On the p-order Semismoothness of the Metric Projection onto Slices of the Positive Semidefinite Cone

Ruoning Chen · Jiaming Ma · Defeng Sun

Abstract The metric projection onto the positive semidefinite (PSD) cone is strongly semismooth, a property that guarantees local quadratic convergence for many powerful algorithms in semidefinite programming. In this paper, we investigate whether this essential property holds for the metric projection onto an affine slice of the PSD cone, which is the operator implicitly used by many algorithms that handle linear constraints directly. Although this property is known to be preserved for the second-order cone, we conclusively demonstrate that this is not the case for the PSD cone. Specifically, we provide a constructive example that for any $p > 0$, there exists an affine slice of a PSD cone for which the metric projection operator fails to be p-order semismooth. This finding establishes a fundamental difference between the geometry of the second-order cone and the PSD cone and necessitates new approaches for both analysis and algorithm design for linear semidefinite programming problems.

Keywords Semismoothness · LMI-representable set · Slice of PSD cone

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1 Introduction

Semidefinite programming (SDP) has emerged as a powerful framework for modeling a wide range of problems across engineering, control theory, and combinatorial optimization [17]. A central challenge in the field is the development of robust and efficient numerical methods capable of solving these problems with high precision. For this purpose, Newton-type methods are particularly desirable because of their characteristically rapid local convergence. However, key operators in SDP, such as the metric projection onto the feasible set [20], are nonsmooth, which prevents the direct application of classical Newton-based approaches. This challenge has spurred the development of a broader analytical framework based on generalized derivatives, giving rise to the powerful class of nonsmooth Newton methods [7,10].

The development of semismooth Newton methods, in particular, has demonstrated that establishing the semismoothness of operators involved can lead to proofs of local superlinear or even quadratic convergence [10]. For a given positive integer n, let S_+^n be the cone of n by n real symmetric and positive semidefinite (PSD) matrices. A PSD cone refers to S_+^n for some integer $n \geq 1$. In the context of SDP, a foundational result is that the metric projection operator onto S_+^n, denoted by $\Pi_{S_+^n}$, is strongly semismooth [15]. This favorable property has become the basis for many solvers such as the Newton-CG augmented Lagrangian methods based solvers for solving large-scale SDPs [19, 21].

However, the feasible set of a standard SDP problem is not the entire PSD cone, but rather a slice of it, i.e., the intersection of the PSD cone with an affine subspace L. The practical performance of many algorithms therefore depends not on $\Pi_{S_+^n}$ but on the metric projection operator onto this constrained set $\Pi_{S_+^n \cap L}$. This naturally leads to a central and consequential question: Does the strong semismoothness of the metric projection onto the PSD cone carry over to the metric projection onto its affine slices?

An affirmative answer to this question would have significant implications, suggesting that a broad class of algorithms enjoys local quadratic convergence when applied directly to linear SDPs, without moving the affine constraint $A(x) = b$ from the feasible set into a penalty term in the objective function. Support for such a conjecture can be drawn from the analogous situation for the second-order cone (SOC). It is a well-established result that the metric projection onto the SOC is strongly semismooth [2]. Crucially, this property has been shown to be preserved for its affine slices; that is, the metric projection onto the intersection of an SOC and an affine subspace also remains strongly semismooth [18]. Therefore, it is natural to conjecture that an analogous result may hold in the SDP setting.

However, the geometric structure of the PSD cone is known to be substantially more complex than that of the SOC. The facial structure is more intricate, and the boundary exhibits a higher degree of algebraic complexity, which often complicates the extension of results from the SOC setting [9]. On the other hand, the metric projection onto a slice of PSD cone can be written

in a constraint convex quadratic SDP problem. When the constraint nondegeneracy holds at the desired point y, the KKT solution mapping $(x(\cdot), \lambda(\cdot))$ is strongly semismooth at y [14,13,8] and thus, the metric projection part $x(\cdot)$ is strongly semismooth at y. Apart from this ideal nondegenerate case, what happens outside is far less understood.

In this paper, we provide a definitive but negative answer to this question. We demonstrate that, unlike the second-order cone case, the metric projection onto an affine slice of the PSD cone can suffer a dramatic loss of strong semismoothness. Specifically, for any $p > 0$, we construct an explicit example where the metric projection operator $\Pi_{S_+^n \cap L}$ fails to be p-order semismooth. This finding reveals a fundamental geometric disparity between the second-order cone and the positive semidefinite cone and implies that the analytical foundation for rapid superlinear convergence (when p near 1) of standard semismooth Newton methods does not hold universally for linear SDPs. Our result therefore clarifies a key theoretical boundary in conic optimization and underscores the need for more nuanced approaches to both the analysis and design of algorithms for this subclass of semidefinite programs.

2 Notation and preliminaries

Let $\mathcal{E}$ and $\mathcal{F}$ be two finite-dimensional real Hilbert spaces each endowed with an inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. For a vector $z \in \mathcal{E}$ (or a subspace $\mathcal{E}_0 \subseteq \mathcal{E}$), $z^\perp$ (or $\mathcal{E}_0^\perp$) denotes its orthogonal complement in $\mathcal{E}$. Given a cone $C \subseteq \mathcal{E}$, $C^\circ := \{v \in \mathcal{E} \mid \langle v, z \rangle \leq 0 \ \forall z \in C\}$ is the polar cone of C. For a linear operator $\mathcal{A} : \mathcal{E} \to \mathcal{F}$, we use $\mathcal{A}^*, \mathsf{rge}(\mathcal{A})$ and $\mathsf{ker}(\mathcal{A})$ to denote its adjoint, range space, and null space, respectively. Note that $\mathsf{rge}(\mathcal{A}) = (\mathsf{ker}(\mathcal{A}^*))^\perp$. Given a set $C \subseteq \mathcal{E}$, we use $\mathsf{bdry}(C)$ to denote the strong topological boundary of C.

Given a matrix $A \in \mathbb{R}^{l \times q}$, we use A_{ik} or $A_{i,k}$ to denote the entry at the i-th row and the k-th column of A. Given two matrices $A, B \in \mathbb{R}^{l \times q}$, the inner product of A, B is defined by $\langle A, B \rangle = \mathsf{trace}(A^\top B)$, and $\|A\| = \sqrt{\langle A, A \rangle}$ is the Frobenius norm. The set of all symmetric matrices in $\mathbb{R}^{l \times l}$ is denoted by S^l. The direct sum of two matrices A and B, denoted as $A \oplus B$, is the block diagonal matrix defined as:

$$A \oplus B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

Let $C \subset \mathcal{E}$ be a closed convex set. For a given $x \in \mathcal{E}$, its metric projection onto C, denoted by $\Pi_C(x)$, is the unique point in C that has the minimum distance from x. It is well known [20] that for $x \in \mathcal{E}$, it holds that

$$\bar{x} := \Pi_C(x) \iff \langle x - \bar{x}, u - \bar{x} \rangle \leq 0 \ \forall u \in C.$$

The normal cone of C at $\bar{z}$ [11] is defined by

$$N_C(\bar{z}) = \begin{cases} \{v \in \mathcal{E} \mid \langle v, z - \bar{z} \rangle \leq 0 \ \forall z \in C\}, & \text{if } \bar{z} \in C, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Note that $N_{C^\circ}(z) = (C^\circ)^\circ \cap z^\perp = C \cap z^\perp$ when C is also a cone. To facilitate the analysis in Section 3, we state the following well-known Hölder's inequality and its equality condition.

Lemma 2.1 (Hölder's Inequality [5]) *Let $p, q \in (1, \infty)$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$. For any vectors $x, y \in \mathbb{R}^n$, it holds that*

$$\sum_{i=1}^n |x_i y_i| \leq \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \left(\sum_{i=1}^n |y_i|^q \right)^{1/q},$$

where the equality holds if and only if there exists a constant $c \geq 0$ such that $|x_i|^p = c|y_i|^q$ for all $i \in \{1, \ldots, n\}$.

2.1 Generalized differentials

Suppose that a function $f : \mathcal{E} \to \mathcal{F}$ is locally Lipschitz continuous around some point $\bar{x} \in \mathcal{E}$. Then, by Rademacher's theorem [12, Theorem 9.60], f is almost everywhere differentiable in a neighborhood $\mathcal{V}$ of $\bar{x}$. We use $D_f \subseteq \mathcal{V}$ to denote the set of points at which f is differentiable. The Bouligand subdifferential of f at $\bar{x}$ is defined by

$$\partial_B f(\bar{x}) := \{v \in \mathbb{R}^{m \times n} \mid \exists x_k \xrightarrow{D_f} \bar{x} \text{ with } f'(x_k) \to v\}.$$

Moreover, the Clarke generalized Jacobian of f at $\bar{x}$ is the convex hull of $\partial_B f(\bar{x})$, i.e.,

$$\partial f(\bar{x}) := \mathsf{conv}(\partial_B f(\bar{x})).$$

With this preparation, we can present the definition of p-order semismoothness [10].

Definition 2.1 Let $f : \mathcal{E} \to \mathcal{F}$ be a locally Lipschitz continuous function, and let $\partial F(x)$ denote the Clarke generalized Jacobian of F at $x \in \mathcal{E}$. Let $p > 0$. We say that F is *p-order semismooth* at x if:

1. f is directionally differentiable at x.
2. for any $V \in \partial F(x + h)$, as $h \to 0$,

$$\|f(x + h) - f(x) - Vh\| = \mathcal{O}(\|h\|^{1+p}). \tag{2.1}$$

We say that the function f is *strongly semismooth* at x if it is 1-order semismooth at x.

According to [10, Theorem 2.3], we know that (2.1) in Definition 2.1 can be replaced by

$$f(x + h) - f(x) - f'(x + h; h) = \mathcal{O}(\|h\|^{1+p}), \tag{2.2}$$

where $f'(x + h; h)$ means the directional derivative of f at $x + h$ along the direction h. Note that a function $\phi(t)$ is $\mathcal{O}(t)$ if there exists a constant c such that $\|\phi(t)\| \leq c|t|$ for all sufficiently small $|t|$; it is $\Theta(t)$ if there exist constants $c_1, c_2 > 0$ such that $c_1|t| \leq \|\phi(t)\| \leq c_2|t|$ for all sufficiently small $|t|$.

2.2 LMI-representable sets

Following [6, Equation (1.3)], we introduce the definition of LMI-representable set, which is the core tool of this research.

Definition 2.2 (LMI-representable set/Spectrahedron) A set $K \subseteq \mathbb{R}^n$ is said to be LMI (Linear Matrix Inequality) representable or is a spectrahedron if there exist $m > 0$ and $A_0, \ldots, A_n \in S^m$ such that

$$K = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid A_0 + A_1 x_1 + \cdots + A_n x_n \in S_+^m\}.$$

LMI-representable sets are closely related to the slices of PSD cone. To illustrate this, consider the spectrahedron $K_1 = \{x \in \mathbb{R}^n \mid \mathcal{A}(x) + A_0 \in S_+^m\}$ for a linear operator $\mathcal{A} : \mathbb{R}^n \to S^m$. If $\mathcal{A}$ is injective, then K_1 is linearly isomorphic to the set $K_2 = S_+^m \cap \{z \in S^m \mid \mathcal{B}z = b\}$ for some linear operator $\mathcal{B}$ and vector b. The set K_2 is a slice of S_+^m. We have already known that the metric projection onto the PSD cone $\Pi_{S_+^m}$ is strongly semismooth [15, Corollary 4.15]. It is of our interest to study the strong semismoothness of Π_{K_2}, the metric projection onto a slice of PSD cone. This operator Π_{K_2} is apparently a semi-algebraic Lipschitz function. Thus, for any fixed $\bar{x}$ we have a rational number $\gamma_{\bar{x}} > 0$ such that $\Pi_{K_2}(\cdot)$ is $\gamma_{\bar{x}}$-order semismooth at $\bar{x}$ [1, Proposition 1]. Therefore, it is of curiosity for a given $p > 0$, whether there exist $m > 0, \mathcal{B}$ and b such that Π_{K_2} fails to be p-order semismooth at some point. To answer this, we first study the semismoothness of Π_{K_1} for an injective $\mathcal{A}$, and take advantage of the linear isomorphism between K_1 and K_2. For this purpose, we first introduce the following lemma.

Lemma 2.2 *Let $\mathcal{A} : \mathcal{E} \to \mathcal{F}$ be an injective linear operator, and $K \subseteq \mathcal{E}$ be a closed convex set. For any given $p > 0$, the p-order semismoothness of Π_K is equivalent to the p-order semismoothness of $\Pi_{\mathcal{A}K}$.*

Proof Since $\mathcal{A}$ is an injective linear operator between finite-dimensional spaces and K is a closed convex set, its image $\mathcal{A}K$ is also closed and convex. This ensures that the metric projection $\Pi_{\mathcal{A}K}$ is well-defined.

Next, we show that the p-order semismoothness of $\Pi_{\mathcal{A}K}$ on $\mathcal{F}$ is equivalent to that of its restriction to $\mathsf{rge}(\mathcal{A})$. Since $\mathcal{A}K \subseteq \mathsf{rge}(\mathcal{A})$, we have $\Pi_{\mathcal{A}K}(x) = \Pi_{\mathcal{A}K}(\mathcal{B}x)$ for any $x \in \mathcal{F}$, where $\mathcal{B} := \Pi_{\mathsf{rge}\mathcal{A}}$ is a linear operator. From the definition of the directional derivative, we know that for any $x, h \in \mathcal{F}$,

$$\Pi'_{\mathcal{A}K}(x; h) = \Pi'_{\mathcal{A}K}(\mathcal{B}x; \mathcal{B}h)$$

and

$$\Pi_{\mathcal{A}K}(x + h) - \Pi_{\mathcal{A}K}(x) - \Pi'_{\mathcal{A}K}(x; h)$$
$$= \Pi_{\mathcal{A}K}(\mathcal{B}x + \mathcal{B}h) - \Pi_{\mathcal{A}K}(\mathcal{B}x) - \Pi'_{\mathcal{A}K}(\mathcal{B}x; \mathcal{B}h),$$

which demonstrates that the expression depends only on the components within $\mathsf{rge}(\mathcal{A})$. Therefore, $\Pi_{\mathcal{A}K}$ is p-order semismooth on $\mathcal{F}$ if and only if its restriction to $\mathsf{rge}(\mathcal{A})$ is p-order semismooth. This allows us to assume, without

loss of generality, that $\mathsf{rge}(\mathcal{A}) = \mathcal{F}$, which implies that $\mathcal{A}$ is an invertible linear operator.

For $x \in \mathcal{F}$, let $y = \Pi_{\mathcal{A}K}(x)$. By definition, y is the unique point in $\mathcal{A}K$ satisfying the following inequality:

$$\langle x - y, c - y \rangle \leq 0 \quad \forall\, c \in \mathcal{A}K.$$

Since $y \in \mathcal{A}K$ and $\mathcal{A}$ is a bijection, there exists a unique $z \in K$ such that $y = \mathcal{A}z$. Similarly, any $c \in \mathcal{A}K$ can be written as $c = \mathcal{A}w$ for some $w \in K$. Substituting these into the above inequality gives:

$$\langle x - \mathcal{A}z, \mathcal{A}w - \mathcal{A}z \rangle \leq 0 \quad \forall\, w \in K.$$

Using the linearity of $\mathcal{A}$ and the property of the adjoint operator $\mathcal{A}^*$, we obtain:

$$\langle \mathcal{A}^*\mathcal{A}z - \mathcal{A}^*x, w - z \rangle \geq 0 \quad \forall w \in K,$$

which is the first-order optimality condition of the strongly convex program

$$\min_{u \in K} \frac{1}{2} \langle \mathcal{A}^*\mathcal{A}u, u \rangle - \langle \mathcal{A}^*x, u \rangle,$$

whose unique solution is denoted by $z(x)$, and we have $\Pi_{\mathcal{A}K}(x) = \mathcal{A}(z(x))$.

Let $0 < \lambda_{\min} \leq \cdots \leq \lambda_{\max}$ be the eigenvalues of $\mathcal{A}^*\mathcal{A}$ and $0 < \gamma < 1/\lambda_{\max}$. Consider the function $F : \mathcal{E} \times \mathcal{F} \to \mathcal{E}$:

$$F(z, x) := z - \Pi_K \left[z - \gamma(\mathcal{A}^*\mathcal{A}z - \mathcal{A}^*x) \right].$$

It is clear that $F(z(x), x) = 0$ for any $x \in \mathcal{F}$. First suppose that Π_K is p-order semismooth. Since F is a composition of affine operators and Π_K, it is p-order semismooth with respect to the joint variable (z, x). Let $\pi_z \partial F(z, x)$ be the canonical projection of $\partial F(z, x)$ onto the z-part. Since $I - \gamma\mathcal{A}^*\mathcal{A}$ is invertible, an element $V \in \pi_z \partial F(z, x)$ has the form $V = I - W(I - \gamma\mathcal{A}^*\mathcal{A})$, where $W \in \partial \Pi_K[z - \gamma(\mathcal{A}^*\mathcal{A}z - \mathcal{A}^*x)]$. Since Π_K is non-expansive, we have $\|W\| \leq 1$. Thus, we have

$$\|W(I - \gamma\mathcal{A}^*\mathcal{A})\| \leq \|W\| \|I - \gamma\mathcal{A}^*\mathcal{A}\| \leq \max(|1 - \gamma\lambda_{\min}|, |1 - \gamma\lambda_{\max}|) < 1,$$

which implies that V is invertible.

By noting that the implicit function theorem for the (strongly) semismooth function [13, Theorem 2.1] can easily be extended to the p-order case, as pointed out at the end of its proof, we know that F satisfies the conditions of the implicit function theorem and $F(z(x), x) = 0$ for any $x \in \mathcal{F}$. Therefore, $z(x)$ is p-order semismooth. Recall that $\Pi_{\mathcal{A}K}(x) = \mathcal{A}(z(x))$. We have that $\Pi_{\mathcal{A}K}$ is p-order semismooth. Since $\mathcal{A}$ is invertible, the p-order semismoothness of $\Pi_{\mathcal{A}K}$ also implies the p-order semismoothness of Π_K. This completes the proof. $\quad\square$

3 A *p*-order semismooth example

In this section, we first present that for any $p > 0$, there exists an LMI-representable set K such that Π_K fails to be p-order semismooth. Then combined with Lemma 2.2, we extend this result to the slices of the PSD cone.

Let $n \geq 2$ be an integer. Consider the closed convex cone

$$K_n := \Big\{ (x_1, x_2, x_3, y_1, \ldots, y_{n-1}, z_1, \ldots, z_{n-1}) \mid x_3^2 \geq y_1^2 + z_1^2,\ x_3 \geq 0,$$
$$x_3 y_i \geq y_{i+1}^2 \text{ for } i = 1, \ldots, n-2, \quad x_3 y_{n-1} \geq x_1^2,$$
$$x_3 z_i \geq z_{i+1}^2 \text{ for } i = 1, \ldots, n-2, \quad x_3 z_{n-1} \geq x_2^2 \Big\}.$$

Define the following linear operators from $\mathbb{R}^{2n+1}$ to S^2:

$$M_0(p) = \begin{pmatrix} x_3 + y_1 & z_1 \\ z_1 & x_3 - y_1 \end{pmatrix},$$
$$M_i^y(p) = \begin{pmatrix} x_3 & y_{i+1} \\ y_{i+1} & y_i \end{pmatrix} \text{ for } i = 1, \ldots, n-2, \quad M_{n-1}^y(p) = \begin{pmatrix} x_3 & y_{n-1} \\ y_{n-1} & x_1 \end{pmatrix},$$
$$M_i^z(p) = \begin{pmatrix} x_3 & z_{i+1} \\ z_{i+1} & z_i \end{pmatrix} \text{ for } i = 1, \ldots, n-2, \quad M_{n-1}^z(p) = \begin{pmatrix} x_3 & z_{n-1} \\ z_{n-1} & x_2 \end{pmatrix},$$

where $p := (x_1, x_2, x_3, y_1, \ldots, y_{n-1}, z_1, \ldots, z_{n-1}) \in \mathbb{R}^{2n+1}$. It is easy to check that K_n can be written in the following form:

$$K_n = \left\{ p \in \mathbb{R}^{2n+1} \mid M_0(p) \oplus \left(\bigoplus_{i=1}^{n-1} M_i^y(p) \right) \oplus \left(\bigoplus_{i=1}^{n-1} M_i^z(p) \right) \in S_+^{4n-2} \right\}.$$
$$(3.1)$$

One can see that K_n is an LMI-representable set. For notational simplicity, we denote $\kappa_n := 2^n$. Consider the closed convex cone S_n that represents the projection of K_n onto the space of the first three variables, i.e.,

$$S_n = \{ (x_1, x_2, x_3) \mid x_1^{\kappa_n} + x_2^{\kappa_n} \leq x_3^{\kappa_n},\ x_3 \geq 0 \}$$
$$= \{ (x_1, x_2, x_3) \mid \exists y_i, z_i \text{ for } i = 1, \ldots, n-1, \quad (3.2)$$
$$\text{s.t. } (x_1, x_2, x_3, y_1, \ldots, y_{n-1}, z_1, \ldots, z_{n-1}) \in K_n \}.$$

Let $\lambda_n := 2^n / (2^n - 1)$. Since κ_n-norm and λ_n-norm are dual norms, we have

$$S_n^\circ = \{ (u_1, u_2, u_3) \mid u_1^{\lambda_n} + u_2^{\lambda_n} \leq u_3^{\lambda_n},\ u_3 \leq 0 \}.$$

With (3.2), it is easy to know

$$S_n^\circ = \{ (u_1, u_2, u_3) \mid (u_1, u_2, u_3, 0, \ldots, 0) \in K_n^\circ \}.$$

We have the following lemma about $N_{K_n^\circ}$.

Lemma 3.1 *Let $u_1, u_2 \geq 0$ and $v := (u_1, u_2, -1, 0, \ldots, 0) \in \mathsf{bdry}(K_n^\circ)$. Then*

$$N_{K_n^\circ}(v) = \mathsf{cone}(w),$$

where $w := (u_1^{\lambda_n/2^n}, u_2^{\lambda_n/2^n}, 1, u_1^{\lambda_n/2^1}, \ldots, u_1^{\lambda_n/2^{n-1}}, u_2^{\lambda_n/2^1}, \ldots, u_2^{\lambda_n/2^{n-1}}).$

Proof Since K_n and K_n° are closed convex cones, we have

$$N_{K_n^\circ}(v) = (K_n^\circ)^\circ \cap v^\perp = K_n \cap v^\perp.$$

Let $(x_1, x_2, x_3, y_1, \ldots, y_{n-1}, z_1, \ldots, z_{n-1}) \in N_{K_n^\circ}(v)$. If $x_3 = 0$, by the definition of K_n, we have $x_1 = x_2 = 0$, $y_i = 0$ and $z_i = 0$. So, we can assume that $x_3 = 1$ since $x_3 \geq 0$. Then

$$x_1 u_1 + x_2 u_2 = 1.$$

Note that $1/\kappa_n + 1/\lambda_n = 1$. By Hölder's inequality in Lemma 2.1, we have

$$x_1 u_1 + x_2 u_2 \leq (x_1^{\kappa_n} + x_2^{\kappa_n})^{1/\kappa_n}(u_1^{\lambda_n} + u_2^{\lambda_n})^{1/\lambda_n} \leq 1.$$

From the equality condition of Hölder's inequality in Lemma 2.1, we have $x_1^{\kappa_n} = u_1^{\lambda_n}$ and $x_2^{\kappa_n} = u_2^{\lambda_n}$. So $x_1 = u_1^{\lambda_n/2^n}$ and $x_2 = u_2^{\lambda_n/2^n}$ since $u_1, u_2 \geq 0$ and $x_1 u_1 + x_2 u_2 = 1$. By using the definition of K_n and noting that $x_1^{\kappa_n} + x_2^{\kappa_n} = 1$, we have $y_i = u_1^{\lambda_n/2^i}$ and $z_i = u_2^{\lambda_n/2^i}$. This completes the proof. $\square$

Now, we are ready to state our main result as in the following theorem.

Theorem 3.1 *For any $p > 0$, there exists an LMI-representable set K such that Π_K fails to be p-order semismooth.*

Proof Consider the following curve:

$$v(t) := (t, (1 - t^{\lambda_n})^{1/\lambda_n}, -1, 0, \ldots, 0) \in K_n^\circ, \quad t \in [0, 1]. \qquad (3.3)$$

Let

$$w(t) := (t^{\lambda_n/\kappa_n}, (1 - t^{\lambda_n})^{1/\kappa_n}, 1, y_1(t), \ldots y_{n-1}(t), z_1(t), \ldots z_{n-1}(t)), \quad (3.4)$$

where $y_i(t) = t^{\lambda_n/2^i}$ and $z_i(t) = (1 - t^{\lambda_n})^{1/2^i}$. So $N_{K_n^\circ}(v(t)) = \mathsf{cone}(w(t))$ by Lemma 3.1. Letting $t \to 0$, we evaluate the semismoothness as follows:

$$v(t) - v(0) - \Pi'_{K_n^\circ}(v(t); v(t) - v(0))$$

$$\overset{(a)}{=} v(t) - v(0) - \Pi_{T_{K_n^\circ}(v(t))}(v(t) - v(0))$$

$$= v(t) - v(0) - \left(v(t) - v(0) - \frac{\langle v(t) - v(0), w(t) \rangle}{\|w(t)\|^2} w(t)\right)$$

$$= \frac{\langle v(t) - v(0), w(t) \rangle}{\|w(t)\|^2} w(t)$$

$$\overset{(b)}{=} \Theta(t^{\lambda_n} + (1 - t^{\lambda_n})^{1/\kappa_n} - 1) w(t)$$

$$\overset{(c)}{=} \Theta(t^{\lambda_n}),$$

where (a) follows from [20, Equation (1.2)], (b) follows from (3.3) and (3.4), and (c) comes from $(1 + x)^a - 1 = \Theta(x)$ when $x \to 0$ and $a > 0$. Note that $\|v(t) - v(0)\| = \Theta(t)$, we have that $\Pi_{K_n^\circ}$ is at most $(\lambda_n - 1)$-order semismooth from (2.2). So, Π_{K_n} is also at most $(\lambda_n - 1)$-order semismooth. Therefore, Theorem 3.1 holds following by $\lambda_n \to 1$ as $n \to \infty$. $\quad\square$

Recall that K_n in (3.1) can be written in the form $\{x \in \mathbb{R}^{2n+1} \mid \mathcal{A}x \in S_+^{4n-2}\}$ where $\mathcal{A} : \mathbb{R}^{2n+1} \to S^{4n-2}$ is a linear operator. It is directly to check that $\mathcal{A}$ is injective. Let $T_n := \mathcal{A}K_n \subset S_+^{4n-2}$. So $T_n = S_+^{4n-2} \cap \mathsf{rge}(\mathcal{A})$. Thus, there exists a linear operator $\mathcal{B} : S^{4n-2} \to \mathbb{R}^m$ such that $T_n = \{X \in S_+^{4n-2} \mid \mathcal{B}X = 0\}$. So, by Lemma 2.2, we have that Π_{T_n} is at most $(\lambda_n - 1)$-order semismooth.

Note that $T_n = S_+^{4n-2} \cap \mathsf{ker}(\mathcal{B})$ is the intersection of the strongly semismooth closed convex cone [15] and a linear subspace. Therefore, we have

$$
\begin{aligned}
T_n^\circ &\overset{(a)}{=} S_-^{4n-2} + \mathsf{ker}(\mathcal{B})^\circ \\
&\overset{(b)}{=} S_-^{4n-2} + \mathsf{rge}(\mathcal{B}^*),
\end{aligned}
\tag{3.5}
$$

where (a) comes from [12, Theorem 6.42] and (b) comes from $(\mathsf{ker}(\mathcal{B}))^\perp = \mathsf{rge}(\mathcal{B}^*)$. From (3.5), T_n° is the Minkowski sum of a subspace in S^{4n-2} and a strongly semismooth convex closed cone. However, $\Pi_{T_n^\circ}$ shares the same semismooth order with Π_{T_n}, and thus is at most $(\lambda_n - 1)$-order semismooth. Therefore, we have the following corollaries.

Corollary 3.1 *For any $p > 0$, there exists a slice K of PSD cone such that Π_K fails to be p-order semismooth.*

Corollary 3.2 *For any $p > 0$, there exists a set K such that K is the Minkowski sum of a nonpolyhedral strongly semismooth set and a polyhedral convex set but Π_K fails to be p-order semismooth.*

4 Conclusion

In this paper, we have demonstrated that the metric projection onto a slice of the PSD cone may fail to be p-semismooth for any $p > 0$. Building on this finding, we conclude our paper by raising the following two natural open questions:

Question 1. According to [16], the metric projection onto a symmetric cone (i.e., a self-dual homogeneous cone) is strongly semismooth. Moreover, every homogeneous cone or symmetric cone is linearly isomorphic to a slice of the PSD cone [3,4]. By Lemma 2.2, this implies that there exist many special slices of the PSD cone for which the metric projection is strongly semismooth. A fundamental open problem is to identify sufficient (and necessary) conditions under which the metric projection onto a slice of the PSD cone is strongly semismooth. For instance, is the metric projection onto an arbitrary homogeneous cone always strongly semismooth?

Question 2. In our constructions, we need a higher-dimensional PSD cone to get lower-order semismoothness. On the other hand, the metric projection onto any slice of S_+^2 (which is linearly isomorphic to an SOC) must be strongly semismooth. This raises the possibility that the order of semismoothness is intrinsically linked to the dimension of the underlying PSD cone. An important question is therefore to establish the lower bound on the attainable order of semismoothness as a function of the dimension of the PSD cone.

Declarations

Conflict of interest The authors have no relevant financial or non-financial interests to disclose.

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