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Alternative theorems for nonlinear projection equations and applications to generalized complementarity problems ☆

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1. Introduction

Let K be a closed convex subset in R^n, and let f, g and h be three functions from R^n into itself. We consider the following nonlinear projection equation (NPE):

$$h(x) = \Pi_K(g(x) - f(x)), \quad x \in R^n, \tag{1}$$

where $\Pi_K(\cdot)$ is the orthogonal projection operator onto the set K. This equation provides a unified formulation of several interesting and important special cases. In the case when $h(x) = g(x)$ for any $x \in R^n$, then (1) reduces to the equation studied by Pang and Yao [17] under the name of generalized normal equation which is equivalent to the following generalized variational inequality (GVI):

$$g(x) \in K, \quad (y - g(x))^{\mathrm{T}} f(x) \geq 0 \quad \text{for all } y \in K. \tag{2}$$

Moreover, if $g(x) = x$, then the above problem reduces to the variational inequality problem (VIP), i.e.,

$$x \in K, \quad (y - x)^{\mathrm{T}} f(x) \geq 0 \quad \text{for all } y \in K, \tag{3}$$

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which was extensively studied in the last several decades. In the case when $K = R^n_+$ (the nonnegative orthant), problems (2) and (3) are further reduced to the following generalized complementarity problem (GCP):

$$g(x) \geq 0, \qquad f(x) \geq 0, \qquad f(x)^{\mathrm{T}} g(x) = 0,$$

and complementarity problem (NCP):

$$x \geq 0, \qquad f(x) \geq 0, \qquad x^{\mathrm{T}} f(x) = 0,$$

respectively.

Given functions f, g and h, the existence of a solution to NPE is not always assured. In this paper, we study the existence of a solution to NPE by using new concepts, that is, the exceptional families for NPE. Based on these concepts, two important alternative theorems for NPE are established, which state that there exists either a solution or an exceptional family for NPE. Thus, the condition "there exists no exceptional family for NPE" is sufficient for the existence of a solution to NPE. Due to the importance of complementarity problems in practical applications (see [4,7]), in Section 3 of this paper, we use one of the alternative theorems to develop several new existence conditions for GCP.

So far, a large number of existence results have been proved for complementarity problems with different classes of the functions. Most of these results assume monotonicity or certain generalized monotonicity of the functions [15,10,6,2,3,19,20,17,5,21, 9,24,23]. These concepts of generalized monotonicity were emerged from the pseudo-monotonicity introduced by Karamardian [10,11]. He showed that the complementarity problem over a pointed, solid closed convex cone K in R^n has a solution if f is a continuous pseudo-monotone function and satisfies the strictly feasible condition, i.e., there exists an $x \in K$ such that $f(x)$ is an interior point of K^*, the dual cone of K. Particularly, when $K = R^n_+$, this strictly feasible condition reduces to the following: There exists a point $u \geq 0$ such that $f(u) > 0$. Later, several generalizations of Karamardian result have been developed. Under the same pseudo-monotonicity assumption, Cottle and Yao [3] generalized the Karamardian result to the case of a solid, closed and convex cone in Hilbert space. Harker and Pang [6] and Yao [19,20] extended the Karamardian result to variational inequality problems. The class of quasi-monotone maps is larger than the pseudo-monotone maps. Hadjisavvas and Schaible [5] established an existence result for quasi-monotone VIP in reflexive Banach space. When restricted to NCP, their result states that if the strictly feasible condition holds, there exists a solution to the complementarity problem with a quasi-monotone map. The concept of monotonicity is also generalized in other directions, for instance, the class of nonlinear P_*-maps. It is easy to give examples to show that a P_*-map need not to be a quasi-monotone map, and the vice versa. Zhao and Han [23] showed that if the strictly feasible condition is satisfied then there exists a solution to NCP with a nonlinear P_*-map.

In this paper, we introduce several new classes of nonlinear functions including so-called quasi-P_*, quasi-P_*^M and $P(\tau, \alpha, \beta)$-maps. Each of these classes can be viewed as the generalization of P_*-maps. For these maps, our main results state that the complementarity problem has a solution if it is strictly feasible. Since quasi-P_*-maps and quasi-P_*^M-maps include the union of quasi-monotone maps and P_*-maps, the existence results established in the paper significantly generalize several previous results, including those of Moré [15], Karamardian [10], Hadjisavvas and Schaible [5], and Zhao and Han [23], in the framework of complementarity problems.

In Section 2, the concepts of exceptional families (A) and (B) for NPE are introduced and several special cases are also discussed. By using the two concepts, we prove two alternative theorems on the existence of a solution to NPE. In Section 3, we define several new classes of functions and develop several new existence conditions for GCPs. Conclusions are given in the last section.

2. Alternative theorems for NPE

Throughout the paper, $\|\cdot\|$ denotes the Euclidean-norm, R^n_+ the nonnegative orthant, and $\Pi_K(\cdot)$ the orthogonal projection operator on the convex set K, that is, for any $z \in R^n$, $\Pi_K(z)$ is the unique solution to the following problem:

$$\min_y \{\|y - z\|: \ y \in K\}.$$

Let D be an open bounded set of R^n. We denote by $\bar{D}$ and $\mathscr{B}(D)$ the closure and boundary of D, respectively. Let f be a continuous function from $\bar{D}$ into R^n. For and $y \in R^n$ such that $y \notin f(\mathscr{B}(D))$, the notation $\deg(f, D, y)$ is the topological degree associated with f, D and y. See [13,16] for a detailed discussion on degree theory. The following three lemmas play a very important role in our analysis.

Lemma 2.1 (Lloyd [13]; Ortega and Rheinboldt [16]). *Let $D \subset R^n$ be an open bounded set and F, G be two continuous functions from $\bar{D}$ into R^n. The homotopy $H(x,t)$ is defined by*

$$H(x,t) = tG(x) + (1-t)F(x), \quad 0 \le t \le 1.$$

Let y be an arbitrary point in R^n. If

$$y \notin \{H(x,t): x \in \mathscr{B}(D) \text{ and } t \in [0,1]\},$$

then

$$deg(G, D, y) = deg(F, D, y).$$

Lemma 2.2 (Lloyd [13]; Ortega and Rheinboldt [16]). *Let D and F be given as in Lemma* 2.1. *If $y \notin F(\mathscr{B}(D))$ and $deg(F, D, y) \neq 0$, then the system of equations $F(x) = y$ has a solution in D.*

Lemma 2.3 (Clarke [1]). *Let $v: R^n \to R$ be a local Lipschitz continuous function, and let K be a closed convex set in R^n. If x^* is a solution to the following problem*:

$$\min_{x \in K} v(x),$$

then

$$0 \in \partial v(x^*) + \mathscr{N}_K(x^*),$$

where $\partial v(x^*)$ denotes the subdifferential of v at x^*, and $\mathcal{N}_K(x^*)$ the normal cone of K at x^*.

Now, we are ready to define the concept of exceptional family (A) for NPE.

Definition 2.1. Let f, g, and h be three functions from R^n into itself, and let $\hat{x}$ be an arbitrary point in R^n. A sequence $\{x^r\}_{r>0} \subset R^n$ is said to be an exceptional family (A) with respect to $\hat{x}$ for the NPE if $\|x^r\| \to \infty$ as $r \to \infty$, and for each x^r there exists a positive scalar α_r such that

$$e(x^r, \alpha_r) := \alpha_r((g+h)(x^r) - (g+h)(\hat{x})) + h(x^r) \in K \tag{4}$$

and

$$-f(x^r) + (1-\alpha_r)g(x^r) - (1+\alpha_r)h(x^r) + \alpha_r(g(\hat{x}) + h(\hat{x})) \in \mathcal{N}_K(e(x^r, \alpha_r)), \tag{5}$$

where $\mathcal{N}_K(e(x^r, \alpha_r))$ is the normal cone of K at $e(x^r, \alpha_r)$.

Using the above concept, we have the following result.

Theorem 2.1. *Let f, g, and h be three continuous functions from R^n into itself, and let $g+h$ be an injective mapping. Let $\hat{x}$ be an arbitrary point in R^n. Then there exists either a solution for NPE or an exceptional family (A) with respect to $\hat{x}$ for NPE.*

Proof. Denote $\phi : R^n \to R^n$ by

$$\phi(x) = h(x) - \Pi_K(g(x) - f(x)). \tag{6}$$

By the continuity of h, g, f and the property of the projection operator, $\phi(x)$ is a continuous function from R^n into itself. We now consider the homotopy between the mapping $[(g+h)(x) - (g+h)(\hat{x})]/2$ and $\phi(x)$, that is

$$H(x,t) = t\left[\frac{(g+h)(x)}{2} - \frac{(g+h)(\hat{x})}{2}\right] + (1-t)\phi(x), \quad t \in [0,1].$$

Consider the following family of open sets:

$$D_r = \{x \in R^n \colon \|x - \hat{x}\| < r\}.$$

Then the boundary $\mathscr{B}(D_r) = \{x \in R^n \colon \|x - \hat{x}\| = r\}$. There are only two cases.

Case 1: There exists a scalar $r > 0$ such that

$$0 \notin \{H(x,t) \colon x \in \mathscr{B}(D_r) \text{ and } t \in [0,1]\}. \tag{7}$$

By Lemma 2.1, we have

$$deg\left(\frac{(g+h)(x)}{2} - \frac{(g+h)(\hat{x})}{2}, D_r, 0\right) = deg(\phi(x), D_r, 0). \tag{8}$$

Since $g+h$ is injective, we have (see [13,16])

$$\left|deg\left(\frac{(g+h)(x)}{2} - \frac{(g+h)(\hat{x})}{2}, D_r, 0\right)\right| = 1.$$

Thus, it follows from (8) and the above relation that $deg(\phi(x), D_r, 0) \neq 0$. By Lemma 2.2, the equation $\phi(x) = 0$, i.e., the NPE, has at least a solution.

Case 2: For each $r > 0$, there exist some point $x^r \in \mathscr{B}(D_r)$ and $t_r \in [0,1]$ such that

$$0 = H(x^r, t_r) = t_r\left(\frac{(g+h)(x^r)}{2} - \frac{(g+h)(\hat{x})}{2}\right) + (1-t_r)\phi(x^r). \tag{9}$$

In this case, if $t_r = 0$, it follows from (9) that $\phi(x^r) = 0$, i.e., x^r is a solution to NPE. On the other hand, we have that $t_r \neq 1$. Indeed, if $t_r = 1$, from (9) we obtain

$$g(x^r) + h(x^r) = g(\hat{x}) + h(\hat{x}).$$

Since $g + h$ is an injective mapping, we deduce that $x^r = \hat{x}$ which contradicts that $x^r \in \mathscr{B}(D_r)$, i.e., $\|x^r - \hat{x}\| = r > 0$. Therefore, in the rest of this proof, it is sufficient to consider the case $t_r \in (0,1)$. We show that in this case NPE has an exceptional family (A) with respect to $\hat{x}$. Combining (6) and (9) yields

$$\frac{t_r}{1-t_r}\left(\frac{(g+h)(x^r)}{2} - \frac{(g+h)(\hat{x})}{2}\right) + h(x^r) = \Pi_K(g(x^r) - f(x^r)). \tag{10}$$

Let $\alpha_r = t_r/[2(1-t_r)]$ and

$$e(x^r, \alpha_r) := \alpha_r((g+h)(x^r) - (g+h)(\hat{x})) + h(x^r).$$

Then (10) can be rewritten as

$$e(x^r, \alpha_r) = \Pi_K(g(x^r) - f(x^r)),$$

which implies that $e(x^r, \alpha_r) \in K$, and that $e(x^r, \alpha_r)$ is the unique solution to the following convex program:

$$\min_{y \in K} Q(y) := \tfrac{1}{2}\|y - (g(x^r) - f(x^r))\|^2. \tag{11}$$

Clearly, $Q(y)$ is differentiable and therefore must be locally Lipschitz continuous, thus by Lemma 2.3, we have

$$-\nabla Q(e(x^r, \alpha^r)) = -e(x^r, \alpha_r) + (g(x^r) - f(x^r)) \in \mathscr{N}_K(e(x^r, \alpha_r)).$$

Therefore,

$$-f(x^r) + (1-\alpha_r)g(x^r) - (1+\alpha_r)h(x^r) + \alpha_r(g(\hat{x}) + h(\hat{x})) \in \mathscr{N}_K(e(x^r, \alpha_r)).$$

Furthermore, by noting that $x^r \in \mathscr{B}(D_r)$ which implies that $\|x^r\| \to \infty$ as $r \to \infty$, we deduce that $\{x^r\}$ is an exceptional family (A) with respect to $\hat{x}$ for NPE. □

It should be pointed out that the injectivity of the map $g + h$ can be removed by using other versions of the exceptional families.

Definition 2.2. Given $\hat{x}$, a sequence $\{x^r\}_{r>0} \subset R^n$ is said to be an exceptional family (B) with respect to $\hat{x}$ for the nonlinear projection equation (NPE) if $\|x^r\| \to \infty$ as $r \to \infty$ and for each x^r there exists a positive scalar α_r such that

$$w(x^r, \alpha_r) := \alpha_r(x^r - \hat{x}) + h(x^r) \in K \tag{12}$$

and

$$-f(x^r) + g(x^r) - h(x^r) - \alpha_r(x^r - \hat{x}) \in \mathscr{N}_K(w(x^r, \alpha_r)). \tag{13}$$

We now prove the following result without the assumption of the injectivity of $g+h$.

Theorem 2.2. *Let f, g, and h be three continuous functions from R^n into itself, and let $\hat{x}$ be an arbitrary point in R^n. Then there exists either a solution for NPE or an exceptional family (B) with respect to $\hat{x}$ for NPE.*

Proof. Let $\phi(x)$ and the family of open sets $\{D_r\}$ be defined as in the proof of Theorem 2.1. We consider the following homotopy between the maps $x-\hat{x}$ and $\phi(x)$, that is:

$$H(x,t) = t(x-\hat{x}) + (1-t)\phi(x), \quad t \in [0,1].$$

It is sufficient to consider two cases.

Case 1: There exists a scalar $r > 0$ such that (7) holds. Since $|deg(x-\hat{x}, D_r, 0)| = 1$, by the same argument of the "Case 1" of the proof of Theorem 2.1, we can show that NPE has a solution.

Case 2: For each $r > 0$, there exists a vector $x^r \in \mathscr{B}(D_r)$ and scalar $t_r \in [0,1]$ such that

$$0 = H(x^r, t_r) = t_r(x^r - \hat{x}) + (1-t_r)\phi(x^r). \tag{14}$$

Clearly, $t_r \neq 1$ since $x^r \neq \hat{x}$. If $t_r = 0$, then $\phi(x) = 0$, and hence x^r is a solution to NPE. Thus, in the rest of the proof, we only consider the case $t_r \in (0,1)$. We show that for this case there exists an exceptional family (B) with respect to $\hat{x}$ for NPE. Indeed, combining (6) and (14) leads to

$$\frac{t_r}{1-t_r}(x^r - \hat{x}) + h(x^r) = \Pi_K(g(x^r) - f(x^r)). \tag{15}$$

Let $\alpha_r = t_r/(1-t_r) > 0$. Then from the above equation we have

$$w(x^r, \alpha_r) := \alpha_r(x^r - \hat{x}) + h(x^r) \in K.$$

Clearly, $w(x^r, \alpha_r)$ is the unique solution to the following convex program:

$$\min_{x \in K} \tfrac{1}{2}\|y - (g(x^r) - f(x^r))\|^2.$$

Thus, by Lemma 2.3 we have

$$-f(x^r) + g(x^r) - h(x^r) - \alpha_r(x^r - \hat{x}) \in \mathscr{N}_K(w(x^r, \alpha_r)).$$

Therefore, $\{x^r\}_{r>0}$ is an exceptional family with respect to $\hat{x}$ for NPE. □

The result below is an immediate consequence of Theorems 2.1 and 2.2.

Corollary 2.1. *Let f, g, and h be three continuous functions from R^n into itself.*

(a) *If $g+h$ is an injective map and there exists a point $\hat{x} \in R^n$ such that the NPE has no exceptional family (A) with respect to $\hat{x}$, then there exists a solution to NPE.*

(b) *If there exists a point $\hat{x} \in R^n$ such that the NPE has no exceptional family (B) with respect to $\hat{x}$, then there exists a solution to NPE.*

In Section 3, we use Theorem 2.1 to prove several new practical existence results for generalized complementarity problems. Our idea is to develop the conditions under which the problem has no exceptional family (A) with respect to some point. While Theorem 2.2 does not require the injectivity of $g+h$, for a general NPE we are not clear at present what conditions can guarantee that the problem is without exceptional family (B) with respect to certain point. However, for some special cases such as VIP and NCP, it is easy to see that the concept of exceptional family (A) with respect to $\hat{x}$ coincides with exceptional family (B) with respect to the same $\hat{x}$. Thus, for VIP both Definitions 2.1 and 2.2 reduce to the following notion.

Definition 2.3 (*Zhao* [22]). Let f be a mapping from R^n into itself and $\hat{x} \in R^n$. A sequence $\{x^r\}$ is said to be an exceptional family with respect to $\hat{x}$ for VIP if $\|x^r\| \to \infty$ as $r \to \infty$ and for each x^r there exists a positive scalar α_r such that

$$\pi(x^r, \alpha_r) := (1+\alpha_r)x^r - \alpha_r\hat{x} \in K \tag{16}$$

and

$$-f(x^r) - a_r(x^r - \hat{x}) \in \mathcal{N}_K(\pi(x^r, \alpha_r)). \tag{17}$$

The results of Theorems 2.1 and 2.2 reduce to the following.

Corollary 2.2 (Zhao [22]). *Let f be a continuous function. Given any $\hat{x} \in R^n$, then there exists either a solution to VIP or an exceptional family with respect to $\hat{x}$ for VIP.*

In most situations, K is represented by a system of inequalities or equations. In this case, (5) and (13) can be further written as a system of equations by using the Lagrangian multipliers involving the inequalities and equations of K. Indeed, we assume that K is given as follows:

$$K = \{x \in R^n\colon C(x) \le 0,\ E(x) = 0\} \tag{18}$$

where $C(x) = (C_1(x), \ldots, C_m(x))^{\mathrm{T}}$ and $E(x) = (E_1(x), \ldots, E_l(x)))^{\mathrm{T}}$. Each component C_i is a continuous differentiable convex function from R^n into R, and each E_i is an affine function from R^n into R. We also assume that some standard constraint qualifications hold, for instance, there exists some point x^0 such that $C(x^0) < 0$ and $E(x^0) = 0$ (the Slater constraint qualification).

Since $e(x^r, \alpha_r)$ is the unique solution to the problem (11), it satisfies the Karush–Kukn–Tuker optimality condition. That is, there exist some vectors $\mu_r \in R^m_+$ and $\lambda_r \in R^l$ such that (5) is reduced to the following:

$$\begin{aligned} &f(x^r) + (\alpha_r - 1)g(x^r) + (1+\alpha_r)h(x^r) - \alpha_r(g+h)(\hat{x}) \\ &\quad -\nabla C(e(x^r, \alpha_r))\mu_r - \nabla E(e(x^r, \alpha_r))\lambda_r = 0 \end{aligned} \tag{19}$$

and

$$C(e(x^r, \alpha_r))^{\mathrm{T}}\mu_r = 0. \tag{20}$$

Similarly, when K is given by (18), there exist $\mu_r \in R^n_+$ and $\lambda_r \in R^l$ such that (13) can be rewritten as

$$f(x^r) - g(x^r) + h(x^r) + \alpha_r(x^r - \hat{x}) - \nabla C(w(x^r, \alpha_r))\mu_r$$
$$-\nabla E(w(x^r, \alpha_r))\lambda_r = 0$$

and

$$C(w(x^r, \alpha_r))^{\mathrm{T}} \mu_r = 0.$$

Clearly, for VIP, the above relations can be written as

$$f(x^r) + \alpha_r(x^r - \hat{x}) - \nabla C(\pi(x^r, \alpha_r))\mu_r - \nabla E(\pi(x^r, \alpha_r)\lambda_r = 0 \tag{21}$$

and

$$C(\pi(x^r, \alpha_r))^{\mathrm{T}} \mu_r = 0, \tag{22}$$

where $\pi(x^r, \alpha_r)$ is given by (16). This version of exceptional family was first introduced by Zhao and Han [23] for VIP. Other versions of exceptional families for VIP can be found in [21,24]. Furthermore, when $K = R^n_+$ and $\hat{x} = 0$, it is easy to see that (21) and (22) reduce to

$$f_i(x^r) \begin{cases} = -\alpha_r x_i^r & \text{if } x_i^r > 0, \\ \geq 0 & \text{if } x_i^r = 0, \end{cases}$$

which is the concept of exceptional family of elements for continuous f defined by Isac et al. [8,9]. This concept includes the Smith concept [18] as a particular case.

3. Solvability of generalized complementarity problems

An important special case of NPE is that $h(x) = g(x)$ and $K = R^n_+$, i.e., the generalized complementarity problem (GCP)

$$g(x) \geq 0, \qquad f(x) \geq 0, \qquad g(x)^{\mathrm{T}} f(x) = 0.$$

In what follows, we establish several new existence results for this problem by using Theorem 2.1. To show these results, we first tail the Definition 2.1 and Theorem 2.1 to our needs. Setting $h(x) = g(x)$ and $C(x) = -x$ in (19) and (20), we have the following concept for GCP.

Definition 3.1. Let f and g be two continuous functions from R^n into itself, and let $\hat{x}$ be an arbitrary point in R^n. The sequence $\{x^r\} \subset R^n$ is said to be an exceptional family (A) with respect to $\hat{x}$ for GCP if $\|x^r\| \to \infty$ as $r \to \infty$, and for each x^r there exists a positive scalar α_r such that

$$e(x^r, \alpha_r) := (1 + \alpha_r)g(x^r) - \alpha_r g(\hat{x}) \in R^n_+$$

and

$$f_i(x^r) = -\alpha_r(g_i(x^r) - g_i(\hat{x})) \quad \text{if } e_i(x^r, \alpha_r) > 0, \tag{23}$$

$$f_i(x^r) \geq -\alpha_r(g_i(x^r) - g_i(\hat{x})) \quad \text{if } e(x^r, \alpha_r) = 0. \tag{24}$$

The following corollary is an immediate consequence of Theorem 2.1.

Corollary 3.1. *Let f and g be two continuous functions and g be an injective map. If there exists a point $\hat{x}$ such that GCP has no exceptional family (A) with respect to $\hat{x}$, then there exists a solution to GCP.*

The first existence result below is related to the so-called quasi-P_*^M-property of the pair (f,g). The following definition makes this property precise.

Definition 3.2. We say that the function $f:R^n \to R^n$ is a quasi-P_*^M-mapping with respect to $g:R^n \to R^n$ if there exists a constant $\tau \geq 0$ such that for any distinct pair x, y in R^n,

$$f(y)^{\mathrm{T}}(g(x) - g(y)) - \tau \max_{1 \leq i \leq n}(g_i(x) - g_i(y))(f_i(x) - f_i(y)) > 0$$

implies that

$$f(x)^{\mathrm{T}}(g(x) - g(y)) \geq 0.$$

Clearly, when g is the identity mapping and $\tau = 0$, the above concept reduces to the well-known notion of quasi-monotone map [11,6]. On the other hand, it is easy to show that a P_*-map is a quasi-P_*-map. We recall that f is said to be a P_*-map if there exists a constant $\tau \geq 0$ such that

$$(f(x) - f(y))^{\mathrm{T}}(x - y) + \tau \sum_{i \in I_+(x,y)} (x_i - y_i)(f_i(x) - f_i(y)) \geq 0,$$

where

$$I_+(x,y) = \{i : (x_i - y_i)(f_i(x) - f_i(y)) > 0\}.$$

The nonlinear P_*-map is the generalization of the concept of a P_*-matrix defined by Kojima et al. [12]. The following notion is utilized to prove our main results.

Definition 3.3 (*Pang and Yao [17]*). A function $g:R^n \to R^n$ is said to be proper with respect to a set S if for every bounded subset S' of S, $g^{-1}(S')$ is bounded.

We now prove the following result.

Theorem 3.1. *Let f and g be two continuous functions. Suppose that g is an injective map, and g is proper with respect to $K = R^n_+$. Suppose that f is a quasi-P_*^M-map. If there exists some point $\hat{x}$ such that $g(\hat{x}) \geq 0$ and $f(\hat{x}) > 0$, then GCP has a solution.*

Proof. By Corollary 3.1, it suffices to show that there is no exceptional family (A) with respect to $\hat{x}$. Assume the contrary that GCP has an exceptional family (A) with respect to $\hat{x}$, denoted by $\{x^r\}$. We now derive a contradiction.

We first show that for each $i \in \{1, 2, \ldots, n\}$ the following inequality holds:

$$(g_i(x^r) - g_i(\hat{x}))(f_i(x^r) - f_i(\hat{x})) \leq g_i(\hat{x}) f_i(\hat{x}). \tag{25}$$

Indeed, if $e_i(x^r, \alpha_r) = (1+\alpha_r)g_i(x^r) - \alpha_r g_i(\hat{x}) > 0$, by using (23) we have

$$\begin{aligned}(g_i(x^r) - g_i(\hat{x}))(f_i(x^r) - f_i(\hat{x}))\\ &= (g_i(x^r) - g_i(\hat{x}))(-\alpha_r(g_i(x^r) - g_i(\hat{x})) - f_i(\hat{x}))\\ &= -\alpha_r(g_i(x^r) - g_i(\hat{x}))^2 - (g_i(x^r) - g_i(\hat{x}))f_i(\hat{x}).\end{aligned} \tag{26}$$

Moreover, it follows from $e_i(x^r, \alpha_r) > 0$ that

$$(1+\alpha_r)(g_i(x^r) - g_i(\hat{x})) > -g_i(\hat{x}).$$

Thus, from (26) and the fact that $f(\hat{x}) > 0$ we deduce that

$$\begin{aligned}(g_i(x^r) - g_i(\hat{x}))(f_i(x^r) - f_i(\hat{x})) &\le -(g_i(x^r) - g_i(\hat{x}))f_i(\hat{x})\\ &\le g_i(\hat{x})f_i(\hat{x})/(1+\alpha_r)\\ &\le g_i(\hat{x})f_i(\hat{x}).\end{aligned}$$

If $e_i(x^r, \alpha_r) = (1+\alpha_r)g_i(x^r) - \alpha_r g_i(\hat{x}) = 0$, then $g_i(x^r) - g_i(\hat{x}) = -g_i(\hat{x})/(1+\alpha_r)$. For this case, by (24) we have

$$\begin{aligned}(g_i(x^r) - g_i(\hat{x}))(f_i(x^r) - f_i(\hat{x}))\\ &= [-g_i(\hat{x})/(1+\alpha_r)](f_i(x^r) - f_i(\hat{x}))\\ &\le [-g_i(\hat{x})/(1+\alpha_r)](-\alpha_r(g_i(x^r) - g_i(\hat{x})) - f_i(\hat{x}))\\ &= -\frac{\alpha_r}{(1+\alpha_r)^2}(g_i(\hat{x}))^2 + \frac{1}{1+\alpha_r}g_i(\hat{x})f_i(\hat{x})\\ &\le g_i(\hat{x})f_i(\hat{x}).\end{aligned}$$

Thus, the inequality (25) holds. Since $\|x^r\| \to \infty$ as $r \to \infty$ and g is proper with respect to R^n_+, the sequence $\{\|g(x^r)\|\}$ must be unbounded as $r \to \infty$. Without loss of generality, we assume that $\|g(x^r)\| \to \infty$. Thus, $\|e(x^r, \alpha_r)\| \to \infty$ as $r \to \infty$. Since $\{e(x^r, \alpha_r)\} \subset R^n_+$, there is a component, denoted by $e_q(x^r, \alpha_r)$, tends to ∞ as $r \to \infty$, and thus $g_q(x^r) - g_q(\hat{x}) \to \infty$ as $r \to \infty$. On the other hand, there exists a subsequence of $\{x^r\}$, denoted by $\{x^{r_j}\}$ $(j = 1, 2, \ldots)$, such that for some fixed index m,

$$\begin{aligned}(g_m(x^{r_j}) - g_m(\hat{x}))(f_m(x^{r_j}) - f_m(\hat{x}))\\ &= \max_{1\le i\le n}(g_i(x^{r_j}) - g_i(\hat{x}))(f_i(x^{r_j}) - f_i(\hat{x})).\end{aligned} \tag{27}$$

Therefore, by (25) and (27), for all sufficiently large j we have

$$\begin{aligned}f(\hat{x})^{\mathrm{T}}(g(x^{r_j}) - g(\hat{x})) - \tau \max_{1\le i\le n}(g_i(x^{r_j}) - g_i(\hat{x}))(f_i(x^{r_j}) - f_i(\hat{x}))\\ &= f(\hat{x})^{\mathrm{T}}(g(x^{r_j}) - g(\hat{x})) - \tau(g_m(x^{r_j}) - g_m(\hat{x}))(f_m(x^{r_j}) - f_m(\hat{x}))\\ &\ge f(\hat{x})^{\mathrm{T}}(g(x^{r_j}) - g(\hat{x})) - \tau g_m(\hat{x})f_m(\hat{x})\end{aligned}$$

$$= f_q(\hat{x})(g_q(x^{r_j}) - g_q(\hat{x})) - \tau g_m(\hat{x}) f_m(\hat{x}) + \sum_{i \neq q} f_i(\hat{x})(g_i(x^{r_j}) - g_i(\hat{x}))$$

$$\geq f_q(\hat{x})(g_q(x^{r_j}) - g_q(\hat{x})) - \tau g_m(\hat{x}) f_m(\hat{x}) + \sum_{i \neq q} - f_i(\hat{x}) g_i(\hat{x})/(1+\alpha_r)$$

$$> 0. \tag{28}$$

The last inequality follows from the facts that $f_q(\hat{x}) > 0$ and $g_q(x^{r_j}) - g_q(\hat{x}) \to \infty$ as $j \to \infty$. Since $f(x)$ is a quasi-P_*^M-mapping, the above inequality implies that

$$f(x^{r_j})^{\mathrm{T}}(g(x^{r_j}) - g(\hat{x})) \geq 0 \tag{29}$$

for all sufficiently large j.

On the other hand, for each i such that $e_i(x^{r_j}, \alpha_{r_j}) = 0$, we have from (24) that

$$f_i(x^{r_j}) \geq -\alpha_{r_j}(g_i(x^{r_j}) - g_i(\hat{x})) \geq \alpha_{r_j} g_i(\hat{x})/(1+\alpha_{r_j}).$$

Therefore,

$$f(x^{r_j})^{\mathrm{T}}(g(x^{r_j}) - g(\hat{x}))$$

$$= \sum_{i \in \{i\,:\, e_i(x^{r_j}, \alpha_{r_j}) > 0\}} -\alpha_{r_j}(g_i(x^{r_j}) - g_i(\hat{x}))^2 + \sum_{i \in \{i\,:\, e_i(x^{r_j}, \alpha_{r_j}) = 0\}} \frac{-g_i(\hat{x}) f_i(x^{r_j})}{1+\alpha_{r_j}}$$

$$\leq \sum_{i \in \{i: e_i(x^{r_j}, \alpha_{r_j}) > 0\}} -\alpha_{r_j}(g_i(x^{r_j}) - g_i(\hat{x}))^2 - \sum_{i \in \{i: e_i(x^{r_j}, \alpha_{r_j}) = 0\}} \frac{\alpha_{r_j}(g_i(\hat{x}))^2}{1+\alpha_{r_j}}$$

$$< 0. \tag{30}$$

The last strict inequality follows from that $g_q(x^{r_j}) - g_q(\hat{x}) \to \infty$ as $j \to \infty$. The above inequality contradicts (29). The proof is complete. □

The concept of quasi-P_*^M-map can be further extended to the following concept.

Definition 3.4. A mapping $f: R^n \to R^n$ is said to be a quasi-P_*-mapping with respect to g if there exists a constant $\kappa \geq 0$ such that for any distinct x, y in R^n

$$f(y)^{\mathrm{T}}(g(x) - g(y)) - \kappa \sum_{i \in I_+(x,y)} (g_i(x) - g_i(y))(f_i(x) - f_i(y)) > 0$$

implies that

$$f(x)^{\mathrm{T}}(g(x) - g(y)) \geq 0,$$

where

$$I_+(x, y) = \{i : (g_i(x) - g_i(y))(f_i(x) - f_i(y)) > 0\}.$$

It is easy to see that a quasi-P_*^M-map must be a quasi-P_*-map, but the converse is not true. For quasi-P_*-maps, we have the following result.

Theorem 3.2. *Let f and g be two continuous functions from R^n into itself. Suppose that g is an injective mapping and proper with respect to the set R^n_+, and f is quasi-P_*-function with respect to g. If there is a point $\hat{x} \in R^n$ such that $g(\hat{x}) \geq 0$ and $f(\hat{x}) > 0$, then GCP has a solution.*

Proof. By the same argument as in the proof of Theorem 3.1, (25)–(27) remain valid. If $I_+(x^{r_j}, \hat{x}) = \emptyset$, then it is easy to show

$$f(\hat{x})^{\mathrm{T}}(g(x^{r_j}) - g(\hat{x})) \geq f_q(\hat{x})(g_q(x^{r_j}) - g_q(\hat{x})) + \sum_{i \neq q} -f_i(\hat{x})g_i(x^{r_j}) > 0 \tag{31}$$

for all sufficiently large j. If $I_+(x^{r_j}, \hat{x}) \neq \emptyset$, by noticing that

$$\sum_{i \in I_+(x^{r_j}, \hat{x})} (g_i(x^{r_j}) - g_i(\hat{x}))(f_i(x^{r_j}) - f_i(\hat{x}))$$

$$\leq n \max_{1 \leq i \leq n} (g_i(x^{r_j}) - g_i(\hat{x}))(f_i(x^{r_j}) - f_i(\hat{x}))$$

and by the same argument of (28), we have

$$f(\hat{x})^{\mathrm{T}}(g(x^{r_j}) - g(\hat{x})) - \tau \sum_{i \in I_+(x,y)} (g_i(x^{r_j}) - g_i(\hat{x}))(f_i(x^{r_j}) - f_i(\hat{x}))$$

$$\geq f(\hat{x})^{\mathrm{T}}(g(x^{r_j}) - g(\hat{x})) - (\tau n) \max_{1 \leq i \leq n} (g_i(x^{r_j}) - g_i(\hat{x}))(f_i(x^{r_j}) - f_i(\hat{x}))$$

$$> 0. \tag{32}$$

Since f is a quasi-P_*-map with respect to g, (31) and (32) imply that (28) holds. Similarly, the inequality (30) remains valid. A contradiction. □

The results of Theorems 3.1 and 3.2 are new even for NCPs. Notice that any quasi-monotone maps and P_*-maps are quasi-P_*^M-maps, the following two corollaries are immediate consequence of Theorems 3.1 or 3.2.

Corollary 3.2 (Hadjisavvas and Schaible [5]). *If the strictly feasible condition holds and f is a continuous quasi-monotone map, then NCP has a solution.*

The above result includes the results on monotone maps [15] and pseudo-monotone maps [10] as special cases.

Corollary 3.3 (Zhao and Han [23]). *If the strictly feasible condition holds and f is a continuous P_*-map, then NCP has a solution.*

In what follows, we introduce a concept of nonlinear $P(\tau, \alpha, \beta)$-mapping which is also the generalization of a P_*-map.

Definition 3.5. A mapping f is said to be a $P(\tau, \alpha, \beta)$-map with respect to g, if there exist constants $\tau \geq 0$, $\alpha \geq 0$ and $1 > \beta \geq 0$ such that the following inequality

$$(1+\tau)\max_{1\le i\le n}(g_i(x)-g_i(y))(f_i(x)-f_i(y))$$
$$+\min_{1\le i\le n}(g_i(x)-g_i(y))(f_i(x)-f_i(y))\ge -\alpha\|x-y\|^{\beta} \tag{33}$$

holds for any $x, y \in R^n$.

Particularly, if $\alpha = 0$, then (33) reduces to

$$(1+\tau)\max_{1\le i\le n}(g_i(x)-g_i(y))(f_i(x)-f_i(y))$$
$$+\min_{1\le i\le n}(g_i(x)-g_i(y))(f_i(x)-f_i(y))\ge 0. \tag{34}$$

It is not difficult to show that f is a P_*-mapping if and only if there exists some scalar $\tau\ge 0$ such that (34) holds with $g(x) = x$. Thus, the class of $P(\tau,\alpha,\beta)$-maps include P_*-maps as special cases. We end this section by showing the following result.

Theorem 3.3. *Let f and g be two continuous mappings from R^n to R^n. Suppose that g is injective and proper with respect to R^n_+, and f is a continuous $P(\tau,\alpha,\beta)$-map with respect to g. If there exists a point $\hat{x}$ such that $g(\hat{x}) \ge 0$ and $f(\hat{x}) > 0$, then there is a solution to GCP.*

Proof. By Corollary 3.1, it is sufficient to show that GCP has no exceptional family (A) with respect to $\hat{x}$. Assume the contrary that GCP has an exceptional family (A) with respect to $\hat{x}$, denoted by $\{x^r\}$. By Definition 3.1, $\|x^r\| \to \infty$ as $r \to \infty$ and for each x^r there exists a positive scalar α_r such that (23) and (24) hold. It is evident that the inequality (25) remains valid.

Since

$$e(x^r,\alpha_r) = (1+\alpha_r)g(x^r) - \alpha_r g(\hat{x}) \in R^n_+,$$

we have

$$g(x^r) \ge \frac{\alpha_r}{1+\alpha_r} g(\hat{x}) \ge 0,$$

i.e., $\{g(x^r)\} \subset R^n_+$. Since $\|x^r\| \to \infty$ and g is proper with respect to R^n_+, the sequence $\{g(x^r)\}$ must be unbounded, without loss of generality, we assume that $\|g(x^r)\| \to \infty$ as $r \to \infty$. There exists a subsequence of $\{x^r\}$, denoted by $\{x^{r_j}\}(j = 1,2,\ldots)$, and a fixed index m such that

$$e_m(x^{r_j},\alpha_{r_j}) = \max_{1\le i\le n} e_i(x^{r_j},\alpha_{r_j}) \to \infty \quad \text{as } j \to \infty, \tag{35}$$

i.e.,

$$(1+\alpha_r)(g_m(x^{r_j}) - g_m(\hat{x})) + g_m(\hat{x}) \to \infty$$

which implies that

$$g_m(x^{r_j}) - g_m(\hat{x}) \to \infty.$$

Thus, by (26) we have

$$(g_m(x^{r_j}) - g_m(\hat{x}))(f_m(x^{r_j}) - f_m(\hat{x})) \to \infty \quad \text{as } j \to \infty. \tag{36}$$

There exists a subsequence of $\{x^{r_j}\}$, denoted also by $\{(x^{r_j})\}$, such that for some fixed index p we have

$$\begin{aligned}(g_p(x^{r_j}) - g_p(\hat{x}))&(f_p(x^{r_j}) - f_p(\hat{x})) \\ &= \max_{1\le i\le n} (g_i(x^{r_j}) - g_i(\hat{x}))(f_i(x^{r_j}) - f_i(\hat{x})).\end{aligned} \tag{37}$$

Therefore, by using (33), (37) and (25), for all sufficiently large j we have

$$\begin{aligned}&(g_m(x^{r_j}) - g_m(\hat{x}))(f_m(x^{r_j}) - f_m(\hat{x})) \\ &\quad\ge \min_{1\le i\le n} (g_i(x^{r_j}) - g_i(\hat{x}))(f_i(x^{r_j}) - f_i(\hat{x})), \\ &\quad\ge -(1+\tau)\max_{1\le i\le n} (g_i(x^{r_j}) - g_i(\hat{x}))(f_i(x^{r_j}) - f_i(\hat{x})) - \alpha\|g_i(x^{r_j}) - g_i(\hat{x})\|^\beta, \\ &\quad= -(1+\tau)(g_p(x^{r_j}) - g_p(\hat{x}))(f_p(x^{r_j}) - f_p(\hat{x})) - \alpha\|g_i(x^{r_j}) - g_i(\hat{x})\|^\beta, \\ &\quad\ge -(1+\tau)g(\hat{x})^{\mathrm{T}} f(\hat{x}) - \alpha\|g_i(x^{r_j}) - g_i(\hat{x})\|^\beta.\end{aligned} \tag{38}$$

It follows from (35) that $e_m(x^{r_j}, \alpha_{r_j}) > 0$, thus by (26), the inequality (38) can be written as

$$\begin{aligned}&-\alpha_{r_j}(g_m(x^{r_j}) - g_m(\hat{x}))^2 - (g_m(x^{r_j}) - g_m(\hat{x}))f_m(\hat{x}) \\ &\quad\ge -(1+\tau)g(\hat{x})^{\mathrm{T}} f(\hat{x}) - \alpha\|g(x^{r_j}) - g(\hat{x})\|^\beta.\end{aligned} \tag{39}$$

Multiplying both sides of the above by $1/(g_m(x^{r_j}) - g_m(\hat{x}))$, and rearranging the terms, we have

$$\begin{aligned}&-\alpha_{r_j}(g_m(x^{r_j}) - g_m(\hat{x})) \\ &\quad\ge f_m(\hat{x}) - \frac{(1+\tau)g(\hat{x})^{\mathrm{T}} f(\hat{x})}{g_m(x^{r_j}) - g_m(\hat{x})} - \frac{\alpha\|g(x^{r_j}) - g(\hat{x})\|^\beta}{g_m(x^{r_j}) - g_m(\hat{x})}.\end{aligned} \tag{40}$$

Notice that for each $i \in \{1, 2, \ldots, n\}$,

$$\begin{aligned}\left|\frac{g_i(x^{r_j}) - g_i(\hat{x})}{g_m(x^{r_j}) - g_m(\hat{x})}\right| &= \left|\frac{(1+\alpha_{r_j})(g_i(x^{r_j}) - g_i(\hat{x}))}{(1+\alpha_{r_j})(g_m(x^{r_j}) - g_m(\hat{x}))}\right| \\ &= \frac{|e_i(x^{r_j}, \alpha_{r_j}) - g_i(\hat{x})|}{(1+\alpha_{r_j})(g_m(x^{r_j}) - g_m(\hat{x}))} \\ &\le \frac{e_i(x^{r_j}, \alpha_{r_j}) + g_i(\hat{x})}{(1+\alpha_{r_j})(g_m(x^{r_j}) - g_m(\hat{x}))} \\ &\le \frac{2g_m(\hat{x})}{(1+\alpha_{r_j})(g_m(x^{r_j}) - g_m(\hat{x}))} + 1.\end{aligned}$$

Thus, for all sufficiently large j, we have

$$-2 \leq \frac{g_i(x^{r_j}) - g_i(\hat{x})}{g_m(x^{r_j}) - g_m(\hat{x})} \leq 2.$$

Therefore, for all sufficiently large j, we obtain

$$\begin{aligned}\frac{\|g(x^{r_j}) - g(\hat{x})\|^{\beta}}{g_m(x^{r_j}) - g_m(\hat{x})} &= \left[\frac{\|g(x^{r_j}) - g(\hat{x})\|^2}{(g_m(x^{r_j}) - g_m(\hat{x}))^2}\right]^{\beta/2} \frac{1}{(g_m(x^{r_j}) - g_m(\hat{x}))^{1-\beta}} \\ &= \left[\sum_{i=1}^{n}\left(\frac{g_i(x^{r_j}) - g_i(\hat{x})}{g_m(x^{r_j}) - g_m(\hat{x})}\right)^2\right]^{\beta/2} \frac{1}{(g_m(x^{r_j}) - g_m(\hat{x}))^{1-\beta}} \\ &\leq \frac{(4n)^{\beta/2}}{(g_m(x^{r_j}) - g_m(\hat{x}))^{1-\beta}} \to 0.\end{aligned}$$

Hence, the right-hand side of (40) is a positive number for all sufficiently large j, however, the left-hand side of (40) is a negative number. This is a contradiction. The proof is complete. $\square$

It is worth noting that in Theorems 3.1–3.3 the strictly feasible condition "$g(\hat{x}) \geq 0$ and $f(\hat{x}) > 0$" cannot be replaced by the feasible condition "$g(\hat{x}) \geq 0$ and $f(\hat{x}) \geq 0$" because the monotone maps are contained in the intersection of the class of quasi-P_*, quasi-P_*^M and $P(\tau, \alpha, \beta)$-maps. An example was given by Megiddo [14] to show that a nonlinear monotone complementarity problem satisfying feasible condition may have no solution.

4. Conclusions

In this paper, two new concepts of exceptional families with respect to certain points for the NPE are introduced. Based on the concepts, two alternative theorems concerning the existence of a solution to NPE are proved. The existence results on GCP presented here generalize several known existence theorems in the literature. From these results, we conclude that the analysis method presented in this paper is a powerful tool for the study of solvability of NPE and its various special cases. The nonlinear quasi-P_*, $P(\tau, \alpha, \beta)$ and quasi-P_*^M-maps defined in the paper are important classes of functions because they include several interesting particular cases such as P_*-maps and quasi-monotone maps.

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