On the Equivalence of Inexact Proximal ALM and ADMM for a Class of Convex Composite Programming

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DIMACS Workshop on ADMM and
Proximal Splitting Methods in Optimization
June 13, 2018

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The multi-block convex composite optimization problem

$$\underbrace{\min_{y\in\mathcal{Y},z\in\mathcal{Z}}}_{w\in\mathcal{W}} \Big\{ \underbrace{p(y_1) + f(y) - \langle b, z\rangle}_{\Phi(w)} \mid \underbrace{\mathcal{F}^* y + \mathcal{G}^* z = c}_{\mathcal{A}^* w = c} \Big\}$$

- $\mathcal{X}$, $\mathcal{Z}$ and $\mathcal{Y}_i$ $(i = 1, \ldots, s)$: finite-dimensional real Hilbert spaces each endowed with $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, $\mathcal{Y} := \mathcal{Y}_1 \times \cdots \times \mathcal{Y}_s$
- $p : \mathcal{Y}_1 \to (-\infty, +\infty]$: a (possibly nonsmooth) closed proper convex function; $f : \mathcal{Y} \to (-\infty, +\infty)$: a continuously differentiable convex function with Lipschitz gradient
- $\mathcal{F}^*$ and $\mathcal{G}^*$: the adjoints of the given linear mappings $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{G} : \mathcal{X} \to \mathcal{Z}$
- $b \in \mathcal{Z}$, $c \in \mathcal{X}$: the given data

Too simple? It covers many important classes of convex optimization problems that are best solved in this (dual) form!

A quintessential example

The convex composite quadratic programming (CCQP)

$$\min_{x} \left\{ \psi(x) + \frac{1}{2}\langle x, \mathcal{Q}x\rangle - \langle c, x\rangle \;\middle|\; \mathcal{A}x = b \right\} \tag{1}$$

- $\psi : \mathcal{X} \to (-\infty, +\infty]$: a closed proper convex function
- $\mathcal{Q} : \mathcal{X} \to \mathcal{X}$: a self-adjoint positive semidefinite linear operator

The dual (minimization form):

$$\min_{y_1, y_2, z} \left\{ \psi^*(y_1) + \frac{1}{2}\langle y_2, \mathcal{Q}y_2\rangle - \langle b, z\rangle \;\middle|\; y_1 + \mathcal{Q}y_2 - \mathcal{A}^* z = c \right\} \tag{2}$$

ψ^* is the conjugate of ψ, $y_1 \in \mathcal{X}$, $y_2 \in \mathcal{X}$, $z \in \mathcal{Z}$

- Many problems are subsumed under the convex composite quadratic programming model (1).
- E.g., the important classes of convex quadratic programming (QP), the convex quadratic semidefinite programming (QSDP)...

Convex QSDP

$$\min_{X \in \mathbb{S}^n} \left\{ \frac{1}{2}\langle X, \mathbf{Q}X\rangle - \langle C, X\rangle \;\middle|\; \mathcal{A}_E X = b_E,\ \mathcal{A}_I X \geq b_I,\ X \in \mathbb{S}^n_+ \right\}$$

$\mathbb{S}^n$ is the space of $n \times n$ real symmetric matrices, $\mathbb{S}^n_+$ is the closed convex cone of positive semidefinite matrices in $\mathbb{S}^n$, $\mathbf{Q} : \mathbb{S}^n \to \mathbb{S}^n$ is a positive semidefinite linear operator, $C \in \mathbb{S}^n$ is the given data, and $\mathcal{A}_E$ and $\mathcal{A}_I$ are linear maps from $\mathbb{S}^n$ to certain finite dimensional Euclidean spaces containing b_E and b_I, respectively

- QSDPNAL[1]: a two-phase augmented Lagrangian method in which the first phase is an inexact block sGS decomposition based multi-block proximal ADMM
- The solution generated in the first phase is used as the initial point to warm-start the second phase algorithm

[1] Li, Sun, Toh: QSDPNAL: A two-phase augmented Lagrangian method for convex quadratic semidefinite programming. MPC online (2018)

Penalized and Constrained Regression Models

The penalized and constrained (PAC) regression often arises in high-dimensional generalized linear models with linear equality and inequality constraints, e.g.,

$$\min_{x \in \mathbb{R}^n} \left\{ p(x) + \frac{1}{2\lambda} \|\Phi x - \eta\|^2 \,\middle|\, A_E x = b_E,\ A_I x \geq b_I \right\} \tag{3}$$

- $\Phi \in \mathbb{R}^{m \times n}$, $A_E \in \mathbb{R}^{r_E \times n}$, $A_I \in \mathbb{R}^{r_I \times n}$, $\eta \in \mathbb{R}^m$, $b_E \in \mathbb{R}^{r_E}$ and $b_I \in \mathbb{R}^{r_I}$ are the given data
- p is a proper closed convex regularizer such as $p(x) = \|x\|_1$
- $\lambda > 0$ is a parameter.
- Obviously, the dual of problem (3) is a particular case of CCQP

The augmented Lagrangian function[2]

$$\min_{y\in\mathcal{Y},z\in\mathcal{Z}} \{p(y_1)+f(y)-\langle b,z\rangle \mid \mathcal{F}^*y+\mathcal{G}^*z=c\} \text{ or } \min_{w\in\mathcal{W}} \{\Phi(w) \mid \mathcal{A}^*w=c\}$$

Let $\sigma > 0$ be the penalty parameter. The augmented Lagrangian function:

$$\mathcal{L}_\sigma(y,z;x) := \underbrace{p(y_1)+f(y)-\langle b,z\rangle}_{\Phi(w)} + \underbrace{\langle x, \mathcal{F}^*y+\mathcal{G}^*z-c\rangle}_{\langle x,\mathcal{A}^*w-c\rangle} + \frac{\sigma}{2}\underbrace{\|\mathcal{F}^*y+\mathcal{G}^*z-c\|^2}_{\|\mathcal{A}^*w-c\|^2},$$

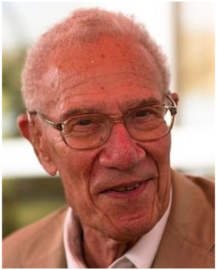
$$\forall w=(y,z)\in\mathcal{W}:=\mathcal{Y}\times\mathcal{Z},\ x\in\mathcal{X}$$

[2] Arrow, K.J., Solow, R.M.: Gradient methods for constrained maxima with weakened assumptions. In: Arrow, K.J., Hurwicz, L., Uzawa, H., (eds.) Studies in Linear and Nonlinear Programming. Stanford University Press, Stanford, pp. 165-176 (1958)

K. Arrow and R. Solow

Kenneth Joseph "Ken" Arrow
(23 August 1921 – 21 February 2017)

John Bates Clark Medal (1957); Nobel Prize in Economics (1972); von Neumann Theory Prize (1986); National Medal of Science (2004); ForMemRS (2006)



Robert Merton Solow
(August 23, 1924 –)

John Bates Clark Medal (1961); Nobel Memorial Prize in Economic Sciences (1987); National Medal of Science (1999); Presidential Medal of Freedom (2014); ForMemRS (2006)

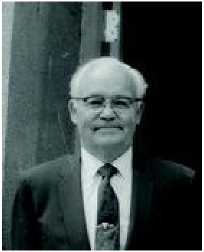
The augmented Lagrangian method[3] (ALM)

$\mathcal{L}_\sigma(y, z; x) = p(y_1) + f(y) - \langle b, z\rangle + \langle x, \mathcal{F}^* y + \mathcal{G}^* z - c\rangle + \frac{\sigma}{2}\|\mathcal{F}^* y + \mathcal{G}^* z - c\|^2$

Starting from $x^0 \in \mathcal{X}$, performs for $k = 0, 1, \ldots$

(1) $\underbrace{(y^{k+1}, z^{k+1})}_{w^{k+1}} \Leftarrow \min_{y,z} \mathcal{L}_\sigma(\underbrace{y, z}_{w}; x^k)$ (approximately)

(2) $x^{k+1} := x^k + \tau\sigma(\mathcal{F}^* y^{k+1} + \mathcal{G}^* z^{k+1} - c)$ with $\tau \in (0, 2)$



Magnus Rudolph Hestenes

(February 13 1906 – May 31 1991)

Michael James David Powell

(29 July 1936 – 19 April 2015)

[3] Also known as the method of multipliers

ALM and variants

- ALM has the desirable asymptotically superlinear convergence (or linearly convergent of an arbitrary order) property.
- While one would really want to $\min_{y,z} \mathcal{L}_\sigma(y, z; x^k)$ without modifying the augmented Lagrangian, it can be expensive due to the coupled quadratic term in y and z.
- In practice, unless the ALM subproblems can be solved efficiently, one would generally want to replace the augmented Lagrangian subproblem with an easier-to-solve surrogate by modifying the augmented Lagrangian function to decouple the minimization with respect to y and z.
- Such a modification is especially desirable during the initial phase of the ALM when the local superlinear convergence phase of ALM has yet to kick in.

ALM to proximal ALM[4] (PALM)

Minimize the augmented Lagrangian function plus a quadratic **proximal term**:

$$w^{k+1} \approx \arg\min_{w} \mathcal{L}_\sigma(w; x^k) + \frac{1}{2}\|w - w^k\|_{\mathcal{D}}^2$$

- $\mathcal{D} = \sigma^{-1}\mathcal{I}$ in the seminal work of Rockafellar (in which inequality constraints are considered). Note that $\mathcal{D} \to 0$ as $\sigma \to \infty$, which is critical for superlinear convergence.
- It is a primal-dual type proximal point algorithm (PPA).

[4] Also known as the proximal method of multipliers

Modification and decomposition

The obvious modification with $\mathcal{D} = \sigma(\lambda^2\mathcal{I} - \mathcal{A}\mathcal{A}^*)$ is generally too drastic and has the undesirable effect of significantly slowing down the convergence of the proximal ALM.

- $\mathcal{D}$ could be positive semidefinite (a kind of PPAs), i.e., the obvious approach:

$$\mathcal{D} = \sigma(\lambda^2\mathcal{I} - \mathcal{A}\mathcal{A}^*) = \sigma(\lambda^2\mathcal{I} - (\mathcal{F};\mathcal{G})(\mathcal{F};\mathcal{G})^*)$$

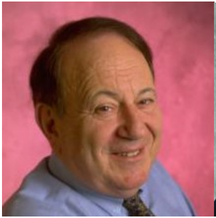
  with λ being the largest singular value of $(\mathcal{F};\mathcal{G})$
- $\mathcal{D}$ can be indefinite (typically used together with the majorization technique)
- **What is an appropriate proximal term** to add so that
  - The PALM subproblem is easier to solve
  - Less drastic than the obvious choice

Decomposition based ADMM

One the other hand, decomposition based approach is available, i.e,

$$y^{k+1} \approx \arg\min_{y}\{\mathcal{L}_\sigma(y, z^k; x^k)\}, \;\; z^{k+1} \approx \arg\min_{z}\{\mathcal{L}_\sigma(y^{k+1}, z; x^k)\}$$

- The **two-block ADMM**
- Allows $\tau \in (0, (1+\sqrt{5})/2)$ if the convergence of the full (primal & dual) sequence is required (Glowinski)
- The case with $\tau = 1$ is a kind of PPA (Gabay + Bertsekas-Eckstein)
- Many variants (proximal/inexact/generalized/parallel etc.)



A part of the result

An equivalent property:

Add an **appropriately designed proximal term** to $\mathcal{L}_\sigma(y, z; x^k)$, we reduce the computation of the modified ALM subproblem to sequentially updating y and z without adding a proximal term, which is **exactly the same** as the two-block ADMM

- A **difference**: one can prove convergence for the step-length τ in the **range** $(0, 2)$ whereas the classic two-block ADMM only admits $(0, (1 + \sqrt{5})/2)$.

For multi-block problems

Turn back to the **multi-block** problem, the subproblem to y can still be difficult due to the coupling of $y_1, \ldots, y_s$

- A successful multi-block ADMM-type algorithm must not only possess **convergence** guarantee but also should numerically **perform** at least as fast as the directly extended ADMM (the Gauss-Seidel iterative fashion) when it does converge.

Algorithmic design

- Majorize the function $f(y)$ at y^k with a quadratic function
- Add an extra proximal term that is derived based on the symmetric Gauss-Seidel (sGS) decomposition theorem to update the sub-blocks in y individually and successively in an sGS fashion
- **The resulting algorithm:**
  A block sGS decomposition based (inexact) majorized multi-block indefinite proximal ADMM with $\tau \in (0, 2)$, which is **equivalent** to an *inexact* majorized proximal ALM

An inexact majorized indefinite proximal ALM

Consider

$$\min_{w \in \mathcal{W}} \Phi(w) := \varphi(w) + h(w) \quad \text{s.t.} \quad \mathcal{A}^* w = c,$$

- The Karush-Kuhn-Tucker (KKT) system:

$$0 \in \partial\varphi(w) + \nabla h(w) + \mathcal{A}x, \qquad \mathcal{A}^* w - c = 0$$

- The gradient of h is Lipschitz continuous, which implies a self-adjoint positive semidefinite linear operator $\widehat{\Sigma}_h : \mathcal{W} \to \mathcal{W}$, such that for any $w, w' \in \mathcal{W}$,

$$h(w) \leq \hat{h}(w, w') := h(w') + \langle \nabla h(w'), w - w' \rangle + \frac{1}{2}\|w - w'\|^2_{\widehat{\Sigma}_h},$$

which is called a majorization of h at w'.

Let $\sigma > 0$. The majorized augmented Lagrangian function is defined, for any $(w, x, w') \in \mathcal{W} \times \mathcal{X} \times \mathcal{W}$, by

$$\widehat{\mathcal{L}}_\sigma(w; (x, w')) := \varphi(w) + \hat{h}(w, w') + \langle \mathcal{A}^* w - c, x \rangle + \frac{\sigma}{2} \|\mathcal{A}^* w - c\|^2.$$

Assumption

The solution set to the KKT system is nonempty and $\mathcal{D} : \mathcal{W} \to \mathcal{W}$ is a given self-adjoint (not necessarily positive semidefinite) linear operator such that

$$\mathcal{D} \succeq -\frac{1}{2}\widehat{\Sigma}_h \quad \textit{and} \quad \frac{1}{2}\widehat{\Sigma}_h + \sigma \mathcal{A}\mathcal{A}^* + \mathcal{D} \succ 0. \tag{4}$$

- **$\mathcal{D}$ is not necessarily to be positive semidefinite!**

Algorithm: an inexact majorized indefinite proximal ALM

Let $\{\varepsilon_k\}$ be a summable sequence of nonnegative numbers. Choose an initial point $(x^0, w^0) \in \mathcal{X} \times \mathcal{W}$. For $k = 0, 1, \ldots,$

1 Compute

$$w^{k+1} \approx \arg\min_{w \in \mathcal{W}} \left\{ \widehat{\mathcal{L}}_\sigma(w; (x^k, w^k)) + \frac{1}{2}\|w - w^k\|_{\mathcal{D}}^2 \right\}$$

such that there exists d_k satisfying $\|d^k\| \leq \varepsilon_k$ and

$$d^k \in \partial_w \widehat{\mathcal{L}}_\sigma(w^{k+1}; (x^k, w^k)) + \mathcal{D}(w^{k+1} - w^k)$$

2 Update $x^{k+1} := x^k + \tau\sigma(\mathcal{A}^* w^{k+1} - c)$ with $\tau \in (0, 2)$

Theorem

The sequence $\{(x^k, w^k)\}$ generated by the above Algorithm converges to a solution to the KKT system.

Multi-block: Majorization and decomposition

The gradient of f is Lipschitz continuous $\Rightarrow$ there exists a self-adjoint linear operator $\widehat{\Sigma}^f : \mathcal{Y} \to \mathcal{Y}$ such that $\widehat{\Sigma}^f \succeq 0$ and for any $y, y' \in \mathcal{Y}$,

$$f(y) \leq \widehat{f}(y, y') := f(y') + \langle \nabla f(y'), y - y' \rangle + \frac{1}{2}\|y - y'\|^2_{\widehat{\Sigma}^f}$$

- Denote for any $y \in \mathcal{Y}$,

$$y_{<i} := (y_1; \ldots; y_{i-1}) \quad \text{and} \quad y_{>i} := (y_{i+1}; \ldots; y_s)$$

- Decompose $\widehat{\Sigma}^f$ as

$$\widehat{\Sigma}^f = \begin{pmatrix} \widehat{\Sigma}^f_{11} & \widehat{\Sigma}^f_{12} & \cdots & \widehat{\Sigma}^f_{1s} \\ (\widehat{\Sigma}^f_{12})^* & \widehat{\Sigma}^f_{22} & \cdots & \widehat{\Sigma}^f_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ (\widehat{\Sigma}^f_{1s})^* & (\widehat{\Sigma}^f_{2s})^* & \cdots & \widehat{\Sigma}^f_{ss} \end{pmatrix}$$

with $\widehat{\Sigma}^f_{ij} : \mathcal{Y}_j \to \mathcal{Y}_i, \ \forall 1 \leq i \leq j \leq s$

Basic assumptions / Majorized augmented Lagrangian

(a) The self-adjoint linear operators $\mathcal{S}_i : \mathcal{Y}_i \to \mathcal{Y}_i, i = 1, \ldots, s$, are chosen such that

$$\frac{1}{2}\widehat{\Sigma}^f_{ii} + \sigma \mathcal{F}_i \mathcal{F}_i^* + \mathcal{S}_i \succ 0 \text{ and } \mathcal{S} := \mathrm{Diag}(\mathcal{S}_1, \ldots, \mathcal{S}_s) \succeq -\frac{1}{2}\widehat{\Sigma}^f$$

(b) The linear operator $\mathcal{G}$ is surjective;

(c) A nonempty solution set to the KKT system:

$$0 \in \begin{pmatrix} \partial p(y_1) \\ 0 \end{pmatrix} + \nabla f(y) + \mathcal{F}x,\ \mathcal{G}x - b = 0,\ \mathcal{F}^* y + \mathcal{G}^* z = c$$

(d) $\{\tilde{\varepsilon}_k\}$ is a summable sequence of nonnegative real numbers

Let $\sigma > 0$. The *majorized* augmented Lagrangian function:

$$\begin{aligned} \widehat{\mathcal{L}}_\sigma(y, z; (x, y')) := \ & p(y_1) + \widehat{f}(y, y') - \langle b, z \rangle \\ & + \langle \mathcal{F}^* y + \mathcal{G}^* z - c, x \rangle + \frac{\sigma}{2} \|\mathcal{F}^* y + \mathcal{G}^* z - c\|^2 \end{aligned}$$

The algorithm sGS-imPADMM

An inexact block sGS based indefinite Proximal ADMM

$(x^0, y^0, z^0) \in \mathcal{X} \times \operatorname{dom} p \times \mathcal{Y}_2 \times \cdots \times \mathcal{Y}_s \times \mathcal{Z}$. For $k = 0, 1, \ldots,$

1 Compute for $\boxed{i = s, \ldots, 2}$

$$y_i^{k+\frac{1}{2}} \approx \arg\min_{y_i \in \mathcal{Y}_i} \Big\{ \widehat{\mathcal{L}}_\sigma \big(y_{\leq i-1}^k, y_i, y_{\geq i+1}^{k+\frac{1}{2}}, z^k; (x^k, y^k) \big) + \frac{1}{2} \| y_i - y_i^k \|_{\mathcal{S}_i}^2 \Big\}$$

2 Compute for $\boxed{i = 1, \ldots, s}$

$$y_i^{k+1} \approx \arg\min_{y_i \in \mathcal{Y}_i} \Big\{ \widehat{\mathcal{L}}_\sigma \big(y_{\leq i-1}^{k+1}, y_i, y_{\geq i+1}^{k+1/2}, z^k; (x^k, y^k) \big) + \frac{1}{2} \| y_i - y_i^k \|_{\mathcal{S}_i}^2 \Big\}$$

3 Compute

$$z^{k+1} \approx \arg\min_{z \in \mathcal{Z}} \big\{ \widehat{\mathcal{L}}_\sigma (y^{k+1}, z; (x^k, y^k)) \big\}$$

4 Compute $x^{k+1} := x^k + \tau\sigma(\mathcal{F}^* y^{k+1} + \mathcal{G}^* z^{k+1} - c)$, $\boxed{\tau \in (0, 2)}$

Criteria for inexact solutions in sGS-imPADMM

1. For $i = s, \ldots, 2$, the approximate solution $y_i^{k+\frac{1}{2}}$ is chosen such that there exists $\tilde{\delta}_i^k$ satisfying $\|\tilde{\delta}_i^k\| \leq \tilde{\varepsilon}_k$ and

$$\tilde{\delta}_i^k \in \partial_{y_i} \widehat{\mathcal{L}}_\sigma\big(y_{\leq i-1}^k, y_i^{k+\frac{1}{2}}, y_{\geq i+1}^{k+\frac{1}{2}}, z^k; (x^k, y^k)\big) + \mathcal{S}_i(y_i^{k+\frac{1}{2}} - y_i^k)$$

2. For $i = 1, \ldots, s$, the approximate solution y_i^{k+1} is chosen such that there exists δ_i^k satisfying $\|\delta_i^k\| \leq \tilde{\varepsilon}_k$ and

$$\delta_i^k \in \partial_{y_i} \widehat{\mathcal{L}}_\sigma\big(y_{\leq i-1}^{k+1}, y_i^{k+1}, y_{\geq i+1}^{k+1/2}, z^k; (x^k, y^k)\big) + \mathcal{S}_i(y_i^{k+1} - y_i^k)$$

3. The approximate solution z^{k+1} is chosen such that $\|\gamma^k\| \leq \tilde{\varepsilon}_k$ with

$$\begin{aligned} \gamma^k : \quad &= \nabla_z \widehat{\mathcal{L}}_\sigma\big(y^{k+1}, z^{k+1}; (x^k, y^k)\big) \\ &= \mathcal{G}x^k - b + \sigma\mathcal{G}(\mathcal{F}^* y^{k+1} + \mathcal{G}^* z^{k+1} - c) \end{aligned}$$

Comments on the sGS-imPADMM algorithm

- The sGS-imPADMM is a versatile framework, one can implement it in different routines
- We are more interested in the previous iteration scheme:
  - The theoretical improvement
  - The practical merit it features for solving large scale problems (especially when the dominating computational cost is in performing the evaluations associated with the linear mappings $\mathcal{G}$ and $\mathcal{G}^*$)

A particular case in point is the following problem:

$$\min_{x \in \mathcal{X}} \left\{ \psi(x) + \frac{1}{2}\langle x, \mathcal{Q}x \rangle - \langle c, x \rangle \;\middle|\; \mathcal{A}_1 x = b_1,\ \mathcal{A}_2 x \geq b_2 \right\},$$

$\mathcal{Q}$, ψ, and c are as the previous; $\mathcal{A}_1 : \mathcal{X} \to \mathcal{Z}_1$ and $\mathcal{A}_2 : \mathcal{X} \to \mathcal{Z}_2$ are the given linear mappings, and $b = (b_1; b_2) \in \mathcal{Z} := \mathcal{Z}_1 \times \mathcal{Z}_2$ is a given vector.

Details

By introducing a slack variable $x' \in \mathcal{Z}_2$, one gets

$$\min_{x\in\mathcal{X},x'\in\mathcal{Z}_2} \left\{ \psi(x) + \frac{1}{2}\langle x, \mathcal{Q}x\rangle - \langle c, x\rangle \;\middle|\; \begin{pmatrix} \mathcal{A}_1 & 0 \\ \mathcal{A}_2 & \mathcal{I} \end{pmatrix} \begin{pmatrix} x \\ x' \end{pmatrix} = b,\ x' \leq 0 \right\},$$

The corresponding dual problem in the minimization form:

$$\min_{y,y',z} \left\{ p(y) + \frac{1}{2}\langle y', \mathcal{Q}y'\rangle - \langle b, z\rangle \;\middle|\; y + \begin{pmatrix} \mathcal{Q} \\ 0 \end{pmatrix} y' - \begin{pmatrix} \mathcal{A}_1^* & \mathcal{A}_2^* \\ 0 & \mathcal{I} \end{pmatrix} z = \begin{pmatrix} c \\ 0 \end{pmatrix} \right\}$$

with $y := (u, v) \in \mathcal{X} \times \mathcal{Z}_2$, $p(y) = p(u, v) = \psi_1^*(u) + \delta_+(v)$, and δ_+ is the indicator function of the nonnegative orthant in $\mathcal{Z}_2$.

- It is clear that with a large number of inequality constraints, the dimension of z can be much larger than that of y'.
- For such a scenario, the adopted iteration scheme is more preferable since the more difficult subproblem involving z is solved only once in each iteration.

inexact block sGS decomposition

Define $\mathcal{H} := \widehat{\Sigma}^f + \sigma \mathcal{F}\mathcal{F}^* + \mathcal{S} = \mathcal{H}_d + \mathcal{H}_u + \mathcal{H}_u^*$ with
$\mathcal{H}_d := \mathrm{Diag}(\mathcal{H}_{11}, \ldots, \mathcal{H}_{ss})$, $\mathcal{H}_{ii} := \widehat{\Sigma}^f_{ii} + \sigma \mathcal{F}_i \mathcal{F}_i^* + \mathcal{S}_i$ and

$$\mathcal{H}_u := \begin{pmatrix} 0 & \mathcal{H}_{12} & \cdots & \mathcal{H}_{1s} \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \mathcal{H}_{(s-1)s} \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \quad \mathcal{H}_{ij} = \widehat{\Sigma}^f_{ij} + \sigma \mathcal{F}_i \mathcal{F}_j^*$$

For convenience, we denote for each $k \geq 0$, $\tilde{\delta}^1_k := \delta^1_k$, $\tilde{\delta}^k := (\tilde{\delta}^k_1, \tilde{\delta}^2_k \ldots, \tilde{\delta}^k_s)$ and $\delta^k := (\delta^k_1, \ldots, \delta^k_s)$
Define the sequence $\{\Delta^k\} \in \mathcal{Y}$ by

$$\Delta^k := \delta^k + \mathcal{H}_u \mathcal{H}_d^{-1}(\delta^k - \tilde{\delta}^k)$$

Moreover, we can define the linear operator

$$\widehat{\mathcal{H}} := \mathcal{H}_u \mathcal{H}_d^{-1} \mathcal{H}_u^*$$

Result by the block sGS decomposition theorem [5]

The iterate y^{k+1} in Step 2 of sGS-imPADMM is the unique solution to a proximal minimization problem given by

$$y^{k+1} = \arg\min_{y} \Big\{ \underbrace{\widehat{\mathcal{L}}_\sigma(y, z^k; (x^k, y^k)) + \frac{1}{2}\|y - y^k\|^2_{\mathcal{S}+\widehat{\mathcal{H}}}}_{\text{strongly convex}} - \langle \Delta^k, y \rangle \Big\}.$$

Moreover, it holds that

$$\mathcal{H} + \widehat{\mathcal{H}} = (\mathcal{H}_d + \mathcal{H}_u)\mathcal{H}_d^{-1}(\mathcal{H}_d + \mathcal{H}_u^*) \succ 0.$$

- Recall that $\mathcal{H} := \widehat{\Sigma}^f + \sigma\mathcal{F}\mathcal{F}^* + \mathcal{S}$
- Linearly transported error: $\Delta^k = \delta^k + \mathcal{H}_u\mathcal{H}_d^{-1}(\delta^k - \tilde{\delta}^k)$

[5] X.D. Li, D.F. Sun, and K.-C Toh, A block symmetric Gauss-Seidel decomposition theorem for convex composite quadratic programming and its applications, MP online [DOI: 10.1007/s10107-018-1247-7]

The equivalence property

Recall that $\mathcal{W} = \mathcal{Y} \times \mathcal{Z}$. Define $\widehat{\Sigma}_h : \mathcal{W} \to \mathcal{W}$ by

$$\widehat{\Sigma}_h := \begin{pmatrix} \widehat{\Sigma}^f & \\ & 0 \end{pmatrix}$$

For $w = (y; z)$ and $w' = (y'; z')$, denote

$$\widehat{\mathcal{L}}_\sigma(w; (x, w')) := \widehat{\mathcal{L}}_\sigma(y, z; (x, y'))$$

Define the error term

$$\widehat{\Delta}^k := \Delta^k - \mathcal{F}\mathcal{G}^*(\mathcal{G}\mathcal{G}^*)^{-1}(\gamma^{k-1} - \gamma^k - \mathcal{G}(x^{k-1} - x^k)) \in \mathcal{Y}$$

with the convention that

$$\begin{cases} x^{-1} := x^0 - \tau\sigma(\mathcal{F}^* y^0 + \mathcal{G}^* z^0 - c), \\ \gamma^{-1} = -b + \mathcal{G}x^{-1} + \sigma\mathcal{G}(\mathcal{F}^* y^0 + \mathcal{G}^* z^0 - c) \end{cases}$$

The equivalence property

Define the block-diagonal linear operator

$$\mathcal{T} := \begin{pmatrix} \mathcal{S} + \widehat{\mathcal{H}} + \sigma \mathcal{F}\mathcal{G}^*(\mathcal{G}\mathcal{G}^*)^{-1}\mathcal{G}\mathcal{F}^* & \\ & 0 \end{pmatrix} \quad \boxed{\mathcal{W} \to \mathcal{W}}$$

Theorem

Let $\{(x^k, w^k)\}$ with $w^k := (y^k; z^k)$ be the sequence generated by sGS-imPADMM. Then, for any $k \geq 0$, it holds that

(i) *the linear operators $\mathcal{T}$, $\mathcal{A}$ and $\widehat{\Sigma}_h$ satisfy*
$\mathcal{T} \succeq -\frac{1}{2}\widehat{\Sigma}_h$ *and* $\frac{1}{2}\widehat{\Sigma}_h + \sigma\mathcal{A}\mathcal{A}^* + \mathcal{T} \succ 0$;

(ii)

$$w^{k+1} \approx \underset{w \in \mathcal{W}}{\arg\min} \left\{ \widehat{\mathcal{L}}_\sigma\big(w; (x^k, w^k)\big) + \frac{1}{2}\|w - w^k\|^2_{\mathcal{T}} \right\}$$

in the sense that $(\widehat{\Delta}^k; \gamma^k) \in \partial_w \widehat{\mathcal{L}}_\sigma((w^{k+1}; (x^k, w^k)) + \mathcal{T}(w^{k+1} - w^k)$ *and* $\|(\widehat{\Delta}^k, \gamma^k)\| \leq \widehat{\varepsilon}_k$ *with* $\{\widehat{\varepsilon}_k\}$ *being a summable sequence of nonnegative numbers.*

sGS-imPADMM convergence

One can readily get the following convergence theorem

Theorem

The sequence $\{(x^k, y^k, z^k)\}$ *generated by the Algorithm converges to a solution to the KKT system of the problem. Thus,* $\{(y^k, z^k)\}$ *converges to a solution to this problem and* $\{x^k\}$ *converges to a solution of its dual.*

Two-block case

Let $\mathcal{Y} = \mathcal{Y}_1$ and f be vacuous, i.e.,

$$\min\{p(y) - \langle b, z\rangle \,|\, \mathcal{F}^* y + \mathcal{G}^* z = c\} \tag{5}$$

- sGS-imPADMM without proximal terms is reduced to a two-block ADMM
- Assume that $\mathcal{G}$ is surjective and that the KKT system of this problem admits a nonempty solution set K
- This two-block ADMM or its inexact variants with $\tau \in (0, 2)$ (in the order that the y-subproblem is solved before the z-subproblem) converges to K if either $\mathcal{F}$ is surjective or p is strongly convex

Comments on the two-block case

- The assumptions we made for problem (5) are apparently weaker than those in original work of Gabay and Mercier[6], where $\mathcal{F}$ is assumed to be the identity operator and p is assumed to be strongly convex
- In Gabay and Mercier (1976), Theorem 3.1, only the convergence of the primal sequence $\{(y^k, z^k)\}$ is obtained while the dual sequence $\{x^k\}$ is only proven to be bounded
- In Sun *et al.*[7], a similar result to ours has been derived with the requirements that the initial multiplier x^0 satisfies $\mathcal{G}x^0 - b = 0$ and all the subproblems are solved exactly

[6] Gabay, D. and Mercier, B.: A dual algorithm for the solution of nonlinear variational problems via finite element approximation. Comput. Math. Appl. **2**(1), 17–40 (1976)

[7] Sun, D.F., Toh, K.-C. and Yang, L.Q.: A convergent proximal alternating direction method of multipliers for conic programming with 4-block constraints. SIAM J. Optim. **25**(2), 882–915 (2015)

Numerical Experiments

Solving dual linear SDP problems via the two-block ADMM with step-length taking values beyond the standard restriction of $(1+\sqrt{5})/2$. The aim is two-fold.

- As ADMM is among the useful first-order algorithms for solving SDP problems, it is of importance to know to what extent can the numerical efficiency be improved if the equivalence proved in this paper is incorporated.
- As the upper bound of the step-length has been enlarged, it is also important to see whether a step-length that is very close to the upper bound will lead to better or worse numerical performance.

Solving $\min_X \{\langle C, X\rangle \mid \mathcal{A}X = b, X \in \mathbb{S}^n_+\}$,

The dual of the above linear SDP is given by

$$\min_{Y,z} \left\{\delta_{\mathbb{S}^n_+}(Y) - \langle b, z\rangle \mid Y + \mathcal{A}^* z = C\right\},$$

$\mathcal{A} : \mathbb{S}^n \to \mathbb{R}^m$ is linear map, $b \in \mathbb{R}^m$ and $C \in \mathbb{S}^n$ are given data.

ADMM has been incorporated in solving dual SDP for a few years

- ADMM with unit step-length was first employed in Povh *et al.* [Comput. 78 (2006)] under the name of boundary point method for solving the dual SDP (Later extended in Malick *et al.* [SIOPT 20 (2009)] with a convergence proof)
- ADMM was used in the software SDPNAL developed by Zhao et al. [SIOPT 20 (2010)] to warm-start a semismooth Newton ALM for dual SDP
- SDPAD by Wen *et al.*[MPC 2 (2010)]: ADMM solver on dual SDP (used SDPNAL template)

ADMM for dual SDP

Let $\sigma > 0$. The augmented Lagrangian function:

$$\mathcal{L}_\sigma(S, z; X) \;=\; \delta_{\mathbb{S}^n_+}(S) - \langle b, z\rangle + \langle X, S + \mathcal{A}^* z - C\rangle + \frac{\sigma}{2}\|S + \mathcal{A}^* z - C\|^2$$

At the k-th step of the two-block ADMM:

$$\begin{cases} S^{k+1} = \Pi_{\mathbb{S}^n_+}(C - \mathcal{A}^* z^k - X^k/\sigma), \\ z^{k+1} = (\mathcal{A}\mathcal{A}^*)^{-1}(\mathcal{A}(C - S^{k+1}) - (\mathcal{A}X^k - b)/\sigma), \\ X^{k+1} = X^k + \tau\sigma(S^{k+1} + \mathcal{A}^* z^{k+1} - C), \end{cases}$$

where $\tau \in (0, 2)$. We emphasize again that this is in contrast to the usual interval of $(0, (1+\sqrt{5})/2)$.

Stopping Criteria: DIMACS[8] rule

Based on relative residuals of priam/dual feasibility and complementarity

We terminate all the tested algorithms if

$$\eta_{\mathsf{SDP}} := \max\{\eta_D, \eta_P, \eta_S\} \leq 10^{-6},$$

where

$$\eta_D = \frac{\|\mathcal{A}^* z + S - C\|}{1+\|C\|}, \eta_P = \frac{\|\mathcal{A}X - b\|}{1+\|b\|}, \eta_S = \max\left\{\frac{\|X - \Pi_{\mathbb{S}^n_+}(X)\|}{1+\|X\|}, \frac{|\langle X, S\rangle|}{1+\|X\|+\|S\|}\right\}$$

with the maximum number of iterations set at 10^6

In addition, we also measure the duality gap:

$$\eta_{\text{gap}} := \frac{\langle C, X\rangle - \langle b, z\rangle}{1 + |\langle C, X\rangle| + |\langle b, z\rangle|}$$

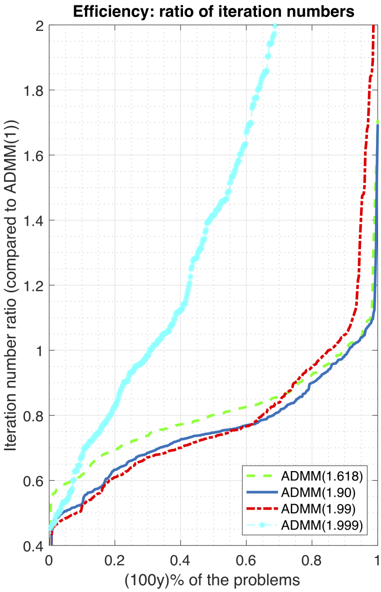
[8] http://dimacs.rutgers.edu/archive/Challenges/Seventh/Instances/error_report.html

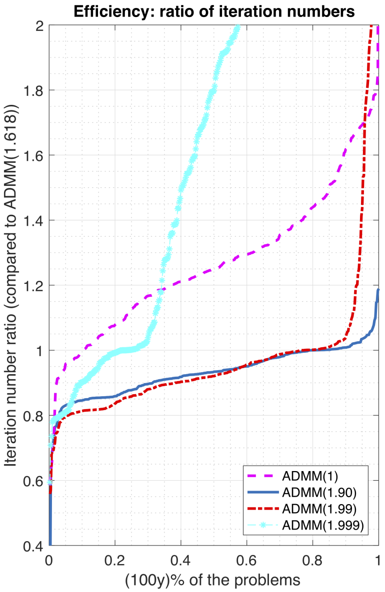
Numerical Experiment: details

- Only consider the cases where $\tau \geq 1$
- We tested five choices of the step-length, i.e., $\tau = 1$, $\tau = 1.618$, $\tau = 1.90$, $\tau = 1.99$ and $\tau = 1.999$
- All these algorithms are tested by running the Matlab package SDPNAL+ (version 1.0)[9]
- We test 6 categories of SDP problems
- In general it is a good idea to use a step-length that is larger than 1, e.g., $\tau = 1.618$, when solving linear SDP problems
- We can even set the step-length to be larger than 1.618, say $\tau = 1.9$, to get better numerical performance

[9] `http://www.math.nus.edu.sg/~mattohkc/SDPNALplus.html`

Numerical result





Conclusions

- For a class of convex composite programming problems, a block sGS decomposition based (inexact) multi-block majorized (proximal) ADMM is equivalent to an inexact proximal ALM.
- An inexact majorized indefinite proximal ALM framework.
- Provide a very general answer to the question on whether the whole sequence generated by the two-block classic ADMM with $\tau \in (0, 2)$, with one linear part, is convergent.
- One can achieve even better numerical performance of the ADMM if the step-length is chosen to be larger than the conventional upper bound of $(1 + \sqrt{5})/2$.
- More insightful theoretical studies on the ADMM-type algorithms are needed for achieving better numerical performance.
- The proximal ALM (with a large proximal term) interpretation of the ADMM may explain why it often converges slow after some iterations.