Bounding Option Prices of Multi-Assets: A Semidefinite Programming Approach

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Abstract

Recently, semidefinite programming has been used to bound the price of a single-asset European call option at a fixed time. Given the first n moments, a tight bound can be obtained by solving a single semidefinite programming problem of dimension $n+1$. In this paper, we study the multi-asset case, which is generally more practical than the single-asset case. We construct a sequence of semidefinite programming relaxations. As the dimension of the semidefinite relaxations increases, the bound becomes more accurate and converges to the tight bound. Some numerical results are reported to illustrate the method.

Key words: Bound of option price, semidefinite programming relaxation, the moment problem.

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1 Introduction

The Black-Scholes [5] formula for the European call option price on a single asset at a fixed time can be obtained only under the conditions that the market is arbitrage-free and that the asset price return follows a lognormal distribution. Unfortunately, the latter condition is usually not satisfied in reality, which may result in pricing biases [16]. Some alternative models, based on some other distributions, have been studied, which include the jump diffusion process of Merton [20], the pure jump model of Cox et al. [11], and the compound option model of Geske [13]. Those models have a unique option price, but they still have pricing biases in some sense.

Without assuming an arbitrage-free market or a specific distribution of the underlying asset price, several authors tried to use tools from mathematical programming, such as linear programming and semidefinite programming, to obtain bounds on option prices [1, 8, 28, 29]. Given the first and the second order moments of the asset, Lo [19] derived an upper bound for the price of a single-asset European option by using a classical result of Scarf [33]. Recently, Lo's result has been extended by Bertsimas and Popescu [4] and Gotoh and Konno [15] to the case where the first n moments of the asset are known. A tight upper or lower bound can be obtained via solving a single semidefinite programming problem of dimensions $n+1$ [4, 15].

In many applications, the payoff on a derivative security often depends on more than one underlying asset. An example is the commodity index bonds issued by producers of a particular commodity. The coupon payments on the bond can depend on both a commodity price and some economic variables [9, 10]. Other examples of this kind can be found in [7, 22].

For European options of two underlying assets, Stulz [37] proposed a closed-form solution. Tilley and Latainer [39] and Johnson [17] extended Stulz's results to the case of an arbitrary number of underlying assets. However, all those results were derived under the assumption that the underlying assets follow the multivariate lognormal distribution, which is a strong assumption.

Boyle and Lin [8] derived distribution-free semi-parametric bounds on a European call on the maximum of any number of assets, which is an extension of Lo's results [19]. Similar to Lo's results, Boyle and Lin only needed the first and the second order moments (means and covariance matrix of the returns of m underlying assets) of a distribution. They suggested to obtain the bound via solving a semidefinite programming problem of dimension $(m+1)(m+2)/2$.

The upper bounds obtained by Boyle and Lin are better than those obtained by applying Lo's method directly to the multi-asset case. However, the upper bound from Boyle and Lin's method is still higher than the exact value (see [8, Table 3]).

In this paper, we propose to approximate the tight upper bound of the price of a multi-asset European call option by solving a sequence of semidefinite programming problems. The method can be viewed as a generalization of the method of Boyle and Lin [8] in the sense that the first n moments, rather than the first two moments are given. The method is also a generalization of the methods of Bertsimas and Popescu [4] and Gotoh and Konno [15] in the sense that we treat multi-assets rather than a single asset. As the dimension of the semidefinite programs increases, the solutions of the semidefinite programming problems converge to the tight bound under suitable assumptions.

It should be noted that Bertsimas and Popescu [2, 3] have also considered the multi-asset problem for a special case. They proved that the problem can be solved in polynomial time under the following three conditions; 1. the first two moments are given; 2. the underlying domain is the whole space R^m; and 3. the payoff function is a piecewise quadratic function. If one of the conditions is not satisfied, then the problem is NP-hard. Zuluaga and Peña [41] have also considered a semidefinite programming approximation problem for generalized Tchebycheff inequalities that includes the problem considered here as a special case. Their method is an outer approximation of the cone of moment sequences and can be viewed as the primal approach used by Lasserre [18]. The method in this paper is an inner approximation of the cone of positive polynomials associated with the technique of sums of squares. It can be viewed as the dual approach used by Shor, Nesterov, and Parrilo [23, 24, 36].

Notation and convention. Throughout the paper, R^m denotes the m dimensional Euclidean space and $R^m_+ := \{x \in R^m \mid x_i \geq 0,\ i = 1, \cdots, m\}$ is the nonnegative orthant of R^m. Similarly, Z^m denotes the set of m-tuples of integers and $Z^m_+ := \{z \in Z^m \mid z_i \geq 0,\ i = 1, \cdots, m\}$ is the set of m-tuples of nonnegative integers. Let e_j be the jth unit coordinate vector. We write "$X \succeq 0$" if a symmetric matrix X is positive semidefinite and "$X \succ 0$" if it is positive definite. We will use T to denote the exercise time while K denote the exercise price. In addition, $S_t := (S_{1t}, \ldots, S_{mt})^\top$ denote the price of m assets at time t. Let $i = (k_1, \ldots, k_m)^\top \in Z^m_+$, $x^i = x_1^{k_1} \ldots x_m^{k_m}$, $p_i = p_{k_1 \ldots k_m}$, $I_n := \{i \in Z^m_+ \mid \sum_{j=1}^{m} k_j \leq n\}$, and $|I_n|$ be the number of elements in I_n. Then any real-coefficient polynomial of degree k can be written in the compact form

$$p(x) = \sum_{i \in I_k} p_i x^i.$$

We will use $B \subseteq R^m$ to denote the domain under consideration, and $M^+(B)$ to denote the nonnegative regular Borel measure on B. For any $A \subseteq R^m$, $\mathrm{int}\, A$ denotes the interior of A.

The paper is organized as follows. In Section 2, we state the problem under consideration. We derive its dual problem and dicuss the relation between the primal and the dual. In Section 3, we present our semidefinite programming relaxation method and prove its convergence. Its relation to the method of Boyle and Lin [8] is discussed in Section 4. We illustrate our methods by some numerical results in Section 5 and conclude the paper in Section 6.

2 Formulation: Primal and Dual

Let $x \in R^m_+$ be a nonnegative random vector and $x_1, \cdots, x_m$ be its correlated components. Let $f:\ R^m \to R$ be a real valued function. Suppose we know the first n moments $m_i,\ i \in I_n$ of the probability measure μ. We are interested in the upper bound of the call option price

$$\text{(UB-P)} \begin{cases} \max\limits_{\mu(x)} & \int_B f(x) d\mu(x), \\ \text{subject to} & \int_B x^i d\mu(x) = m_i,\ i \in I_n, \\ & \mu \in M^+(B), \end{cases} \tag{1}$$

where $m_0 = 1$ corresponds to the probability mass constraint. If

$$f(x) = \max\{\max\{x_1, \cdots, x_m\} - K, 0\}, \tag{2}$$

and $x_1, \cdots, x_m$ are the prices of m assets at a fixed time T, then $f(x)$ is a European call on the maximum of these m assets with strike price K. When $x_1, \cdots, x_m$ represent the prices of an asset at different times $t_1 < t_2 < \cdots < t_m$, $f(x)$ is the payoff of a discrete lookback option. Furthermore, if $n = 2$, then (UB-P) reduces to the case studied in [8]. If $m = 1$, then (1) reduces to the case of single asset at a fixed time, which was studied in, e.g., [4, 15, 19]. Thus, model (1) is quite general.

To derive the dual of (1), let's first rewrite it as

$$\begin{array}{ll} \max\limits_{\mu} & \langle f(x), \mu(x) \rangle \\ \text{subject to} & \langle x^i, \mu(x) \rangle = m_i,\ i \in I_n, \\ & \mu(x) \in M^+(B), \end{array} \tag{3}$$

Similar to finite linear programming, we associate a dual variable $y_i,\ i \in I_n$ to the equality constraints, then the dual can be defined as

$$\text{(UB-D)} \begin{cases} \min\limits_{y} & \sum\limits_{i \in I_n} m_i y_i \\ \text{subject to} & \sum\limits_{i \in I_n} x^i y_i \geq f(x),\ \forall x \in B. \end{cases} \tag{4}$$

This is a semi-infinite programming problem in the sense that a linear objective function in the finite Euclidean space $R^{|I_n|}$ is minimized subject to an infinite number of linear constraints. Let V(UB-P) and V(UB-D) be the optimal values of the primal and the dual problems, respectively, which might be $\pm\infty$ in general. It is easy to establish the weak duality.

Theorem 2.1 (Weak duality) *V(UB-D)$\geq V$(UB-P).*

To establish strong duality, we need some additional conditions. Let S be a set and let cone(S) be the convex cone generated by S in the sense of convex analysis [30]. Consider the cone

$$\begin{aligned} M_{|I_n|+1} : &= \text{cone}\left(\left\{\begin{pmatrix} x^i \\ f(x) \end{pmatrix} \mid i \in I_n, x \in B\right\}\right) \\ &= \left\{\begin{pmatrix} \int_B x^i d\mu(x) \\ \int_B f(x) d\mu(x) \end{pmatrix} \mid i \in I_n, \mu \in M^+(B)\right\} \\ &\subseteq R^{|I_n|+1}. \end{aligned} \tag{5}$$

For the equality (5), see [32] for the case that B is compact and [35, Lemma 3.1] for the general case (See also [6, 31]).

Based on conic duality theory, Shapiro proved that ([35, Proposition 3.1],):

Theorem 2.2 (Strong duality) *Suppose that V(UB-P) is finite and the cone $M_{|I_n|+1}$ is closed in the standard topology of $R^{|I_n|+1}$. Then, V(UB-P)=V(UB-D) and the primal problem has an optimal solution.*

Consider also the cone of feasible moment vectors

$$\begin{aligned} M_{|I_n|} : &= \text{cone}\left\{x^i, \mid i \in I_n\ \ x \in B\right\} \\ &= \left\{\int_B x^i d\mu(x),\ i \in I_n \mid \mu \in M^+(B)\right\} \subseteq R^{|I_n|}. \end{aligned}$$

Then, under the following so-called *Slater* condition, strong duality also holds ([35, Proposition 3.4]):

Theorem 2.3 *If the moment vector* $m = (m_i,\ i \in I_n)^\top$ *satisfies*

$$m \in int(M_{|I_n|}), \tag{6}$$

then V(UB-P)=V(UB-D).

If the strong duality holds, then by solving the dual problem (UB-D), the exact upper bound of the original problem (UB-P) is obtained.

A similar relation also holds between the lower bound problem

$$\text{(LB-P)} \begin{cases} \min\limits_{\mu(x)} & \int_B f(x) d\mu(x), \\ \text{subject to} & \int_B x^i d\mu(x) = m_i,\ i \in I_n, \\ & \mu \in M^+(B), \end{cases} \tag{7}$$

and its dual

$$\text{(LB-D)} \begin{cases} \max\limits_{y} & \sum\limits_{i \in I_n} m_i y_i \\ \text{subject to} & \sum\limits_{i \in I_n} x^i y_i \leq f(x),\ \forall x \in B. \end{cases} \tag{8}$$

That is, the weak duality V(LB-D)$\leq$ V(LB-P) holds and under suitable conditions such as those in Theorem 2.2 and 2.3, the equality holds.

3 The SDP Relaxation

In the following, we restrict our domain B under consideration to $B = [0, B_i]_{i=1}^m$ and the function f to a polynomial. The case that f is a piecewise polynomial (e.g., the maximum function (2)) can be discussed in a similar way.

Since the constraints in (UB-D) involve nonnegative polynomials, we first investigate the conditions that guarantee a polynomial to be nonnegative. A univariate polynomial $f(x) = \sum\limits_{i=0}^{2n} y_i x^i$ is nonnegative on R if and only if there is a matrix $A = [a_{kj}]_{k,j=0,\cdots,n}$, such that [2, 23, 36]

$$\begin{aligned} y_i &= \sum_{k+j=i} a_{kj},\ i = 1, \cdots, 2n, \\ A &\succeq 0. \end{aligned}$$

A univariate polynomial $f(x) = \sum_{i=0}^{n} y_i x^i$ is nonnegative on R_+ if and only if there is a matrix $A = [a_{kj}]_{k,j=0,\cdots,n}$, such that [2, 23]

$$\begin{aligned} y_i &= \sum_{k+j=2i} a_{kj},\ i = 0, \cdots, n, \\ 0 &= \sum_{k+j=2i-1} a_{kj},\ i = 1, \cdots, n, \\ A &\succeq 0. \end{aligned}$$

Thus, when $m = 1$, the problem (UB-D) is equivalent to a semidefinite programming problem, which can be solved efficiently by *interior point methods* [21]. If in addition there is no duality gap between (UB-P) and (UB-D) (Theorems 2.2 and 2.3), then the problem (UB-P) is also solved [4, 15].

In the multivariate case ($m \geq 2$), the problem is generally very hard. Clearly, if a polynomial can be written as a *sum of squares* (sos for short) of other polynomials, then, it is nonnegative. The converse is not true, as shown by the following example [27]

$$M(x, y, z) = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2,$$

where the nonnegativity follows from the arithmetic-geometric inequality and the nonexistence of an sos decomposition follows from standard algebraic manipulations [27].

A polynomial $p(x)$ is nonnegative on a semialgebraic set $D \subseteq R^m$ defined by polynomial inequalities

$$D := \{x \in R^m \mid p_j(x) \geq 0,\ j = 1, \cdots, l\} \tag{9}$$

if it can be written as

$$p(x) = s_0(x) + \sum_{i=1}^{l} p_i(x)s_i(x), \tag{10}$$

where $s_i(x),\ i = 0, \cdots, l$ are all sums of squares. Same as the unconstrained case, the above decomposition is not necessary; that is, a polynomial $p(x)$ that is nonnegative on D may not be represented as (10). However, Putinar [26] proved that

Theorem 3.1 *Suppose that the semialgebraic set D defined by (9) is compact and there is a polynomial $p:\ R^m \to R$*

$$p(x) = s_0(x) + \sum_{i=1}^{l} p_i(x)s_i(x),\ \forall x \in R^m$$

such that the set

$$\{x \in R^m \mid p(x) \geq 0\}$$

is compact and the polynomials $s_i(x),\ i=0,\ \cdots,\ l$ are all sums of squares. Then any polynomial $v(x)$, strictly positive on D, can be written as

$$v(x) = u_0(x) + \sum_{i=1}^{l} p_i(x)u_i(x),\ \forall x \in R^m$$

for some polynomials $u_i(x),\ i=0,\ \cdots,\ l$ that are all sums of squares. □

The conditions in Theorem 3.1 are satisfied in many cases. For example [26],

1. There is one polynomial $p_j(x)$ such that the set $\{x \mid p_j(x) \geq 0\}$ is compact. In this case, we can take $u_i(x) \equiv 0$ for all $i \neq j$ and $u_j(x) = 1$;
2. All $p_i(x)$ are linear, $i = 1, \cdots, l$ and D is compact;
3. For 0-1 programs.

Recall that our problem (UB-D) is

$$\text{(UB-D)} \begin{cases} \min\limits_{y} & \sum\limits_{i \in I_n} m_i y_i \\ \text{subject to} & \sum\limits_{i \in I_n} x^i y_i - f(x) \geq 0,\ \forall x \in B. \end{cases} \tag{11}$$

Based on the above discussion, a simple relaxation of our problem is

$$\text{(UB-R)} \begin{cases} \min\limits_{y} & \sum\limits_{i \in I_n} m_i y_i \\ \text{subject to} & \sum\limits_{i \in I_n} x^i y_i - f(x) = s^0(x) + \sum\limits_{i=1}^{m} x_i s^{1i}(x) + \sum\limits_{i=1}^{m} (B_i - x_i) s^{2i}(x), \\ & s^0(x), s^{ij}(x) \text{ are sos},\ i = 1, 2,\ j = 1, \cdots, m. \end{cases} \tag{12}$$

The following lemma shows that the sos condition is equivalent to positive semidefiniteness of a certain matrix.

Lemma 3.1 *The polynomial $g(x) = \sum_{r=0}^{2k} y_r x^r$ is sum of squares if and only if there is a positive semidefinite matrix M, such that*

$$g(x) = d_k(x)^\top M d_k(x), \tag{13}$$

where

$$d_k(x) = (1, x_1, \ldots, x_m, x_1^2, x_1 x_2, \ldots, x_m^2, \ldots, x_1^k, \ldots, x_m^k)^\top$$

is a basis vector for polynomials of m variables of degree at most k.

Proof. " $\Longrightarrow$ ". Suppose that there is $M \succeq 0$, such that (13) holds. Let $M = \sum_i a_i w_i w_i^\top$ be its eigenvalue-eigenvector decomposition. Since $M \succeq 0$, $a_i \geq 0$, $\forall i$. We have

$$g(x) = d_k(x)^\top M d_k(x) = \sum_i a_i d_k(x)^\top w_i w_i^\top d_k(x) = \sum_i a_i \left(\sum_j w_{ij} (d_k(x))_j \right)^2 .$$

" $\Longleftarrow$ ". Suppose that g is a sum of squares, i.e.,

$$g(x) = \sum_i (f_i(x))^2.$$

Let F_i be the vector of the coefficients of $f_i(x)$ under the basis $d_k(x)$. Then,

$$g(x) = \sum_i (F_i^\top d_k(x))^2 = \sum_i d_k(x)^\top F_i F_i^\top d_k(x),$$

and (13) holds with $M = \sum_i F_i F_i^\top$. □

Let $\alpha = \lceil \deg(f)/2 \rceil$ be the smallest integer larger than $\deg(f)/2$, half of the degree of f, $\beta = \lceil n/2 \rceil$ and $\gamma = \max\{\alpha, \beta\}$. Denoting

$$s^0(x) = d_{\gamma+N}(x)^\top S^0 d_{\gamma+N}(x),\ s^{il}(x) = d_{\gamma+N-1}(x)^\top S^{il} d_{\gamma+N-1}(x),\ i = 1, 2,\ l = 1, \cdots, m,$$

where $d_0(x) := 0$, $S^0 \succeq 0$, and $S^{il} \succeq 0$, $i = 1, 2$, $l = 1, \cdots, m$, by Lemma 3.1 we can rewrite problem (12) as for a certain N

$$\text{(UB-D-R(N))} \begin{cases} \min\limits_{y} & \sum\limits_{i \in I_n} m_i y_i \\ \text{subject to} & \sum\limits_{i \in I_n} x^i y_i - f(x) \\ & \quad = d_{\gamma+N}(x)^\top S^0 d_{\gamma+N}(x) + \sum\limits_{i=1}^m x_i d_{\gamma+N-1}(x)^\top S^{1i} d_{\gamma+N-1}(x) \\ & \qquad + \sum\limits_{i=1}^m (B_i - x_i) d_{\gamma+N-1}(x)^\top S^{2i} d_{\gamma+N-1}(x) \\ & S^0 \succeq 0,\ S^{jl} \succeq 0, j = 1, 2,\ l = 1, \cdots, m. \end{cases}$$

Theorem 3.2 *(UB-D-R(N)) is equivalent to the following semidefinite program*

$$\text{(UB-D-SDP(N))}\left\{\begin{array}{ll} \min\limits_{y} & \sum\limits_{i\in I_n} m_i y_i \\ \text{subject to} & y_i - f_i = s_i^0 + \sum\limits_{k=1}^{m}(s_{i-e_i}^{1k} - s_{i-e_i}^{2k}) + \sum\limits_{k=1}^{m} s_i^{2k}B_i,\ \forall i \in I_{\gamma+N} \\ & S^0 = [S_{ij}^0]_{i,j\in I_{\gamma+N}} \succeq 0 \\ & s_k^0 = \sum\limits_{i,j\in I_{\gamma+N}, i+j=k} S_{ij}^0 \\ & S^{1l} = [S_{ij}^{1l}]_{i,j\in I_{\gamma+N-1}} \succeq 0 \\ & s_k^{1l} = \sum\limits_{i,j\in I_{\gamma+N-1}, i+j=k} S_{ij}^{1l},\ l = 1,\cdots,m \\ & S^{2l} = [S_{ij}^{2l}]_{i,j\in I_{\gamma+N-1}} \succeq 0 \\ & s_k^{2l} = \sum\limits_{i,j\in I_{\gamma+N-1}, i+j=k} S_{ij}^{2l},\ l = 1,\cdots,m. \end{array}\right.$$

Proof. By equating terms in (UB-D-R(N)), we obtain the results immediately. □

Let

$$F := \left\{ y \in R^{|I_n|} \mid \sum_{i\in I_n} y_i x^i - f(x) \geq 0,\ \forall x \in B \right\}$$

denote the feasible set of (UB-D) and let

$$F_N := \{y \in R^{|I_n|} \mid \exists S^0,\ S^{il}, \text{ such that the constraints in (UB-D-SDP(N)) hold}\}$$

denote the projection of the feasible set of (UB-D-SDP(N)) to $R^{|I_n|}$. Then, from the construction we have that

$$F_1 \subseteq \cdots \subseteq F_N \subseteq F_{N+1} \subseteq \cdots \subseteq F. \tag{14}$$

For any $y \in F$ and any small positive number ϵ, we have

$$\sum_{i\in I_n} x^i y_i - f(x) + \epsilon > 0,\ \forall x \in B.$$

It then follows from Putinar's theorem (Theorem 3.1) that there exists a number L such that

$$y + \epsilon e_0 \in F_L,$$

where $e_0 \in R^{|I_n|}$ with its $0th$ component 1 and all others 0. Since $\epsilon > 0$ is arbitrary, we conclude that

$$F = \bigcup_N F_N. \tag{15}$$

Theorem 3.3 *Let V_N(UB-R) be the optimal value of the problem (UB-D-SDP(N)) and V(UB-D) be the optimal value of problem (UB-D), then*

$$V_1(UB\text{-}R) \geq \cdots \geq V_N(UB\text{-}R) \geq V_{N+1}(UB\text{-}R) \geq \cdots \geq V(UB\text{-}D), \tag{16}$$

and

$$\lim_{N\to\infty} V_N(UB\text{-}R) = V(UB\text{-}D). \tag{17}$$

Proof. The relation (16) follows from (14) and (17) follows from (15). □

Let $M = \{p_{i_1}p_{i_2}\cdots p_{i_k} \mid i_1, i_2, \cdots i_k \in \{1, \cdots l\}\}$ be the set of all partial products of p_{i_j}, where p_{i_j} is a defining polynomial of D, $i_j = 1, \cdots, l$. Without the assumptions in Theorem 3.1, there is another representation of polynomials strictly positive on D [34], i.e.,

$$p(x) > 0,\ \forall x \in D \Longrightarrow p(x) = s^0(x) + \sum_{v_i \in M} v_i(x)s^i(x),$$

where $i = (i_1, \cdots, i_k), i_j \in \{1, \cdots, l\}$, such that $v_i(x) = p_{i_1}(x)\cdots p_{i_k}(x)$, $s^0(x)$, and $s^i(x)$ are all sum of squares. Then we still have semidefinite programming relaxation of (UB-D).

If f is not a polynomial but a 'piecewise' polynomial in the sense that we can do a partition of the underlying domain $B = \bigcup_{j=1}^{k} C_j$, such that each C_j is a semi-algebraic set and

$$f(x) = f_j(x),\ \forall x \in C_j,\ j = 1, \cdots, k,$$

then (UB-D) can be written as

$$\begin{array}{ll} \min\limits_{y} & \sum\limits_{i \in I_n} m_i y_i \\ \text{subject to} & \sum\limits_{i \in I_n} x^i y_i \geq f_j(x),\ \forall x \in C_j,\ j = 1, \cdots, k, \end{array} \tag{18}$$

and we can first define a similar sums of squares relaxation for (18) then solve the resulted semidefinite programs to get an upper bound of (UB-P).

In summary, our method for solving (UB-D) solves the semidefinite programming relaxations (UB-D-SDP(N)), starting with $N = 1$. If the solution of the semidefinite programming problem is good enough, then stop; otherwise, increase N and solve the next semidefinite programming problem to get a better approximation. Theorem 3.3 guarantees that a suitable bound will be obtained for some N. As to whether the solution of a semidefinite programming problem is good enough, one can refer to [25].

4 Relation to the method of Boyle and Lin

In this section, we discuss the relation between our method and the semidefinite programming relaxation method of Boyle and Lin [8].

Boyle and Lin considered the option price of a European call on the maximum of any number of assets, given the means, variances and covariance matrix of asset prices at the time to option maturity T. That is

$$f(x) = \max\{\max\{x_1, \cdots, x_m\} - K, 0\},$$

with strike price K. Let $B = R_+^m$, then the problem is

$$(\text{UB-P}) \begin{cases} \max\limits_{\mu(x)} & \int_{R_+^m} \max\{\max\{x_1, \cdots, x_m\} - K, 0\} d\mu(x), \\ \text{subject to} & \int_{R_+^m} x^i d\mu(x) = m_i, \ i \in I_2, \\ & \mu \in M^+(R_+^m), \end{cases}$$

The dual problem then becomes

$$(\text{UB-D}) \begin{cases} \min\limits_{y} & \sum\limits_{i \in I_2} m_i y_i \\ \text{subject to} & \sum\limits_{i \in I_2} x^i y_i \geq \max\{\max\{x_1, \cdots, x_m\} - K, 0\}, \ \forall x \in R_+^m. \end{cases}$$

Let

$$\mathcal{E}_0 := \{x \mid 0 \leq x_i \leq K, \ i = 1, \cdots, m\}$$

and

$$\mathcal{E}_i := \{x \mid x_i \geq K, \ 0 \leq x_j \leq x_i, \ j \neq i\}.$$

Then, the dual problem can be written equivalently as

$$\begin{aligned} \min_{y} \quad & \sum_{i \in I_2} m_i y_i \\ \text{s.t.} \quad & \sum_{i \in I_2} x^i y_i \geq 0, \ \forall x \in \mathcal{E}_0 && (19) \\ & \sum_{i \in I_2} x^i y_i - x_j + K \geq 0, \ \forall x \in \mathcal{E}_j, \ j = 1, \cdots, m. && (20) \end{aligned}$$

Boyle and Lin [8] first relaxed the conditions in (19) and (20) to R^m,

$$\begin{aligned} \min_{y} \quad & \sum_{i \in I_2} m_i y_i \\ \text{s.t.} \quad & \sum_{i \in I_2} x^i y_i \geq 0, \ \forall x \in R^m \\ & \sum_{i \in I_2} x^i y_i - x_j + K \geq 0, \ \forall x \in R^m \ j = 1, \cdots, m. \end{aligned}$$

Since the polynomial is quadratic, this problem is in fact equivalent to the sos problem [27]

$$\begin{array}{ll} \min\limits_{y} & \sum\limits_{i \in I_2} m_i y_i \\ \text{s.t.} & \sum\limits_{i \in I_2} x^i y_i \ \text{ is sos} \\ & \sum\limits_{i \in I_2} x^i y_i - x_j + K \ \text{ is sos } j = 1, \cdots, m, \end{array}$$

which, by Lemma 3.1, is equivalent to

$$(BL) \left\{ \begin{array}{ll} \min\limits_{y} & \sum\limits_{i \in I_2} m_i y_i \\ \text{s.t.} & \begin{bmatrix} y_{00} & \frac{1}{2}y_{01} & \cdots & \frac{1}{2}y_{0m} \\ \frac{1}{2}y_{01} & y_{11} & \cdots & \frac{1}{2}y_{1m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}y_{0m} & \frac{1}{2}y_{1m} & \cdots & y_{mm} \end{bmatrix} \succeq 0 \\ & \begin{bmatrix} y_{00}+K & \frac{1}{2}y_{01} & \cdots & \frac{1}{2}y_{0j}-1/2 & \cdots & \frac{1}{2}y_{0m} \\ \frac{1}{2}y_{01} & y_{11} & \cdots & \cdots & \cdots & \frac{1}{2}y_{1m} \\ \vdots & \vdots & \cdots & \ddots & \cdots & \vdots \\ \frac{1}{2}y_{0j}-1/2 & \frac{1}{2}y_{1j} & \cdots & \cdots & \cdots & \frac{1}{2}y_{jm} \\ \vdots & \vdots & \cdots & \ddots & \cdots & \vdots \\ \frac{1}{2}y_{0m} & \frac{1}{2}y_{1m} & \cdots & \cdots & \cdots & y_{mm} \end{bmatrix} \succeq 0, \ j = 1, \cdots, m. \end{array} \right.$$

A simple comparison between (BL) and (UB-D-R(N)) shows that

$$\text{(BL)} \Longleftrightarrow \text{(UB-D-R(1))},$$

that is, the relaxation of Boyle and Lin serves as our first level relaxation. Boyle and Lin then solve the following *nonlinear* semidefinite programming problem to get the upper bound:

$$\begin{array}{ll} \min & \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{m} a_{ij} \left[\rho_{ij}\sigma_i\sigma_j + (\mu_i - K + \frac{1}{4}b_i)(\mu_j - K + \frac{1}{4}b_j) \right] \\ \text{s.t.} & A = [a_{ij}] \succ 0, \end{array}$$

where b_i is the ith diagonal entry of A^{-1}, ρ_{ij} is the correlation coefficient between x_i and x_j, σ_i is the standard variance and μ_i is mean of x_i.

5 Numerical Results

We now illustrate the application of the proposed method to some problems of bounding the option price on several assets. The first example is an artificial one to check the reliability of the method with $f(x) = (x_1 + x_2)^2$. The second involves a European call option where f is the maximum function as (2), which is the first example considered in [8]. We show that we can get better bounds with the third level relaxation than those obtained from Boyle and Lin's method. The third is the two asset rainbow and the fourth is the basket option on two currencies. We use the sum of squares optimization toolbox, a free Matlab software developed by Prajna, Papachristodoulou and Parrilo [25] based on the method of Parrilo [24] and available at http://www.cds.caltech.edu/sostools to solve (UB-D-R(N)), in which the semidefinite programming problem (UB-D-SDP(N)) is solved with SeDuMi version 1.03 developed by Sturm [38]. The computations are done on a Pentium IV 2.4G Hz PC with 256M RAM.

In applying our method, we do not need to make any assumption on the distribution of the underlying assets. For convenience of comparison, however, we assume in the following that the underlying assets follow a multivariate lognormal distribution. Under this assumption, the k-th moment can be computed by

$$\begin{aligned} m_k &= E\left[\prod_{i=1}^{m} S_i(T)^{k_i}\right] \\ &=\prod_{i=1}^{m} S_i(t)^{k_i} \exp\left(\left[kr - \sum_{i=1}^{m} k_i q_i + \frac{1}{2}\sum_{i=1}^{m} k_i(k_i-1)\sigma_i^2 + \sum_{i=1}^{m-1}\sum_{j>i}^{m} k_i k_j \rho_{ij}\sigma_i\sigma_j\right](T-t)\right), \end{aligned} \tag{21}$$

where $k_i = 0, 1, \cdots, k$, $\sum_{i=1}^{m} k_i = k$, $S_i(t)$ is the price of asset i at time t, T is the maturity, r is the interest rate, q_i and σ_i are the dividend and volatility of asset i, respectively, and ρ_{ij} is the correlation of the assets.

5.1 A small example with known solution

To check the reliability of our method, we consider the case of two assets, the underlying domain $B = R^2$ and the payoff function $f = (S_1 + S_2)^2$. In this case, the payoff of the derivative can be expressed as

$$\begin{aligned} &v(t, S_1, S_2) \\ &= S_1(t)^2 e^{(r-2q_1+\sigma_1^2)(T-t)} + S_2(t)^2 e^{(r-2q_2+\sigma_2^2)(T-t)} + 2S_1(t)S_2(t)e^{(r-q_1-q_2+\rho\sigma_1\sigma_2)(T-t)}. \end{aligned} \tag{22}$$

Table 1. Semidefinite relaxation values for $\rho = 0.5, 0, -0.5$

ρ	Exact Value	$N = 1$	$N = 2$	$N = 3$	$N = 4$	$N = 5$
0.5	539.4928	562.0363	562.0281	562.0008	560.4683	540.5069
0	529.8517	557.0212	557.0085	556.8446	556.2763	529.3202
-0.5	520.5655	552.0880	552.1005	550.7934	550.7692	520.0693

The riskless interest rate r is assumed to be 5%, the time to option maturity, T, is one year, the current price is $S_1(0) = 10$ and $S_2(0) = 12$, $q_1 = q_2 = 0$ and the volatilities are $\sigma_1 = 25\%$ and $\sigma_2 = 30\%$, respectively. Suppose that we know the first four moments, that is, $n = 4$.

The computational results are listed in Table 1, where the first column is the values of ρ, the second column is the exact value computed via (22), and the third to the seventh columns are the SDP values of the first to the fifth level of relaxation, respectively.

we can see from Table 1 that with the increase of the level of the semidefinite programming relaxation, we obtain closer and closer upper bound of the exact value. We plot the exact value and the computational results in Figures 1, 2, and 3, where $\rho = 0.5, 0$, and -0.5, respectively.

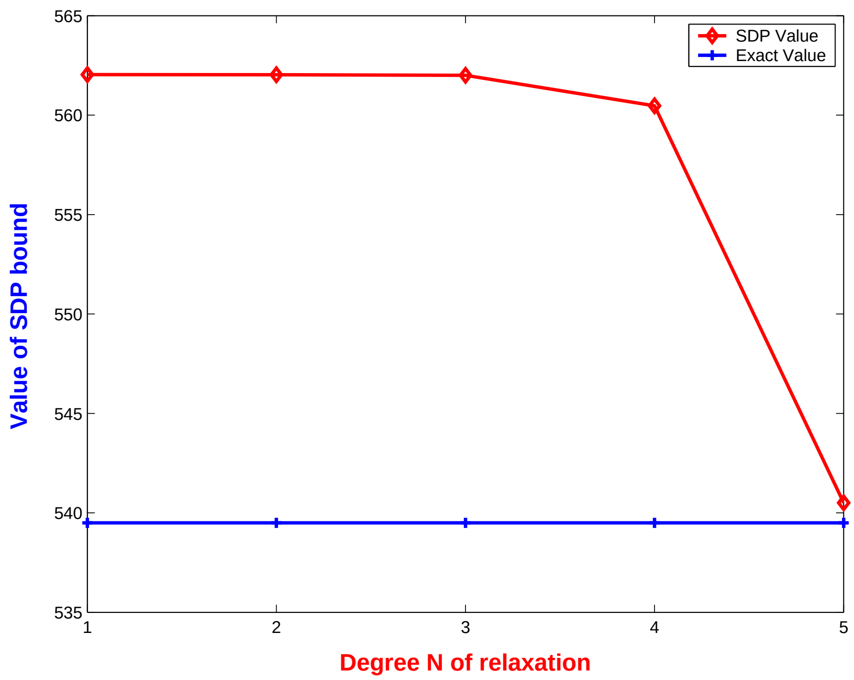

Figure 1: Computational results for $\rho = 0.5$ (moments up to 4 order)

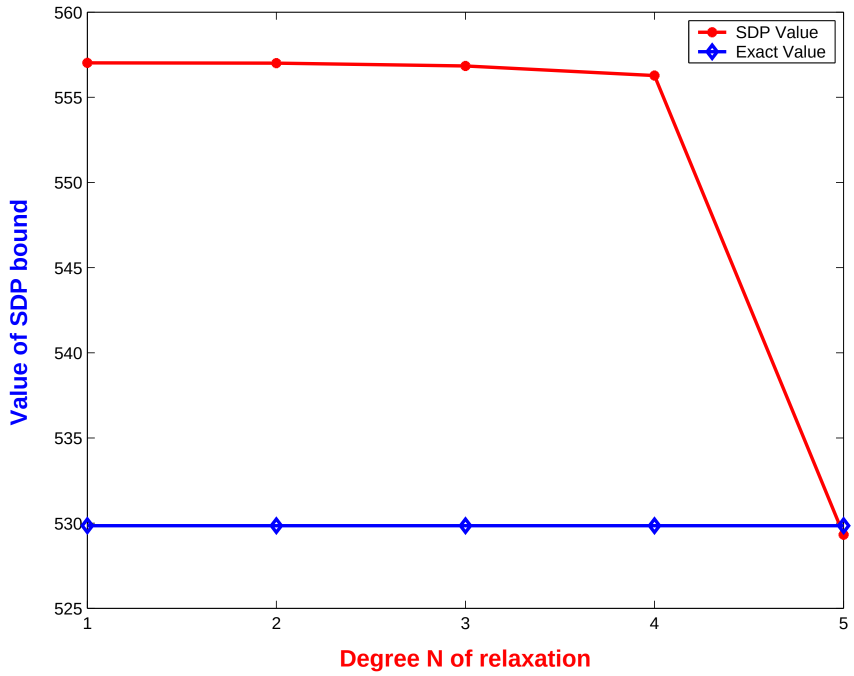

Figure 2: Computational results for $\rho = 0$ (moments up to 4 order)

5.2 An example of Boyle and Lin

In this subsection, we consider example 1 from Boyle and Lin [8], which is a call option on the maximum of three assets. We use the same data. That is, the current price $S_1(0) = S_2(0) = S_3(0) = 40$, $r = 10\%$, $q_1 = q_2 = q_3 = 0$, $\sigma_1 = \sigma_2 = \sigma_3 = 30\%$, and $\rho_{12} = \rho_{13} = \rho_{23} = 0.9$. We get the means and covariance from [8, Table 2].

The computational results of the upper and lower bounds are reported in Table 2 and Table 3, respectively. From Table 2 we can observe that the results of the third level semidefinite relaxation are better than the upper bounds obtained in [8].

Figures 4 and 5 plot the results in Table 2 and Table 3, respectively, and Figure 6 plots the exact value, the upper bound and the lower bound of the third level relaxation with the strike price.

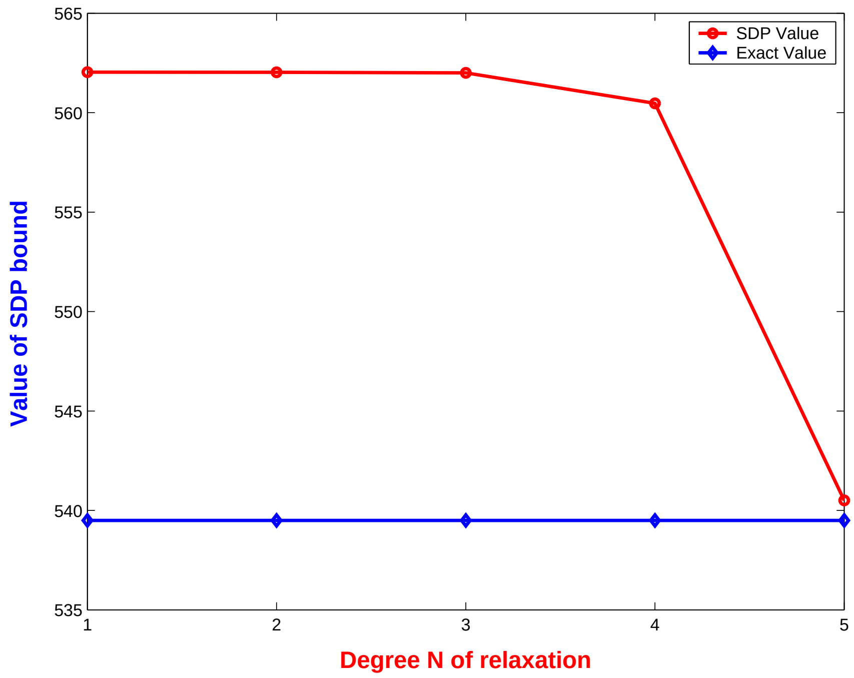

Figure 3: Computational results for $\rho = -0.5$ (moments up to 4th order)

Table 2. Upper bound from the semidefinite relaxation with up to 2nd order moments

Strike price	Exact value	Upper bound from this paper			Upper bound of BL
		$N = 1$	$N = 2$	$N = 3$	
30	16.35	21.5136	20.2105	17.3405	19.39
35	12.38	17.1727	14.5008	13.9102	15.52
40	8.98	13.2100	11.7852	10.7630	11.94
45	6.27	9.8520	7.5018	6.6347	8.88
50	4.23	7.3095	4.7313	4.3380	6.59

Table 3. Lower bound from the semidefinite relaxation with up to 2nd order moments

Strike price	Exact value	Lower bound from this paper		
		$N = 1$	$N = 2$	$N = 3$
30	16.35	14.2100	14.5870	16.2565
35	12.38	9.2096	9.7160	12.1891
40	8.98	4.2096	5.9368	8.1489
45	6.27	0.0000	4.1017	5.6945
50	4.23	0.0000	1.7886	2.1857

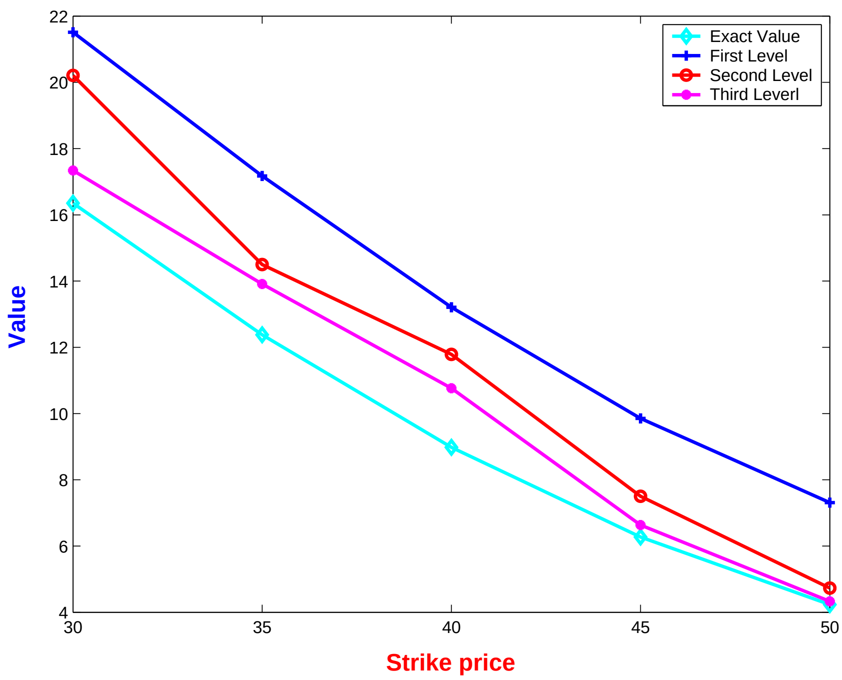

Figure 4: Computational results: Upper bound

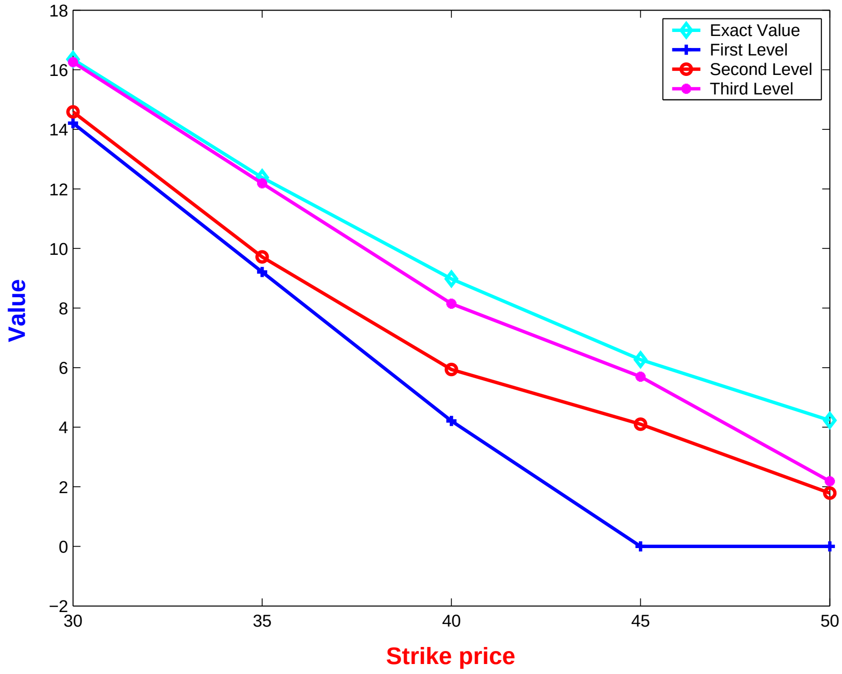

Figure 5: Computational results: Lower bound

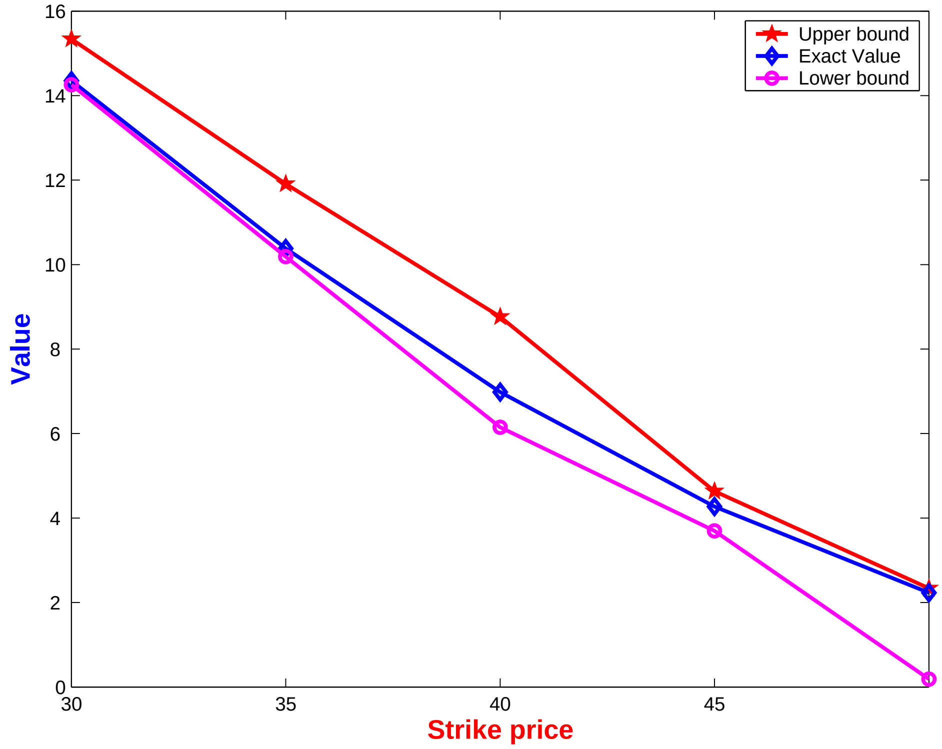

Figure 6: Computational results: Upper bound and lower bound of the third relaxation

Table 4. Semidefinite relaxation value for two asset rainbow with up to 2nd order moments

Exact Value	$N=1$	$N=2$	$N=3$	$N=4$	$N=5$
8.26	15.1096	15.1017	14.5737	13.0403	8.3667

5.3 Two asset rainbow

The two assets satisfy, $S_1(0) = S_2(0) = 100$, $\sigma_1 = \sigma_2 = 10\%$, $\rho = 0$, $q_1 = \ln(1.05)$ and $q_2 = 0$. We price an at-the-money basket option $S_1(T) + S_2(T)$, with $K = 200$, $T = 0.5$ and the interest rate $r = \ln(1.1)$. For this basket option, our method with the fifth level relaxation yields a price of \$8.3667, while Rubinstein's quasi-binomial method yields a price of \$8.26 and the high-order Gauss-Hermite integration method of [40] yields a price of \$8.2612. See Table 4 and Figure 7

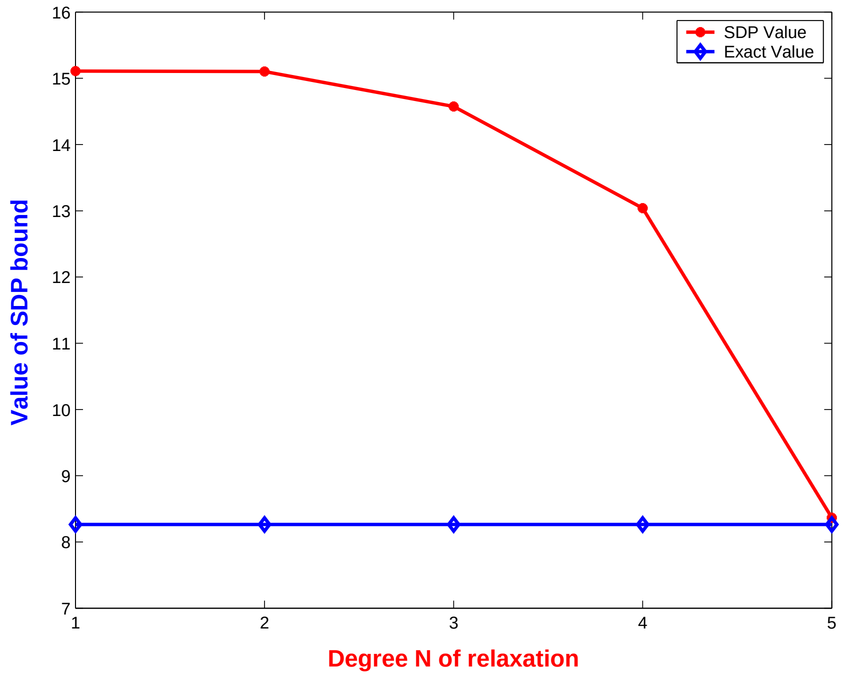

Figure 7: Computational results for two asset rainbow

5.4 Basket options on two currencies

We consider several basket options on two currencies from [14], wherein some present works are reviewed and benchmarked. The nominal underlying currency is US Dollar. The parameters are as follows. The interest rate in US Dollar is $r = 0.04$, the Yen amount S_1 is 1,250,000 with an

Table 5. Basket call options on two currencies.

Correlation $\rho(\%)$	Strike price	Rubinstein	Vorst/Gentle	Our Method
-50	27,000	2,402.19	2,402.46	2,408.1
-50	29,400	492.74	490.68	493.7
-50	31,000	61.69	59.21	65.0
0	27,000	2,432.70	2,432.70	2,438.9
0	29,400	654.74	652.97	656.7
0	31,000	152.83	151.90	153.4
50	27,000	2,476.98	2,476.71	2,479.3
50	29,400	782.63	781.75	782.2
50	31,000	244.28	243.66	244.4

exchange rate of \$0.008/Yen, with its interest rate playing the role of dividend $q_1 = 3.5\%$, and the Sterling amount S_2 is 10,000 with the exchange rate of \$2/Sterling and its interest rate playing the role of dividend $q_2 = 10\%$. The volatility of Yen expressed in US Dollar is $\sigma_1 = 12\%$ and that of Sterling is $\sigma_2 = 10\%$. The maturity time is six months, and the correlation matrix is given by $\rho(S_1, S_2)$. The results are in Table 5, where the column of Rubinstein is obtained by a simulation method, hence could somehow be thought of as the approximate exact value. The column of Vorst/Gentle shows a lowerbound in [14] and the column of our methods shows an upperbound obtained with $N = 5$.

6 Conclusions

We have considered the problem of bounding the option prices on multi-asset, under the condition of knowing the first n moments. We proposed to approximate the dual problem, a semi-infinite programming problem, by using the technique of sum of squares of polynomials, which is equivalent to a semidefinite programming problem. Under suitable conditions, we proved the asymptotic convergence of the method. Some numerical results were reported, which indicated that we can get quite good bounds after a few iterations.

The method can be extended to the problem where f is a piecewise polynomial. The discussion is similar to the polynomial case. However, this is true only if the number of the pieces is small from the numerical point of view since the scale of the resulted semidefinite programming problem may be numerically prohibitive if the number of the pieces is large.

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