Matrix Cones and Spectral Operators of Matrices

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Based on joint works with Chao Ding, Jie Sun, and Kim-Chuan Toh
Let $S^n$ be set of $n$ by $n$ symmetric matrices in $\mathbb{R}^{m \times n}$ and $S^n_+$ be the cone of positive semidefinite matrices in $S^n$.

Let $X \in S^n$ have the following spectral decomposition

$$X = P\Lambda P^T = \sum_{i=1}^{n} \lambda_i p_i p_i^T,$$

where $\Lambda$ is the diagonal matrix of eigenvalues $\lambda_1, \ldots, \lambda_n$ of $X$ and $P$ is a corresponding orthogonal matrix of orthonormal eigenvectors. Then

$$X_+ := \Pi_{S^n_+}(X) = P\Lambda_+ P^T = \sum_{i=1}^{n} (\lambda_i)_+ p_i p_i^T.$$

Here $\Pi_{S^n_+}(X)$ is the unique optimal solution to

$$\min \frac{1}{2} \|Z - X\|^2_F \quad \text{s.t.} \quad Z \in S^n_+.$$
Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a scalar function. The corresponding Löwner operator $F : S^n \rightarrow S^n$ is defined by

$$F(X) := \sum_{i=1}^{n} f(\lambda_i)p_ip_i^T, \quad X \in S^n$$

Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be an odd scalar function satisfying $g(-t) = -g(t)$ for all $t \geq 0$ (naturally $g(0) = 0$). One may define Löwner’s operator $G : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^{m \times n}$ (assuming $m \leq n$) by

$$G(Z) := \sum_{i=1}^{m} g(\sigma_i(Z))u_iv_i^T, \quad Z \in \mathbb{R}^{m \times n},$$

where for any given $Z \in \mathbb{R}^{m \times n}$, $\sigma_1(Z) \geq \sigma_2(Z) \geq \ldots \geq \sigma_m(Z)$ denotes the singular values of $Z$ (always nonnegative and counting multiplicity) and $\sigma(Z)$ denotes the vector of the singular values of $Z$; $\mathbb{O}^{m,n}(Z)$ denotes the set of matrix pairs $(U, V) \in \mathbb{O}^m \times \mathbb{O}^n$ satisfying the singular value decomposition

$$Z = U [\Sigma(Z) \ 0] V^T,$$

where $\Sigma(Z)$ is an $m \times m$ diagonal matrix whose $i$-th diagonal entry is $\sigma_i(Z) \geq 0$.

Let $X \in \mathbb{R}^{m \times n}$ admit the following singular value decomposition:

$$X = \bar{U} [\Sigma(X) \ 0] \bar{V}^T = \bar{U} [\Sigma(X) \ 0] [\bar{V}_1 \ \bar{V}_2]^T = \bar{U} \Sigma(X) \bar{V}^T_1,$$

(1)

where $\bar{U} \in \mathcal{O}^m$, $\bar{V} \in \mathcal{O}^n$ and $\bar{V}_1 \in \mathbb{R}^{n \times m}$, $\bar{V}_2 \in \mathbb{R}^{n \times (n-m)}$ and $\bar{V} = [\bar{V}_1 \ \bar{V}_2]$. The set of such matrices $(U, V)$ in the singular value decomposition (1) is denoted by $\mathcal{O}^{m,n}(X)$, i.e.,

$$\mathcal{O}^{m,n}(X) := \{(U, V) \in \mathbb{R}^{m \times m} \times \mathbb{R}^{n \times n} \mid X = U [\Sigma(X) \ 0] V^T\}.$$
For any positive constant \( \varepsilon > 0 \), denote the closed convex cone \( D^n_\varepsilon \) by

\[
D^n_\varepsilon := \{(t, x) \in \mathbb{R} \times \mathbb{R}^n \mid \varepsilon^{-1} t \geq x_i, \ i = 1, \ldots, n\}.
\] (2)

Let \( \Pi_{D^n_\varepsilon} (\cdot) \) be the metric projector over \( D^n_\varepsilon \) under the Euclidean inner product in \( \mathbb{R}^n \). That is, for any \( (t, x) \in \mathbb{R} \times \mathbb{R}^n \), \( \Pi_{D^n_\varepsilon} (t, x) \) is the unique optimal solution to the following convex optimization problem

\[
\begin{align*}
\text{min} & \quad \frac{1}{2} ((\tau - t)^2 + \|y - x\|^2) \\
\text{s.t.} & \quad \varepsilon^{-1} \tau \geq y_i, \ i = 1, \ldots, n.
\end{align*}
\] (3)
For any $x \in \mathbb{R}^n$, let $x^\downarrow$ be the vector of components of $x$ being arranged in the non-increasing order $x^\downarrow_1 \geq \ldots \geq x^\downarrow_n$. Let $\text{sgn}(x)$ be the sign vector of $x$, i.e., $(\text{sgn})_i(x) = 1$ if $x_i \geq 0$ and $-1$ otherwise. We use “$\circ$” to denote the Hadamard product operation either for two vectors or two matrices of the same dimensions.

**Proposition**

Assume that $\varepsilon > 0$ and $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ are given. Let $\pi$ be a permutation of $\{1, \ldots, n\}$ such that $x^\downarrow = x_\pi$, i.e., $x^\downarrow_i = x_{\pi(i)}$, $i = 1, \ldots, n$ and $\pi^{-1}$ the inverse of $\pi$. For convenience, write $x^\downarrow_0 = +\infty$ and $x^\downarrow_{n+1} = -\infty$. Let $\bar{k}$ be the smallest integer $k \in \{0, 1, \ldots, n\}$ such that

$$
x^\downarrow_{k+1} \leq \left( \sum_{j=1}^{k} x^\downarrow_j + \varepsilon t \right) / \left( k + \varepsilon^2 \right) < x^\downarrow_k.
$$

(4)
Define $\bar{y} \in \mathbb{R}^n$ and $\bar{\tau} \in \mathbb{R}_+$, respectively, by

$$
\bar{y}_i := \begin{cases} 
\left( \sum_{j=1}^{\bar{\kappa}} x_j^+ + \varepsilon t \right) / (\bar{\kappa} + \varepsilon^2) & \text{if } 1 \leq i \leq \bar{\kappa}, \\
x_i^- & \text{otherwise}
\end{cases}
$$

and

$$
\bar{\tau} := \varepsilon \bar{y}_1 = \varepsilon \left( \sum_{j=1}^{\bar{k}} x_j^+ + \varepsilon t \right) / (\bar{k} + \varepsilon^2).
$$

The metric projection $\Pi_{D_{\varepsilon}^n}(t, x)$ is computed by $\Pi_{D_{\varepsilon}^n}(t, x) = (\bar{\tau}, \bar{y}_{\pi-1})$. 
For any positive constant $\varepsilon > 0$, define the matrix cone $\mathcal{M}_n^\varepsilon$ in $S^n$ as the epigraph of the convex function $\varepsilon \lambda_{\max}(\cdot)$, i.e.,

$$\mathcal{M}_n^\varepsilon := \{(t, X) \in \mathbb{R} \times S^n \mid \varepsilon^{-1}t \geq \lambda_{\max}(X)\}.$$  

(5)

**Proposition**

Assume that $(t, X) \in \mathbb{R} \times S^n$ is given. Let $X$ have the eigenvalue decomposition

$$X = \overline{P} \text{diag}(\lambda(X)) \overline{P}^T,$$

(6)

where $\overline{P} \in O^n$. Let $\Pi_{\mathcal{M}_n^\varepsilon}(\cdot, \cdot)$ be the metric projector over $\mathcal{M}_n^\varepsilon$ under Frobenius norm in $S^n$. Then,

$$\Pi_{\mathcal{M}_n^\varepsilon}(t, X) = (\bar{t}, \bar{P} \text{diag}(\bar{y}) \bar{P}^T) \quad \forall (t, X) \in \mathbb{R} \times S^n,$$

(7)

where $(\bar{t}, \bar{y}) = \Pi_{\mathcal{D}_n^\varepsilon}(t, \lambda(X)) \in \mathbb{R} \times \mathbb{R}^n$. 

For any positive constant $\varepsilon > 0$, denote the closed convex cone $C^n_\varepsilon$ by

$$
C^n_\varepsilon := \{(t, x) \in \mathbb{R} \times \mathbb{R}^n \mid \varepsilon^{-1} t \geq \|x\|_\infty\}.
$$

(8)

Let $\Pi_{C^n_\varepsilon}(\cdot, \cdot)$ be the metric projector over $C^n_\varepsilon$ under the Euclidean inner product in $\mathbb{R}^n$. That is, for any $(t, x) \in \mathbb{R} \times \mathbb{R}^n$, $\Pi_{C^n_\varepsilon}(t, x)$ is the unique optimal solution to the following convex optimization problem

$$
\begin{align*}
\min & \quad \frac{1}{2}((\tau - t)^2 + \|y - x\|^2) \\
\text{s.t.} & \quad \varepsilon^{-1}\tau \geq \|y\|_\infty.
\end{align*}
$$

(9)

In the following discussions, we frequently drop $n$ from $C^n_\varepsilon$ when its size can be found from the context.
Assume that $\varepsilon > 0$ and $(t, x) \in \mathbb{R} \times \mathbb{R}^n$ are given. Let $\pi$ be a permutation of $\{1, \ldots, n\}$ such that $|x|_\downarrow = |x|_\pi$, i.e., $|x|_i = |x|_{\pi(i)}$, $i = 1, \ldots, n$ and $\pi^{-1}$ be the inverse of $\pi$. Let $|x|_0^\downarrow = +\infty$ and $|x|_{n+1}^\downarrow = 0$. Let $s_0 = 0$ and $s_k = \sum_{i=1}^k |x|_i^\downarrow$, $k = 1, \ldots, n + 1$. Let $\overline{k}$ be the smallest integer $k \in \{0, 1, \ldots, n\}$ such that

$$|x|_{k+1}^\downarrow \leq (s_k + \varepsilon t)/(k + \varepsilon^2) < |x|_k^\downarrow \quad (10)$$

or $\overline{k} = n + 1$ if such an integer does not exist. Denote

$$\theta^\varepsilon(t, x) := (s_{\overline{k}} + \varepsilon t)/(\overline{k} + \varepsilon^2) \quad (11)$$
Let $\alpha, \beta$ and $\gamma$ be the index sets of $|x|^\downarrow$ as

$$\alpha := \{ i \mid |x|^\uparrow_i > \theta^\varepsilon(t, x) \}, \quad \beta := \{ i \mid |x|^\uparrow_i = \theta^\varepsilon(t, x) \}$$

(12)

and

$$\gamma := \{ i \mid |x|^\uparrow_i < \theta^\varepsilon(t, x) \}.$$  

(13)

Define $\bar{y} \in \mathbb{R}^n$ and $\bar{\tau} \in \mathbb{R}_+$, respectively, by

$$\bar{y}_i := \begin{cases} \max\{\theta^\varepsilon(t, x), 0\} & \text{if } i \in \alpha, \\ |x|^\uparrow_i & \text{otherwise} \end{cases}$$

and

$$\bar{\tau} := \varepsilon \max\{\theta^\varepsilon(t, x), 0\}.$$
Proposition

Assume that \( \varepsilon > 0 \) and \( (t, x) \in \mathbb{R} \times \mathbb{R}^n \) are given. The metric projection \( \Pi_{\mathcal{C}_\varepsilon}(t, x) \) of \( (t, x) \) onto \( \mathcal{C}_\varepsilon \) can be computed as follows

\[
\Pi_{\mathcal{C}_\varepsilon}(t, x) = (\bar{\tau}, \text{sgn}(x) \circ \bar{y}_{\pi - 1}).
\] (14)

Theorem

Assume that \( (t, X) \in \mathbb{R} \times \mathbb{R}^{m \times n} \) is given. Let \( X \) have the singular value decomposition (1). Let \( \Pi_{\mathcal{K}_\varepsilon}(:, :) \) be the metric projector over \( \mathcal{K}_\varepsilon \) under Frobenius norm in \( \mathbb{R}^{m \times n} \), where

\[
\mathcal{K}_\varepsilon := \{(t, X) \in \mathbb{R} \times \mathbb{R}^{m \times n} | \varepsilon^{-1}t \geq \|X\|_2\}.
\] (15)

For any \( (t, X) \in \mathbb{R} \times \mathbb{R}^{m \times n} \), we have

\[
\Pi_{\mathcal{K}_\varepsilon}(t, X) = \left(\bar{t}, \bar{U} \begin{bmatrix} \text{diag}(\bar{y}) & 0 \end{bmatrix} \bar{V}^T\right),
\] (16)

where

\[
(\bar{t}, \bar{y}) = \Pi_{\mathcal{C}_\varepsilon}(t, \sigma(X)) \in \mathbb{R} \times \mathbb{R}^m.
\]
Löwner operators are inadequate for applications

For a given unitarily invariant proper closed convex function $f : \mathcal{X} \to (-\infty, \infty]$, in matrix optimization one often considers the proximal mapping of $f$ at $X$:

$$P_f(X) := \arg\min_{Y \in \mathcal{X}} \left\{ f(Y) + \frac{1}{2} \|Y - X\|^2 \right\}, \quad X \in \mathcal{X}, \quad (17)$$

where $\mathcal{X}$ is either the real vector subspace $\mathbb{S}^m$ of $m \times m$ real symmetric (or complex) Hermitian matrices, or the real vector subspace $\mathbb{V}^{m \times n}$ of $m \times n$

For example, for $f(Y) = \|Y\|_2 = \sigma_{\text{max}}(Y)$, the spectral norm of $Y$, $P_f(\cdot)$ is no longer the Löwner operator [it is the Löwner operator for $f(Y) = \|Y\|_* = \sum_{i=1}^{m} \sigma_i(Y)$].

If $f(\cdot)$ is the indicator function of a matrix cone, then the proximal mapping $P_f(\cdot)$ is the metric projector over the corresponding matrix cone.
The Setting

Let $s$ be a positive integer and $0 \leq s_0 \leq s$ be a nonnegative integer. For given positive integers $m_1, \ldots, m_s$ and $n_{s_0+1}, \ldots, n_s$, define the real vector space $\mathcal{X}$ by

$$\mathcal{X} := S^{m_1} \times \ldots \times S^{m_{s_0}} \times V^{m_{s_0+1} \times n_{s_0+1}} \times \ldots \times V^{m_s \times n_s}.$$  \hfill (18)

Without loss of generality, we assume that $m_k \leq n_k$, $k = s_0 + 1, \ldots, s$.

For any $X = (X_1, \ldots, X_s) \in \mathcal{X}$, we have for $1 \leq k \leq s_0$, $X_k \in S^{m_k}$ and $s_0 + 1 \leq k \leq s$, $X_k \in V^{m_k \times n_k}$. Denote

$$\mathcal{Y} := \mathbb{R}^{m_1} \times \ldots \times \mathbb{R}^{m_{s_0}} \times \mathbb{R}^{m_{s_0}} \times \ldots \times \mathbb{R}^{m_s}.$$ \hfill (19)

For any $X \in \mathcal{X}$, define $\kappa(X) \in \mathcal{Y}$ by

$$\kappa(X) := (\lambda(X_1), \ldots, \lambda(X_{s_0}), \sigma(X_{s_0+1}), \ldots, \sigma(X_s)).$$
Define the set $\mathcal{P}$ by

$$
\mathcal{P} := \{(Q_1, \ldots, Q_s) \mid Q_k \in \mathbb{P}^{m_k}, \ 1 \leq k \leq s_0 \text{ and } Q_k \in \pm \mathbb{P}^{m_k},\ s_0 + 1 \leq k \leq s\}.
$$

Let $g : \mathcal{Y} \to \mathcal{Y}$ be a given mapping. For any $x = (x_1, \ldots, x_s) \in \mathcal{Y}$ with $x_k \in \mathbb{R}^{m_k}$, we write $g(x) \in \mathcal{Y}$ in the form $g(x) = (g_1(x), \ldots, g_s(x))$ with $g_k(x) \in \mathbb{R}^{m_k}$ for $1 \leq k \leq s$.

**Definition**

The given mapping $g : \mathcal{Y} \to \mathcal{Y}$ is said to be *mixed symmetric*, with respect to $\mathcal{P}$, at $x = (x_1, \ldots, x_s) \in \mathcal{Y}$ with $x_k \in \mathbb{R}^{m_k}$, if

$$
g(Q_1 x_1, \ldots, Q_s x_s) = (Q_1 g_1(x), \ldots, Q_s g_s(x)) \quad \forall \ (Q_1, \ldots, Q_s) \in \mathcal{P}. \quad (20)
$$

The mapping $g$ is said to be mixed symmetric, with respect to $\mathcal{P}$, over a set $\mathcal{D} \subseteq \mathcal{Y}$ if (20) holds for every $x \in \mathcal{D}$. We call $g$ a *mixed symmetric* mapping, with respect to $\mathcal{P}$, if (20) holds for every $x \in \mathcal{Y}$.
Note that for each $k \in \{1, \ldots, s\}$, the function value $g_k(x) \in \mathbb{R}^{m_k}$ is dependent on all $x_1, \ldots, x_s$. When there is no danger of confusion, in later discussions we often drop the phrase “with respect to $P$” from Definition 1. Let $\mathcal{N}$ be a given nonempty set in $\mathcal{X}$. Define $\kappa_{\mathcal{N}} := \{\kappa(X) \in \mathcal{Y} \mid X \in \mathcal{N}\}$. The following definition of the spectral operator with respect to a mixed symmetric mapping $g$.

**Definition**

Suppose that $g : \mathcal{Y} \to \mathcal{Y}$ is mixed symmetric on $\kappa_{\mathcal{N}}$. The spectral operator $G : \mathcal{N} \to \mathcal{X}$ with respect to $g$ is defined as $G(X) := (G_1(X), \ldots, G_s(X))$ for $X = (X_1, \ldots, X_s) \in \mathcal{N}$ such that

$$G_k(X) := \begin{cases} P_k \text{Diag}(g_k(\kappa(X))) P_k^T & \text{if } 1 \leq k \leq s_0, \\ U_k \left[ \text{Diag}(g_k(\kappa(X))) \right] V_k^T & \text{if } s_0 + 1 \leq k \leq s, \end{cases}$$

where $P_k \in \mathbb{O}^{m_k}(X_k)$, $1 \leq k \leq s_0$, $(U_k, V_k) \in \mathbb{O}^{m_k,n_k}(X_k)$, $s_0 + 1 \leq k \leq s$. 
Next, we will focus on the study of spectral operators for the case that $\mathcal{X} \equiv \mathbb{V}^{m \times n}$. The corresponding extensions for the spectral operators defined on the general Cartesian product of several matrix spaces can be considered in a similar fashion.

Let $\mathcal{N}$ be a given nonempty open set in $\mathbb{V}^{m \times n}$. Suppose that $g : \mathbb{R}^m \rightarrow \mathbb{R}^m$ is mixed symmetric with respect to $\mathcal{P} \equiv \pm \mathbb{P}^m$ (i.e., absolutely symmetric), on an open set $\widehat{\sigma}_N$ in $\mathbb{R}^m$ containing $\sigma_N := \{\sigma(X) \mid X \in \mathcal{N}\}$. The spectral operator $G : \mathcal{N} \rightarrow \mathbb{V}^{m \times n}$ with respect to $g$ then takes the form of

$$G(X) = U \begin{bmatrix} \text{Diag}(g(\sigma(X))) & 0 \end{bmatrix} V^T, \quad X \in \mathcal{N},$$

where $(U, V) \in \mathbb{O}^{m,n}(X)$. For a given $\overline{X} \in \mathcal{N}$, consider the singular value decomposition (SVD) of $\overline{X}$, i.e.,

$$\overline{X} = \overline{U} \begin{bmatrix} \Sigma(\overline{X}) & 0 \end{bmatrix} \overline{V}^T,$$

where $\Sigma(\overline{X})$ is an $m \times m$ diagonal matrix whose $i$-th diagonal entry is $\sigma_i(\overline{X})$, $\overline{U} \in \mathbb{O}^m$ and $\overline{V} = [\overline{V}_1 \overline{V}_2] \in \mathbb{O}^n$ with $\overline{V}_1 \in \mathbb{V}^{m \times m}$ and $\overline{V}_2 \in \mathbb{V}^{n \times (n-m)}$. 

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Let \( X, Y \) be two finite-dimensional real Euclidean spaces and \( F : X \to Y \) a locally Lipschitz continuous function.

Since \( F \) is almost everywhere differentiable [Rademacher, 1912], we can define

\[ \partial_B F(x) := \left\{ \lim F'(x_k) : x_k \to x, x_k \in D_F \right\}. \]

Here \( D_F \) is the set of points where \( F \) is differentiable. Hence, Clarke’s generalized Jacobian of \( F \) at \( x \) is given by

\[ \partial F(x) = \text{conv} \ \partial_B F(x). \]
A brief review on nonsmooth Newton methods

Definition

Let $\mathcal{K} : X \rightrightarrows L(X, Y)$ be a nonempty, compact valued and upper-semicontinuous multifunction. We say that $F$ is semismooth at $x \in X$ with respect to $\mathcal{K}$ if (i) $F$ is directionally differentiable at $x$; and (ii) for any $\Delta x \in X$ and $V \in \mathcal{K}(x + \Delta x)$ with $\Delta x \to 0$,

$$F(x + \Delta x) - F(x) - V(\Delta x) = o(\|\Delta x\|) \quad (g\text{-semismooth}). \quad (22)$$

Furthermore, if (22) is replaced by

$$F(x + \Delta x) - F(x) - V(\Delta x) = O(\|\Delta x\|^{1+\gamma}), \quad (23)$$

where $\gamma > 0$ is a constant, then $F$ is said to be $\gamma$-order (strongly if $\gamma = 1$) semismooth at $x$ with respect to $\mathcal{K}$. 
Nonsmooth (local) Newton’s method

Assume that $F(\bar{x}) = 0$.

Given $x^0 \in X$. For $k = 0, 1, \ldots$

Main Step  Choose an arbitrary $V_k \in K(x^k)$. Solve

$$F(x^k) + V_k (x^{k+1} - x^k) = 0$$

Rates of Convergence: Assume that $K(\bar{x})$ is nonsingular and that $x^0$ is sufficiently close to $\bar{x}$. If $F$ is $g$-semismooth at $\bar{x}$, then

$$\|x^{k+1} - \bar{x}\| = \| V_k^{-1} [F(x^k) - F(\bar{x}) - V_k (x^k - \bar{x})] \| = o(\|x^k - \bar{x}\|) \text{ bounded}$$

$$g\text{-semismooth} \quad \text{superlinear}$$

It takes $o(\|x^k - \bar{x}\|^{1+\gamma})$ if $F$ is $\gamma$-order $g$-semismooth at $\bar{x}$ [the directional differentiability of $F$ is not needed in the above local convergence analysis]
The nonsmooth equation approach is popular in the complementarity and variational inequalities (nonsmooth equations) community (Robinson, Pang, ...)


Kummer (1988, 1992) gave a sufficient condition (22) to extend Kojima and Shindo’s work.

L. Qi and J. Sun (1993) proved what we know now.

Since then, many exciting developments, in particular in the large-scale settings ...

Why nonsmooth Newton methods important in solving large-scale optimization problems? We illustrate this with an example.
Consider the nearest correlation matrix (NCM) problem:

\[
\min \left\{ \frac{1}{2} \| X - G \|_F^2 \mid X \succeq 0, \ X_{ii} = 1, \ i = 1, \ldots, n \right\}.
\]

The dual of the above problem can be written as (in its minimization format)

\[
\min \frac{1}{2} \| \Xi \|_2^2 - \langle b, y \rangle - \frac{1}{2} \| G \|_2^2 \\
\text{s.t.} \quad S - \Xi + A^*y = -G, \quad S \succeq 0
\]

or via eliminating \( \Xi \) and \( S \succeq 0 \), the following

\[
\min \left\{ \varphi(y) := \frac{1}{2} \| \Pi S_n^+ (A^*y + G) \|_2^2 - \langle b, y \rangle - \frac{1}{2} \| G \|_2^2 \right\},
\]

which is equivalent to the strongly semismooth system (S. & Sun, 02) of equations

\[
\nabla \varphi(y) = A\Pi S_n^+ (A^*y + G) - b = 0.
\]
Numerical results for the NCM

Test the second order nonsmooth Newton-CG method [H.-D. Qi & S. 06] ([X,y] = \texttt{CorrelationMatrix}(G,b,\tau,tol) in Matlab from Sun’s webpage) and two popular first order methods (FOMs) [APG of Nesterov; ADMM of Glowinski (steplength 1.618)] all to the dual forms for the NCM with real financial data:

\( G: \text{Cor3120, } n = 3, 120, \) obtained from [N. J. Higham & N. Strabić, SIMAX, 2016] [Optimal sol. rank = 3, 025, high rank]

<table>
<thead>
<tr>
<th>( n = 3, 120 )</th>
<th>Newton-CG</th>
<th>ADMM</th>
<th>APG</th>
</tr>
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<td>Rel. KKT Res.</td>
<td>2.7-8</td>
<td>2.9-7</td>
<td>9.2-7</td>
</tr>
<tr>
<td>time (s)</td>
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<td>246.4</td>
<td>459.1</td>
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<td>avg-time/iter</td>
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<td>4.3</td>
<td>4.1</td>
</tr>
</tbody>
</table>

Newton’s method only takes at most 40% time more than ADMM & APG (or FISTA) per iteration (Newton will take less time on average per iteration if it took more iterations).
### Theorem

Suppose that $\bar{X} \in \mathcal{N}$ has the SVD (21). The spectral operator $G$ is continuous at $\bar{X}$ if and only if $g$ is continuous at $\sigma(\bar{X})$.

### Theorem

Suppose that $\bar{X}$ has the SVD (21). The spectral operator $G$ is locally Lipschitz continuous near $\bar{X}$ if and only if $g$ is locally Lipschitz continuous near $\bar{\sigma} = \sigma(\bar{X})$. 
Let $\eta(\sigma) \in \mathbb{R}^m$ be the vector defined by $(i \in \{1, \ldots, m\})$

\[
(\eta(\sigma))_i := \begin{cases} 
(g'(\sigma))_{ii} - (g'(\sigma))_{ij} & \text{if } \exists j \in \{1, \ldots, m\} \text{ and } j \neq i \text{ such that } \sigma_i = \sigma_j, \\
(g'(\sigma))_{ii} & \text{otherwise,}
\end{cases}
\]  

(24)

Define the corresponding divided difference matrix $E_1(\sigma) \in \mathbb{R}^{m \times m}$, the divided addition matrix $E_2(\sigma) \in \mathbb{R}^{m \times m}$, the division matrix $F(\sigma) \in \mathbb{R}^{m \times (n-m)}$, respectively, by

\[
(E_1(\sigma))_{ij} := \begin{cases} 
g_i(\sigma) - g_j(\sigma) & \text{if } \sigma_i \neq \sigma_j, \\
 \sigma_i - \sigma_j & (\eta(\sigma))_i \text{otherwise},
\end{cases} \quad i, j \in \{1, \ldots, m\},
\]  

(25)

\[
(E_2(\sigma))_{ij} := \begin{cases} 
g_i(\sigma) + g_j(\sigma) & \text{if } \sigma_i + \sigma_j \neq 0, \\
 \sigma_i + \sigma_j & (g'(\sigma))_{ii} \text{otherwise},
\end{cases} \quad i, j \in \{1, \ldots, m\},
\]  

(26)

\[
(F(\sigma))_{ij} := \begin{cases} 
g_i(\sigma) & \text{if } \sigma_i \neq 0, \\
 \sigma_i & (g'(\sigma))_{ii} \text{otherwise},
\end{cases} \quad i \in \{1, \ldots, m\}, \quad j \in \{1, \ldots, n-m\}.
\]  

(27)
Define the matrix $\mathcal{C}(\sigma) \in \mathbb{R}^{m \times m}$ to be the difference between $g'(\sigma)$ and $\text{Diag}(\eta(\sigma))$, i.e.,

$$\mathcal{C}(\sigma) := g'(\sigma) - \text{Diag}(\eta(\sigma)).$$

(28)

When the dependence of $\eta$, $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{F}$ and $\mathcal{C}$ on $\sigma$ is clear from the context, we often drop $\sigma$ from the corresponding notations. Note that the divided difference matrix $\mathcal{E}_1(\sigma)$ is the same with the commonly defined for the symmetric matrix case. The divided addition matrix $\mathcal{E}_2(\sigma)$ and the division matrix $\mathcal{F}(\sigma)$ are particular to general non-Hermitian matrices.

Denote $\overline{\eta} = \eta(\overline{\sigma}) \in \mathbb{R}^m$ to be the vector defined by (24). Let $\overline{\mathcal{E}}_1$, $\overline{\mathcal{E}}_2$, $\overline{\mathcal{F}}$ and $\overline{\mathcal{C}}$ be the real matrices defined in (25)–(28) with respect to $\overline{\sigma}$.
Theorem

Suppose that the given matrix $\overline{X} \in N$ has the SVD (21). Then the spectral operator $G$ is $F$-differentiable at $\overline{X}$ if and only if $g$ is $F$-differentiable at $\overline{\sigma}$. In that case, the derivative of $G$ at $\overline{X}$ is given by

$$G'(\overline{X})H = \overline{U}[\overline{E}_1 \circ S(A) + \text{Diag}(\overline{C}\text{diag}(S(A)))] + \overline{E}_2 \circ T(A) \quad \overline{F} \circ B]V^T \quad \forall H \in \mathbb{V}^{m \times n},$$

(29)

where $A := \overline{U}^T H \overline{V}_1$, $B := \overline{U}^T H \overline{V}_2$ and for any $X \in \mathbb{V}^{m \times m}$, $\text{diag}(X)$ denotes the column vector consisting of all the diagonal entries of $X$ being arranged from the first to the last.

Here the two linear matrix operators $S : \mathbb{V}^{p \times p} \rightarrow \mathbb{S}^p$ and $T : \mathbb{V}^{p \times p} \rightarrow \mathbb{V}^{p \times p}$ are given by

$$S(Y) := \frac{1}{2}(Y + Y^T), \quad T(Y) := \frac{1}{2}(Y - Y^T), \quad Y \in \mathbb{V}^{p \times p}.$$  

(30)
Let $\mathcal{Z}$ be a finite dimensional real Euclidean space equipped with an inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. Let $\mathcal{O}$ be an open set in $\mathcal{Z}$ and $\mathcal{Z}'$ be another finite dimensional real Euclidean space. The function $F : \mathcal{O} \subseteq \mathcal{Z} \to \mathcal{Z}'$ is said to be $B(ouligand)$-differentiable at $z \in \mathcal{O}$ if for any $h \in \mathcal{Z}$ with $h \to 0$,

$$F(z + h) - F(z) - F'(z; h) = o(\|h\|).$$

A stronger notion than $B$-differentiability is $\rho$-order $B$-differentiability with $\rho > 0$. The function $F : \mathcal{O} \subseteq \mathcal{Z} \to \mathcal{Z}'$ is said to be $\rho$-order $B$-differentiable at $z \in \mathcal{O}$ if for any $h \in \mathcal{Z}$ with $h \to 0$,

$$F(z + h) - F(z) - F'(z; h) = O(\|h\|^{1+\rho}).$$

**Theorem**

Suppose that $\overline{X} \in \mathcal{N}$ has the SVD (21). Let $0 < \rho \leq 1$ be given.

(i) If $g$ is locally Lipschitz continuous near $\sigma(\overline{X})$ and $\rho$-order $B$-differentiable at $\sigma(\overline{X})$, then $G$ is $\rho$-order $B$-differentiable at $\overline{X}$.

(ii) If $G$ is $\rho$-order $B$-differentiable at $\overline{X}$, then $g$ is $\rho$-order $B$-differentiable at $\sigma(\overline{X})$. 
**Theorem**

Suppose that $\overline{X} \in \mathcal{N}$ has the singular value decomposition (21). Let $0 < \rho \leq 1$ be given. $G$ is $\rho$-order $g$-semismooth at $\overline{X}$ if and only if $g$ is $\rho$-order $g$-semismooth at $\overline{\sigma}$. 
Assume that $g$ is locally Lipschitz continuous. Then since the spectral operator $G$ is locally Lipschitz continuous near $\overline{X}$, $\Psi = G'(\overline{X}; \cdot)$ is globally Lipschitz continuous if exists. In that case, $\partial_B \Psi(0)$ and $\partial \Psi(0)$ are well-defined. Furthermore, we have the following characterization of the $B$-subdifferential and Clarke’s subdifferential of the spectral operator $G$ at $\overline{X}$.

**Theorem**

Suppose that the given $\overline{X} \in \mathcal{N}$ has the decomposition (21). Suppose that there exists an open neighborhood $B \subseteq \mathbb{R}^m$ of $\overline{\sigma}$ in $\hat{\sigma}_\mathcal{N}$ such that $g$ is differentiable at $\sigma \in B$ if and only if $g'(\overline{\sigma}; \cdot)$ is differentiable at $\sigma - \overline{\sigma}$. Assume further that the function $d : \mathbb{R}^m \to \mathbb{R}^m$ defined by

$$d(h) := g(\sigma + h) - g(\sigma) - g'(\overline{\sigma}; h), \quad h \in \mathbb{R}^m$$

(31)

is strictly differentiable at zero. Then, we have

$$\partial_B G(\overline{X}) = \partial_B \Psi(0) \quad \text{and} \quad \partial G(\overline{X}) = \partial \Psi(0).$$

Many more to be developed ...
