Nonsmooth Matrix Valued Functions Defined by Singular Values

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Abstract. A class of matrix valued functions defined by singular values of nonsymmetric matrices are shown to have many properties analogous to matrix valued functions defined by eigenvalues of symmetric matrices. In particular, the (smoothed) matrix valued Fischer-Burmeister function is proved to be strongly semismooth everywhere. This result is also used to show the strong semismoothness of the (smoothed) vector valued Fischer-Burmeister function associated with the second order cone. The strong semismoothness of singular values of a nonsymmetric matrix is discussed and used to analyze the quadratic convergence of Newton's method for solving the inverse singular value problem.

Keywords: Fischer-Burmeister function, SVD, strong semismoothness, inverse singular value problem

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1 Introduction

Let $\mathcal{M}_{p,q}$ be the linear space of $p \times q$ real matrices. We denote the ijth entry of $A \in \mathcal{M}_{p,q}$ by A_{ij}. For any two matrices A and B in $\mathcal{M}_{p,q}$, we write

$$A \bullet B := \sum_{i=1}^{p} \sum_{j=1}^{q} A_{ij} B_{ij} \, = \, \mathrm{tr}(AB^T)$$

for the *Frobenius inner product* between A and B, where "tr" denotes the trace of a matrix. The *Frobenius norm* on $\mathcal{M}_{p,q}$ is the norm induced by the above inner product:

$$\| A \|_{\mathcal{F}} := \sqrt{A \bullet A} \, = \, \sqrt{\sum_{i=1}^{p} \sum_{j=1}^{q} A_{ij}^2} \, .$$

The identity matrix in $\mathcal{M}_{p,p}$ is denoted by I.

Let $\mathcal{S}^p$ be the linear space of $p \times p$ real symmetric matrices; let $\mathcal{S}^p_+$ denote the cone of $p \times p$ symmetric positive semidefinite matrices. For any vector $y \in \Re^p$, let $\mathrm{diag}(y_1, \ldots, y_p)$ denote the $p \times p$ diagonal matrix with its ith diagonal entry y_i. We write $X \succeq 0$ to mean that X is a symmetric positive semidefinite matrix. Under the Frobenius norm, the projection $\Pi_{\mathcal{S}^p_+}(X)$ of a matrix $X \in \mathcal{S}^p$ onto the cone $\mathcal{S}^p_+$ is the unique minimizer of the following convex program in the matrix variable Y:

$$\begin{array}{ll} \text{minimize} & \| Y - X \|_{\mathcal{F}} \\ \text{subject to} & Y \in \mathcal{S}^p_+ \, . \end{array}$$

Throughout this paper, we let X_+ denote the (Frobenius) projection of $X \in \mathcal{S}^p$ onto $\mathcal{S}^p_+$. The projection X_+ has an explicit representation. Namely, if

$$X \, = \, P \Lambda(X) P^T \, , \tag{1}$$

where $\Lambda(X)$ is the diagonal matrix of eigenvalues of X and P is a corresponding orthogonal matrix of orthonormal eigenvectors, then

$$X_+ \, = \, P \Lambda(X)_+ P^T,$$

where $\Lambda(X)_+$ is the diagonal matrix whose diagonal entries are the nonnegative parts of the respective diagonal entries of $\Lambda(X)$. If $X \in \mathcal{S}^p_+$, then we use

$$\sqrt{X} := P \sqrt{\Lambda(X)} \, P^T$$

to denote the square root of X, where X has the spectral decomposition as in (1) and $\sqrt{\Lambda(X)}$ is the diagonal matrix whose diagonal entries are the square root of the (nonnegative) eigenvalues of X.

A function $\Phi^{\rm sdc} : \mathcal{S}^p \times \mathcal{S}^p \to \mathcal{S}^p$ is called a semidefinite cone complementarity function (SDC C-function for short) if

$$\Phi^{\rm sdc}(X,Y) \,=\, 0 \iff \mathcal{S}^p_+ \,\ni\, X \,\perp\, Y \,\in\, \mathcal{S}^p_+\,, \tag{2}$$

where the $\perp$ notation means "perpendicular under the above matrix inner product"; i.e., $X \perp Y \Leftrightarrow X \bullet Y = 0$ for any two matrices X and Y in $\mathcal{S}^p$. Of particular interest are two SDC C-functions

$$\Phi^{\rm sdc}_{\min}(X,Y) := \, X - (X-Y)_+ \tag{3}$$

and

$$\Phi^{\rm sdc}_{\rm FB}(X,Y) := \, X + Y - \sqrt{X^2 + Y^2}\,. \tag{4}$$

The SDC C-function $\Phi^{\rm sdc}_{\min}$ defined by (3) is called the matrix valued min function. It is well known that $\Phi^{\rm sdc}_{\min}$ is globally Lipschitz continuous [42, 40]. However, $\Phi^{\rm sdc}_{\min}$ is in general not continuously differentiable. A result of Bonnans, Cominetti, and Shapiro [2] on the directional differentiability of $\Pi_{\mathcal{S}^p_+}$ implies that $\Phi^{\rm sdc}_{\min}$ is directionally differentiable. More recently, it is proved in [36] that $\Phi^{\rm sdc}_{\min}$ is actually strongly semismooth (see [30] and Section 2 for the definition of strong semismoothness.) This property plays a fundamental role in proving the quadratic convergence of Newton's method for solving systems of nonsmooth equations [26, 28, 30]. Newton-type methods for solving the semidefinite programming and the semidefinite complementarity problem (SDCP) based on the smoothed form of $\Phi^{\rm sdc}_{\min}$ are discussed in [5, 6, 21, 38]. Semismooth homeomorphisms for the SDCP are established in [27].

The SDC C-function (4) is called the matrix valued Fischer-Burmeister function due to the fact that when $p = 1$, $\Phi^{\rm sdc}_{\rm FB}$ reduces to the scalar valued Fischer-Burmeister function

$$\phi_{\rm FB}(a,b) := \, a + b - \sqrt{a^2 + b^2}\,, \;\, a, b \in \Re\,,$$

which is first introduced by Fischer [14]. In [40], Tseng proves that $\Phi^{\rm sdc}_{\rm FB}$ satisfies (2). In a recent book [3], Borwein and Lewis also suggest a proof on this in Exercise 11 of Section 5.2. A desirable property of $\Phi^{\rm sdc}_{\rm FB}$ is that $\|\, \Phi^{\rm sdc}_{\rm FB} \,\|^2_{\mathcal{F}}$ is continuously differentiable [40]. While the strong semismoothness of the scalar valued Fischer-Burmeister function $\phi_{\rm FB}$ can be checked easily by the definition [11, 12, 29], the strong semismoothness of its counter part $\Phi^{\rm sdc}_{\rm FB}$ in the matrix form has not been proved yet. For other properties related to SDC C-functions, see [40, 41, 18].

The primary motivation of this paper is to prove that $\Phi^{\rm sdc}_{\rm FB}$ is globally Lipschitz continuous, directionally differentiable and strongly semismooth. In order to achieve these, we first introduce a matrix valued function defined by singular values of a real matrix, which is in general neither symmetric nor square, and then relate its properties to those studied for the symmetric matrix valued functions defined by eigenvalues of a symmetric matrix [5, 6, 36]. We then proceed to study important properties of vector valued C-functions associated with the second order cone (SOC). Finally, we discuss the inverse singular value problem.

2 Basic Concepts and Properties

2.1 Semismoothness

Let $\theta : \Re^s \to \Re^q$ [we regard the $r \times r$ (respectively, symmetric) matrix space as a special case of $\Re^s$ with (respectively, $s = r(r+1)/2$) $s = r^2$. Hence the discussions of this subsection apply to matrix variable and/or matrix valued functions as well.] Let $\| \cdot \|$ denote the l_2 norm in finite dimensional Euclidean spaces. Recall that θ is said to be locally Lipschitz continuous around $x \in \Re^s$ if there exist a constant κ and an open neighborhood $\mathcal{N}$ of x such that

$$\| \theta(y) - \theta(z) \| \;\leq\; \kappa \| y - z \| \quad \forall\, y, z \in \mathcal{N} \,.$$

We call θ a locally Lipschitz function if it is locally Lipschitz continuous around every point of $\Re^s$. Moreover, if the above inequality holds for $\mathcal{N} = \Re^s$, then θ is said to be globally Lipschitz continuous with Lipschitz constant κ.

The function θ is said to be directionally differentiable at x if the directional derivative

$$\theta'(x; h) := \lim_{t \downarrow 0} \frac{\theta(x + th) - \theta(x)}{t}$$

exists in every direction $h \in \Re^s$. θ is said to be differentiable (in the sense of Fréchet) at $x \in \Re^s$ with a (Fréchet) derivative $\theta'(x) \in \mathcal{M}_{q,s}$ if

$$\theta(x + h) - \theta(x) - \theta'(x)(h) \;=\; o(\| h \|) \,.$$

Assume that $\theta : \Re^s \to \Re^q$ is locally Lipschitz continuous around $x \in \Re^s$. Then, according to Rademacher's Theorem, θ is differentiable almost everywhere in an open set $\mathcal{N}$ containing x. Let D_θ be the set of differentiable points of θ on $\mathcal{N}$. Denote

$$\partial_B \theta(x) := \{ V \in \mathcal{M}_{q,s} \,|\, V = \lim_{x^k \to x} \theta'(x^k), \; x^k \in D_\theta \} \,.$$

Then Clarke's generalized Jacobian [10] of θ at x is

$$\partial \theta(x) \;=\; \mathrm{conv}\{ \partial_B \theta(x) \} \,, \tag{5}$$

where "conv" stands for the convex hull in the usual sense of convex analysis [31].

Extending Mifflin's definition for a scalar function [25], Qi and Sun [30] introduced the semismoothness property for a vector valued function.

Definition 2.1 *Suppose that $\theta : \Re^s \to \Re^q$ is locally Lipschitz continuous around $x \in \Re^s$. θ is said to be semismooth at x if θ is directionally differentiable at x and for any $V \in \partial \theta(x + \Delta x)$,*

$$\theta(x + \Delta x) - \theta(x) - V(\Delta x) \;=\; o(\| \Delta x \|) \,.$$

θ is said to be $\gamma-$order $(0 < \gamma < \infty)$ semismooth at x if θ is semismooth at x and

$$\theta(x + \Delta x) - \theta(x) - V(\Delta x) \;=\; O(\| \Delta x \|^{1+\gamma}) \,.$$

In particular, θ is said to be strongly semismooth at x if θ is 1-order semismooth at x.

A function $\theta : \Re^s \to \Re^q$ is said to be a semismooth (respectively, γ-order semismooth) function if it is semismooth (respectively, γ-order semismooth) everywhere in $\Re^s$. Semismooth functions include smooth functions, piecewise smooth functions, and convex and concave functions. It is also true that the composition of (strongly) semismooth functions is still a (strongly) semismooth function (see [25, 15]). A similar result also holds for γ-order semismoothness. These results are summarized in the next proposition.

Proposition 2.2 *Let $\gamma \in (0, \infty)$. Suppose that $\xi : \Re^\tau \to \Re^q$ is semismooth (respectively, γ-order semismooth) at $x \in \Re^s$ and $\theta : \Re^s \to \Re^\tau$ is semismooth (respectively, γ-order semismooth) at $\xi(x)$. Then, the composite function $\theta \circ \xi$ is semismooth (γ-order semismooth) at x.*

The next result provides a convenient tool for proving the semismoothness and γ-order semismoothness of θ. Its proof follows virtually from [36, Thm. 3.7], where the γ-order semismoothness is discussed under the assumption that θ is directionally differentiable in a neighborhood of $x \in \Re^s$. A closer look at the proof of [36, Thm. 3.7] reveals that the directional differentiability of θ in a neighborhood of $x \in \Re^s$ is not needed.

Proposition 2.3 *Suppose that $\theta : \Re^s \to \Re^q$ is locally Lipschitz continuous around $x \in \Re^s$. Then for any $\gamma \in (0, \infty)$, the following two statements are equivalent:*

(i) *for any $V \in \partial\theta(x + \Delta x)$,*

$$\theta(x + \Delta x) - \theta(x) - V(\Delta x) \;=\; o(\| \Delta x \|) \text{ (respectively, } O(\| \Delta x \|^{1+\gamma})) \,; \qquad (6)$$

(ii) *for any $x + \Delta x \in D_\theta$,*

$$\theta(x + \Delta x) - \theta(x) - \theta'(x + \Delta x)(\Delta x) \;=\; o(\| \Delta x \|) \text{ (respectively, } O(\| \Delta x \|^{1+\gamma})) \,. \qquad (7)$$

2.2 Strong semismoothness of eigenvalues of a symmetric matrix

For a symmetric matrix $X \in \mathcal{S}^p$, let $\lambda_1(X) \geq \ldots \geq \lambda_p(X)$ be the p eigenvalues of X arranged in the decreasing order. Let $\omega_k(X)$ be the sum of the k largest eigenvalues of $X \in \mathcal{S}^p$. Then, Fan's maximum principle [13] says that for each $i = 1, \cdots, p$, $\omega_i(\cdot)$ is a convex function on $\mathcal{S}^p$. This result implies that

- $\lambda_1(\cdot)$ is a convex function and $\lambda_p(\cdot)$ is a concave function;
- For $i = 2, \cdots, p-1$, $\lambda_i(\cdot)$ is the difference of two convex functions.

Since convex and concave functions are semismooth and the difference of two semismooth functions is still a semismooth function [25], Fan's result shows that $\lambda_1(\cdot), \cdots, \lambda_p(\cdot)$ are all semismooth functions. Sun and Sun [37] further prove that all these functions are strongly semismooth.

Proposition 2.4 *The functions $\omega_1(\cdot), \ldots, \omega_p(\cdot)$ and $\lambda_1(\cdot), \ldots, \lambda_p(\cdot)$ are strongly semismooth functions on $\mathcal{S}^p$.*

The proof of the above proposition uses an upper Lipschitz continuous property of (normalized) eigenvectors of symmetric matrices. This Lipschitz property is obtained by Chen and Tseng [6, Lemma 3] based on a so called "sin(Θ)" theorem in [35, Thm. 3.4] and is also implied in [36, Lemma 4.12] in the proof of strong semismoothness for the matrix valued function $\sqrt{X^2}, X \in \mathcal{S}^p$. Very recently, based on an earlier result of Shapiro and Fan [34], Shapiro [33], among others, provides a different proof to Proposition 2.4. Second order directional derivatives of eigenvalue functions $\lambda_1(\cdot), \ldots, \lambda_p(\cdot)$ are discussed in [24, 32, 39]. For a survey on general nonsmooth analysis involving eigenvalues of symmetric matrices, see Lewis [22] and Lewis and Overton [23].

2.3 Properties of matrix functions over symmetric matrices

In this subsection, we shall list several useful properties of matrix valued functions over symmetric matrices. Let $f : \Re \to \Re$ be a scalar function. The matrix valued function $F^{\text{mat}} : \mathcal{S}^p \to \mathcal{S}^p$ can be defined as

$$F^{\text{mat}}(X) := P\text{diag}(f(\lambda_1(X)), \ldots, f(\lambda_p(X)))P^T = P \begin{bmatrix} f(\lambda_1(X)) & & \\ & \ddots & \\ & & f(\lambda_p(X)) \end{bmatrix} P^T, \tag{8}$$

where for each $X \in \mathcal{S}^p$, X has the spectral decomposition as in (1) and $\lambda_1(X), \ldots, \lambda_p(X)$ are eigenvalues of X. It is well known that F^{mat} is well defined independent of the ordering of $\lambda_1(X), \ldots, \lambda_p(X)$ and the choice of P, see [1, Chapter V] and [20, Section 6.2].

The matrix function F^{mat} inherits many properties from the scalar function f. Here we summarize those properties needed in the discussion of this paper in the next proposition. Part (i) of Proposition 2.5 can be found in [20, p. 433] and [5, Prop. 4.1]; part (ii) is shown in [5, Prop. 4.3] and is also implied in [24, Thm. 3.3] for the case that $f = h'$ for some differentiable function $h : \Re \to \Re$; the "if" part of (iii) is proved in [6, Lemma 4] while the "only if" part is shown in [5, Prop. 4.4]; parts (iv)-(vii) are proved in Propositions 4.6, 4.2, and 4.10 of [5], respectively; and part (viii), which generalizes the strong semismoothness of $F^{\rm mat}$ for cases $f(t) = |t|$ and $f(t) = \max\{0, t\}$ derived in [36], follows directly from the proof of [5, Prop. 4.10], and Proposition 2.3.

Proposition 2.5 *For any function $f : \Re \to \Re$, the following results hold:*

(i) $F^{\rm mat}$ *is continuous at $X \in \mathcal{S}^p$ with eigenvalues $\lambda_1(X), \ldots, \lambda_p(X)$ if and only if f is continuous at every $\lambda_i(X), i = 1, \ldots, p$;*

(ii) $F^{\rm mat}$ *is differentiable at $X \in \mathcal{S}^p$ with eigenvalues $\lambda_1(X), \ldots, \lambda_p(X)$ if and only if f is differentiable at every $\lambda_i(X), i = 1, \ldots, p$;*

(iii) $F^{\rm mat}$ *is continuously differentiable at $X \in \mathcal{S}^p$ with eigenvalues $\lambda_1(X), \ldots, \lambda_p(X)$ if and only if f is continuously differentiable at every $\lambda_i(X), i = 1, \ldots, p$;*

(iv) $F^{\rm mat}$ *is locally Lipschitz continuous around $X \in \mathcal{S}^p$ with eigenvalues $\lambda_1(X), \ldots, \lambda_p(X)$ if and only if f is locally Lipschitz continuous around every $\lambda_i(X), i = 1, \ldots, p$;*

(v) $F^{\rm mat}$ *is globally Lipschitz continuous (with respect to $\|\cdot\|_{\mathcal{F}}$) with Lipschitz constant κ if and only if f is Lipschitz continuous with Lipschitz constant κ;*

(vi) $F^{\rm mat}$ *is directionally differentiable at $X \in \mathcal{S}^p$ with eigenvalues $\lambda_1(X), \ldots, \lambda_p(X)$ if and only if f is directionally differentiable at every $\lambda_i(X), i = 1, \ldots, p$;*

(vii) $F^{\rm mat}$ *is semismooth at $X \in \mathcal{S}^p$ with eigenvalues $\lambda_1(X), \ldots, \lambda_p(X)$ if and only if f is semismooth at every $\lambda_i(X), i = 1, \ldots, p$;*

(viii) $F^{\rm mat}$ *is $\min\{1, \gamma\}$-order semismooth at $X \in \mathcal{S}^p$ with eigenvalues $\lambda_1(X), \ldots, \lambda_p(X)$ if f is γ-order semismooth $(0 < \gamma < \infty)$ at every $\lambda_i(X), i = 1, \ldots, p$.*

It is noted that although $F^{\rm mat}$ inherits many properties from f, it does not inherit all of them. For instance, even if f is a piecewise linear function, i.e., f is a continuous selection of a finite number of linear functions, $F^{\rm mat}$ may not be piecewise smooth unless $p = 1$ (taking $f(t) = |t|$ for a counter example.)

3 Matrix Functions over Nonsymmetric Matrices

The matrix valued function $F^{\rm mat}$ defined by (8) needs X to be symmetric. To study the strong semismoothness of $\Phi_{\rm FB}^{\rm sdc}$ and beyond, we need to define a matrix valued function over nonsymmetric matrices.

Let $g : \Re \to \Re$ be a scalar function satisfying the property that $g(t) = g(-t)\ \forall\ t \in \Re$, i.e., g is an even function. Let $A \in M_{n,m}$ and $n \le m$ [there is no loss of generality by assuming $n \le m$ because the case $n \ge m$ can be discussed similarly.] Then there exist orthogonal matrices $U \in \mathcal{M}_{n,n}$ and $V \in \mathcal{M}_{m,m}$ such that A has the following singular value decomposition (SVD)

$$U^T A V = [\Sigma(A) \ \ 0] \tag{9}$$

where $\Sigma(A) = \mathrm{diag}(\sigma_1(A), \ldots, \sigma_n(A))$ and $\sigma_1(A) \ge \sigma_2(A) \ge \ldots \ge \sigma_n(A) \ge 0$ are singular values of A [19, Chapter 2]. It is then natural to define the following matrix valued function $G^{\mathrm{mat}} : \mathcal{M}_{n,m} \to \mathcal{S}^n$ by

$$G^{\mathrm{mat}}(A) := U\mathrm{diag}(g(\sigma_1(A)), \ldots, g(\sigma_n(A)))U^T = U \begin{bmatrix} g(\sigma_1(A)) & & \\ & \ddots & \\ & & g(\sigma_n(A)) \end{bmatrix} U^T. \tag{10}$$

Based on the well known relationships between the SVD of A and the spectral decompositions of the symmetric matrices AA^T, A^TA, and $\begin{bmatrix} 0 & A \\ A^T & 0 \end{bmatrix}$ [19, Section 8.6], we shall study some important properties of the matrix function G^{mat}. In particular, we shall prove that when $g(t) = |t|$, G^{mat} is strongly semismooth everywhere. This implies that $\sqrt{X^2 + Y^2}$ is strongly semismooth at any $(X, Y) \in \mathcal{S}^n \times \mathcal{S}^n$ by taking $A = [X \ \ Y]$. The strong semismoothness of the matrix valued Fischer-Burmeister function $\Phi_{\mathrm{FB}}^{\mathrm{sdc}}$ then follows easily (see Corollary 3.5 below).

First, by noting the fact that

$$\sqrt{AA^T} = \sqrt{U\Sigma^2(A)U^T} = U\mathrm{diag}(\sigma_1(A), \ldots, \sigma_n(A))U^T,$$

we know that by taking $f(t) = g(t)$,

$$\begin{aligned} G^{\mathrm{mat}}(A) &= U\mathrm{diag}(g(\sigma_1(A)), \ldots, g(\sigma_n(A)))U^T \\ &= U\mathrm{diag}(f(\sigma_1(A)), \ldots, f(\sigma_n(A)))U^T \\ &= U\mathrm{diag}\left(f(\sqrt{\lambda_1(AA^T)}), \ldots, f(\sqrt{\lambda_n(AA^T)})\right)U^T \\ &= F^{\mathrm{mat}}(\sqrt{AA^T}), \end{aligned} \tag{11}$$

where $\lambda_1(AA^T) \ge \ldots \ge \lambda_n(AA^T)$ are eigenvalues of AA^T arranged in the decreasing order. This, together with the well definedness of F^{mat}, implies that (10) is well defined. In particular, when $f(t) = g(t) = |t|$, (11) becomes

$$\begin{aligned} G^{\mathrm{mat}}(A) &= U\mathrm{diag}(\sigma_1(A), \ldots, \sigma_n(A))U^T \\ &= U\mathrm{diag}\left(\sqrt{\lambda_1(AA^T)}, \ldots, \sqrt{\lambda_n(AA^T)}\right)U^T \\ &= F^{\mathrm{mat}}(\sqrt{AA^T}) \\ &= \sqrt{AA^T}. \end{aligned} \tag{12}$$

Note that the strong semismoothness of the matrix valued Fischer-Burmeister function $\Phi_{\rm FB}^{\rm sdc}$ does not follow from (12) directly because $\sqrt{|t|}$ is not locally Lipschitz continuous around $t = 0$, let alone strongly semismooth at $t = 0$.

For any $W \in \mathcal{S}^{n+m}$, we define

$$\Lambda(W) := \, \mathrm{diag}(\lambda_1(W), \ldots, \lambda_n(W), \lambda_{n+m}(W), \ldots, \lambda_{n+1}(W)) \, ,$$

where $\lambda_1(W) \geq \ldots \geq \lambda_{n+m}(W)$ are eigenvalues of W arranged in the decreasing order. Note that the first n diagonal entries of $\Lambda(W)$, which are the n largest eigenvalues of W, are arranged in the decreasing order and the last m diagonal entries of $\Lambda(W)$, which are the m smallest eigenvalues of W, are arranged in the increasing order. We shall see shortly that this special arrangement has its convenience.

Define the linear operator $\Xi : \mathcal{M}_{n,m} \to \mathcal{S}^{n+m}$ by

$$\Xi(B) := \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix} , \quad B \in \mathcal{M}_{n,m} \, . \tag{13}$$

Write $V \in \mathcal{M}_{m,m}$ in the form

$$V \; = \; [V_1 \;\; V_2] \, ,$$

where $V_1 \in \mathcal{M}_{m,n}$ and $V_2 \in \mathcal{M}_{m,m-n}$. We define the orthogonal matrix $Q \in \mathcal{M}_{n+m,n+m}$ by

$$Q := \frac{1}{\sqrt{2}} \begin{bmatrix} U & U & 0 \\ V_1 & -V_1 & \sqrt{2} V_2 \end{bmatrix} .$$

Then, by [19, Section 8.6], we have the following result.

Proposition 3.1 *Suppose that $A \in \mathcal{M}_{n,m}$ has an SVD as in (9). Then the matrix $\Xi(A)$ has the following spectral decomposition:*

$$\Xi(A) \; = \; Q(\Lambda(\Xi(A)))Q^T \; = \; Q \begin{bmatrix} \Sigma(A) & 0 & 0 \\ 0 & -\Sigma(A) & 0 \\ 0 & 0 & 0 \end{bmatrix} Q^T \, ,$$

i.e., the eigenvalues of $\Xi(A)$ are $\pm\sigma_i(A), i = 1, \ldots, n$, and 0 of multiplicity $m - n$.

The next result shows that the singular value functions are strongly semismooth everywhere.

Proposition 3.2 *The singular value functions $\sigma_1(\cdot), \ldots, \sigma_n(\cdot)$ are strongly semismooth everywhere in $\mathcal{M}_{n,m}$.*

Proof. By Proposition 3.1, we know that

$$\sigma_i(A) \;=\; \lambda_i(\Xi(A)),\; i \;=\; 1, \ldots, n\,.$$

This, together with Propositions 2.4 and 2.2, shows that $\sigma_1(\cdot), \ldots, \sigma_n(\cdot)$ are strongly semismooth everywhere in $\mathcal{M}_{n,m}$. Q.E.D.

We define another linear operator $\pi : \mathcal{S}^{n+m} \to \mathcal{S}^n$ by

$$(\pi(W))_{ij} := \; W_{ij}\,,\; i\,, j \;=\; 1, \ldots, n\,,\; W \in \mathcal{S}^{n+m},$$

i.e., for any $W \in \mathcal{S}^{n+m}$, $\pi(W)$ denotes the top left corner $n \times n$ submatrix of W. The next result establishes the relationship between $G^{\rm mat}(A)$ and $\pi(F^{\rm mat}(\Xi(A)))$.

Proposition 3.3 *If $f(t) = g(t), t \in \Re$, then it holds that*

$$F^{\rm mat}(\Xi(A)) \;=\; \begin{bmatrix} U\Sigma_g(A)U^T & 0 \\ 0 & V_1\Sigma_g(A)V_1^T + g(0)V_2V_2^T \end{bmatrix} \tag{14}$$

and

$$G^{\rm mat}(A) \;=\; U\Sigma_g(A)U^T \;=\; \pi(F^{\rm mat}(\Xi(A)))\,, \tag{15}$$

where $\Sigma_g(A) :=\; {\rm diag}(g(\sigma_1(A)), \ldots, g(\sigma_n(A)))$.

Proof. By Proposition 3.1, the definition of $F^{\rm mat}$, and the assumption that $f(t) = g(t)$, we have

$$\begin{aligned} F^{\rm mat}(\Xi(A)) \;&=\; Q \begin{bmatrix} \Sigma_g(A) & 0 & 0 \\ 0 & \Sigma_g(A) & 0 \\ 0 & 0 & g(0)I \end{bmatrix} Q^T \\ &=\; \frac{1}{2} \begin{bmatrix} U & U & 0 \\ V_1 & -V_1 & \sqrt{2}V_2 \end{bmatrix} \begin{bmatrix} \Sigma_g(A) & 0 & 0 \\ 0 & \Sigma_g(A) & 0 \\ 0 & 0 & g(0)I \end{bmatrix} \begin{bmatrix} U^T & V_1^T \\ U^T & -V_1^T \\ 0 & \sqrt{2}V_2^T \end{bmatrix} \\ &=\; \begin{bmatrix} U\Sigma_g(A)U^T & 0 \\ 0 & V_1\Sigma_g(A)V_1^T + g(0)V_2V_2^T \end{bmatrix}, \end{aligned}$$

which, completes the proof of the proposition. Q.E.D.

The next theorem is our main result of this paper.

Theorem 3.4 *Let $g : \Re \to \Re$ be an even function. Define another even function $\bar{g} : \Re \to \Re$ by*

$$\bar{g}(t) := g(\,\sqrt{|t|}\,)\,,\; t \in \Re\,. \tag{16}$$

Then for any $A \in \mathcal{M}_{n,m}$ with singular values $\sigma_1(A) \geq \ldots \geq \sigma_n(A)$, the following results hold:

(i) $G^{\rm mat}$ *is continuous at* $A \in \mathcal{M}_{n,m}$ *if and only if* g *is continuous at every* $\sigma_i(A), i = 1, \dots, n$;

(ii) *If either* g *is differentiable at every* $\sigma_i(A), i = 1, \dots, n$, *and* 0 *or* $\bar{g}$ *is differentiable at every* $\sigma_i^2(A), i = 1, \dots, n$, *then* $G^{\rm mat}$ *is differentiable at* $A \in \mathcal{M}_{n,m}$;

(iii) *If either* g *is continuously differentiable at every* $\sigma_i(A), i = 1, \dots, n$, *and* 0 *or* $\bar{g}$ *is continuously differentiable at every* $\sigma_i^2(A), i = 1, \dots, n$, *then* $G^{\rm mat}$ *is continuously differentiable at* $A \in \mathcal{M}_{n,m}$;

(iv) *If either* g *is locally Lipschitz continuous around every* $\sigma_i(A), i = 1, \dots, n$, *and* 0 *or* $\bar{g}$ *is locally Lipschitz continuous around every* $\sigma_i^2(A), i = 1, \dots, n$, *then* $G^{\rm mat}$ *is locally Lipschitz continuous around* $A \in \mathcal{M}_{n,m}$;

(v) *If* g *is globally Lipschitz continuous with Lipschitz constant* κ, *then* $G^{\rm mat}$ *is globally Lipschitz continuous* (*with respect to* $\|\cdot\|_{\mathcal{F}}$) *with Lipschitz constant* $\sqrt{2}\kappa$;

(vi) *If* g *is directionally differentiable at every* $\sigma_i(A), i = 1, \dots, n$, *and* 0, *then* $G^{\rm mat}$ *is directionally differentiable at* $A \in \mathcal{M}_{n,m}$ *with directional derivative given by*

$$(G^{\rm mat})'(A; H) \;=\; \pi[(F^{\rm mat})'(\Xi(A); \Xi(H))] \quad \forall\, H \,\in\, \mathcal{M}_{n,m};$$

(vii) *If* g *is semismooth at every* $\sigma_i(A), i = 1, \dots, n$, *and* 0, *then* $G^{\rm mat}$ *is semismooth at* $A \in \mathcal{M}_{n,m}$;

(viii) *If* g *is* γ*-order semismooth at every* $\sigma_i(A), i = 1, \dots, n$, *and* 0, *then* G *is* $\min\{1, \gamma\}$*-order semismooth at* $A \in \mathcal{M}_{n,m}$, *where* $\gamma \in (0, \infty)$.

Proof. In the proof of this theorem, we always assume that $f(t) = g(t)$, $t \in \Re$. Then, by Proposition 3.1, the eigenvalues of $W := \Xi(A)$ are $\pm\sigma_i(A)$, $i = 1, \dots, n$, and 0 of multiplicity $m - n$.

(i) This result follows directly from the proof of [5, Prop. 4. 1] and Proposition 3.3.

(ii)-(iv) The conclusions follow directly from (15) in Proposition 3.3, the fact that

$$\begin{aligned} G^{\rm mat}(A) &= U{\rm diag}(\bar{g}(\sigma_1(A)^2), \dots, \bar{g}(\sigma_n(A)^2))U^T \\ &= U{\rm diag}(\bar{g}(\lambda_1(AA^T)), \dots, \bar{g}(\lambda_n(AA^T)))U^T\,, \end{aligned}$$

and parts (ii)-(iv) of Proposition 2.5, respectively.

(v) If g is globally Lipschitz continuous with Lipschitz constant κ, then by part (v) of Proposition 2.5, and Proposition 3.3 , for any $A, B \in \mathcal{M}_{n,m}$, we have

$$\begin{aligned}\| G^{\rm mat}(A) - G^{\rm mat}(B) \|_{\mathcal{F}} &= \| \pi[F^{\rm mat}(\Xi(A)) - F^{\rm mat}(\Xi(B))] \|_{\mathcal{F}} \\ &\leq \| F^{\rm mat}(\Xi(A)) - F^{\rm mat}(\Xi(B)) \|_{\mathcal{F}} \\ &\leq \kappa \| \Xi(A) - \Xi(B) \|_{\mathcal{F}} \\ &= \kappa \sqrt{2 \| A - B \|_{\mathcal{F}}^2} \\ &= \sqrt{2}\kappa \| A - B \|_{\mathcal{F}} \,,\end{aligned}$$

which, proves that $G^{\rm mat}$ is globally Lipschitz continuous with Lipschitz constant $\sqrt{2}\kappa$;

(vi) By part (vi) of Proposition 2.5, $F^{\rm mat}$ is directionally differentiable at $\Xi(A)$. Hence, by Proposition 3.3, the conclusion follows.

(vii) This follows directly from part (vii) of Proposition 2.5, and Propositions 2.2 and 3.3.

(viii) By part (viii) of Proposition 2.5, $F^{\rm mat}$ is $\min\{1,\gamma\}$-order semismooth at $\Xi(A)$. Hence, by using the fact that linear mappings are γ-order semismooth, and Propositions 2.2 and 3.3, we know that then $G(\cdot) = \pi(F(E(\cdot)))$ is $\min\{1,\gamma\}$-order semismooth at $A \in \mathcal{M}_{n,m}$. Q.E.D.

Let the matrix valued Fischer-Burmeister function $\Phi_{\rm FB}^{\rm sdc} : \mathcal{S}^p \times \mathcal{S}^p \to \mathcal{S}^p$ be defined as in (4).

Corollary 3.5 *The matrix valued Fischer-Burmeister function* $\Phi_{\rm FB}^{\rm sdc} : \mathcal{S}^p \times \mathcal{S}^p \to \mathcal{S}^p$ *has the following properties*:

(i) $\Phi_{\rm FB}^{\rm sdc}$ *is globally Lipschitz continuous with Lipschitz constant* $2\sqrt{2}$;

(ii) $\Phi_{\rm FB}^{\rm sdc}$ *is continuously differentiable at any* $(X, Y) \in \mathcal{S}^p \times \mathcal{S}^p$ *if* $[X \;\; Y]$ *is of full row rank*;

(iii) $\Phi_{\rm FB}^{\rm sdc}$ *is directionally differentiable at any* $(X, Y) \in \mathcal{S}^p \times \mathcal{S}^p$;

(iv) $\Phi_{\rm FB}^{\rm sdc}$ *is a strongly semismooth function.*

Proof. Define two linear mappings $\Phi_1 : \mathcal{S}^p \times \mathcal{S}^p \to \mathcal{S}^p$ by

$$\Phi_1(X, Y) := X + Y$$

and $\Phi_2 : \mathcal{S}^p \times \mathcal{S}^p \to \mathcal{M}_{p,2p}$ by

$$\Phi_2(X, Y) := [X \;\; Y] \,,$$

respectively, where $(X, Y) \in \mathcal{S}^p \times \mathcal{S}^p$. Let $g(t) = |t|, t \in \Re$ and let $G^{\rm mat}$ be defined as in (12). We then obtain for any $(X, Y) \in \mathcal{S}^p \times \mathcal{S}^p$ that

$$\Phi_{\rm FB}^{\rm sdc}(X, Y) \;=\; \Phi_1(X, Y) \;-\; G^{\rm mat}(\Phi_2(X, Y)) \,. \tag{17}$$

(i) This follows from equation (17) and part (v) of Theorem 3.4.

(ii) This is a consequence of part (iii) of Theorem 3.4 and the fact that all singular values of $\Phi_2(X,Y)$ are positive numbers under the assumption that $[X \;\; Y]$ is of full row rank.

(iii) The directional differentiability of $\Phi_{\rm FB}^{\rm sdc}$ follows directly from equation (17) and part (vi) of Theorem 3.4.

(iv) The strong semismoothness of $\Phi_{\rm FB}^{\rm sdc}$ is a direct application of part (viii) of Theorem 3.4, and Proposition 2.2 to equation (17). Q.E.D.

A smoothed version of $\Phi_{\rm FB}^{\rm sdc}$ is defined as

$$\Psi_{\rm FB}^{\rm sdc}(\varepsilon, X, Y) := X + Y - \sqrt{\varepsilon^2 I + X^2 + Y^2}, \; (\varepsilon, X, Y) \in \Re \times \mathcal{S}^p \times \mathcal{S}^p . \tag{18}$$

Smoothing Newton-type methods based on $\Psi_{\rm FB}^{\rm sdc}$ are discussed in [6, 21]. The following result can be proved similarly to that of Corollary 3.5. We omit the details here.

Corollary 3.6 *The smoothed matrix valued Fischer-Burmeister function* $\Psi_{\rm FB}^{\rm sdc} : \Re \times \mathcal{S}^p \times \mathcal{S}^p \to \mathcal{S}^p$ *has the following properties*:

(i) $\Psi_{\rm FB}^{\rm sdc}$ *is globally Lipschitz continuous with Lipschitz constant* $\sqrt{2}(1+\sqrt{n})$;

(ii) $\Psi_{\rm FB}^{\rm sdc}$ *is continuously differentiable at* $(\varepsilon, X, Y) \in \Re \times \mathcal{S}^p \times \mathcal{S}^p$ *if* $[\varepsilon I \;\; X \;\; Y]$ *is of full row rank, in particular, if* $\varepsilon \neq 0$;

(iii) $\Psi_{\rm FB}^{\rm sdc}$ *is directionally differentiable at any* $(\varepsilon, X, Y) \in \Re \times \mathcal{S}^p \times \mathcal{S}^p$;

(iv) $\Psi_{\rm FB}^{\rm sdc}$ *is a strongly semismooth function.*

4 Vector Functions Associated with the Second Order Cone

The second order cone (SOC) in $\Re^n$ $(n \geq 2)$, also called the Lorentz cone or ice-cream cone, is defined by

$$\mathcal{K}^n = \{(x_1, x_2^T)^T \mid x_1 \in \Re, x_2 \in \Re^{n-1} \text{ and } x_1 \geq \|x_2\|\} . \tag{19}$$

Here and below, $\|\cdot\|$ denotes the l_2-norm in $\Re^n$. If there is no ambiguity, for convenience, we write $x = (x_1, x_2)$ instead of $x = (x_1, x_2^T)^T$.

For any $x = (x_1, x_2), y = (y_1, y_2) \in \Re \times \Re^{n-1}$, we define the Jordan product as

$$x \cdot y := \begin{bmatrix} x^T y \\ y_1 x_2 + x_1 y_2 \end{bmatrix} . \tag{20}$$

Denote

$$e = (1, 0, \ldots, 0)^T \in \Re^n .$$

Any $x = (x_1, x_2) \in \Re \times \Re^{n-1}$ has the following spectral decomposition [17]:

$$x = \lambda_1(x)\, u^{(1)} + \lambda_2(x)\, u^{(2)}\,, \tag{21}$$

where $\lambda_1(x)$, $\lambda_2(x)$ and $u^{(1)}$, $u^{(2)}$ are the *spectral values* and the associated *spectral vectors* of x, with respect to $\mathcal{K}^n$, given by

$$\lambda_i(x) \;=\; x_1 + (-1)^i \| \, x_2 \, \| \tag{22}$$

and

$$u^{(i)} \;=\; \begin{cases} \frac{1}{2}\left(1, (-1)^i \dfrac{x_2}{\| \, x_2 \, \|}\right), & \text{if } x_2 \neq 0, \\ \frac{1}{2}\left(1, (-1)^i \dfrac{w}{\| \, w \, \|}\right), & \text{otherwise,} \end{cases} \tag{23}$$

where $i = 1, 2$ and w is any vector in $\Re^{n-1}$ satisfying $\| \, w \, \| = 1$. In [17], for any scalar function $f : \Re \to \Re$, the following vector valued function associated with the SOC is introduced

$$f^{\rm soc}(x) := \; f(\lambda_1(x))u^{(1)} + f(\lambda_2(x))u^{(2)}\,. \tag{24}$$

For convenience of discussion, we denote

$$x_+ := \; (\lambda_1(x))_+ u^{(1)} + (\lambda_2(x))_+ u^{(2)}$$

and

$$|x| := \; |\lambda_1(x)| u^{(1)} + |\lambda_2(x)| u^{(2)},$$

where for any scalar $\alpha \in \Re$, $\alpha_+ = \max\{0, \alpha\}$. That is, x_+ and $|x|$ are equal to $f^{\rm soc}(x)$ with $f(t) = t_+$ and $f(t) = |t|$, $t \in \Re$, respectively. For any $x \in \mathcal{K}^n$, since $\lambda_1(x)$ and $\lambda_2(x)$ are nonnegative, we define

$$\sqrt{x} = x^{1/2} := \; (\lambda_1(x))^{1/2} u^{(1)} + (\lambda_2(x))^{1/2} u^{(2)}\,.$$

For $x \in \Re^n$, let $x^2 = x \cdot x$. It has been shown in [17] that the following results hold.

Proposition 4.1 *Suppose that $x \in \Re^n$ has the spectral decomposition as in (21). Then*

(i) $|x| \;=\; (x^2)^{1/2}$;

(ii) $x^2 \;=\; (\lambda_1(x))^2 u^{(1)} + (\lambda_2(x))^2 u^{(2)}$;

(iii) *x_+ is the orthogonal projection of x onto $\mathcal{K}^n$ and* $x_+ \;=\; (x + |x|)/2$;

(iv) $x, y \in \mathcal{K}^n$ *and* $x^T y = 0 \iff x, y \in \mathcal{K}^n$ *and* $x \cdot y \;=\; 0 \iff x - (x - y)_+ \;=\; 0\,.$

A function $\phi^{\rm soc} : \Re^n \times \Re^n \to \Re^n$ is called an SOC C-function if

$$\phi^{\rm soc}(x,y) \,=\, 0 \iff \mathcal{K}^n \,\ni\, x \,\perp\, y \,\in\, \mathcal{K}^n\,, \tag{25}$$

where the $\perp$ notation means "perpendicular under the above Jordan product", i.e., $x \perp y \Leftrightarrow x \cdot y = 0$ for any two vectors x and y in $\Re^n$. Part (iv) of Proposition 4.1 shows that the following function

$$\phi^{\rm soc}_{\min}(x,y) :=\, x - (x-y)_+ \tag{26}$$

is an SOC C-function. In [17], it is shown that the following vector valued Fischer-Burmeister function

$$\phi^{\rm soc}_{\rm FB}(x,y) :=\, x + y - \sqrt{x^2+y^2} \tag{27}$$

is also an SOC C-function. Smoothed forms of $\phi^{\rm soc}_{\min}$ and $\phi^{\rm soc}_{\rm FB}$ are defined by

$$\psi^{\rm soc}_{\min}(\varepsilon,x,y) :=\, \frac{1}{2}(\,x+y-\sqrt{\varepsilon^2 e + (x-y)^2}\,)\,, \quad (\varepsilon,x,y) \in \Re\times\Re^n\times\Re^n \tag{28}$$

and

$$\psi^{\rm soc}_{\rm FB}(\varepsilon,x,y) :=\, x + y - \sqrt{\varepsilon^2 e + x^2 + y^2}\,, \quad (\varepsilon,x,y) \in \Re\times\Re^n\times\Re^n \tag{29}$$

in [17], respectively. It is shown in [7] that both $\phi^{\rm soc}_{\min}$ and $\psi^{\rm soc}_{\min}$ are strongly semismooth functions. By making use of a relationship between the vector function $f^{\rm soc}$ and the corresponding matrix function $F^{\rm mat}$, Chen, Chen and Tseng [4], among others, provide a shorter (indirect) proof to the above result. Luo, Fukushima and Tseng [17] have discussed many properties of $\phi^{\rm soc}_{\rm FB}$ and $\psi^{\rm soc}_{\rm FB}$ including the continuous differentiability of $\psi^{\rm soc}_{\rm FB}$ at any $(\varepsilon,x,y) \in \Re\times\Re^n\times\Re^n$ for $\varepsilon \neq 0$. In this section, we shall prove that $\phi^{\rm soc}_{\rm FB}$ and $\psi^{\rm soc}_{\rm FB}$ are globally Lipschitz continuous, directionally differentiable and strongly semismooth everywhere.

For any $x = (x_1,x_2) \in \Re\times\Re^{n-1}$, let $L(x), M(x) \in \mathcal{S}^n$ be defined by

$$L(x) := \begin{bmatrix} x_1 & x_2^T \\ x_2 & x_1 I \end{bmatrix} \tag{30}$$

and

$$M(x) := \begin{bmatrix} 0 & 0^T \\ 0 & N(x_2) \end{bmatrix}, \tag{31}$$

where for any $z \in \Re^{n-1}$, $N(z) \in \mathcal{S}^{n-1}$ denotes

$$N(z) :=\, \|\,z\,\|\,(I - zz^T/\|\,z\,\|^2) \,=\, \|\,z\,\|\,I - zz^T/\|\,z\,\| \tag{32}$$

and the convention "$\frac{0}{0} = 0$" is adopted.

The next lemma presents some useful properties about the operators L and M.

Lemma 4.2 *For any $x = (x_1,x_2) \in \Re\times\Re^{n-1}$, the following results hold:*

(i) $L(x^2) \;=\; (L(x))^2 + (M(x))^2\,;$

(ii) *M is globally Lipschitz continuous with Lipschitz constant $\sqrt{n-2}$;*

(iii) *M is at least twice continuously differentiable at x if $x_2 \neq 0$;*

(iv) *M is directionally differentiable everywhere in $\Re^n$;*

(v) *M is strongly semismooth everywhere in $\Re^n$.*

Proof. (i) By a direct calculation, we have

$$L(x^2) \;=\; (L(x))^2 + \begin{bmatrix} 0 & 0^T \\ 0 & \| x_2 \|^2 - x_2 x_2^T I \end{bmatrix}$$

which, together with the fact that

$$(M(x))^2 = \begin{bmatrix} 0 & 0^T \\ 0 & \| x_2 \|^2 - x_2 x_2^T I \end{bmatrix},$$

implies that

$$L(x^2) \;=\; (L(x))^2 + (M(x))^2\,.$$

(ii) By noting the fact that for any $x = (x_1, x_2) \in \Re \times \Re^{n-1}$ and $y = (y_1, y_2) \in \Re \times \Re^{n-1}$,

$$\| M(x) - M(y) \|_{\mathcal{F}} \;=\; \| N(x_2) - N(y_2) \|_{\mathcal{F}}\,,$$

we only need to show that that N is globally Lipschitz continuous with Lipschitz constant $\sqrt{n-2}$.

Suppose that $z^{(1)}, z^{(2)}$ are two arbitrary points in $\Re^{n-1}$. If the line segment $[z^{(1)}, z^{(2)}]$ connecting $z^{(1)}$ and $z^{(2)}$ contains the origin 0, then

$$\begin{aligned} & \| N(z^{(1)}) - N(z^{(2)}) \|_{\mathcal{F}} \\ \leq\; & \| N(z^{(1)}) - N(0) \|_{\mathcal{F}} + \| N(z^{(2)}) - N(0) \|_{\mathcal{F}} \\ =\; & \| z^{(1)} \| \| [I - z^{(1)} {z^{(1)}}^T / \| z^{(1)} \|^2] \|_{\mathcal{F}} + \| z^{(2)} \| \| [I - z^{(2)} {z^{(2)}}^T / \| z^{(2)} \|^2] \|_{\mathcal{F}} \\ \leq\; & \sqrt{n-2} \| z^{(1)} \| + \sqrt{n-2} \| z^{(2)} \| \;=\; \sqrt{n-2} \| z^{(1)} - z^{(2)} \|\,. \end{aligned}$$

If the line segment $[z^{(1)}, z^{(2)}]$ does not contain the origin 0, then by the mean value theorem we have

$$\begin{aligned} & \| N(z^{(1)}) - N(z^{(2)}) \|_{\mathcal{F}} \\ =\; & \left\| \int_0^1 N'(z^{(1)} + t[z^{(2)} - z^{(1)}])(z^{(2)} - z^{(1)}) dt \right\|_{\mathcal{F}} \\ \leq\; & \int_0^1 \| N'(z^{(1)} + t[z^{(2)} - z^{(1)}])(z^{(2)} - z^{(1)}) \|_{\mathcal{F}} dt\,, \end{aligned}$$

which, together with the fact that for any $z \neq 0$, N is differentiable at z with

$$N'(z)(\Delta z) \;=\; \frac{(\Delta z)^T z}{\| z \|}[I + zz^T/\| z \|^2] - \frac{1}{\| z \|}[z(\Delta z)^T + (\Delta z)z^T] \tag{33}$$

and

$$\| N'(z)(\Delta z) \|_{\mathcal{F}} \;\leq\; \sqrt{n-2}\| \Delta z \| \;\; \forall\, \Delta z \in \Re^{n-1} \,,$$

implies that

$$\| N(z^{(1)}) - N(z^{(2)}) \|_{\mathcal{F}} \;\leq\; \sqrt{n-2}\| z^{(1)} - z^{(2)} \| \,.$$

Therefore, N is globally Lipschitz continuous with Lipschitz constant $\sqrt{n-2}$.

(iii) By equation (33), we know that N is at least twice continuously differentiable at any $z \neq 0$. Therefore, M is at least twice continuously differentiable at x if $x_2 \neq 0$.

(iv) By part (iii), we only need to show that M is directionally differentiable at x with $x_2 = 0$. This can be achieved by showing that N is directionally differentiable at $x_2 = 0$. Note that N is a positive homogeneous mapping, i.e., for any $t \geq 0$ and $z \in \Re^{n-1}$,

$$N(tz) \;=\; tN(z) \,.$$

Hence, N is directionally differentiable at $x_2 = 0$ and for any $z \in \Re^{n-1}$,

$$N'(x_2; z) \;=\; N(z) \,.$$

(v) By part (iii), we only need to show that M is strongly semismooth at x with $x_2 = 0$. This can be done by showing that N is strongly semismooth at $x_2 = 0$. For any $0 \neq z \in \Re^{n-1}$, by the positive homogeneity of N, we have

$$N'(z)(z) \;=\; N(z) \,.$$

Actually, the above result can also be derived directly by (33). Therefore, for any $0 \neq z \in \Re^{n-1}$,

$$\begin{aligned} & N(x_2 + z) - N(0) - N'(x_2 + z)(z) \\ =\; & N(z) - N(0) - N'(z)(z) \\ =\; & 0 \,, \end{aligned}$$

which, together with Proposition 2.3, the Lipschitz continuity and the directional differentiability of N, shows that N is strongly semismooth at $x_2 = 0$. Q.E.D.

The following lemma, which relates $f^{\rm soc}$ to $F^{\rm mat}$, is obtained recently by Chen, Chen and Tseng [4, Lemma 4.1][1].

[1]P. Tseng first presented this result in "The Third International Conference on Complementarity Problems", held in Cambridge University, United Kingdom, July 29 -August 1, 2002.

Lemma 4.3 *For any $x = (x_1, x_2) \in \Re \times \Re^{n-1}$, let $\lambda_1(x), \lambda_2(x)$ be its spectral values given in (22). The following results hold:*

(i) *For any $t \in \Re$, the matrix $L(x) + t\widetilde{L}(x_2)$ has eigenvalues $\lambda_1(x), \lambda_2(x)$, and $x_1 + t$ of multiplicity $n-2$, where*

$$\widetilde{L}(x_2) := \begin{bmatrix} 0 & 0 \\ 0 & I - x_2 x_2^T / \|x_2\|^2 \end{bmatrix};$$

(ii) *For any $f : \Re \to \Re$ and any $t \in \Re$, we have*

$$f^{\rm soc}(x) \; = \; F^{\rm mat}(L(x) + t\widetilde{L}(x_2))\, e\,,$$

where $F^{\rm mat}$ is defined by (8).

For any $a^{(1)}, \ldots, a^{(p)} \in \Re^n$, we write

$$\chi(a^{(1)}, \ldots, a^{(p)}) := \sqrt{\sum_{i=1}^{p} (a^{(i)})^2} \tag{34}$$

and

$$\Gamma(a^{(1)}, \ldots, a^{(p)}) := [L(a^{(1)}) \; \ldots \; L(a^{(p)}) \;\; M(a^{(1)}) \; \ldots \; M(a^{(p)})]\,, \tag{35}$$

where the operators L and M are defined by (30) and (31), respectively. The relationship between χ and Γ is revealed in the next result.

Lemma 4.4 *For any $a^{(1)}, \ldots, a^{(p)} \in \Re^n$,*

$$L(v) \; = \; \Gamma(a^{(1)}, \ldots, a^{(p)})(\Gamma(a^{(1)}, \ldots, a^{(p)}))^T$$

and

$$\chi(a^{(1)}, \ldots, a^{(p)}) \; = \; \left(\sqrt{\Gamma(a^{(1)}, \ldots, a^{(p)}) \left(\Gamma(a^{(1)}, \ldots, a^{(p)})\right)^T} \right) e\,,$$

where $v := \sum_{i=1}^{p} (a^{(i)})^2$.

Proof. By part (i) of Lemma 4.2, we obtain that

$$\begin{aligned} L(v) \; &= \; L\left(\sum_{i=1}^{p}(a^{(i)})^2\right) \; = \; \sum_{i=1}^{p} L((a^{(i)})^2) \; = \; \sum_{i=1}^{p} \left[\left(L(a^{(i)})\right)^2 + \left(M(a^{(i)})\right)^2 \right] \\ &= \sum_{i=1}^{p} \left(L(a^{(i)})\right)^2 + \sum_{i=1}^{p} \left(M(a^{(i)})\right)^2 \; = \; \Gamma(a^{(1)}, \ldots, a^{(p)}) \left(\Gamma(a^{(1)}, \ldots, a^{(p)})\right)^T . \end{aligned}$$

By taking $f(t) = \sqrt{|t|}, t \in \Re$ in part (ii) of Lemma 4.3, for any $x \in \Re^n$ we have

$$\sqrt{|x|} = f^{\rm soc}(x) = F^{\rm mat}(L(|x|))e \; = \; \left(\sqrt{L(|x|)}\right) e\,,$$

which, together with the fact that $v \in \mathcal{K}^n$, implies

$$\chi(a^{(1)},\dots,a^{(p)}) \;=\; \sqrt{v} \;=\; \left(\sqrt{L(v)}\right) e \;=\; \left(\sqrt{\Gamma(a^{(1)},\dots,a^{(p)})\,(\Gamma(a^{(1)},\dots,a^{(p)}))^T}\right) e\,.$$

This completes the proof. Q.E.D.

For the rest of this section, let $g:\Re\to\Re$ be defined by $g(t)=|t|, t\in\Re$; let the corresponding matrix function $G^{\rm mat}$ be defined by (10). Then, by Lemma 4.4 and equation (12), for any $a^{(1)},\dots,a^{(p)}\in\Re^n$ we have

$$\chi(a^{(1)},\dots,a^{(p)}) \;=\; G^{\rm mat}(\Gamma(a^{(1)},\dots,a^{(p)}))\,e\,. \tag{36}$$

Theorem 4.5 *For any $a^{(1)},\dots,a^{(p)}\in\Re^n$, let $\chi(a^{(1)},\dots,a^{(p)})$ be defined by (34). Then the following results hold*:

(i) *χ is globally Lipschitz continuous with Lipschitz constant $2\sqrt{n-1}$;*

(ii) *χ is at least once continuously differentiable at any $(a^{(1)},\dots,a^{(p)})$ with $a^{(i)}\in\Re^n\,, i=1,\dots,p$ if $v_1\neq\|\,v_2\|$, where $v=(v_1,v_2)\in\Re\times\Re^{n-1}$ and $v:=\sum_{i=1}^p(a^{(i)})^2$;*

(iii) *χ is directionally differentiable at any $(a^{(1)},\dots,a^{(p)})$ with $a^{(i)}\in\Re^n\,, i=1,\dots,p$;*

(iv) *χ is a strongly semismooth function.*

Proof. (i) For any $(a^{(1)},\dots,a^{(p)})\in\Re^n\times\ldots\times\Re^n$ and $(b^{(1)},\dots,b^{(p)})\in\Re^n\times\ldots\times\Re^n$, by equation (36), part (v) of Theorem 3.4, and Lemma 4.2, we have

$$\begin{aligned}
&\|\,\chi(a^{(1)},\dots,a^{(p)})-\chi(b^{(1)},\dots,b^{(p)})\,\|\\
=\;&\|\,[G^{\rm mat}(\Gamma(a^{(1)},\dots,a^{(p)}))-G^{\rm mat}(\Gamma(b^{(1)},\dots,b^{(p)}))]e\,\|\\
\le\;&\|\,G^{\rm mat}(\Gamma(a^{(1)},\dots,a^{(p)}))-G^{\rm mat}(\Gamma(b^{(1)},\dots,b^{(p)}))\,\|_{\mathcal{F}}\|\,e\,\|\\
\le\;&\sqrt{2}\|\,\Gamma(a^{(1)},\dots,a^{(p)})-\Gamma(b^{(1)},\dots,b^{(p)})\,\|_{\mathcal{F}}\\
=\;&\sqrt{2}\sqrt{\sum_{i=1}^p\|\,L(a^{(i)})-L(b^{(i)})\,\|_{\mathcal{F}}^2+\sum_{i=1}^p\|\,M(a^{(i)})-M(b^{(i)})\,\|_{\mathcal{F}}^2}\\
\le\;&\sqrt{2}\sqrt{\sum_{i=1}^p n\|\,a^{(i)}-b^{(i)}\,\|^2\;+\;\sum_{i=1}^p(n-2)\|\,a^{(i)}-b^{(i)}\,\|^2}\\
=\;&2\sqrt{n-1}\sqrt{\sum_{i=1}^p\|\,a^{(i)}-b^{(i)}\,\|^2}\,,
\end{aligned}$$

which, proves that χ is globally Lipschitz continuous with Lipschitz constant $2\sqrt{n-1}$.

(ii) This result follows directly from (36), Lemma 4.4, part (iii) of Theorem 3.4, and the fact that

$$\Gamma(a^{(1)}, \ldots, a^{(m)}) \left(\Gamma(a^{(1)}, \ldots, a^{(m)})\right)^T = L(v)$$

is positive definite when $v_1 \neq \| v_2 \|$.

(iii) It follows directly from (36), part (iv) of Lemma 4.2, and part (vi) of Theorem 3.4.

(iv) This property follows from (36), part (viii) of Theorem 3.4, Proposition 2.2, and the fact that the mapping Γ is strongly semismooth by part (v) of Lemma 4.2. Q.E.D.

Theorem 4.5 generalizes the results discussed in [7] from the absolute value function $|x|$ to the function χ. By noting the fact that for any $\varepsilon \in \Re$ and $(x, y) \in \Re^n \times \Re^n$,

$$\phi_{\mathrm{FB}}^{\mathrm{soc}}(x, y) = x + y - \chi(x, y)$$

and

$$\psi_{\mathrm{FB}}^{\mathrm{soc}}(\varepsilon, x, y) = x + y - \chi(\varepsilon e, x, y),$$

we have the following results, which do not need a proof.

Corollary 4.6 *The vector valued Fischer-Burmeister function $\phi_{\mathrm{FB}}^{\mathrm{soc}} : \Re^n \times \Re^n \to \Re^n$ has the following properties*:

(i) *$\phi_{\mathrm{FB}}^{\mathrm{soc}}$ is globally Lipschitz continuous with Lipschitz constant $\sqrt{2} + 2\sqrt{n-1}$;*

(ii) *$\phi_{\mathrm{FB}}^{\mathrm{soc}}$ is at least once continuously differentiable at any $(x, y) \in \Re^n \times \Re^n$ if $v_1 \neq \| v_2 \|$, where $v := x^2 + y^2$;*

(iii) *$\phi_{\mathrm{FB}}^{\mathrm{soc}}$ is directionally differentiable at any $(x, y) \in \Re^n \times \Re^n$;*

(iv) *$\phi_{\mathrm{FB}}^{\mathrm{soc}}$ is a strongly semismooth function.*

Corollary 4.7 *The smoothed vector valued Fischer-Burmeister function $\psi_{\mathrm{FB}}^{\mathrm{soc}} : \Re \times \Re^n \times \Re^n \to \Re^n$ has the following properties*:

(i) *$\psi_{\mathrm{FB}}^{\mathrm{soc}}$ is globally Lipschitz continuous with Lipschitz constant $\sqrt{2} + 2\sqrt{n-1}$;*

(ii) *$\psi_{\mathrm{FB}}^{\mathrm{soc}}$ is at least once continuously differentiable at any $(\varepsilon, x, y) \in \Re \times \Re^n \times \Re^n$ if $\varepsilon \neq 0$ or $v_1 \neq \| v_2 \|$, where $v := x^2 + y^2$;*

(iii) *$\psi_{\mathrm{FB}}^{\mathrm{soc}}$ is directionally differentiable at any $(\varepsilon, x, y) \in \Re \times \Re^n \times \Re^n$;*

(iv) *$\psi_{\mathrm{FB}}^{\mathrm{soc}}$ is a strongly semismooth function.*

5 Inverse Singular Value Problems

Given a family of matrices $D(c) \in \mathcal{M}_{n,m}$ $(n \leq m)$ with $c \in \Re^n$ and nonnegative real numbers $\lambda_1^* \geq \ldots \geq \lambda_n^* \geq 0$ arranged in the decreasing order, the inverse singular value problem (ISVP) is to find a parameter $c^* \in \Re^n$ such that

$$\sigma_i(D(c^*)) = \lambda_i^*, \; i = 1, \ldots, n,$$

where for any $A \in \mathcal{M}_{n,m}$, $\sigma_1(A) \geq \ldots \geq \sigma_n(A)$ are the singular values of A. See Chu [9] for a comprehensive survey on the inverse eigenvalue problems, which include the ISVP.

Define $\theta : \Re^n \to \Re^n$ by

$$\theta(c) := \begin{bmatrix} \sigma_1(D(c)) - \lambda_1^* \\ \vdots \\ \sigma_n(D(c)) - \lambda_n^* \end{bmatrix}. \tag{37}$$

Then the ISVP is equivalent to finding $c^* \in \Re^n$ such that $\theta(c^*) = 0$. In general, the function θ is not continuously differentiable. So classical Newton's method for solving smooth equations can not be applied directly to solve $\theta(c) = 0$. However, the recent research reveals that quadratic convergence of generalized Newton's method for solving nonsmooth equations does not need the continuous differentiability of the function involved [30, 28]. What is needed is the strong semismoothness property. Next, we show that θ has such a property.

Theorem 5.1 *Suppose that $D : \Re^n \to \mathcal{M}_{n,m}$ is twice continuously differentiable. Let $\theta : \Re^n \to \Re^n$ be defined by (37). Then θ is a strongly semismooth function.*

Proof. Since D is twice continuously differentiable, D is strongly semismooth everywhere. Then, from Propositions 2.2 and 3.2, we know that θ is strongly semismooth everywhere. Q.E.D.

In [37], based on the strong semismoothness of eigenvalues of symmetric matrices, we established the quadratic convergence of generalized Newton's method for solving the (generalized) inverse eigenvalue problem for symmetric matrices. This approach, which is different from the one given in [16], relys on the convergent theory of generalized Newton's method for solving nonsmooth equations [30, 28]. Analogously, by using Theorem 5.1, we can establish the quadratic convergence of generalized Newton's method for solving the ISVP. Again, this approach is different from the one given in [8] for the ISVP, where D is assumed to be a linear mapping and the convergence analysis is based on [16]. Here, we omit the details because it can be worked out similarly as in [37] for the inverse eigenvalue problem for symmetric matrices.

6 Conclusions

In this paper, based on the singular values of nonsymmetric matrices, we defined a matrix valued function G^{mat} over nonsymmetric matrices. We showed that G^{mat} inherits many properties from its base scalar valued function g. In particular, we showed that the (smoothed) matrix valued Fischer-Burmeister function is strongly semismooth everywhere. By using a recent result of Chen, Chen and Tseng [4], we also established the strong semismoothness of the (smoothed) vector valued Fischer-Burmeister function associated with the second order cone. Finally, we briefly mentioned the quadratic convergence of generalized Newton's method for solving the inverse singular value problem.

References

[1] R. Bhatia, *Matrix Analysis*, Springer-Verlag, New York, 1997.

[2] J. F. Bonnans, R. Cominetti and A. Shapiro, "Second order optimality conditions based on parabolic second order tangent sets," *SIAM Journal on Optimization*, **9** (1999) 466–493.

[3] J. M. Borwein and A. S. Lewis, *Convex Analysis and Nonlinear Optimization: Theory and Examples*, Springer-Verlag, New York, 2000.

[4] J. Chen, X. Chen and P. Tseng, "Analysis of nonsmooth vector-valued functions associated with second-order cones," Manuscript, Department of Mathematics, University of Washington, USA, November 2002.

[5] X. Chen, H. Qi, and P. Tseng, "Analysis of nonsmooth symmetric-matrix functions with applications to semidefinite complementarity problems," Manuscript, Department of Mathematics, University of Washington, Seattle, Washington 98195, USA, October 2000; revised, March 2002. To appear in *SIAM Journal on Optimization*.

[6] X. Chen and P. Tseng, "Non-interior continuation methods for solving semidefinite complementarity problems," Manuscript, Department of Mathematics, University of Washington, Seattle, Washington 98195, USA, October 1999; revised, January 2002. To appear in *Mathematical Programming*.

[7] X. D. Chen, D. Sun and J. Sun, "Complementarity functions and numerical experiments on some smoothing Newton methods for second-order-cone complementarity problems," Manuscript, Department of Mathematics, National University of Singapore, Republic of Sinpaore, February 2002; revised, July 2002. To appear in *Computational Optimization and Applications*.

[8] M. T. Chu, "Numerical methods for inverse singular value problems," *SIAM Journal on Numerical Analysis*, **29** (1992) 885–903.

[9] M. T. Chu, "Inverse eigenvalue problems," *SIAM Review*, **40** (1998) 1–39.

[10] F. Clarke, *Optimization and Nonsmooth Analysis,* John Wiley and Sons, New York, 1983.

[11] T. De Luca, F. Facchinei and C. Kanzow, "A semismooth equation approach to the solution of nonlinear complementarity problems," *Mathematical Programming*, **75** (1996) 407–439.

[12] F. Facchinei and J. Soares, "A new merit function for nonlinear complementarity problems and a related algorithm," *SIAM Journal on Optimization*, **7** (1997) 225–247.

[13] K. Fan, "On a theorem of Weyl concerning eigenvalues of linear transformations. I.," *Proc. Nat. Acad. Sci. U. S. A.*, **35** (1949) 652–655.

[14] A. Fischer, "A special Newton-type optimization method," *Optimization*, **24** (1992) 269–284.

[15] A. Fischer, "Solution of monotone complementarity problems with locally Lipschitzian functions," *Mathematical Programming,* **76** (1997) 513–532.

[16] S. Friedland, J. Nocedal and M. L. Overton, "The formulation and analysis of numerical methods for inverse eigenvalue problems," *SIAM Journal on Numerical Analysis*, **24** (1987) 634–667.

[17] M. Fukushima, Z. Q. Luo and P. Tseng, "Smoothing functions for second-order-cone complementarity problems," *SIAM Journal on Optimization*, **12** (2002) 436–460.

[18] R. M. B. Goes and P. R. Oliveira, "A new class of merit functions for the semidefinite complementarity problem," Technical Report ES-569, Programa de Engenharia de sistemas e Computacão, COPPE-Federal University of Rio de Janeiro, Brazil, January 2002.

[19] G. H. Golub and C. F. Van Loan, *Matrix Computations*, The Johns Hopkins University Press, Baltimore, USA, Third Edition, 1996.

[20] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, UK, 1991.

[21] C. Kanzow and C. Nagel, "Semidefinite programs: new search directions, smoothing-type methods, and numerical results," Technical Report, Department of Mathematics, University of Hamburg, Germany, June 2001. To appear in *SIAM Journal on Optimization.*

[22] A. S. Lewis, “Nonsmooth analysis of eigenvalues,” *Mathematical Programming*, **84** (1999) 1–24.

[23] A. S. Lewis and M. L. Overton, “Eigenvalue optimization,” *Acta Numerica*, **5** (1996) 149–190.

[24] A. S. Lewis and H. S. Sendov, “Twice differentiable spectral functions,” *SIAM Journal on Matrix Analysis and Applications*, **23** (2001) 368–386.

[25] R. Mifflin, “Semismooth and semiconvex functions in constrained optimization,” *SIAM Journal on Control and Optimization*, **15** (1977) 957–972.

[26] J. -S. Pang and L. Qi, “Nonsmooth equations: motivation and algorithms,” *SIAM Journal on Optimization*, **3** (1993), 443–465.

[27] J. -S. Pang, D. Sun and J. Sun, “Semismooth homeomorphisms and strong stability of semidefinite and Lorentz cone complementarity problems,” Preprint, Department of Mathematics, National University of Singapore, Republic of Singapore, April 2002; revised, October 2002. To appear in *Mathematics of Operations Research.*

[28] L. Qi, “Convergence analysis of some algorithms for solving nonsmooth equations,” *Mathematics of Operations Research*, **18** (1993) 227–244.

[29] L. Qi and H. Jiang, “Semismooth Karush-Kuhn-Tucker equations and convergence analysis of Newton and quasi-Newton methods for solving these equations,” *Mathematics of Operations Research*, **22** (1997) 301–325.

[30] L. Qi and J. Sun, “A nonsmooth version of Newton’s method,” *Mathematical Programming*, **58** (1993) 353–367.

[31] R. T. Rockafellar, *Convex Analysis,* Princeton University Press, New Jersey, 1970.

[32] A. Shapiro, “Perturbation theory of nonlinear programs when the set of optimal solutions is not a singleton,” *Applied Mathematics and Optimization*, **18** (1988) 215–229.

[33] A. Shapiro, “On a class of nonsmooth composite functions,” Technical Report, School of Industrial and Systems Engineering, Georgia Institute of Technology, USA, July 2002.

[34] A. Shapiro and M. K. H. Fan, “On eigenvalue optimization,” *SIAM Journal on Optimization*, **5** (1995) 552–569.

[35] G. W. Stewart and J. -G. Sun, *Matrix Perturbation Theory*, Academic Press, New York, 1990.

[36] D. Sun and J. Sun, “Semismooth matrix valued functions,” *Mathematics of Operations Research*, **27** (2002) 150–169.

[37] D. Sun and J. Sun, "Strong semismoothness of eigenvalues of symmetric matrices and its application to inverse eigenvalue problems," Preprint, Department of Mathematics, National University of Singapore, Republic of Singapore, August 2001; revised, May 2002. To appear in *SIAM Journal on Numerical Analysis*.

[38] J. Sun, D. Sun and L. Qi, "Quadratic convergence of a smoothing Newton method for nonsmooth matrix equations and its application in semidefinite optimization problems," Preprint, Department of Decision Sciences, National University of Singapore, Republic of Singapore, October 2000; revised, April 2002.

[39] M. Torki, "Second-order directional derivatives of all eigenvalues of a symmetric matrix," *Nonlinear Analysis*, **46** (2001) 1133–1150.

[40] P. Tseng, "Merit functions for semidefinite complementarity problems," *Mathematical Programming*, **83** (1998) 159–185.

[41] N. Yamashita and M. Fukushima, "A new merit function and a descent method for semidefinite complementarity problems," in M. Fukushima and L. Qi (eds.), *Reformulation - Nonsmooth, Piecewsie Smooth, Semismooth and Smoothing Methods*, Boston, Kluwer Academic Publishers, pp. 405–420, 1999.

[42] E. H. Zarantonello, "Projections on convex sets in Hilbert space and spectral theory I and II," in E.H. Zarantonello, editor, *Contributions to Nonlinear Functional Analysis*, Academic Press (New York 1971) pp. 237–424.