarrow \infty$, which is a contradiction. The other case is that there exists a convergent subsequence $v_{k_i} \rightarrow v \neq 0$. Note that $\langle u_0, x^\delta - x_0 \rangle = 0$, since $u_0 \in L^\perp$ and $x^\delta - x_0 \in L$. By $v_{k_i} / \|v_{k_i}\| \rightarrow (x^\delta - x_0) / \delta$, we have $\langle v, x^\delta - x_0 \rangle > 0$. So $\langle u_0 - v, x^\delta - x_0 \rangle < 0$. However, by the outer semicontinuity of N_K , we have $u_0 - v = \lim \hat{u}_{k_i}^\delta \in N_K(x^\delta)$, which implies $\langle u_0 - v, x^\delta - x_0 \rangle \geq 0$. This is a contradiction. Therefore, $v_k \rightarrow 0$ and $\hat{u}_k^\delta \rightarrow u_0$. Moreover, we have $\hat{u}_k^\delta \in N_K(x_k^\delta)$ and $\dim N_K(x_k^\delta) \leq \dim N_{K \cap D^\delta}(x_k^\delta) < d$.

Therefore, for any $\delta > 0$, there exist $\hat{x}_k^\delta \rightarrow x^\delta$ and $\hat{u}_k^\delta \rightarrow u_0$ such that $\|x^\delta - x_0\| \leq \delta$, $\hat{u}_k^\delta \in N_K(\hat{x}_k^\delta)$ and $\dim N_K(\hat{x}_k^\delta) < d$. For each i , let $\delta_i = 1/i$, then we can choose x_i and u_i such that $u_i \in N_K(x_i)$, $\|x_i - x_0\| < 2\delta_i$, $\|u_i - u_0\| < \delta_i$ and $\dim N_K(x_i) < d$. So $x_i \rightarrow x_0$ and $u_i \rightarrow u_0$. $\square$

The assumption of a nonempty interior in Lemma 9, which simplifies the proof, can be relaxed. The core argument can be applied to the relative interior of any nonempty closed convex set within its affine hull. This yields a more general result, which we state as a corollary due to its potential independent interest.

Corollary 1. *Let $K \subset \mathbb{R}^n$ be a closed convex set. Then for any $x_0 \in \partial K$ and $u_0 \in \text{rb } N_K(x_0)$, there exist sequences $\{x_i\}$ and $\{u_i\}$ converging to x_0 and u_0 , respectively, such that $u_i \in N_K(x_i)$ and $\dim N_K(x_i) < \dim N_K(x_0)$ for each i .*

Proof. The proof follows by applying Lemma 9 to K within its affine hull, and the details are omitted as it is not essential for the main results of this paper. $\square$

Using Lemma 9, we have the following crucial lemma.

Lemma 10. *Let $K \subset \mathbb{R}^n$ be a closed convex set with nonempty interior and A be an n by n matrix. Let $N(x) = A(x - \Pi_K(x)) + \Pi_K(x)$ for $x \in \mathbb{R}^n$. Assume that N is open around $x_0 \in \partial K$. Then*

$$AN_K(x_0) \cap \text{int } T_K(x_0) = \emptyset.$$

Proof. Let U be an open neighborhood of x_0 such that $N|_U$ is an open map. For any $x \in \partial K \cap U$, we claim that if $AN_K(x) \cap \text{int } T_K(x) \neq \emptyset$, then $A \text{rb } N_K(x) \cap \text{int } T_K(x) \neq \emptyset$. For the sake of contradiction, suppose that there exists a point $\bar{x} \in \partial K \cap U$ such that

$$AN_K(\bar{x}) \cap \text{int } T_K(\bar{x}) \neq \emptyset \quad \text{and} \quad A \text{rb } N_K(\bar{x}) \cap \text{int } T_K(\bar{x}) = \emptyset.$$

Let L be the smallest linear subspace containing $N_K(\bar{x})$ and let P be the orthogonal projection matrix onto L . Then $y \in \text{int } T_K(\bar{x})$ if and only if $Py \in \text{ri}(T_K(\bar{x}) \cap L)$, since $\text{int } T_K(\bar{x}) = \text{ri}(T_K(\bar{x}) \cap L) + L^\perp$. So we have

$$PAN_K(\bar{x}) \cap \text{ri}(T_K(\bar{x}) \cap L) \neq \emptyset \quad \text{and} \quad PA \text{rb } N_K(\bar{x}) \cap \text{ri}(T_K(\bar{x}) \cap L) = \emptyset.$$

Note that $N_K(\bar{x})$ is full-dimensional relative to L and contains no line. So we have $PAL = L$, otherwise, for any $u \in PAN_K(\bar{x}) \cap \text{ri}(T_K(\bar{x}) \cap L)$, there exists $u' \in \text{rb } N_K(\bar{x})$ such that $PAu' = u$, which is contradict to $PA \text{rb } N_K(\bar{x}) \cap \text{ri}(T_K(\bar{x}) \cap L) = \emptyset$. Thus, $PAN_K(\bar{x})$ is a closed convex cone and full-dimensional relative to L , and $PA \text{rb } N_K(\bar{x}) = \text{rb } PAN_K(\bar{x})$. Therefore, we have $(T_K(\bar{x}) \cap L) \subset PAN_K(\bar{x})$. By noting that

$$\dim L = \dim PAN_K(\bar{x}) \leq \dim AN_K(\bar{x}) \leq \dim N_K(\bar{x}) = \dim L,$$

we have $\dim AL = \dim L$ and $AN_K(\bar{x}) \cap L^\perp = \{0\}$. So, $AN_K(\bar{x})$ is a closed convex cone that contains no line, and there exists a closed half-space $H \subset \mathbb{R}^n$ such that $AN_K(\bar{x}) \setminus \{0\} \subset \text{int } H$ and $L^\perp \subset \partial H$. Thus, we have

$$T_K(\bar{x}) = (T_K(\bar{x}) \cap L) + L^\perp \subset PAN_K(\bar{x}) + L^\perp = AN_K(\bar{x}) + L^\perp \subset H.$$

Consider the continuous function $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ given by $f(x) := A(x - \Pi_K(x))$. Note that $\bar{x} \in f^{-1}(H)$. We claim that $f^{-1}(H)$ is a neighborhood of $\bar{x}$. Suppose our claim is false. Then there exists a sequence $\{x_i\}$ converging to $\bar{x}$ such that $x_i \notin f^{-1}(H)$ for each i . It is clear that $x_i - \Pi_K(x_i) \neq 0$ for each i and $\Pi_K(x_i) \rightarrow$

$\bar{x}$. Let $u_i := (x_i - \Pi_K(x_i))/\|x_i - \Pi_K(x_i)\| \in N_K(\Pi_K(x_i))$, and we have $Au_i = f(x_i)/\|x_i - \Pi_K(x_i)\| \notin H$ for each i , since H is a cone. Without loss of generality, assume $u_i \rightarrow \bar{u}$. So $A\bar{u} \notin \text{int } H$. But the upper semicontinuity of N_K implies $A\bar{u} \in AN_K(\bar{x}) \setminus \{0\} \subset \text{int } H$, where $A\bar{u} \neq 0$ since $\bar{u} \in L \setminus \{0\}$ and $\dim AL = \dim L$. Therefore, $f^{-1}(H)$ is a neighborhood of $\bar{x}$ and there exists an open neighborhood $V \subset f^{-1}(H) \cap U$ of $\bar{x}$. We have $\bar{x} = N(\bar{x}) \in N(V)$ and

$$N(V) = \bigcup_{x \in V} (\Pi_K(x) + f(x)) \subset K + H \subset \bar{x} + T_K(\bar{x}) + H = \bar{x} + H.$$

It is clear that $\bar{x}$ is not an interior point of $N(V)$. However, since $N|_U$ is an open map, $N(V)$ should be an open neighborhood of $\bar{x}$. This is a contradiction.

So for any $x \in \partial K \cap U$, $AN_K(x) \cap \text{int } T_K(x) \neq \emptyset$ implies $A \text{rb } N_K(x) \cap \text{int } T_K(x) \neq \emptyset$. We will prove the lemma by contradiction. Suppose that

$$AN_K(x_0) \cap \text{int } T_K(x_0) \neq \emptyset.$$

Then $A \text{rb } N_K(x_0) \cap \text{int } T_K(x_0) \neq \emptyset$. Let $d := \dim N_K(x_0) \geq 1$. If $d = 1$, we have $\text{rb } N_K(x_0) = \{0\}$ and $A \text{rb } N_K(x_0) \cap \text{int } T_K(x_0) = \emptyset$, which is a contradiction. Assume $d > 1$. Then there exists a normal vector $u_0 \in \text{rb } N_K(x_0) \setminus \{0\}$ such that $Au_0 \in \text{int } T_K(x_0)$. By the inner semicontinuity of the convex-valued map T_K and [18, Theorem 5.9], there is an open neighborhood W of (x_0, u_0) , such that $Au \in \text{int } T_K(x)$ for any $(x, u) \in W$. So for any $(x, u) \in \text{gph } N_K$ sufficiently close to (x_0, u_0) , we have $AN_K(x) \cap \text{int } T_K(x) \neq \emptyset$. Therefore, by Lemma 9, we can find $x'_0 \in \partial K \cap U$ such that

$$AN_K(x'_0) \cap \text{int } T_K(x'_0) \neq \emptyset \quad \text{and} \quad \dim N_K(x'_0) < \dim N_K(x_0).$$

Then, we can repeat the same process with x'_0 as was done with x_0 , eventually finding $x''_0 \in \partial K \cap U$ such that $AN_K(x''_0) \cap \text{int } T_K(x''_0) \neq \emptyset$ and $\dim N_K(x''_0) = 1$, which contradicts the previous argument for $d = 1$. Therefore, we have $AN_K(x_0) \cap \text{int } T_K(x_0) = \emptyset$. $\square$

In Lemma 10, we prove that there exists an open neighborhood of x_0 such that N is open around each $x \in \partial K \cap U$. So for each $x \in \partial K \cap U$, we have $AN_K(x) \cap \text{int } T_K(x) = \emptyset$, which implies $(x + N_K(x)) \cap N^{-1}(\text{int } K) = \emptyset$. Thus, if N is discrete at x_0 , we can prove $|\text{ind}(N, x_0)| = 1$ by the local constancy of the degree. In fact, we have the following theorem for N in a more general form.

Theorem 1. *Let $K \subset \mathbb{R}^n$ be a nonempty closed convex set and A, B be n by n matrices. Let*

$$N(x) = A(x - \Pi_K(x)) + B\Pi_K(x), \quad x \in \mathbb{R}^n.$$

Assume that N is open around $x_0 \in K$ and discrete at x_0 . Then $|\text{ind}(N, x_0)| = 1$.

Proof. Proof Let $N_0(x) := N(x + x_0) - N(x_0) = A(x - \Pi_{K-x_0}(x)) + B\Pi_{K-x_0}(x)$ for $x \in \mathbb{R}^n$. Then N_0 is open around and discrete at $0 \in K - x_0$, and $\text{ind}(N, x_0) = \text{ind}(N_0, 0)$. So, without loss of generality, we assume that $x_0 = 0$. Then, we have $0 \in K$ and $N(0) = 0$.

Let L be the smallest linear subspace containing K , i.e., the affine hull of K as $0 \in K$. Also let P be the orthogonal projection matrix onto L . By the assumption, there exists a bounded open neighborhood U of 0 such that $N|_U$ is an open map and $N^{-1}(0) \cap U = \{0\}$. Since $0 \in K$, we know that $\text{ri } K \cap U \neq \emptyset$. Note that $N_K(y) = L^\perp$

for any $y \in \text{ri } K \subset L$. Thus, $\Pi_K(x) = Px$ and $N(x) = (A - AP + BP)x$ for any $x \in \Pi_K^{-1}(\text{ri } K)$. By noting that

$$\Pi_K^{-1}(\text{ri } K) = \bigcup_{y \in \text{ri } K} \Pi_K^{-1}(y) = \bigcup_{y \in \text{ri } K} y + L^\perp = \text{ri } K + L^\perp,$$

we know that $\Pi_K^{-1}(\text{ri } K)$ is an open set in $\mathbb{R}^n$. Since the function N is linear on the open set $\Pi_K^{-1}(\text{ri } K)$ and is open around any $x \in \text{ri } K \cap U \subset \Pi_K^{-1}(\text{ri } K)$, we obtain that the matrix $A - AP + BP$ is nonsingular. If $0 \in \text{ri } K$, then

$$|\text{ind}(N, 0)| = |\deg(N|_{\bar{U}}, U, 0)| = |\text{sgn det } N'(0)| = 1.$$

So we only need to consider the case that $0 \in \text{rb } K$. Let $M = (A - AP + BP)^{-1}$. Then

$$(4) \quad MAL^\perp = L^\perp \quad \text{and} \quad MBx = M[(Ax - APx) + BPx] = x, \quad \forall x \in L.$$

Thus, for any $x \in \mathbb{R}^n$,

$$PMN(x) = PMA(x - \Pi_K(x)) + \Pi_K(x) = PMA(Px - \Pi_K(Px)) + \Pi_K(Px).$$

Let $\hat{A} : L \rightarrow L$ be the linear map given by $\hat{A}x = PMAx$, $x \in \mathbb{R}^n$. Define the function $\hat{N} : L \rightarrow L$ by

$$\hat{N}(u) = PMN(u) = \hat{A}(u - \Pi_K(u)) + \Pi_K(u), \quad u \in L.$$

For each open set $V \subset U \cap L$ in the linear subspace L , we have $\hat{N}(V) = PMN((V + L^\perp) \cap U)$, which is open in L , since $N|_U : U \rightarrow \mathbb{R}^n$ and $P : \mathbb{R}^n \rightarrow L$ are open maps. So $\hat{N}|_{U \cap L}$ is an open map. Note that the interior of K relative to L , i.e., $\text{ri } K$, is nonempty. So by applying Lemma 10 to the function $\hat{N} : L \rightarrow L$, we have that for each $x \in \text{rb } K \cap U$,

$$\hat{A}N_K^L(x) \cap \text{int}^L T_K^L(x) = \emptyset,$$

where $N_K^L(x)$ and $T_K^L(x)$ is the normal cone and the tangent cone to K at x relative to L , respectively, and int^L means the interior relative to L . It is clear that $N_K^L(x) = PN_K(x)$ and $\text{int}^L T_K^L(x) = \text{ri } T_K(x)$. Let $x \in \text{rb } K \cap U$ be arbitrarily chosen. Then we have

$$PMAPN_K(x) \cap \text{ri } T_K(x) = \emptyset.$$

Adding $L^\perp$ to both sets in the intersection preserves the empty intersection, that is,

$$(PMAPN_K(x) + L^\perp) \cap (\text{ri } T_K(x) + L^\perp) = \emptyset.$$

Thus, by using (4), we have

$$PMAPN_K(x) + L^\perp = MAPN_K(x) + L^\perp = MA(PN_K(x) + L^\perp) = MAN_K(x),$$

which, further implies $MAN_K(x) \cap (\text{ri } T_K(x) + L^\perp) = \emptyset$. Applying M^{-1} to both sets, and using the identities $M^{-1}\text{ri } T_K(x) = B \text{ri } T_K(x)$ and $M^{-1}L^\perp = AL^\perp$ given by (4), we obtain

$$AN_K(x) \cap (B \text{ri } T_K(x) + AL^\perp) = \emptyset.$$

Consequently, for any $x \in \text{rb } K \cap U$,

$$(5) \quad N(x + N_K(x)) \cap N(\text{ri } K) = (Bx + AN_K(x)) \cap (B \text{ri } K + AL^\perp) = \emptyset,$$

where we use the fact that $\text{ri } K - x \subset \text{ri } T_K(x)$.

By taking $U := U \cap \Pi_K^{-1}(U)$ if necessary, we can assume that $U \subset \Pi_K^{-1}(U)$. Let $y \in \text{ri } K \cap U$. We claim that $N^{-1}(N(y)) \cap U = \{y\}$. To see this, suppose that $y' \in N^{-1}(N(y)) \cap U$. Then there are two cases: $y' \in \text{ri } K$ and $y' \notin \text{ri } K$.

If $y' \in \text{ri } K$, then we have that $By' = N(y') = N(y) = By$, which, by (4), gives $y' = y$. If $y' \notin \text{ri } K$, then $\Pi_K(y') \in \text{rb } K \cap U$ as $y' \in U \subset \Pi_K^{-1}(U)$. Therefore, by using (5), we know that

$$N(\Pi_K(y') + N_K(\Pi_K(y')))) \cap N(\text{ri } K) = \emptyset,$$

which, together with the fact that $y' = \Pi_K(y') + (y' - \Pi_K(y')) \in \Pi_K(y') + N_K(\Pi_K(y'))$, implies $N(y') \notin N(\text{ri } K)$. However, this contradicts our assumption that $N(y') = N(y) \in N(\text{ri } K)$. Thus, y' must be equal to y . Consequently, $N^{-1}(N(y)) \cap U = \{y\}$ for any $y \in \text{ri } K \cap U$.

Let D be a bounded open neighborhood of 0 with $\overline{D} \subset U$. For any $y \in \text{ri } K \cap D$, since $N^{-1}(N(y)) \cap U = \{y\}$, we know that $N(y) \notin N(\partial D)$ and

$$\deg(N|_{\overline{D}}, D, N(y)) = \text{sgn } \det N'(y) = \text{sgn } \det(A - AP + BP).$$

It follows from $N^{-1}(0) \cap U = \{0\}$ that $0 \notin N(\partial D)$. The image $N(\partial D)$ is compact because it is the continuous image of the compact set ∂D . Thus, $\text{dist}(0, N(\partial D)) > 0$. Then from the local constancy of the degree, we obtain that

$$|\text{ind}(N, 0)| = |\deg(N|_{\overline{D}}, D, N(0))| = |\text{sgn } \det(A - AP + BP)| = 1$$

as there exists a sequence $\{y_i\} \subset \text{ri } K \cap D$ converging to $0 \in \text{rb } K \cap D$. $\square$

4. EQUIVALENCE OF THE AUBIN PROPERTY AND THE STRONG REGULARITY

In this section, we establish the main results of the paper, proving the equivalence between the Aubin property and strong regularity for generalized equations over C^2 -cone reducible sets. Our proof strategy proceeds in three main steps. First, in Theorem 2, we tackle a canonical form of the function $N(x) = A(x - \Pi_S(x)) + B\Pi_S(x)$, where the core of our degree-theoretic argument is applied. Second, we extend this result in Theorem 3 to a more general form where the constant matrices A and B are replaced by C^1 functions. Finally, in Theorem 4, we show how the original generalized equation can be transformed to fit the structure of Theorem 3, thus completing the proof of our main claim.

Theorem 2. *Let $S \subset \mathbb{R}^n$ be a C^2 -cone reducible set and A, B be n by n matrices. Let*

$$N(x) = A(x - \Pi_S(x)) + B\Pi_S(x), \quad x \in \mathbb{R}^n.$$

Let N^{-1} have the Aubin property around $(N(x_0), x_0)$. Then N is a local homeomorphism at x_0 .

Proof. Proof According to [5, Theorem 7.2], we know that Π_S is directionally differentiable since S is C^2 -cone reducible. So N is also directionally differentiable. By Lemma 8, N is discrete around x_0 .

According to Lemma 7, there exist a closed convex cone $C \subset \mathbb{R}^n$, open neighborhoods $U \subset \mathbb{R}^n$ of $\Pi_S(x_0)$ and $V \subset \mathbb{R}^n$ of 0 and a C^2 homeomorphism $h : U \rightarrow V$ such that $h(\Pi_S(x_0)) = 0$ and $h(S \cap U) = C \cap V$. The continuity of Π_S implies that $\Pi_S^{-1}(U)$ is an open neighborhood of x_0 . Define a function $g : \Pi_S^{-1}(U) \rightarrow W := g(\Pi_S^{-1}(U)) \subset \mathbb{R}^n$ by

$$g(x) := [\nabla h(\Pi_S(x))]^{-1}(x - \Pi_S(x)) + h(\Pi_S(x)), \quad x \in \Pi_S^{-1}(U).$$

Since $N_S(z) = \nabla h(z)N_C(h(z))$ for each $z \in U$ [18, Exercise 6.7], we have that for each $x \in \Pi_S^{-1}(U)$, $[\nabla h(\Pi_S(x))]^{-1}(x - \Pi_S(x)) \in [\nabla h(\Pi_S(x))]^{-1}N_S(\Pi_S(x)) =$

$N_C(h(\Pi_S(x)))$, and thus

$$\Pi_C(g(x)) = h(\Pi_S(x)).$$

So $\Pi_S(x) = h^{-1}(\Pi_C(g(x)))$. By replacing $\Pi_S(x)$ with $h^{-1}(\Pi_C(g(x)))$ in the definition of g , we know that the inverse $g^{-1} : W \rightarrow \Pi_S^{-1}(U)$ takes the form of

$$g^{-1}(y) = \nabla h(h^{-1}(\Pi_C(y)))(y - \Pi_C(y)) + h^{-1}(\Pi_C(y)), \quad y \in W.$$

Since $\Pi_S(g^{-1}(y)) = h^{-1}(\Pi_C(y))$ for any $y \in W$, we can define a function N_0 by

$$N_0(y) := N(g^{-1}(y)) = A\nabla h(h^{-1}(\Pi_C(y)))(y - \Pi_C(y)) + Bh^{-1}(\Pi_C(y)), \quad y \in W.$$

Let $y_0 = g(x_0)$. Note that g is Lipschitz continuous around x_0 and g^{-1} is Lipschitz continuous around y_0 . So, by definitions, we can easily check that N^{-1} has the Aubin property around $(N(x_0), x_0)$ if and only if N_0^{-1} has the Aubin property around $(N_0(y_0), y_0)$; N is a local homeomorphism at x_0 if and only if N_0 is a local homeomorphism at y_0 ; and N_0 is discrete around y_0 as N is discrete around x_0 . Since $\Pi_C(y_0) = h(\Pi_S(x_0)) = 0$, we know $y_0 \in C^\circ$. By using that fact that $y = \Pi_C(y) + \Pi_{C^\circ}(y)$, we can write N_0 in the following form:

$$N_0(y) = Bh^{-1}(y - \Pi_{C^\circ}(y)) + A\nabla h(h^{-1}(y - \Pi_{C^\circ}(y)))\Pi_{C^\circ}(y), \quad y \in W.$$

Since h^{-1} is continuous differentiable around 0, we have that the following defined function

$$\psi_1(y) := B[(h^{-1})'(0)(y - \Pi_{C^\circ}(y)) - h^{-1}(y - \Pi_{C^\circ}(y))], \quad y \in W$$

is strictly stationary at y_0 by Lemma 5. Let $\widehat{B} := B(h^{-1})'(0)$ and

$$N_1(y) := N_0(y) + \psi_1(y) = \widehat{B}(y - \Pi_{C^\circ}(y)) + A\nabla h(h^{-1}(y - \Pi_{C^\circ}(y)))\Pi_{C^\circ}(y), \quad y \in W.$$

By Lemmas 4 and 6, N_1^{-1} has the Aubin property around $(N_1(y_0), y_0)$, N_1 is discrete at y_0 and $\text{ind}(N_0, y_0) = \text{ind}(N_1, y_0)$. It is clear that $\nabla h \circ h^{-1}$ is continuously differentiable around 0. So we can define a function $\psi_2 : W \rightarrow \mathbb{R}^n$ by

$$\psi_2(y) := A[(\nabla h \circ h^{-1})'(0)(y - \Pi_{C^\circ}(y)) - \nabla h(h^{-1}(y - \Pi_{C^\circ}(y)))]\Pi_{C^\circ}(y).$$

It follows from Lemma 5 that ψ_2 is strictly stationary at y_0 . Let $M : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$ be a linear map such that for any $a, b \in \mathbb{R}^n$, $M(a)b := A(\nabla h \circ h^{-1})'(0)(b)a$, and let $\widehat{A} := A\nabla h(h^{-1}(0))$. Define

$$N_2(y) := N_1(y) + \psi_2(y) = (\widehat{B} + M(\Pi_{C^\circ}(y)))(y - \Pi_{C^\circ}(y)) + \widehat{A}\Pi_{C^\circ}(y), \quad y \in W.$$

By Lemmas 4 and 6, N_2^{-1} has the Aubin property around $(N_2(y_0), y_0)$, N_2 is discrete at y_0 and $\text{ind}(N_1, y_0) = \text{ind}(N_2, y_0)$. Consider the function

$$\psi_3(y) := (M(y_0) - M(\Pi_{C^\circ}(y)))(y - \Pi_{C^\circ}(y)), \quad y \in W.$$

According to Lemma 5, we have that ψ_3 is strictly stationary at y_0 . Define

$$N_3(y) := N_2(y) + \psi_3(y) = (\widehat{B} + M(y_0))(y - \Pi_{C^\circ}(y)) + \widehat{A}\Pi_{C^\circ}(y), \quad y \in W.$$

By Lemmas 4 and 6, N_3^{-1} has the Aubin property around $(N_3(y_0), y_0)$, N_3 is discrete at y_0 and $\text{ind}(N_2, y_0) = \text{ind}(N_3, y_0)$. And Lemma 1 implies that N_3 is open around y_0 . So, according to Theorem 1, we have $|\text{ind}(N_3, y_0)| = 1$. Thus, $|\text{ind}(N_0, y_0)| = 1$. By Lemma 3, N_0 is a local homeomorphism at y_0 . Therefore, N is a local homeomorphism at x_0 . $\square$

By Lemma 5, we can easily extend Theorem 2 to a more general case.

Theorem 3. *Let $S \subset \mathbb{R}^n$ be a C^2 -cone reducible set and $f, g : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be C^1 functions. Let*

$$N(x) = f(x - \Pi_S(x)) + g(\Pi_S(x)), \quad x \in \mathbb{R}^n.$$

If N^{-1} have the Aubin property around $(N(x_0), x_0)$, then N is a local homeomorphism at x_0 .

Proof. Define the function $N_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n$ by $N_0(x) := f'(x_0 - \Pi_S(x_0))(x - \Pi_S(x)) + g'(\Pi_S(x_0))\Pi_S(x)$. By Lemma 5, we know that $N_0 - N$ is strictly stationary at x_0 . The proof is completed by applying Lemma 4 and Theorem 2. $\square$

We are now fully equipped to prove the main theorem of this paper. The following proof demonstrates that the solution map Φ^{-1} for the generalized equation can be related, through a homeomorphism, to a function of the form analyzed in Theorem 3. This allows us to translate the Aubin property of Φ^{-1} into the local homeomorphism of the related function, which in turn implies the strong regularity of Φ^{-1} .

Theorem 4. *Let $S \subset \mathbb{R}^n$ be a C^2 -cone reducible set, and let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a C^1 function. Let $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ be a set-valued map in one of the following forms:*

$$\Phi(x) = \varphi(x) + N_S(x) \quad \text{or} \quad \Phi(x) = \varphi(x) + N_S^{-1}(x), \quad x \in \mathbb{R}^n.$$

Then the Aubin property and the strong regularity for Φ^{-1} around $(y_0, x_0) \in \text{gph } \Phi^{-1}$ are equivalent.

Proof. We only need to prove that the Aubin property of Φ^{-1} around $(y_0, x_0) \in \text{gph } \Phi^{-1}$ implies that Φ^{-1} is locally single-valued around (y_0, x_0) . Assume that Φ^{-1} has the Aubin property around (y_0, x_0) .

Suppose that $\Phi(x) = \varphi(x) + N_S(x)$. Let $N(z) := \varphi(\Pi_S(z)) + (z - \Pi_S(z))$ for any $z \in \mathbb{R}^n$ and $z_0 := x_0 + y_0 - \varphi(x_0)$. Consider the function $h : \text{gph } \Phi \rightarrow \mathbb{R}^n$ given by $h(x, y) := x + y - \varphi(x)$, where $(x, y) \in \text{gph } \Phi$. Since $y - \varphi(x) \in N_S(x)$ for any $(x, y) \in \text{gph } \Phi$, we have $h^{-1}(z) = (\Pi_S(z), N(z)) \in \text{gph } \Phi$ for any $z \in \mathbb{R}^n$, which implies that h is a homeomorphism. Using the fact that $N(z) = y$ if and only if there exists an x such that $h^{-1}(z) = (x, y) \in \text{gph } \Phi$, we have $N^{-1}(y) = \{x + y - \varphi(x) \mid x \in \Phi^{-1}(y)\}$ for any $y \in \mathbb{R}^n$. So, by the Lipschitz continuity of φ and the definition of the Aubin property, we can easily check that the map N^{-1} has the Aubin property around $(N(z_0), z_0)$. Thus, according to Theorem 3, N is a local homeomorphism at z_0 .

So, there exists an open neighborhood $W \subset \mathbb{R}^n$ of z_0 such that $N(W)$ is open and $N|_W : W \rightarrow N(W)$ is a homeomorphism. Therefore, there exist open neighborhoods $U \subset \mathbb{R}^n$ of x_0 and $V \subset \mathbb{R}^n$ of y_0 , such that $(U \times V) \cap \text{gph } \Phi \subset h^{-1}(W)$. It is clear that $W' := h((U \times V) \cap \text{gph } \Phi) \subset W$ is an open neighborhood of z_0 . Let $V' := N(W')$. Then V' is an open neighborhood of y_0 . Let y' be an arbitrary element of V' . Since $N|_W$ is a homeomorphism, there exists a unique $z' \in W'$ such that $y' = N(z')$. We have $h^{-1}(z') = (\Pi_S(z'), y') \in U \times V$. We claim that $\Phi^{-1}(y') \cap U = \{\Pi_S(z')\}$. Consider any $x' \in \Phi^{-1}(y') \cap U$. Then $(x', y') \in (U \times V) \cap \text{gph } \Phi$ and $h(x', y') \in W'$. Note that $N(h(x', y')) = y'$. Thus, $h(x', y')$ must be equal to z' , which implies $x' = \Pi_S(z')$. Therefore, the map $y \mapsto \Phi^{-1}(y) \cap U$ is single-valued on V' , i.e., Φ^{-1} is locally single-valued around (y_0, x_0) .

Next, consider the case that $\Phi(x) = \varphi(x) + N_S^{-1}(x)$, where the argument is similar to the one above. Let $N(z) := \varphi(z - \Pi_S(z)) + \Pi_S(z)$ for any $z \in \mathbb{R}^n$

and $z_0 := x_0 + y_0 - \varphi(x_0)$. Consider the function $h : \text{gph } \Phi \rightarrow \mathbb{R}^n$ given by $h(x, y) := x + y - \varphi(x)$, where $(x, y) \in \text{gph } \Phi$. Since $x \in N_S(y - \varphi(x))$ for any $(x, y) \in \text{gph } \Phi$, we have $h^{-1}(z) = (z - \Pi_S(z), N(z)) \in \text{gph } \Phi$ for any $z \in \mathbb{R}^n$, which implies that h is a homeomorphism. Note that φ is Lipschitz continuous and $N^{-1}(y) = \{x + y - \varphi(x) \mid x \in \Phi^{-1}(y)\}$ for any $y \in \mathbb{R}^n$. We know that the map N^{-1} has the Aubin property around $(N(z_0), z_0)$. Thus, according to Theorem 3, N is a local homeomorphism at z_0 .

So, there exists an open neighborhood $W \subset \mathbb{R}^n$ of z_0 such that $N(W)$ is open and $N|_W : W \rightarrow N(W)$ is a homeomorphism. Therefore, there exist open neighborhoods $U \subset \mathbb{R}^n$ of x_0 and $V \subset \mathbb{R}^n$ of y_0 , such that $(U \times V) \cap \text{gph } \Phi \subset h^{-1}(W)$. It is clear that $W' := h((U \times V) \cap \text{gph } \Phi) \subset W$ is an open neighborhood of z_0 . Let $V' := N(W')$. Then V' is an open neighborhood of y_0 . Let y' be an arbitrary element of V' . Since $N|_W$ is a homeomorphism, there exists a unique $z' \in W'$ such that $y' = N(z')$. Consider any $x' \in \Phi^{-1}(y') \cap U$. Then $(x', y') \in (U \times V') \cap \text{gph } \Phi$ and $h(x', y') \in W'$. Note that $N(h(x', y')) = y'$. Thus, $h(x', y') = z'$ and $x' = z' - \Pi_S(z')$. Therefore, the map $y \mapsto \Phi^{-1}(y) \cap U$ is single-valued on V' , which completes the proof. $\square$

5. CONCLUSIONS

In this paper, we establish the equivalence between the Aubin property and the strong regularity for generalized equations over C^2 -cone reducible sets. This result resolves a long-standing open question in variational analysis and extends the celebrated theorem of Dontchev and Rockafellar [12] beyond the classical polyhedral case to a significantly broader class of non-polyhedral problems.

Our proof strategy represents a departure from traditional variational techniques. By integrating deep insights from convex geometry with powerful tools from algebraic topology, we developed a novel approach to analyze the problem. The results herein provide a unified framework for the stability analysis of important optimization problems, including conventional nonlinear programming, nonlinear second-order cone programming, and nonlinear semidefinite programming, under a single, general condition. For example, consider the p -order cone, i.e., the epigraph of the p -norm, which is the second-order cone when $p = 2$. From our results we can immediately get the equivalence between the Aubin property and the strong regularity for nonlinear $2k$ -order cone constrained optimization problems with any positive integer k , since the $2k$ -order cone is C^2 -cone reducible. By noting that $N_C^{-1} = N_{C^\circ}$ for any closed convex cone C , we also have the same result for nonlinear $2k/(2k-1)$ -order cone (i.e., the dual cone of the $2k$ -order cone) constrained optimization problems with any positive integer k .

It is important to emphasize, however, that our framework relies on the C^2 -cone reducibility of the underlying set S , which is essential for the local transformations in our proof. Whether the equivalence between the Aubin property and the strong regularity continues to hold for generalized equations over an arbitrary closed convex set remains a challenging open problem. The methods developed here do not directly apply to this more general setting, highlighting the need for new techniques and presenting a compelling direction for future research.

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