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Strong Semismoothness of the Fischer-Burmeister SDC and SOC Complementarity Functions

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Abstract. We show that the Fischer-Burmeister complementarity functions, associated to the semidefinite cone (SDC) and the second order cone (SOC), respectively, are strongly semismooth everywhere. Interestingly enough, the proof relys on a relationship between the singular value decomposition of a nonsymmetric matrix and the spectral decomposition of a symmetric matrix.

Key words. Fischer-Burmeister function, SDC, SOC, SVD, strong semismoothness

1. Introduction

Let $\mathcal{M}_{p,q}$ be the linear space of $p \times q$ real matrices. We denote the ijth entry of $A \in \mathcal{M}_{p,q}$ by A_{ij}. For any two matrices A and B in $\mathcal{M}_{p,q}$, we write

$$A \bullet B := \sum_{i=1}^{p} \sum_{j=1}^{q} A_{ij} B_{ij} = \operatorname{tr}(AB^T)$$

for the *Frobenius inner product* between A and B, where "tr" denotes the trace of a matrix. The *Frobenius norm* induced by the above inner product on $\mathcal{M}_{p,q}$ is defined as $\| A \|_{\mathcal{F}} := \sqrt{A \bullet A}$. The identity matrix in $\mathcal{M}_{p,p}$ is denoted by I.

Let $\mathcal{S}^p$ be the linear space of $p \times p$ real symmetric matrices; let $\mathcal{S}^p_+$ denote the cone of $p \times p$ symmetric positive semidefinite matrices. For any vector $y \in \Re^p$, let $\operatorname{diag}(y_1, \ldots, y_p)$ denote the $p \times p$ diagonal matrix with its ith diagonal entry being y_i. We write $X \succeq 0$ to mean that X is a symmetric positive semidefinite matrix. Throughout this paper, we let X_+ denote the (Frobenius) projection of $X \in \mathcal{S}^p$ onto $\mathcal{S}^p_+$. The projection X_+ has an explicit representation; namely, if

$$X = P\Lambda(X)P^T , \tag{1}$$

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where $\Lambda(X) := \operatorname{diag}(\lambda_1, ..., \lambda_p)$ is the diagonal matrix of eigenvalues of X and P is the corresponding orthogonal matrix of orthonormal eigenvectors, then $X_+ = P\Lambda(X)_+P^T$, where $\Lambda(X)_+ := \operatorname{diag}\big(\max(\lambda_1, 0), ..., \max(\lambda_p, 0)\big)$. If $X \in \mathcal{S}^p_+$, then we use $\sqrt{X} := P\sqrt{\Lambda(X)}\,P^T$ to denote the square root of X, where X has the spectral decomposition (1) and $\sqrt{\Lambda(X)} := \operatorname{diag}(\sqrt{\lambda_1}, \dots, \sqrt{\lambda_p})$. For $X \in \mathcal{S}^p$, we let $|X| := \sqrt{X^2}$.

A function $\Phi^{\rm sdc} : \mathcal{S}^p \times \mathcal{S}^p \to \mathcal{S}^p$ is called a semidefinite cone (SDC) complementarity function if

$$\Phi^{\rm sdc}(X, Y) = 0 \Longleftrightarrow \mathcal{S}^p_+ \ni X \perp Y \in \mathcal{S}^p_+, \tag{2}$$

where the symbol $\perp$ means "perpendicular under the Frobenius matrix inner product"; i.e., $X \perp Y \Leftrightarrow X \bullet Y = 0$ for any two matrices X and Y in $\mathcal{S}^p$. Of particular interest are two SDC complementarity functions

$$\Phi^{\rm sdc}_{\min}(X, Y) := X - (X - Y)_+ \tag{3}$$

and

$$\Phi^{\rm sdc}_{\rm FB}(X, Y) := X + Y - \sqrt{X^2 + Y^2}\,. \tag{4}$$

The function $\Phi^{\rm sdc}_{\min}$ is called the matrix-valued min-function. It is known that $\Phi^{\rm sdc}_{\min}$ is globally Lipschitz continuous, directionally differentiable [1], and strongly semismooth [15] (see [14] for the definition of strong semismoothness). Strong semismoothness plays a fundamental role in the analysis of the quadratic convergence of Newton's method for solving systems of nonsmooth equations [13, 14]. Newton-type methods for solving the semidefinite programming and the semidefinite complementarity problem based on a smoothed form of $\Phi^{\rm sdc}_{\min}$ are discussed in [4, 5, 12, 17].

The function $\Phi^{\rm sdc}_{\rm FB}$ is called the matrix-valued Fischer-Burmeister function. When $p = 1$, $\Phi^{\rm sdc}_{\rm FB}$ is reduced to the scalar-valued Fischer-Burmeister function $\phi_{\rm FB}(a, b) := a + b - \sqrt{a^2 + b^2}$, $a, b \in \Re$, which is introduced by Fischer [8]. In [18], Tseng proves that $\Phi^{\rm sdc}_{\rm FB}$ satisfies (2). Borwein and Lewis also suggest a proof in their recent book [2, Exercise 5.2.11]. A desirable property of $\Phi^{\rm sdc}_{\rm FB}$ is its continuous differentiability [18]. For other properties of SDC complementarity functions, see [18, 19].

The primary motivation of this paper is to prove that $\Phi^{\rm sdc}_{\rm FB}$ is globally Lipschitz continuous, directionally differentiable, and strongly semismooth. This goal is achieved in Section 2 by using a relationship between the singular value decomposition of a nonsymmetric matrix and the spectral decomposition of a symmetric matrix in higher dimension and by using the same properties of the function $|Y|$, $Y \in \mathcal{S}^p$, obtained in [15]. We then proceed to study similar properties of the vector-valued complementarity functions associated with the second order cone (SOC) in Section 3.

2. Strong Semismoothness of $\Phi^{\rm sdc}_{\rm FB}$

Let $A \in M_{n,m}$ and assume $n \le m$. Then there exist orthogonal matrices $U \in \mathcal{M}_{n,n}$ and $V \in \mathcal{M}_{m,m}$ such that A has the following singular value decomposition (SVD)

$$U^T A V = [\Sigma(A) \;\; 0], \tag{5}$$

where $\Sigma(A) = \mathrm{diag}(\sigma_1(A), \ldots, \sigma_n(A))$ and $\sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq \sigma_n(A) \geq 0$ are singular values of A [11, Chapter 2]. Write $V \in \mathcal{M}_{m,m}$ in the form $V = [V_1 \;\, V_2]$, where $V_1 \in \mathcal{M}_{m,n}$ and $V_2 \in \mathcal{M}_{m,m-n}$. We define the orthogonal matrix $Q \in \mathcal{M}_{n+m,n+m}$ by

$$Q := \frac{1}{\sqrt{2}} \begin{bmatrix} U & U & 0 \\ V_1 & -V_1 & \sqrt{2}V_2 \end{bmatrix}. \tag{6}$$

Define the following matrix valued function $G^{\mathrm{mat}} : \mathcal{M}_{n,m} \to \mathcal{S}^n$ by

$$G^{\mathrm{mat}}(A) := \sqrt{AA^T} = U\mathrm{diag}(\sigma_1(A), \ldots, \sigma_n(A))U^T, \tag{7}$$

where $A \in \mathcal{M}_{n,m}$ has the SVD as in (5). Define two linear operators $\Xi : \mathcal{M}_{n,m} \to \mathcal{S}^{n+m}$ and $\pi : \mathcal{S}^{n+m} \to \mathcal{S}^n$ by

$$\Xi(B) := \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}, \quad B \in \mathcal{M}_{n,m} \tag{8}$$

and

$$(\pi(W))_{ij} := W_{ij}, \; i, j = 1, \ldots, n, \; W \in \mathcal{S}^{n+m}, \tag{9}$$

respectively. Then, by [11, Section 8.6], when $A \in \mathcal{M}_{n,m}$ has an SVD as in (5) and Q is defined in (6), the matrix $\Xi(A)$ has the following spectral decomposition:

$$\Xi(A) = Q \begin{bmatrix} \Sigma(A) & 0 & 0 \\ 0 & -\Sigma(A) & 0 \\ 0 & 0 & 0 \end{bmatrix} Q^T, \tag{10}$$

i.e., the eigenvalues of $\Xi(A)$ are $\pm\sigma_i(A)$, $i = 1, \ldots, n$, and 0 of multiplicity $m - n$. Thus, $\sigma_i(A) = \lambda_i(\Xi(A))$, $i = 1, \ldots, n$, where $\lambda_i(\Xi(A))$ is the ith largest eigenvalue of $\Xi(A)$. This, together with the linearity of $\Xi(\cdot)$ and Theorem 4.7 in [16] on the strong semismoothness of eigenvalue functions of symmetric matrices, shows that $\sigma_1(\cdot), \ldots, \sigma_n(\cdot)$ are strongly semismooth everywhere in $\mathcal{M}_{n,m}$. In a similar way to [16], the strong semismoothness of the singular value functions can be used to study the quadratic convergence of generalized Newton methods for solving inverse singular value problems. For a survey on inverse eigenvalue and singular value problems, see [7].

Proposition 2.1. *Suppose that $A \in \mathcal{M}_{n,m}$ has an SVD as in (5). Then it holds that*

$$G^{\mathrm{mat}}(A) = \pi(|\Xi(A)|). \tag{11}$$

Proof. By (6) and (10), we have

$$\begin{aligned} |\Xi(A)| &= \frac{1}{2} \begin{bmatrix} U & U & 0 \\ V_1 & -V_1 & \sqrt{2}V_2 \end{bmatrix} \begin{bmatrix} |\Sigma(A)| & 0 & 0 \\ 0 & |-\Sigma(A)| & 0 \\ 0 & 0 & |0| \end{bmatrix} \begin{bmatrix} U^T & V_1^T \\ U^T & -V_1^T \\ 0 & \sqrt{2}V_2^T \end{bmatrix} \\ &= \begin{bmatrix} U\Sigma(A)U^T & 0 \\ 0 & V_1\Sigma(A)V_1^T \end{bmatrix}. \end{aligned}$$

Thus, $\pi(|\Xi(A)|) = U\Sigma(A)U^T = G^{\mathrm{mat}}(A)$. □

The next theorem is our main result of this section.

Theorem 2.2. *The function $G^{\text{mat}} : \mathcal{M}_{n,m} \to \mathcal{S}^n$ defined by (7) is globally Lipschitz continuous, continuously differentiable around any $A \in \mathcal{M}_{n,m}$ of full row rank, and strongly semismooth everywhere in $\mathcal{M}_{n,m}$.*

Proof. First, by Proposition 2.1, for any $A, B \in \mathcal{M}_{n,m}$, we have

$$\left\|G^{\text{mat}}(A) - G^{\text{mat}}(B)\right\|_{\mathcal{F}} = \|\pi\left(|\Xi(A)| - |\Xi(B)|\right)\|_{\mathcal{F}} \leq \sqrt{2\|A - B\|_{\mathcal{F}}^2},$$

which proves that G^{mat} is globally Lipschitz continuous.

Second, the continuous differentiability of G^{mat} around any $A \in \mathcal{M}_{n,m}$ of full row rank can be obtained easily by using [5, Lemma 4], the definition of G^{mat}, and the fact that AA^T is positive definite when A is of full row rank. The details are omitted here.

Finally, it is known that $|Y|$, $Y \in \mathcal{S}^{n+m}$ is strongly semismooth everywhere [15, Theorem 4.12]. Then Proposition 2.1 and the linearity of $\Xi(\cdot)$ imply that G^{mat} is strongly semismooth at any $A \in \mathcal{M}_{n,m}$. □

Let the matrix valued Fischer-Burmeister function $\Phi_{\text{FB}}^{\text{sdc}} : \mathcal{S}^p \times \mathcal{S}^p \to \mathcal{S}^p$ be defined as in (4). By noting the fact that for any $(X, Y) \in \mathcal{S}^p \times \mathcal{S}^p$, $\Phi_{\text{FB}}^{\text{sdc}}(X, Y) = X + Y - G^{\text{mat}}([X \;\; Y])$, we obtain from Theorem 2.2 the following corollary.

Corollary 2.3. *The matrix valued Fischer-Burmeister function $\Phi_{\text{FB}}^{\text{sdc}} : \mathcal{S}^p \times \mathcal{S}^p \to \mathcal{S}^p$ is globally Lipschitz continuous, continuously differentiable around any $(X, Y) \in \mathcal{S}^p \times \mathcal{S}^p$ if $[X \;\; Y]$ is of full row rank, and strongly semismooth everywhere in $\mathcal{S}^p \times \mathcal{S}^p$.*

3. The FB Function Associated with the SOC

The second order cone (SOC) in $\Re^n$ ($n \geq 2$), also called the Lorentz cone or the ice-cream cone, is defined as $\mathcal{K}^n := \{(x_1, x_2^T)^T \mid x_1 \in \Re, x_2 \in \Re^{n-1} \text{ and } x_1 \geq \|x_2\|\}$. Here and below, $\|\cdot\|$ denotes the l_2-norm in $\Re^n$ and, for convenience, we write $x = (x_1, x_2)$ instead of $x = (x_1, x_2^T)^T$. For any $x = (x_1, x_2)$, $y = (y_1, y_2) \in \Re \times \Re^{n-1}$, we define their Jordan product as

$$x \cdot y := \begin{bmatrix} x^T y \\ y_1 x_2 + x_1 y_2 \end{bmatrix}. \tag{12}$$

Denote $e = (1, 0, \ldots, 0)^T \in \Re^n$. Let x_+ be the orthogonal projection of $x \in \Re^n$ onto $\mathcal{K}^n$. Denote $x^2 := x \cdot x$ and $|x| := \sqrt{x^2}$, where for any $y \in \mathcal{K}^n$, $\sqrt{y}$ is the unique vector in $\mathcal{K}^n$ such that $y = \sqrt{y} \cdot \sqrt{y}$. Then, by [10], we know that $x_+ = (x + |x|)/2$.

A function $\phi^{\text{soc}} : \Re^n \times \Re^n \to \Re^n$ is called an SOC complementarity function if

$$\phi^{\text{soc}}(x, y) = 0 \Longleftrightarrow \mathcal{K}^n \ni x \perp y \in \mathcal{K}^n, \tag{13}$$

where $x \perp y \Leftrightarrow x \cdot y = 0$. By [10], both the vector-valued min-function

$$\phi_{\min}^{\text{soc}}(x, y) := x - (x - y)_+ \tag{14}$$

and the vector valued Fischer-Burmeister function

$$\phi_{\rm FB}^{\rm soc}(x,y) := x + y - \sqrt{x^2+y^2} \tag{15}$$

are SOC complementarity functions. The strong semismoothness of $\phi_{\min}^{\rm soc}$ can be checked directly and has been done in [3, 6]. In this section, we shall prove that $\phi_{\rm FB}^{\rm soc}$ is strongly semismooth.

For any $x = (x_1, x_2) \in \Re \times \Re^{n-1}$, let $L(x), M(x) \in \mathcal{S}^n$ be defined by

$$L(x) := \begin{bmatrix} x_1 & x_2^T \\ x_2 & x_1 I \end{bmatrix} \text{ and } M(x) := \begin{bmatrix} 0 & 0^T \\ 0 & N(x_2) \end{bmatrix}, \tag{16}$$

respectively, where for any $z \in \Re^{n-1}$, $N(z) \in \mathcal{S}^{n-1}$ denotes

$$N(z) := \|z\|(I - zz^T/\|z\|^2) = \|z\|I - zz^T/\|z\| \tag{17}$$

and the convention "$\frac{0}{0} = 0$" is adopted. A direct calculation shows that

$$L(x^2) = (L(x))^2 + (M(x))^2, \quad \forall x = (x_1, x_2) \in \Re \times \Re^{n-1}. \tag{18}$$

Lemma 3.1. *The operator $N(\cdot)$ is globally Lipschitz continuous, twice continuously differentiable around any $0 \neq z \in \Re^{n-1}$, and strongly semismooth everywhere in $\Re^{n-1}$.*

Proof. Suppose that $z^{(1)}, z^{(2)}$ are two arbitrary points in $\Re^{n-1}$. If the line segment $[z^{(1)}, z^{(2)}]$ connecting $z^{(1)}$ and $z^{(2)}$ contains the origin 0, then

$$\|N(z^{(1)}) - N(z^{(2)})\|_{\mathcal{F}} \leq \sqrt{n-2}\|z^{(1)}\| + \sqrt{n-2}\|z^{(2)}\| = \sqrt{n-2}\|z^{(1)} - z^{(2)}\|.$$

If the line segment $[z^{(1)}, z^{(2)}]$ does not contain the origin 0, then by the mean value theorem we have

$$\|N(z^{(1)}) - N(z^{(2)})\|_{\mathcal{F}} \leq \int_0^1 \|N'(z^{(1)} + t[z^{(2)} - z^{(1)}])(z^{(2)} - z^{(1)})\|_{\mathcal{F}} dt,$$

which, together with the fact that for any $z \neq 0$, N is differentiable at z with

$$N'(z)(\Delta z) = \frac{(\Delta z)^T z}{\|z\|}[I + zz^T/\|z\|^2] - \frac{1}{\|z\|}[z(\Delta z)^T + (\Delta z)z^T] \tag{19}$$

and

$$\|N'(z)(\Delta z)\|_{\mathcal{F}} \leq \sqrt{n-2}\|\Delta z\| \ \forall\ \Delta z \in \Re^{n-1},$$

implies that

$$\|N(z^{(1)}) - N(z^{(2)})\|_{\mathcal{F}} \leq \sqrt{n-2}\|z^{(1)} - z^{(2)}\|.$$

Therefore, N is globally Lipschitz continuous.

By equation (19), we know that N is at least twice continuously differentiable around any $z \neq 0$, and so strongly semismooth at any $0 \neq z \in \Re^{n-1}$. Now it suffices to show that N is strongly semismooth at $z^* := 0$.

Note that N is a positive homogeneous mapping, i.e., for any $t \geq 0$ and $z \in \Re^{n-1}$, $N(tz) = tN(z)$. Hence, N is directionally differentiable at 0 and for any $0 \neq z \in \Re^{n-1}$, $N'(0; z) = N(z)$. By (19), for any $0 \neq z \in \Re^{n-1}$,

$$N(z^* + z) - N(z^*) - N'(z^* + z)(z) = N(z) - N(0) - N'(z)(z) = 0,$$

which, together with [15, Theorem 3.7], the Lipschitz continuity, and the directional differentiability of N, shows that N is strongly semismooth at $z^* = 0$. □

Suppose that the operators L and M are defined by (16). For any $a^1, \ldots, a^p \in \Re^n$, let

$$\chi(a^1, \ldots, a^p) := \sqrt{\sum_{i=1}^{p} \left(a^i\right)^2} \tag{20}$$

and

$$\Gamma(a^1, \ldots, a^p) := [L(a^1) \ldots L(a^p) M(a^1) \ldots M(a^p)]. \tag{21}$$

By [3, Lemma 4.1][1], for any $x \in \Re^n$ we have $\sqrt{|x|} = \left(\sqrt{L(|x|)}\right) e$. This, together with the fact that $v := \sum_{i=1}^{p} (a^i)^2 \in \mathcal{K}^n$ and (18), implies

$$\chi(a^1, \ldots, a^p) = \sqrt{v} = \left(\sqrt{L(v)}\right) e = \left(\sqrt{\Gamma(a^1, \ldots, a^p)\left(\Gamma(a^1, \ldots, a^p)\right)^T}\right) e. \tag{22}$$

Therefore, by (22), for any $a^1, \ldots, a^p \in \Re^n$, we have

$$\chi(a^1, \ldots, a^p) = G^{\text{mat}}(\Gamma(a^1, \ldots, a^p))e, \tag{23}$$

where G^{mat} is defined by (7)

Theorem 3.2. *For any $a^1, \ldots, a^p \in \Re^n$, let $\chi(a^1, \ldots, a^p)$ be defined by (20). Then χ is globally Lipschitz continuous, continuously differentiable around any $(a^1, \ldots, a^p)$ if $v_1 \neq \|v_2\|$, where $v = (v_1, v_2) \in \Re \times \Re^{n-1}$ and $v := \sum_{i=1}^{p} (a^i)^2$, and strongly semismooth everywhere.*

Proof. First, the global Lipschitz continuity of χ can be obtained directly by Theorem 2.2, Lemma 3.1, and equation (23).

Second, let $a^i \in \Re^n$, $i = 1, \ldots, p$ be such that $v_1 \neq \|v_2\|$, where $v = (v_1, v_2) \in \Re \times \Re^{n-1}$ and $v = \sum_{i=1}^{p} (a^{(i)})^2$. Then, from (23), Theorem 2.2, and the fact that $\Gamma(a^{(1)}, \ldots, a^{(m)}) \left(\Gamma(a^{(1)}, \ldots, a^{(m)})\right)^T = L(v)$ (cf. (18)) is positive definite when $v_1 \neq \|v_2\|$, we know that χ is continuously differentiable around $(a^1, \ldots, a^p)$.

Finally, we know from [9] that the composite of two strongly semismooth functions is strongly semismooth. Hence, by (23), Theorem 2.2, and the fact that the mapping Γ is strongly semismooth (cf. Lemma 3.1), we can draw the conclusion that χ is strongly semismooth everywhere. □

[1] P. Tseng presented this result in "The Third International Conference on Complementarity Problems", held in Cambridge University, United Kingdom, July 29 -August 1, 2002.

Theorem 3.2 generalizes the results discussed in [6] from the absolute value function $|x|$ to the function χ. By Theorems 2.2 and 3.2, we have the following results, which do not require a proof.

Corollary 3.3. *The vector-valued Fischer-Burmeister function* $\phi_{\rm FB}^{\rm soc} : \Re^n \times \Re^n \to \Re^n$ *is globally Lipschitz continuous, continuously differentiable around any* $(x, y) \in \Re^n \times \Re^n$ *if* $v_1 \neq \|v_2\|$, *where* $v := x^2 + y^2$, *and strongly semismooth everywhere.*

Corollary 3.4. *The smoothed version of* $\Phi_{\rm FB}^{\rm sdc}$,

$$\bar{\Phi}_{\rm FB}^{\rm sdc} : \mathcal{S}^p \times \mathcal{S}^p \times \Re \to \mathcal{S}^p, \quad \bar{\Phi}_{\rm FB}^{\rm sdc}(X, Y, \varepsilon) := X + Y - \sqrt{X^2 + Y^2 + \varepsilon^2 I}$$

and the smoothed version of $\phi_{\rm FB}^{\rm soc}$,

$$\bar{\phi}_{\rm FB}^{\rm soc} : \Re^n \times \Re^n \times \Re \to I\!R^n, \quad \bar{\phi}_{\rm FB}^{\rm soc}(x, y, \varepsilon) := x + y - \sqrt{x^2 + y^2 + \varepsilon^2 e}$$

are strongly semismooth everywhere.

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