On the Relationships of ADMM and Proximal ALM for Convex Optimization Problems

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Multi-block convex programming

$$\min_{x,y} \Big\{ p_1(x_1) + f(\underbrace{x_1, \ldots, x_m}_{x}) + q_1(y_1) + g(\underbrace{y_1, \ldots, y_n}_{y}) \,|\, \mathcal{A}^* x + \mathcal{B}^* y = c \Big\} \tag{P}$$

- $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$: finite-dim. real Hilbert spaces endowed with $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$
- $p_1 : \mathcal{X}_1 \to (-\infty, \infty]$ and $q_1 : \mathcal{Y}_1 \to (-\infty, \infty]$ are closed and proper convex functions. Denote $p(x) := p_1(x_1)$ and $q(y) := q_1(y_1)$
- $f : \mathcal{X} \to (-\infty, \infty)$ and $g : \mathcal{Y} \to (-\infty, \infty)$ are convex, continuously differentiable with Lipschitz continuous gradients
- $\mathcal{A}^*$ and $\mathcal{B}^*$ are the adjoints of the linear mappings $\mathcal{A} : \mathcal{Z} \to \mathcal{X}$ and $\mathcal{B} : \mathcal{Z} \to \mathcal{Y}$, $c \in \mathcal{Z}$

Notation

- Let $\mathcal{U}$ and $\mathcal{V}$ be two finite dimensional real Hilbert spaces. For any given linear map $\mathcal{H} : \mathcal{U} \to \mathcal{V}$, we use $\|\mathcal{H}\|$ to denote its spectral norm and $\mathcal{H}^* : \mathcal{V} \to \mathcal{U}$ to denote its adjoint linear operator
- If $\mathcal{U} = \mathcal{V}$ and $\mathcal{H}$ is self-adjoint, for any $u, v \in \mathcal{U}$, define $\langle u, v\rangle_{\mathcal{H}} := \langle u, \mathcal{H}v\rangle$ and $\|u\|_{\mathcal{H}}^2 := \langle u, \mathcal{H}u\rangle$; if $\mathcal{H}$ is also positive semidefinite, there exists a unique self-adjoint positive semidefinite linear operator $\mathcal{H}^{\frac{1}{2}} : \mathcal{U} \to \mathcal{U}$ such that $\mathcal{H}^{\frac{1}{2}}\mathcal{H}^{\frac{1}{2}} = \mathcal{H}$.
- For a closed proper convex function $\theta : \mathcal{U} \to (-\infty, +\infty]$, denote by $\operatorname{dom}\theta$ and $\partial\theta$ for the effective domain and the subdifferential mapping of θ, respectively

Decomposition

Decompose $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2 \times \ldots \times \mathcal{U}_s$, with each $\mathcal{U}_i$ being a finite dimensional real Hilbert space endowed with $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$

Decompose the self-adjoint and positive semidefinite $\mathcal{H}$ as

$$\mathcal{H} = \begin{pmatrix} \mathcal{H}_{11} & \mathcal{H}_{12} & \cdots & \mathcal{H}_{1s} \\ \mathcal{H}_{12}^* & \mathcal{H}_{22} & \cdots & \mathcal{H}_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{H}_{1s}^* & \mathcal{H}_{2s}^* & \cdots & \mathcal{H}_{ss} \end{pmatrix}, \tag{1}$$

where $\mathcal{H}_{ij} : \mathcal{U}_j \to \mathcal{U}_i$, $i, j = 1, \ldots, s$ are linear maps and $\mathcal{H}_{ii}$ are self-adjoint positive definite linear operators ($\mathcal{H}_{ii} \succ 0$), $i = 1, \ldots, s$

We use $\mathcal{H}_d := \mathrm{Diag}(\mathcal{H}_{11}, \ldots, \mathcal{H}_{ss})$ to denote the block-diagonal part of $\mathcal{H}$, and denote the symbolically strictly upper triangular part of $\mathcal{H}$ by $\mathcal{H}_u$. Thus, $\mathcal{H} = \mathcal{H}_d + \mathcal{H}_u + \mathcal{H}_u^*$

One cycle of the block symmetric Gauss-Seidel

Let $\theta_1 : \mathcal{U}_1 \to (-\infty, \infty]$ be a given closed and proper convex function, $b \in \mathcal{U}$ be a given vector, and $h : \mathcal{U} \to (-\infty, \infty)$ be defined by

$$h(u) := \frac{1}{2}\langle u, \mathcal{H}u\rangle - \langle b, u\rangle$$

Suppose that $u^- \in \mathcal{U}$ is a given vector. Define

$$\begin{cases} u_i^{\frac{1}{2}} := \arg\min_{u_i} \Big\{\theta(u_1^-) + h(u_{<i}^-, u_i, u_{>i}^{\frac{1}{2}}) - \langle \tilde{\delta}_i, u_i\rangle\Big\}, \quad i = s, \ldots, 2 \\ u_1^+ := \arg\min_{u_1} \Big\{\theta(u_1) + h(u_1, u_{>1}^{\frac{1}{2}}) - \langle \delta_1, u_1\rangle\Big\}, \\ u_i^+ := \arg\min_{u_i} \Big\{\theta(u_1^+) + h(u_{<i}^+, u_i, u_{>i}^{\frac{1}{2}}) - \langle \delta_i, u_i\rangle\Big\}, \quad i = 2, \ldots, s \end{cases} \tag{sGS}$$

where for any $u = (u_1, \ldots, u_s) \in \mathcal{U}$ and $i \in \{1, \ldots, s\}$, we denote $u_{<i} := \{u_1, \ldots, u_{i-1}\}$, $u_{>i} := \{u_{i+1}, \ldots, u_s\}$

One cycle of the block sGS

Define

$$d(\tilde{\delta}, \delta) := \delta + \mathcal{H}_u \mathcal{H}_d^{-1}(\delta - \tilde{\delta})$$

with $\tilde{\delta}_1 = \delta_1$, $\delta := (\delta_1, \ldots, \delta_s)$ and $\tilde{\delta} := (\tilde{\delta}_1, \ldots, \tilde{\delta}_s)$

Define the self-adjoint positive semidefinite linear operator on $\mathcal{U}$ by

$$\mathrm{sGS}(\mathcal{H}) := \mathcal{H}_u \mathcal{H}_d^{-1} \mathcal{H}_u^* \quad \text{(sGS Splitting Operator)}$$

Consider the following convex composite quadratic programming:

$$\min_{u \in \mathcal{U}} \left\{ \theta(u_1) + h(u) + \frac{1}{2}\|u - u^-\|^2_{\mathrm{sGS}(\mathcal{H})} - \langle d(\tilde{\delta}, \delta),\, u \rangle \right\} \qquad \text{(CQP)}$$

Block sGS decomposition theorem

Theorem

Suppose that $\mathcal{H}_d = \mathrm{Diag}(\mathcal{H}_{11}, \ldots, \mathcal{H}_{ss}) \succ 0$. *Then,*

- (CQP) *is well-defined and admits a unique solution, which is exactly the vector* u^+ *generated by the* (sGS) *procedure*
-
$$\widehat{\mathcal{H}} := \mathcal{H} + \mathrm{sGS}(\mathcal{H}) = (\mathcal{H}_d + \mathcal{H}_u)\mathcal{H}_d^{-1}(\mathcal{H}_d + \mathcal{H}_u^*) \succ 0$$
- *the error vector* $d(\tilde{\delta}, \delta)$ *satisfies*

$$\|\widehat{\mathcal{H}}^{-\frac{1}{2}} d(\tilde{\delta}, \delta)\| \leq \|\mathcal{H}_d^{-\frac{1}{2}}(\delta - \tilde{\delta})\| + \|\mathcal{H}_d^{\frac{1}{2}}(\mathcal{H}_d + \mathcal{H}_u)^{-1}\tilde{\delta}\|$$

Majorization

Problem (P):

$$\min\left\{p_1(x_1) + f(x) + q_1(y_1) + g(y) \mid \mathcal{A}^*x + \mathcal{B}^*y = c\right\}$$

For the two smooth convex functions f and g in problem (P), there exist two self-adjoint positive semidefinite linear operators $\widehat{\Sigma}^f : \mathcal{X} \to \mathcal{X}$ and $\widehat{\Sigma}^g : \mathcal{Y} \to \mathcal{Y}$ such that

$$\begin{cases} f(x) \leq \hat{f}(x; x') := f(x') + \langle \nabla f(x'), x - x' \rangle + \frac{1}{2}\|x - x'\|^2_{\widehat{\Sigma}^f} \\ g(y) \leq \hat{g}(y; y') := g(y') + \langle \nabla g(y'), y - y' \rangle + \frac{1}{2}\|y - y'\|^2_{\widehat{\Sigma}^g} \end{cases}$$

Quadratic on the RHS

Majorized proximal augmented Lagrangian function

For any given $\sigma > 0$, the majorized proximal augmented Lagrangian function associated with problem (P) is defined by

$$\widetilde{\mathcal{L}}_\sigma(x, y; (x', y', z')) :$$

$$= p(x) + \hat{f}(x; x') + q(y) + \hat{g}(y; y') + \langle z', \mathcal{A}^* x + \mathcal{B}^* y - c\rangle$$

$$+ \frac{\sigma}{2}\|\mathcal{A}^* x + \mathcal{B}^* y - c\|^2 + \frac{1}{2}\|x - x'\|^2_{\widetilde{\mathcal{S}}} + \frac{1}{2}\|y - y'\|^2_{\widetilde{\mathcal{T}}},$$

$$\forall\, (x, y) \in \mathcal{X} \times \mathcal{Y} \quad \text{and} \quad \forall\, (x', y', z') \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Z},$$

where $\widetilde{\mathcal{S}} : \mathcal{X} \to \mathcal{X}$ and $\widetilde{\mathcal{T}} : \mathcal{Y} \to \mathcal{Y}$ are self-adjoint (not necessarily positive semidefinite) linear operators

Nonsmooth+Quadratic Terms on RHS

sGS-imiPADMM

An inexact sGS decomposition based majorized indefinite-proximal ADMM

Let $\tau \in (0, (1+\sqrt{5})/2)$ [e.g., $\tau = 1.618$], $\{\tilde{\varepsilon}_k\}_{k\geq 0}$ be a summable nonnegative sequence, $(x^0, y^0, z^0) \in \mathrm{dom}\, p \times \mathrm{dom}\, q \times \mathcal{Z}$ be the initial point

For $k = 0, 1, \ldots,$

1a. Compute for $i = m, \ldots, 2,$

$$\boxed{x_i^{k+\frac{1}{2}} \approx \arg\min_{x_i \in \mathcal{X}_i} \left\{ \widetilde{\mathcal{L}}_\sigma \left((x_{<i}^k, x_i, x_{>i}^{k+\frac{1}{2}}), y^k; (x^k, y^k, z^k) \right) \right\}},$$

$$\tilde{\delta}_i^k \in \partial_{x_i} \widetilde{\mathcal{L}}_\sigma \left((x_{<i}^k, x_i^{k+\frac{1}{2}}, x_{>i}^{k+\frac{1}{2}}), y^k; (x^k, y^k, z^k) \right) \text{ with } \|\tilde{\delta}_i^k\| \leq \tilde{\varepsilon}_k$$

1b. Compute for $i = 1, \ldots, m,$

$$\boxed{x_i^{k+1} \approx \arg\min_{x_i \in \mathcal{X}_i} \left\{ \widetilde{\mathcal{L}}_\sigma \left((x_{<i}^{k+1}, x_i, x_{>i}^{k+\frac{1}{2}}), y^k; (x^k, y^k, z^k) \right) \right\}},$$

$$\delta_i^k \in \partial_{x_i} \widetilde{\mathcal{L}}_\sigma \left((x_{<i}^{k+1}, x_i^{k+1}, x_{>i}^{k+\frac{1}{2}}), y^k; (x^k, y^k, z^k) \right) \text{ with } \|\delta_i^k\| \leq \tilde{\varepsilon}_k$$

sGS-imiPADMM

2a. Compute for $j = n, \ldots, 2$,

$$\boxed{y_j^{k+\frac{1}{2}} \approx \arg\min_{y_j \in \mathcal{Y}_j} \left\{ \widetilde{\mathcal{L}}_\sigma \left(x^{k+1}, (y_{<j}^k, y_j, y_{>j}^{k+\frac{1}{2}}); (x^k, y^k, z^k) \right) \right\}},$$

$$\tilde{\gamma}_j^k \in \partial_{y_j} \widetilde{\mathcal{L}}_\sigma \left(x^{k+1}, (y_{<j}^k, y_j^{k+\frac{1}{2}}, y_{>j}^{k+\frac{1}{2}}); (x^k, y^k, z^k) \right) \text{ with } \|\tilde{\gamma}_j^k\| \leq \tilde{\varepsilon}_k$$

2b. Compute for $j = 1, \ldots, n$,

$$\boxed{y_j^{k+1} \approx \arg\min_{y_j \in \mathcal{Y}_j} \left\{ \widetilde{\mathcal{L}}_\sigma \left(x^{k+1}, (y_{<j}^{k+1}, y_j, y_{>j}^{k+\frac{1}{2}}); (x^k, y^k, z^k) \right) \right\}},$$

$$\gamma_j^k \in \partial_{y_j} \widetilde{\mathcal{L}}_\sigma \left(x^{k+1}, (y_{<j}^{k+1}, y_j^{k+1}, y_{>j}^{k+\frac{1}{2}}); (x^k, y^k, z^k) \right) \text{ with } \|\gamma_j^k\| \leq \tilde{\varepsilon}_k$$

3. Compute $z^{k+1} := z^k + \tau\sigma(\mathcal{A}^* x^{k+1} + \mathcal{B}^* y^{k+1} - c)$

Decompositions

We symbolically decompose the positive semidefinite linear operators $\widehat{\Sigma}^f$ into

$$\widehat{\Sigma}^f = \begin{pmatrix} \widehat{\Sigma}^f_{11} & \widehat{\Sigma}^f_{12} & \cdots & \widehat{\Sigma}^f_{1m} \\ (\widehat{\Sigma}^f_{12})^* & \widehat{\Sigma}^f_{22} & \cdots & \widehat{\Sigma}^f_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ (\widehat{\Sigma}^f_{1m})^* & (\widehat{\Sigma}^f_{2m})^* & \cdots & \widehat{\Sigma}^f_{mm} \end{pmatrix} \tag{2}$$

and decompose $\widehat{\Sigma}^g$ similarly, in consistent with the decompositions of $\mathcal{X}$ and $\mathcal{Y}$.

Define two linear operators $\widetilde{\mathcal{M}} : \mathcal{X} \to \mathcal{X}$ and $\widetilde{\mathcal{N}} : \mathcal{Y} \to \mathcal{Y}$ as follows:

$$\widetilde{\mathcal{M}} := \widehat{\Sigma}^f + \sigma \mathcal{A}\mathcal{A}^* + \widetilde{\mathcal{S}}, \quad \widetilde{\mathcal{N}} := \widehat{\Sigma}^g + \sigma \mathcal{B}\mathcal{B}^* + \widetilde{\mathcal{T}}$$

Just like the decomposition of $\widehat{\Sigma}^f$ and $\widehat{\Sigma}^g$ in (2), we can symbolically decompose $\widetilde{\mathcal{S}}$, $\widetilde{\mathcal{T}}$, $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{N}}$ accordingly.

Decompositions

We use $\widetilde{\mathcal{M}}_d$ and $\widetilde{\mathcal{N}}_d$ to denote the corresponding diagonal parts, and $\widetilde{\mathcal{M}}_u$ and $\widetilde{\mathcal{N}}_u$ to denote the strictly upper triangular parts, respectively. Then,

$$\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}_u + \widetilde{\mathcal{M}}_d + \widetilde{\mathcal{M}}_u^*, \quad \widetilde{\mathcal{N}} = \widetilde{\mathcal{N}}_u + \widetilde{\mathcal{N}}_d + \widetilde{\mathcal{N}}_u^*$$

Decompose $\mathcal{A}$ and $\mathcal{B}$ as

$$\mathcal{A}z = (\mathcal{A}_1 z, \ldots, \mathcal{A}_m z) \quad \text{and} \quad \mathcal{B}z = (\mathcal{B}_1 z, \ldots, \mathcal{B}_n z)$$

where $\mathcal{A}_i z \in \mathcal{X}_i$ and $\mathcal{B}_j z \in \mathcal{Y}_j$, and $z \in \mathcal{Z}$

Define

$$\tilde{\delta}^k := (\tilde{\delta}_1^k, \ldots, \tilde{\delta}_m^k), \;\; \delta^k := (\delta_1^k, \ldots, \delta_m^k)$$

$$\tilde{\gamma}^k := (\tilde{\gamma}_1^k, \ldots, \tilde{\gamma}_n^k), \text{ and } \gamma^k := (\gamma_1^k, \ldots, \gamma_n^k)$$

with the convention that $\tilde{\delta}_1^k := \delta_1^k$ and $\tilde{\gamma}_1^k := \gamma_1^k$.

Decompositions

To apply the block sGS decomposition theorem, we require

$$\widetilde{\mathcal{M}}_{ii} \equiv \widehat{\Sigma}^f_{ii} + \sigma \mathcal{A}_i \mathcal{A}_i^* + \widetilde{\mathcal{S}}_{ii} \succ 0, \quad i = 1, \ldots, m$$

$$\widetilde{\mathcal{N}}_{jj} \equiv \widehat{\Sigma}^g_{jj} + \sigma \mathcal{B}_j \mathcal{B}_j^* + \widetilde{\mathcal{T}}_{jj} \succ 0, \quad j = 1, \ldots, n$$

Define the following linear operators:

$$\begin{cases} \mathcal{S}_{\rm sGS} := \widetilde{\mathcal{S}} + {\rm sGS}(\widetilde{\mathcal{M}}) = \widetilde{\mathcal{S}} + \widetilde{\mathcal{M}}_u \widetilde{\mathcal{M}}_d^{-1} \widetilde{\mathcal{M}}_u^* \\ \mathcal{T}_{\rm sGS} := \widetilde{\mathcal{T}} + {\rm sGS}(\widetilde{\mathcal{N}}) = \widetilde{\mathcal{T}} + \widetilde{\mathcal{N}}_u \widetilde{\mathcal{N}}_d^{-1} \widetilde{\mathcal{N}}_u^* \end{cases} \tag{3}$$

Theorem (via the sGS decomposition theorem)

- $\mathcal{S}_{\rm sGS}$ *and* $\mathcal{T}_{\rm sGS}$ *defined in* (3) *are well-defined, and*

$$\mathcal{M}_{\rm sGS} := \widehat{\Sigma}^f + \sigma \mathcal{A}\mathcal{A}^* + \mathcal{S}_{\rm sGS} \succ 0, \ \mathcal{N}_{\rm sGS} := \widehat{\Sigma}^g + \sigma \mathcal{B}\mathcal{B}^* + \mathcal{T}_{\rm sGS} \succ 0 \tag{4}$$

- *it holds that*

$$\begin{cases} d_x^k \in \partial_x \left\{ \widetilde{\mathcal{L}}_\sigma \left(x^{k+1}, y^k; (x^k, y^k, z^k) \right) + \frac{1}{2} \| x^{k+1} - x^k \|^2_{{\rm sGS}(\widetilde{\mathcal{M}})} \right\}, \\ d_y^k \in \partial_y \left\{ \widetilde{\mathcal{L}}_\sigma \left(x^{k+1}, y^{k+1}; (x^k, y^k, z^k) \right) + \frac{1}{2} \| y^{k+1} - y^k \|^2_{{\rm sGS}(\widetilde{\mathcal{N}})} \right\}, \end{cases}$$

$$d_x^k := \delta^k + \widetilde{\mathcal{M}}_u \widetilde{\mathcal{M}}_d^{-1} (\delta^k - \tilde{\delta}^k) \quad \textit{and} \quad d_y^k := \gamma^k + \widetilde{\mathcal{N}}_u \widetilde{\mathcal{N}}_d^{-1} (\gamma^k - \tilde{\gamma}^k)$$

- *one has* $\| \mathcal{M}_{\rm sGS}^{-\frac{1}{2}} d_x^k \| \le \kappa \tilde{\varepsilon}_k$ *and* $\| \mathcal{N}_{\rm sGS}^{-\frac{1}{2}} d_y^k \| \le \kappa' \tilde{\varepsilon}_k$, *where* κ *and* κ' *are some constants*

Karush-Kuhn-Tucker (KKT)

Recall that the Karush-Kuhn-Tucker (KKT) system of problem (P) is given by

$$0 \in \partial p(x) + \nabla f(x) + \mathcal{A}z, \; 0 \in \partial q(y) + \nabla g(y) + \mathcal{B}z, \; \mathcal{A}^* x + \mathcal{B}^* y = c$$

Denote the solution set of the KKT system for problem (P) by $\overline{\mathcal{W}}$.

Stopping criterion: always use the (relative) KKT residual to stop an algorithm. The (relative) distance of two consecutive iterates CANNOT be used as a reliable stopping criterion.

Assumption

- *The solution set* $\overline{\mathcal{W}}$ *to the KKT system of* (P) *is nonempty*
- $\widetilde{\mathcal{S}}$ *and* $\widetilde{\mathcal{T}}$ *are chosen such that*

$$\widetilde{\mathcal{S}} \succeq -\frac{1}{2}\widehat{\Sigma}^{f} \quad \& \quad \widetilde{\mathcal{T}} \succeq -\frac{1}{2}\widehat{\Sigma}^{g} \tag{5}$$

and

$$\widehat{\Sigma}^{f}_{ii} + \sigma\mathcal{A}_i\mathcal{A}_i^* + \widetilde{\mathcal{S}}_{ii} \succ 0, \quad i = 1, \ldots, m$$
$$\widehat{\Sigma}^{g}_{jj} + \sigma\mathcal{B}_j\mathcal{B}_j^* + \widetilde{\mathcal{T}}_{jj} \succ 0, \quad j = 1, \ldots, n$$

Theorem (Convergence of sGS-imiPADMM)

Suppose that the Assumption holds, and the linear operators $\widetilde{\mathcal{S}}$ and $\widetilde{\mathcal{T}}$ are chosen such that

$$\frac{1}{2}\widehat{\Sigma}^f + \sigma\mathcal{A}\mathcal{A}^* + \mathcal{S}_{\rm sGS} \succ 0 \quad \& \quad \frac{1}{2}\widehat{\Sigma}^g + \sigma\mathcal{B}\mathcal{B}^* + \mathcal{T}_{\rm sGS} \succ 0 \tag{6}$$

Then the whole sequence $\{(x^k, y^k, z^k)\}$ converges to a solution of the KKT system

Remark: The above convergence theorem is fairly general; it covers the classic ADMM and the most recent developments for solving multi-block convex optimization problems. Below we shall use a more specific example to reveal the relations between ADMM and proxiaml ALM.

Convex Composite Quadratic Programming

$$\min_{x\in\mathcal{X}}\left\{\psi(x)+\frac{1}{2}\langle x,\mathcal{Q}x\rangle-\langle c,x\rangle \;\Big|\; \mathcal{A}_E x=b_E,\ \mathcal{A}_I x-b_I\in\mathcal{K}\right\} \tag{7}$$

- $\psi:\mathcal{X}\to(-\infty,+\infty]$ is a closed proper convex function [simple]
- $\mathcal{Q}:\mathcal{X}\to\mathcal{X}$ satisfying $\mathcal{Q}=\mathcal{Q}^*,\ \mathcal{Q}\succeq 0$
- $\mathcal{A}_E:\mathcal{X}\to\mathcal{Z}_1$ and $\mathcal{A}_I:\mathcal{X}\to\mathcal{Z}_2$, given linear mappings
- $b=(b_E;b_I)\in\mathcal{Z}:=\mathcal{Z}_1\times\mathcal{Z}_2$, given vector
- $\mathcal{K}\subseteq\mathcal{Z}_2$ is a closed convex set (cone) [simple]

Equivalently,

$$\min_{x\in\mathcal{X},x'\in\mathcal{Z}_2}\left\{\psi(x)+\delta_{\mathcal{K}}(x')+\frac{1}{2}\langle x,\mathcal{Q}x\rangle-\langle c,x\rangle \;\Big|\; \begin{pmatrix}\mathcal{A}_E & 0\\ \mathcal{A}_I & -\mathcal{I}\end{pmatrix}\begin{pmatrix}x\\ x'\end{pmatrix}=b\right\}$$

CCQP

The dual of the above problem [or equivalently problem (7)] is

$$\min_{w,y,z} \left\{ p(w) + \frac{1}{2}\langle y, \mathcal{Q}y\rangle - \langle b, z\rangle \;\middle|\; \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} \mathcal{Q} \\ 0 \end{pmatrix} y - \begin{pmatrix} \mathcal{A}_E^* & \mathcal{A}_I^* \\ 0 & -\mathcal{I} \end{pmatrix} z = \begin{pmatrix} c \\ 0 \end{pmatrix} \right\}$$

- $w := (u, v) \in \mathcal{X} \times \mathcal{Z}_2$
- $p(w) := p(u, v) = \psi^*(u) + \delta_{\mathcal{K}}^*(v)$
- $\delta_{\mathcal{K}}(\cdot)$ is the indicator function over $\mathcal{K}$

- The dual is about (w, y, z) – three or more blocks
- Nonsmoothness only exists in one block of variables, i.e., the w-block
- It covers many important classes of convex optimization problems that are best solved in this (dual) form!

Deal with Convex Quadratic Constraints

Add additional convex quadratic constraints to problem (7):

$$\langle x, \mathcal{Q}_i x\rangle - \langle c_i', x\rangle \leq b_i', \quad i = 1, \ldots, l$$

where $\mathcal{Q}_i = \mathcal{L}_i \mathcal{L}_i^* \succeq 0$ for a certain linear operator $\mathcal{L}_i$

Write the above constraints as $\|\mathcal{L}_i^* x\|^2 - \langle c_i', x\rangle \leq b_i', \quad i = 1, \ldots, l$, or equivalently

$$\left\| \begin{pmatrix} 1 - b_i' - \langle c_i', x\rangle \\ 2\mathcal{L}_i^* x \end{pmatrix} \right\|_2 \leq 1 + b_i' + \langle c_i', x\rangle, \quad i = 1, \ldots, l$$

We can further rewrite the above as

$$\begin{pmatrix} 1 + b_i' + \langle c_i', x\rangle \\ 1 - b_i' - \langle c_i', x\rangle \\ 2\mathcal{L}_i^* x \end{pmatrix} \in \mathcal{K}_i, \quad i = 1, \ldots, l$$

where $\mathcal{K}_i$ is the second-order-cone of a proper dimension, $i = 1, \ldots, l$

Therefore, convex quadratic constraints can be added to problem (7) without changing its structure

Penalized and Constrained Regression Models

The penalized and constrained (PAC) regression often arises in high-dimensional generalized linear models with linear equality and inequality constraints, e.g.,

$$\min_{x\in\mathbb{R}^n} \left\{ p(x) + \frac{1}{2\lambda}\|\Phi x - \eta\|^2 \,\middle|\, A_E x = b_E,\ A_I x - b_I \in \mathcal{K} \right\}$$

- $p(\cdot)$ is a proper closed convex regularizer such as $p(x) = \|x\|_1,\ \|x\|_*$ [Non-convex counterparts can be dealt with via proximal DC (difference of convex functions) algorithm – another talk]
- $\lambda > 0$ is a parameter
- It is a special case of problem (7)

Multi-block convex composite optimization

$$\min_{y\in\mathcal{Y},z\in\mathcal{Z}}\Big\{\underbrace{p(y_1)+f(\underbrace{y_1,y_2,\ldots,y_s}_{y})-\langle b,z\rangle}_{\Phi(w)}\ \Big|\ \underbrace{\mathcal{F}^*y+\mathcal{G}^*z=c}_{\mathcal{A}^*w=c}\Big\}$$

with $w=(y,z)\in\mathcal{W}:=\mathcal{Y}\times\mathcal{Z}$

- $\mathcal{X}$, $\mathcal{Z}$ and $\mathcal{Y}:=\mathcal{Y}_1\times\cdots\times\mathcal{Y}_s$: finite-dimensional real Hilbert spaces, endowed with $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$
- $p:\mathcal{Y}_1\to(-\infty,+\infty]$: (nonsmooth) closed proper convex function
  $f:\mathcal{Y}\to(-\infty,+\infty)$: continuously differentiable convex function with Lipschitz gradient
- $\mathcal{F}^*$ and $\mathcal{G}^*$: the adjoints of the given linear mappings $\mathcal{F}:\mathcal{X}\to\mathcal{Y}$ and $\mathcal{G}:\mathcal{X}\to\mathcal{Z}$; $b\in\mathcal{Z}$ and $c\in\mathcal{X}$: the given data

The augmented Lagrangian function[1]

Recall the problem

$$\min_{y\in\mathcal{Y},z\in\mathcal{Z}} \{p(y_1) + f(y_1, y_2, \ldots, y_s) - \langle b, z\rangle \mid \mathcal{F}^*y + \mathcal{G}^*z = c\}$$

or

$$\min_{w\in\mathcal{W}} \{\Phi(w) \mid \mathcal{A}^*w = c\}$$

Let $\sigma > 0$ be the penalty parameter. The augmented Lagrangian function:

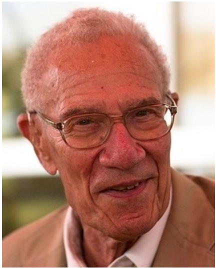
$$\begin{aligned}\mathcal{L}_\sigma(y, z; x) :=\ & \underbrace{p(y_1) + f(y_1, y_2, \ldots, y_s) - \langle b, z\rangle}_{\Phi(w)} \\ & + \underbrace{\langle x, \mathcal{F}^*y + \mathcal{G}^*z - c\rangle}_{\langle x, \mathcal{A}^*w - c\rangle} + \frac{\sigma}{2}\underbrace{\|\mathcal{F}^*y + \mathcal{G}^*z - c\|^2}_{\|\mathcal{A}^*w - c\|^2}, \\ & \qquad \forall w = (y, z) \in \mathcal{W} := \mathcal{Y}\times\mathcal{Z},\ x \in \mathcal{X}\end{aligned}$$

[1] Arrow, K.J., Solow, R.M.: Gradient methods for constrained maxima with weakened assumptions. In: Arrow, K.J., Hurwicz, L., Uzawa, H., (eds.) Studies in Linear and Nonlinear Programming. Stanford University Press, Stanford, pp. 165-176 (1958)

K. Arrow and R. Solow

Kenneth Joseph “Ken” Arrow
(23 August 1921 – 21 February 2017)

John Bates Clark Medal (1957); Nobel Prize in Economics (1972); von Neumann Theory Prize (1986); National Medal of Science (2004); ForMemRS (2006)



Robert Merton Solow
(August 23, 1924 –)

John Bates Clark Medal (1961); Nobel Memorial Prize in Economic Sciences (1987); National Medal of Science (1999); Presidential Medal of Freedom (2014); ForMemRS (2006)

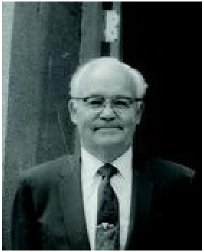
The augmented Lagrangian method[2] (ALM)

$\mathcal{L}_\sigma(y, z; x) = p(y_1) + f(y) - \langle b, z\rangle + \langle x, \mathcal{F}^* y + \mathcal{G}^* z - c\rangle + \frac{\sigma}{2}\|\mathcal{F}^* y + \mathcal{G}^* z - c\|^2$

Starting from $x^0 \in \mathcal{X}$, performs for $k = 0, 1, \ldots$

(1) $\underbrace{(y^{k+1}, z^{k+1})}_{w^{k+1}} \Leftarrow \min_{y,z} \mathcal{L}_\sigma(\underbrace{y, z}_{w}; x^k)$ (approximately)

(2) $x^{k+1} := x^k + \tau\,\sigma(\mathcal{F}^* y^{k+1} + \mathcal{G}^* z^{k+1} - c)$ with $\tau \in (0, 2)$



Magnus Rudolph Hestenes
(February 13 1906 – May 31 1991)

Michael James David Powell
(29 July 1936 – 19 April 2015)

[2]Also known as the method of multipliers

ALM to proximal ALM[3] (PALM)

Minimize the augmented Lagrangian function plus a quadratic **proximal term**:

$$w^{k+1} \approx \arg\min_{w} \mathcal{L}_\sigma(w; x^k) + \frac{1}{2}\|w - w^k\|_{\mathcal{D}}^2$$

- $\mathcal{D} = \sigma^{-1}\mathcal{I}$ in the seminal work of Rockafellar (in which inequality constraints are considered). Note that $\mathcal{D} \to 0$ as $\sigma \to \infty$, which is critical for asymptotically superlinear convergence (for $\tau = 1$)
- It is a primal-dual type proximal point algorithm (PPA)

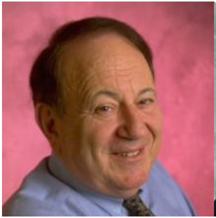
[3] Also known as the proximal method of multipliers

"Decoupling" (or "splitting") based ADMM

One the other hand, "decoupling" (or "splitting") based approach is available, i.e,

$$\begin{cases} y^{k+1} \approx \arg\min\limits_{y}\{\mathcal{L}_\sigma(y, z^k; x^k)\}, \ z^{k+1} \approx \arg\min\limits_{z}\{\mathcal{L}_\sigma(y^{k+1}, z; x^k)\}; \\ x^{k+1} := x^k + \tau\sigma(\mathcal{F}^* y^{k+1} + \mathcal{G}^* z^{k+1} - c), \quad \tau \in (\mathbf{0}, \infty) \end{cases}$$

- The **two-block ADMM**
- Allows $\tau \in (0, (1+\sqrt{5})/2)$ if the convergence of the full (primal & dual) sequence is required (first proven by Glowinski in 1977 at Tata Institute, India)
- The case with $\tau = 1$ is a kind of PPA (Gabay + Bertsekas-Eckstein)



An inexact majorized indefinite proximal ALM

Consider

$$\min_{w\in\mathcal{W}} \Phi(w) := \varphi(w) + h(w) \quad \text{s.t.} \quad \mathcal{A}^* w = c$$

- There exists a self-adjoint positive semidefinite linear operator $\widehat{\Sigma}^h : \mathcal{W} \to \mathcal{W}$, such that for any $w, w' \in \mathcal{W}$,

$$\boxed{h(w) \leq \hat{h}(w, w') := h(w') + \langle \nabla h(w'), w - w' \rangle + \frac{1}{2}\|w - w'\|^2_{\widehat{\Sigma}^h}}$$

  which is called a majorization (or surrogate) of h at w'

Prerequisites

One definition and one assumption

Let $\sigma > 0$. The majorized augmented Lagrangian function is defined, for any $(w, x, w') \in \mathcal{W} \times \mathcal{X} \times \mathcal{W}$, by

$$\widehat{\mathcal{L}}_\sigma(w; (x, w')) := \varphi(w) + \hat{h}(w, w') + \langle \mathcal{A}^* w - c, x \rangle + \frac{\sigma}{2} \|\mathcal{A}^* w - c\|^2$$

Assumption

The solution set K to the KKT system is nonempty and $\mathcal{D} : \mathcal{W} \to \mathcal{W}$ is a given self-adjoint linear operator such that

$$\frac{1}{2}\widehat{\Sigma}^h + \mathcal{D} \succeq 0 \quad \& \quad \frac{1}{2}\widehat{\Sigma}^h + \mathcal{D} + \sigma \mathcal{A}\mathcal{A}^* \succ 0 \tag{8}$$

- **$\mathcal{D}$ is not necessarily to be positive semidefinite!**

Alg. an inexact majorized indefinite proximal ALM

Let $\{\varepsilon_k\}$ be a summable sequence of nonnegative numbers. Choose an initial point $(x^0, w^0) \in \mathcal{X} \times \mathcal{W}$. For $k = 0, 1, \ldots,$

1 Compute

$$w^{k+1} \approx \arg\min_{w \in \mathcal{W}} \left\{ \widehat{\mathcal{L}}_\sigma(w; (x^k, w^k)) + \frac{1}{2}\|w - w^k\|_{\mathcal{D}}^2 \right\}$$

such that there exists d_k satisfying $\|d^k\| \leq \varepsilon_k$ and

$$d^k \in \partial_w \widehat{\mathcal{L}}_\sigma(w^{k+1}; (x^k, w^k)) + \mathcal{D}(w^{k+1} - w^k)$$

2 Update $x^{k+1} := x^k + \tau\sigma(\mathcal{A}^* w^{k+1} - c)$ with $\tau \in (0, 2)$

Theorem

The sequence $\{(x^k, w^k)\}$ generated by the above Algorithm converges to a solution to the KKT system.

Multi-block: Majorization and Splitting

There exists a self-adjoint linear operator $\widehat{\Sigma}^f \succeq 0$ on $\mathcal{Y}$ such that for any $y, y' \in \mathcal{Y}$,

$$f(y) \leq \hat{f}(y, y') := f(y') + \langle \nabla f(y'), y - y' \rangle + \frac{1}{2}\|y - y'\|^2_{\widehat{\Sigma}^f}$$

- Denote $y_{<i} := (y_1; \ldots; y_{i-1})$ and $y_{>i} := (y_{i+1}; \ldots; y_s)$
- Decompose $\widehat{\Sigma}^f = \begin{pmatrix} \widehat{\Sigma}^f_{11} & \widehat{\Sigma}^f_{12} & \cdots & \widehat{\Sigma}^f_{1s} \\ (\widehat{\Sigma}^f_{12})^* & \widehat{\Sigma}^f_{22} & \cdots & \widehat{\Sigma}^f_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ (\widehat{\Sigma}^f_{1s})^* & (\widehat{\Sigma}^f_{2s})^* & \cdots & \widehat{\Sigma}^f_{ss} \end{pmatrix}$

  with

$$\widehat{\Sigma}^f_{ij} : \mathcal{Y}_j \to \mathcal{Y}_i, \ \forall 1 \leq i \leq j \leq s$$

Basic Assumptions

(a) The self-adjoint linear operator $\mathcal{S} : \mathcal{Y} \to \mathcal{Y}$ satisfies

$$\boxed{\widehat{\Sigma}^f_{ii} + \sigma \mathcal{F}_i \mathcal{F}_i^* + \mathcal{S}_{ii} \succ 0 \quad \text{and} \quad \mathcal{S} \succeq -\tfrac{1}{2}\widehat{\Sigma}^f}$$

(b) The linear operator $\mathcal{G}$ is surjective (always true if restricted to its range space)

Let $\sigma > 0$. The *majorized* proximal augmented Lagrangian function:

$$\begin{aligned}
\widetilde{\mathcal{L}}_\sigma(y, z; (x, y')) := \quad & p(y_1) + \hat{f}(y, y') - \langle b, z \rangle \\
& + \langle \mathcal{F}^* y + \mathcal{G}^* z - c, x \rangle + \frac{\sigma}{2} \|\mathcal{F}^* y + \mathcal{G}^* z - c\|^2 \\
& + \frac{1}{2}\|y - y'\|^2_{\mathcal{S}}
\end{aligned}$$

The Algorithm: sGS-imPADMM

$(x^0, y^0, z^0) \in \mathcal{X} \times \operatorname{dom} p \times \mathcal{Y}_2 \times \cdots \times \mathcal{Y}_s \times \mathcal{Z}$. $\{\tilde{\varepsilon}_k\}$ nonnegative and summable. For $k = 0, 1, \ldots,$

1 Compute for $i = s, \ldots, 2,$

$$y_i^{k+\frac{1}{2}} \approx \operatorname*{arg\,min}_{y_i \in \mathcal{Y}_i} \widetilde{\mathcal{L}}_\sigma\Big(y_{<i}^k, y_i, y_{>i}^{k+\frac{1}{2}}, z^k; (x^k, y^k)\Big)$$

2 Compute for $i = 1, \ldots, s,$

$$y_i^{k+1} \approx \operatorname*{arg\,min}_{y_i \in \mathcal{Y}_i} \widetilde{\mathcal{L}}_\sigma\Big(y_{<i}^{k+1}, y_i, y_{>i}^{k+1/2}, z^k; (x^k, y^k)\Big)$$

3 Compute

$$z^{k+1} \approx \operatorname*{arg\,min}_{z \in \mathcal{Z}} \widetilde{\mathcal{L}}_\sigma(y^{k+1}, z; (x^k, y^k))$$

4 Compute $x^{k+1} := x^k + \tau\sigma(\mathcal{F}^* y^{k+1} + \mathcal{G}^* z^{k+1} - c)$, $\tau \in (0, 2)$

Criteria for inexact solutions in sGS-imPADMM

1 For $i = s, \ldots, 2$, the approximate solution $y_i^{k+\frac{1}{2}}$ is chosen such that there exists $\tilde{\delta}_i^k$ satisfying $\|\tilde{\delta}_i^k\| \leq \tilde{\varepsilon}_k$ and

$$\tilde{\delta}_i^k \in \partial_{y_i} \widetilde{\mathcal{L}}_\sigma\Big(y_{<i}^k, y_i^{k+\frac{1}{2}}, y_{>i}^{k+\frac{1}{2}}, z^k; (x^k, y^k)\Big)$$

2 For $i = 1, \ldots, s$, the approximate solution y_i^{k+1} is chosen such that there exists δ_i^k satisfying $\|\delta_i^k\| \leq \tilde{\varepsilon}_k$ and

$$\delta_i^k \in \partial_{y_i} \widetilde{\mathcal{L}}_\sigma\Big(y_{<i}^{k+1}, y_i^{k+1}, y_{>i}^{k+1/2}, z^k; (x^k, y^k)\Big)$$

3 The approximate solution z^{k+1} is chosen such that $\|\gamma^k\| \leq \tilde{\varepsilon}_k$ with

$$\begin{aligned}\gamma^k : \;&= \nabla_z \widetilde{\mathcal{L}}_\sigma\Big(y^{k+1}, z^{k+1}; (x^k, y^k)\Big) \\ &= \mathcal{G}x^k - b + \sigma\mathcal{G}(\mathcal{F}^* y^{k+1} + \mathcal{G}^* z^{k+1} - c)\end{aligned}$$

Inexact block sGS decomposition

Define $\mathcal{H} := \widehat{\Sigma}^f + \sigma\mathcal{F}\mathcal{F}^* + \mathcal{S} = \mathcal{H}_u + \mathcal{H}_d + \mathcal{H}_u^*$ with $\mathcal{H}_d := \text{Diag}(\mathcal{H}_{11}, \ldots, \mathcal{H}_{ss})$ and

$$\mathcal{H}_u := \begin{pmatrix} 0 & \mathcal{H}_{12} & \cdots & \mathcal{H}_{1s} \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \mathcal{H}_{(s-1)s} \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \mathcal{H}_{ij} = \widehat{\Sigma}^f_{ij} + \sigma\mathcal{F}_i\mathcal{F}_j^* + \mathcal{S}_{ij}$$

For convenience, we denote for each $k \geq 0$,

$$\tilde{\delta}_1^k := \delta_1^k, \quad \tilde{\delta}^k := (\tilde{\delta}_1^k, \tilde{\delta}_2^k \ldots, \tilde{\delta}_s^k), \quad \delta^k := (\delta_1^k, \ldots, \delta_s^k)$$

Define the sequence $\{\Delta^k\} \in \mathcal{Y}$ by

$$\Delta^k := \delta^k + \mathcal{H}_u\mathcal{H}_d^{-1}(\delta^k - \tilde{\delta}^k)$$

Moreover, we can define the linear operator

$$\widehat{\mathcal{H}} := \mathcal{H}_u\mathcal{H}_d^{-1}\mathcal{H}_u^* \quad \textbf{(sGS Splitting Operator)}$$

Result by the block sGS decomposition theorem [4]

The iterate y^{k+1} in Step 2 of sGS-imPADMM is the unique solution to a proximal minimization problem given by

$$y^{k+1} = \arg\min_{y} \Big\{ \underbrace{\widehat{\mathcal{L}}_\sigma(y, z^k; (x^k, y^k)) + \frac{1}{2}\|y - y^k\|^2_{\mathcal{S}+\widehat{\mathcal{H}}}}_{\text{strongly convex}} - \langle \Delta^k, y \rangle \Big\}$$

- Recall that $\mathcal{H} := \widehat{\Sigma}^f + \sigma\mathcal{F}\mathcal{F}^* + \mathcal{S}$
- Linearly transported error: $\Delta^k = \delta^k + \mathcal{H}_u \mathcal{H}_d^{-1}(\delta^k - \tilde{\delta}^k)$

[4] X.D. Li, D.F. Sun, and K.-C Toh, A block symmetric Gauss-Seidel decomposition theorem for convex composite quadratic programming and its applications, Math Prog (2019) [DOI: 10.1007/s10107-018-1247-7]

The equivalence property

Recall that $\mathcal{W} = \mathcal{Y} \times \mathcal{Z}$. Define $\widehat{\Sigma}^h : \mathcal{W} \to \mathcal{W}$ by

$$\widehat{\Sigma}^h := \begin{pmatrix} \widehat{\Sigma}^f & \\ & 0 \end{pmatrix}$$

For $w = (y; z)$ and $w' = (y'; z')$, denote

$$\widehat{\mathcal{L}}_\sigma(w; (x, w')) := \widehat{\mathcal{L}}_\sigma(y, z; (x, y'))$$

Define the error term

$$\widehat{\Delta}^k := \Delta^k - \mathcal{F}\mathcal{G}^*(\mathcal{G}\mathcal{G}^*)^{-1}(\gamma^{k-1} - \gamma^k - \mathcal{G}(x^{k-1} - x^k)) \in \mathcal{Y}$$

with the convention that

$$\begin{cases} x^{-1} := x^0 - \tau\sigma(\mathcal{F}^* y^0 + \mathcal{G}^* z^0 - c), \\ \gamma^{-1} := -b + \mathcal{G}x^{-1} + \sigma\mathcal{G}(\mathcal{F}^* y^0 + \mathcal{G}^* z^0 - c) \end{cases}$$

The equivalence property

Define the block-diagonal linear operator

$$\mathcal{T} := \begin{pmatrix} \mathcal{S} + \widehat{\mathcal{H}} + \sigma \mathcal{F}\mathcal{G}^*(\mathcal{G}\mathcal{G}^*)^{-1}\mathcal{G}\mathcal{F}^* & \\ & 0 \end{pmatrix} \quad \boxed{\mathcal{W} \to \mathcal{W}}$$

Theorem

Let $\{(x^k, w^k)\}$ with $w^k := (y^k; z^k)$ be the sequence generated by sGS-imPADMM. Then, for any $k \geq 0$, it holds that

(i) *the linear operators $\mathcal{T}$, $\mathcal{A}$ and $\widehat{\Sigma}^h$ satisfy*

$$\mathcal{T} + \frac{1}{2}\widehat{\Sigma}^h \succeq 0$$

(ii)

$$w^{k+1} \approx \arg\min_{w \in \mathcal{W}} \left\{ \widehat{\mathcal{L}}_\sigma\Big(w; (x^k, w^k)\Big) + \frac{1}{2}\|w - w^k\|^2_{\mathcal{T}} \right\}$$

in the sense that $(\widehat{\Delta}^k; \gamma^k) \in \partial_w \widehat{\mathcal{L}}_\sigma((w^{k+1}; (x^k, w^k)) + \mathcal{T}(w^{k+1} - w^k)$ and $\|(\widehat{\Delta}^k, \gamma^k)\| \leq \widehat{\varepsilon}_k$ with $\{\widehat{\varepsilon}_k\}$ being a summable sequence of nonnegative numbers

sGS-imPADMM convergence

One can readily get the following convergence theorem

Theorem

Suppose that

$$\frac{1}{2}\widehat{\Sigma}^f + \sigma\mathcal{F}\mathcal{F}^* + \mathcal{S} + \mathcal{H}_u\mathcal{H}_d^{-1}\mathcal{H}_u^* \succ 0$$

Then,

$$\mathcal{T} + \frac{1}{2}\widehat{\Sigma}^h + \sigma\mathcal{A}\mathcal{A}^* \succ 0$$

Moreover, the sequence $\{(x^k, y^k, z^k)\}$ *generated by the Algorithm converges to a solution of the KKT system of the problem. Thus,* $\{(y^k, z^k)\}$ *converges to a solution to this problem and* $\{x^k\}$ *converges to a solution of its dual*

The two-block case

Let $\mathcal{Y} = \mathcal{Y}_1$ and f be vacuous (e.g., the dual of linear conic programming), i.e.,

$$\min\{p(y) - \langle b, z\rangle \mid \mathcal{F}^* y + \mathcal{G}^* z = c\} \tag{9}$$

- The two-block ADMM originates from the ALM, but it actually deviates substantially from the ALM!!!
- ADMM (decoupling) is NOT ALM (recoupling)
- Note that $\mathcal{T}$ has a term propositional to σ while in Rockafellar's proximal ALM, the corresponding proximal term is proportional to σ^{-1}. This is the $\underbrace{\text{price to pay}}$ for "decoupling" — loss of the arbitrary linear convergence rate [in the terminology of M.J.D. Powell]

Comments on the two-block case

- The assumptions we made for problem (9) are apparently much weaker than those in original work of Gabay and Mercier[5], where $\mathcal{F}$ is assumed to be the identity operator and p is assumed to be strongly convex

- In Gabay and Mercier (1976), Theorem 3.1, only the convergence of the primal sequence $\{(y^k, z^k)\}$ is obtained while the dual sequence $\{x^k\}$ is only proven to be bounded

- In S., Toh and Yang *et al.*[6], a similar result to ours has been derived with the requirements that the initial multiplier x^0 satisfies $\mathcal{G}x^0 - b = 0$ and all the subproblems are solved exactly

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[5] Gabay, D. and Mercier, B.: A dual algorithm for the solution of nonlinear variational problems via finite element approximation. Comput. Math. Appl. **2**(1), 17–40 (1976)

[6] Sun, D.F., Toh, K.-C. and Yang, L.Q.: A convergent proximal alternating direction method of multipliers for conic programming with 4-type constraints. SIAM J. Optim. **25**(2), 882–915 (2015)

Numerical Experiments

Solving dual linear SDP problems via the two-block ADMM with step-length taking values beyond the classic restriction of $(1+\sqrt{5})/2$

- To know to what extent the numerical efficiency of the ADMM can be improved if the equivalence proved in this paper is incorporated
- To see whether a step-length that is very close to 2 will lead to better or worse numerical performance

Solving $\min_X \{\langle C, X\rangle \mid \mathcal{A}X = b, X \in \mathbb{S}^n_+\}$

The dual is

$$\min_{Y,z} \left\{\delta_{\mathbb{S}^n_+}(Y) - \langle b, z\rangle \mid Y + \mathcal{A}^* z = C\right\}$$

Here $\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ is linear, $b \in \mathbb{R}^m$ and $C \in \mathbb{S}^n$ are given data

ADMM has been incorporated in solving dual SDP for more than a decade:

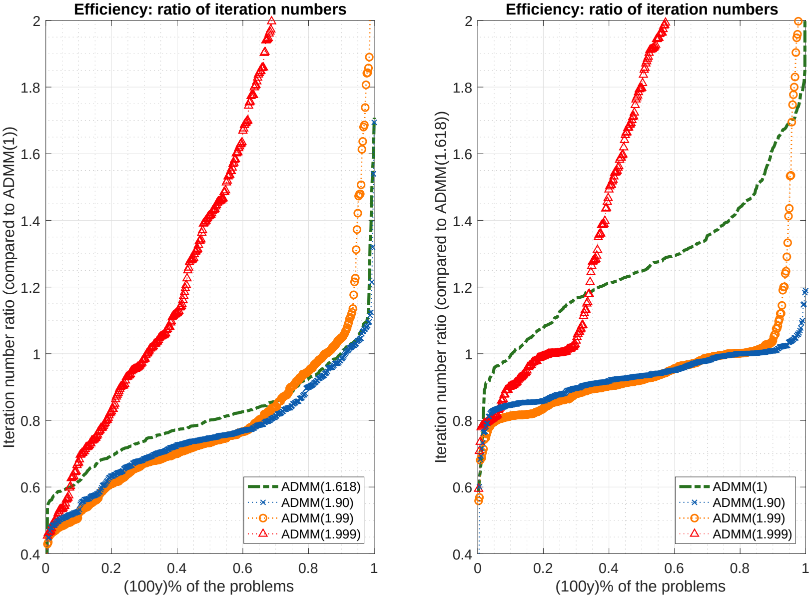
- ADMM with unit step-length was first employed in Povh *et al.* [Comput. 78 (2006)] under the name of boundary point method for solving the dual SDP (Later extended in Malick *et al.* [SIOPT 20 (2009)] with a convergence proof)
- ADMM was used in the software SDPNAL developed by Zhao et al. [SIOPT 20 (2010)] to warm-start a semismooth Newton ALM for dual SDP
- SDPAD by Wen *et al.* [MPC 2 (2010)]: ADMM solver on dual SDP (used SDPNAL template)

Numerical Experiments: details

- Five choices of the step-length, i.e., $\tau = 1$, $\tau = 1.618$, $\tau = 1.90$, $\tau = 1.99$ and $\tau = 1.999$
- Running the Matlab package SDPNAL+ (version 1.0)[7]
- 6 categories of SDP problems
- In general it is a good idea to use a step-length larger than 1, e.g., $\tau = 1.618$
- We can even set the step-length to be larger than 1.618, say $\tau = 1.9$, to get better numerical performance
- Stopping Criteria: DIMACS rule based on relative residuals of primal/dual feasibility and complementarity
- maximum number of iterations: 10^5

[7] awarded the triennial Beale-Orchard–Hays Prize for Excellence in Computational Mathematical Programming by the Mathematical Optimization Society in 2018

Numerical comparisons



Conclusions

- A block sGS decomposition based (exact or inexact) multi-block majorized (proximal or not) ADMM is equivalent to an inexact majorized proximal ALM with $\tau \in (0, 2)$

- ADMM can achieve better numerical performance if the step-length is larger than the conventional upper bound of $(1 + \sqrt{5})/2$ but not too close to 2. It also justifies the safety and effectiveness of choosing $\tau = 1.618$

- The proximal ALM interpretation of the ADMM may explain why it often converges slowly after the initial iterations [the automatically generated proximal term (hidden) is too large]

“Recoupling”?

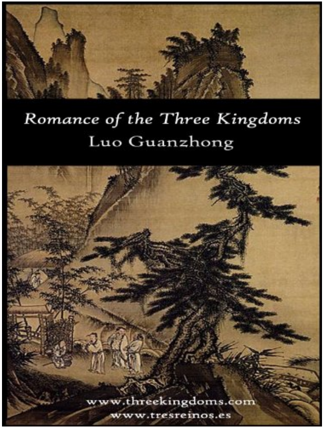
- ALM $\Longrightarrow$ ADMM $\quad\Longleftrightarrow\quad$ “Coupling” $\Longrightarrow$ “Decoupling”

- For big challenging problems, it is time for “Recoupling”?

Any Reason?

天下大事 分久必合 合久必分

罗贯中 《三国演义》



World under heaven, after a long period of division, tends to unite; after a long period of union, tend to divide. This has been so since antiquity.

From **"Romance of the Three Kingdoms"**

a 14th-century historical novel

by **Guanzhong Luo** (Author)

`www.threekingdoms.com` (Editor)

`www.tresreinos.es` (Editor)

C.H. Brewitt Taylor (Translator)