CONVERGENCE OF RENORMALIZED FINITE ELEMENT METHODS FOR HEAT FLOW OF HARMONIC MAPS

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Abstract. A linearly implicit renormalized lumped mass finite element method is considered for solving the equations describing heat flow of harmonic maps, of which the exact solution naturally satisfies the pointwise constraint $|\mathbf{m}| = 1$. At every time level, the method first computes an auxiliary numerical solution by a linearly implicit lumped mass method and then renormalizes it at all finite element nodes before proceeding to the next time level. It is shown that such a renormalized finite element method has an error bound of $O(\tau + h^{r+1})$ for tensor-product finite elements of degree $r \geqslant 1$. The proof of the error estimates is based on a geometric relation between the auxiliary and renormalized numerical solutions. The extension of the error analysis to triangular mesh is straightforward and discussed in the conclusion section.

Key words. heat flow of harmonic maps, finite element methods, renormalization at nodes, lumped mass, error estimates

AMS subject classifications. 65M12, 65M60, 35K55, 35Q35

1. Introduction. We consider the heat flow of harmonic maps in a bounded domain $\Omega \subset \mathbb{R}^d$, with $d \in \{1, 2, 3\}$, described by the partial differential equation (PDE)

$$\partial_t \mathbf{m} = \Delta \mathbf{m} + |\nabla \mathbf{m}|^2 \mathbf{m} \quad \text{in } \Omega \times (0, T], \tag{1.1}$$

$$\partial_\nu \mathbf{m} = \mathbf{0} \quad \text{on } \partial\Omega \times (0, T], \tag{1.2}$$

$$\mathbf{m} = \mathbf{m}^0 \quad \text{in } \Omega \times \{0\}, \tag{1.3}$$

where $\partial_\nu \mathbf{m}$ denotes the normal derivative of $\mathbf{m}$ and the initial value $\mathbf{m}^0$ satisfies

$$|\mathbf{m}^0| = 1 \quad \text{in } \Omega. \tag{1.4}$$

Under condition (1.4), it is known that the solution of problem (1.1)–(1.3) automatically satisfies the pointwise constraint

$$|\mathbf{m}| = 1 \quad \text{in } \Omega \times (0, T]. \tag{1.5}$$

The problem can be viewed as the L^2 gradient flow of the energy functional $E(\mathbf{u}) = \int_\Omega |\nabla \mathbf{u}|^2 \mathrm{d}V$ under constraint (1.5). When the initial value is sufficiently smooth, it is known that the heat flow of harmonic maps has a unique smooth solution in short time and may blow up at some finite time; see [16]. Equation (1.1) appears in many applications, including the Landau–Lifshitz equation of magnetization dynamics (as the limiting case when the damping parameter tends to ∞ [4, Proposition 5.2]; see [28]), the nematic liquid crystals model (coupled with the Navier-Stokes equations to describe the local molecular direction [12]), and color image denoising [29, 30]. The structure of (1.1) also appears in the geometric evolution equations describing mean curvature flow of surfaces [24, 25]. In particular, the normal vector $\mathbf{n}$ on a surface Γ evolving under mean curvature flow satisfies the surface PDE:

$$\partial_t^\bullet \mathbf{n} = \Delta_\Gamma \mathbf{n} + |\nabla_\Gamma \mathbf{n}|^2 \mathbf{n} \quad \text{on } \Gamma \times (0, T], \tag{1.6}$$

where ∇_Γ and Δ_Γ are the tangential gradient and Laplace–Beltrami operators on the surface Γ, respectively, and $\partial_t^\bullet$ denotes material derivative, i.e., the time derivative in the Lagrangian coordinates. The normal vector $\mathbf{n}$ also satisfies the pointwise constraint $|\mathbf{n}| = 1$.

In contrast to the exact solution of (1.1), which satisfies the constraint $|\mathbf{m}| \equiv 1$ automatically, the numerical solutions of (1.1) and related PDEs given by classical finite element methods (FEMs) and commonly used time-stepping schemes are generally not of unit length.

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In this case, the simplest method to restore the unit length is to artificially renormalize the numerical solutions by a post-processing technique, i.e., changing the numerical solution $\mathbf{m}_h^n$ to $\mathbf{m}_h^n/|\mathbf{m}_h^n|$ artificially after solving the equation at every time level, before proceeding to the next time level. This renormalization method is simple to implement and flexible to be combined with general FEMs in space and linearly implicit time-stepping schemes. More importantly, such a simple renormalization at every time level can significantly improve the performance of a numerical method, especially when singularity arises, as shown in the numerical experiments in [24, Figs 4 and 6]. As far as we know, except for the renormalization method, no other linearly implicit methods can preserve the unit length in numerical solutions for the problems mentioned above. However, the error analysis of such a simple renormalization method with commonly used FEMs and time-stepping schemes is still challenging. The main difficulty is the analysis of stability in approximating the time derivative when the renormalization is used at every time level. This is a common difficulty for all related PDEs, including heat flow of harmonic maps, the Landau–Lifshitz equation, and the nematic liquid crystals equations.

Convergence rates of unconstrained FEMs without renormalization were studied in many articles for the related Landau–Lifshitz equation and nematic liquid crystals equations. For the Landau–Lifshitz equation, first-order convergence in time of linearly implicit time discretizations was proved by Cimrák [17]; optimal-order error estimates for fully discrete FEMs with linearly implicit backward Euler and Crank–Nicolson time-stepping schemes were obtained by Gao [21] and An [5], respectively; first-order convergence of a linearly implicit FEM for the Landau–Lifshitz–Gilbert equation coupled with the eddy current equation was gained by Feischl & Tran [20]. For the nematic liquid crystals equations, first-order convergence in time and space of a semi-implicit mixed FEM was derived in [22]; optimal-order convergence of a linearly implicit stabilized FEM was proved in [7]. More recently, Akrivis, Feischl, Kovács & Lubich [2] established optimal-order error estimates for high-order linearly implicit FEMs preserving an energy inequality for the Landau–Lifshitz equation.

Alouges, Kritsikis, Steiner & Toussaint [3] considered an unconditionally stable and second-order method combined with a renormalization stage at every time level, and proved convergence of the method without explicit rates; Chen, Wang & Xie [15] acquired second-order convergence of a *semi-renormalized* method for the Landau–Lifshitz equation with the two-step backward differentiation formula (BDF2), i.e., the renormalization is used in the extrapolation of the right-hand side but not used in the BDF2 approximation to the time derivative.

Convergence of several nonlinearly implicit constrained FEMs for equation (1.1) preserving $|\mathbf{m}_h^n| = 1$ at the finite element nodes was shown in the literature based on compactness arguments. For example, Bartels & Prohl [10] proved convergence of constrained FEMs based on a non-divergence formulation using the discrete Laplacian; Bartels, Lubich & Prohl [9] presented convergence of constrained FEMs based on a variational approach using a discrete Lagrange multiplier; Baňas, Prohl & Schätzle [11] established convergence analysis of constrained FEMs for heat flow into spheres of nonconstant radii. A unified inf-sup stable saddle point approach was proposed by Gutiérrez-Santacreu & Restelli [23] for both Landau–Lifshitz equation and harmonic map heat flows to impose the unit sphere constraint at finite element nodes with a discrete energy law. However, no convergence rates were given for these nonlinearly implicit constraint-preserving methods so far.

More recently, An, Gao & Sun [6] provided the optimal-order convergence $O(\tau + h^2)$ and $O(\tau^2 + h^2)$ for the backward Euler and Crank–Nicolson semi-implicit finite difference projection methods under the conditions $h^2 \leqslant \tau \leqslant h^{1+\varepsilon_1}$ and $c_1 h \leqslant \tau \leqslant c_2 h$, respectively, where $\varepsilon_1 \in (0, 1)$ is any positive constant. For the backward Euler Galerkin FEM with renormalization, An & Sun [8] obtained the optimal order $O(\tau + h^{r+1})$ under the condition $c_1 h \leqslant \tau \leqslant c_2 h$ and $r \geqslant 2$, where c_1 and c_2 are some positive constants. The proof of optimal-order $O(\tau + h^{r+1})$ under a less restrictive condition such as $\tau \geqslant \kappa h^{r+1}$ is still challenging.

In this paper, we present an optimal-order error estimate of $O(\tau + h^{r+1})$ for a linearly implicit renormalized lumped mass FEM on rectangular mesh under the mild condition

$\tau \geqslant \kappa h^{r+1}$ for $r \geqslant 1$, where κ is any positive constant. On a triangular mesh, our analysis would yield $O(\tau + h^r)$ under the condition $\tau \geqslant \kappa h^r$ for $r \geqslant 2$. We focus on the heat flow of harmonic maps, but the techniques in this paper would also work for related PDEs, including the Landau–Lifshitz equation and nematic liquid crystals equations, as the common difficulty for all these equations is the analysis of stability for the renormalization technique in approximating the time derivative. We illustrate our idea and techniques through analyzing the following renormalized lumped mass FEM:

(1) For given $\mathbf{m}_h^{n-1}$ in a finite element space $\mathbf{S}_h^r$, compute an auxiliary numerical solution $\widetilde{\mathbf{m}}_h^n \in \mathbf{S}_h^r$ by

$$\Big(\frac{\widetilde{\mathbf{m}}_h^n - \mathbf{m}_h^{n-1}}{\tau}, \mathbf{v}_h\Big)_h + \big(\nabla\widetilde{\mathbf{m}}_h^n, \nabla\mathbf{v}_h\big) = \big(|\nabla\mathbf{m}_h^{n-1}|^2\widetilde{\mathbf{m}}_h^n, \mathbf{v}_h\big) \quad \forall\, \mathbf{v}_h \in \mathbf{S}_h^r, \tag{1.7}$$

where $(\cdot,\cdot)_h$ denotes the discrete inner product in the lumped mass FEM; see Section 2.1.

(2) Renormalize the auxiliary numerical solution to

$$\mathbf{m}_h^n = I_h\Big(\frac{\widetilde{\mathbf{m}}_h^n}{|\widetilde{\mathbf{m}}_h^n|}\Big), \tag{1.8}$$

where I_h denotes the Lagrange interpolation onto the finite element space $\mathbf{S}_h^r$.

Clearly, the renormalized numerical solution $\mathbf{m}_h^n$ satisfies $|\mathbf{m}_h^n| = 1$ at all finite element nodes. Since the renormalization stage (1.8) only requires us to re-define the finite element function at the nodes, the computation would be easier than renormalizing the numerical solution pointwisely everywhere. The latter would yield a function which is not in the finite element space and therefore would lead to additional quadrature error in approximating the inner products. Since both $\mathbf{m}_h^{n-1}$ and $\widetilde{\mathbf{m}}_h^n$ are finite element functions in the method (1.7)–(1.8), it follows that $|\nabla\mathbf{m}_h^{n-1}|^2\widetilde{\mathbf{m}}_h^n \cdot \mathbf{v}_h$ is a piecewise polynomial and therefore the inner products in (1.7) can be evaluated exactly without additional quadrature error.

If we denote by $\widetilde{\mathbf{e}}_h^{n-1} = I_h\mathbf{m}(t_{n-1}) - \widetilde{\mathbf{m}}_h^{n-1}$ and $\mathbf{e}_h^{n-1} = I_h\mathbf{m}(t_{n-1}) - \mathbf{m}_h^{n-1}$ the errors of the auxiliary numerical solution and numerical solution, respectively, then the main difficulty in proving stability of renormalized FEMs is the conversion of $\|\widetilde{\mathbf{e}}_h^{n-1}\|_{L_h^2}^2$ to $\|\mathbf{e}_h^{n-1}\|_{L_h^2}^2$ without generating an additional coefficient, where $\|\cdot\|_{L_h^2}$ denotes the discrete L^2 norm. The semi-renormalized method in [15] was analyzed by using the equivalence relation

$$\|\mathbf{e}_h^{n-1}\|_{L_h^2} \leqslant C\|\widetilde{\mathbf{e}}_h^{n-1}\|_{L_h^2}. \tag{1.9}$$

This additional constant C prevents the error analysis to be extended to a fully renormalized FEM, unless an additional stepsize restriction $\tau \geqslant h$ is required. We overcome this difficulty by proving and utilizing an improved geometric relation

$$\|\mathbf{e}_h^{n-1}\|_{L_h^2}^2 \leqslant \|\widetilde{\mathbf{e}}_h^{n-1}\|_{L_h^2}^2 + \text{higher-order terms} \tag{1.10}$$

without the additional constant C. Since we only renormalize the numerical solution at the nodes (for the convenience of computation), the geometric relation (1.10) only holds for the discrete L^2 norm (instead of the standard continuous L^2 norm). This requires us to use lumped mass FEM in order to have a desired error estimate. We shall prove optimal-order convergence for the lumped mass FEM based on tensor-product finite elements in a rectangular domain, and discuss the extension to triangular elements in the conclusion section.

The rest of this paper is organized as follows. The basic notation and main theoretical result on the convergence of the numerical method are presented in Section 2. The proof of the main theorem is presented in Section 3. Numerical results are provided in Section 4 to support the theoretical analysis by illustrating the convergence rates of the proposed method. The extension of the error analysis to triangular mesh is discussed in the conclusion section. The proofs of some technical results, including the discrete Sobolev interpolation and embedding inequalities, as well as the superconvergence of the Lagrange interpolation operator, are presented in Appendices.

2. Notation and main results. In this section, we introduce notation and present the numerical scheme for solving problem (1.1)–(1.3) and our main theoretical results. The notation for lumped mass FEM is based on the notation in [26].

2.1. Notation and finite element space. Let $C(\overline{\Omega})$ be the space of continuous functions on $\overline{\Omega}$. For $1 \leqslant p \leqslant \infty$ and integer $k \geqslant 0$, let $W^{k,p}$ be the usual Sobolev space of functions defined in Ω equipped with the norm (see [1])

$$\|f\|_{W^{k,p}} := \begin{cases} \Big(\sum\limits_{|\alpha| \leqslant k} \|D^\alpha f\|_{L^p}^p \Big)^{\frac{1}{p}} & 1 \leqslant p < \infty, \\ \max\limits_{|\alpha| \leqslant k} \|D^\alpha f\|_{L^\infty} & p = \infty, \end{cases}$$

respectively, where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}$ for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$, $\alpha_i \geqslant 0$, $i = 1, \ldots, d$, with $|\alpha| = \alpha_1 + \cdots + \alpha_d$. The semi-norm of $W^{k,p}$ is defined by

$$|f|_{W^{k,p}} := \begin{cases} \Big(\sum\limits_{|\alpha| = k} \|D^\alpha f\|_{L^p}^p \Big)^{\frac{1}{p}} & 1 \leqslant p < \infty, \\ \max\limits_{|\alpha| = k} \|D^\alpha f\|_{L^\infty} & p = \infty. \end{cases}$$

Let $\mathbf{W}^{k,p} := (W^{k,p})^d$ be the d-dimensional vector-valued Sobolev space, with the norm and semi-norm still denoted by $\|\cdot\|_{W^{k,p}}$ and $|\cdot|_{W^{k,p}}$, respectively. As usual, we use the abbreviation $H^k = W^{k,2}$ and $\mathbf{H}^k = \mathbf{W}^{k,2}$.

Next, we define the H^1-conforming tensor-product finite element space on a rectangular domain. Without loss of generality, we consider the case $d = 3$ and $\Omega = (a_x, b_x) \times (a_y, b_y) \times (a_z, b_z)$. For positive integers J_x, J_y, and J_z, let $\mathcal{K}$ be a quasi-uniform partition of Ω into cuboids, denoted by K, and the corresponding mesh sizes are $h_x = \frac{b_x - a_x}{J_x}$, $h_y = \frac{b_y - a_y}{J_y}$, and $h_z = \frac{b_z - a_z}{J_z}$. Then, we define the tensor-product finite element space of degree $r \geqslant 1$ by

$$\mathbf{S}_h^r := \{\mathbf{v} \in \mathbf{H}^1 : \mathbf{v}|_K \in Q_r^d,\ \forall K \in \mathcal{K}\},$$

where Q_r^d is the space of polynomials of degree up to r in every variable on K, defined by

$$Q_r^d := \Big\{ \sum_{0 \leqslant \alpha_1, \alpha_2, \alpha_3 \leqslant r} C_{\alpha_1,\alpha_2,\alpha_3} x^{\alpha_1} y^{\alpha_2} z^{\alpha_3},\ C_{\alpha_1,\alpha_2,\alpha_3} \in \mathbb{R} \Big\}.$$

The tensor-product finite element spaces in one- and two-dimensions can be defined similarly. In a lumped mass FEM, the discrete inner product $(\cdot,\cdot)_h$ is used and defined below.

In the case $d = 1$ and $\Omega = (a, b)$, we consider a partition $a = x_0 < x_r < \cdots < x_{Jr} = b$ with a uniform mesh size $h = \frac{b-a}{J}$ and denote $\mathcal{I}_j := [x_{(j-1)r}, x_{jr}]$, $j = 1, \ldots, J$. Let I_h be the piecewise Lagrange interpolation operator with Gauss–Lobatto points $x_{(j-1)r+k}$, $k = 0, 1, \ldots, r$, on every subinterval $\mathcal{I}_j$. For any two continuous functions $f, g \in C(\overline{\Omega})$, we define

$$(f, g)_h := \int_\Omega I_h(fg) \mathrm{d}x = \sum_{j=1}^J \int_{\mathcal{I}_j} I_h(fg) \mathrm{d}x = \sum_{j=1}^J \sum_{k=0}^r \overline{\alpha}_j^k f(x_{(j-1)r+k}) g(x_{(j-1)r+k}),$$

where $\overline{\alpha}_j^k$ are the Gauss–Lobatto quadrature weights in the subinterval $\mathcal{I}_j$. By taking

$$\alpha_{(j-1)r+k} = \begin{cases} \overline{\alpha}_j^k & \text{for } 1 \leqslant k \leqslant r-1, \\ 2\overline{\alpha}_j^k & \text{for } k = 0, r, \end{cases}$$

we have

$$(f, g)_h = \sum_{i=0}^{Jr} \alpha_i f(x_i) g(x_i).$$

If $\vec{f}$ and $\vec{g}$ are the $(Jr + 1)$-dimensional vectors consisting of the nodal values of the two functions f and g in $\mathbf{S}_h^r$, respectively, then $(f,g)_h = M\vec{f}\cdot\vec{g}$, where $M \in \mathbb{R}^{Jr+1}\times\mathbb{R}^{Jr+1}$ is the diagonal matrix with elements $M_{ij} := \delta_{ij}\alpha_i$, with δ_{ij} denoting the Kronecker symbol.

In the case $d = 2$ and $\Omega = (a_x, b_x)\times(a_y, b_y)$, the Lagrange interpolation operator onto the tensor-product finite element space $\mathbf{S}_h^r$ is given by

$$I_h f = I_{h_x} I_{h_y} f \quad \text{for } f \in C(\overline{\Omega}),$$

where I_{h_x} and I_{h_y} are the one-dimensional Lagrange interpolation operators with respect to the x and y variables (based on Gauss–Lobatto points on every subinterval), respectively. For any two functions $f, g \in \mathbf{S}_h^r$, we define the discrete inner product $(\cdot,\cdot)_h$ by

$$(f,g)_h = \int_\Omega I_{h_x} I_{h_y}(fg)\mathrm{d}x\mathrm{d}y = \sum_{i_1=0}^{J_x r}\sum_{i_2=0}^{J_y r} \alpha_{i_1}\alpha_{i_2} f(x_{i_1}, y_{i_2}) g(x_{i_1}, y_{i_2}),$$

and similarly

$$(f,g)_h = M\vec{f}\cdot\vec{g},$$

where $\vec{f}$ and $\vec{g}$ are $(J_x r+1)(J_y r+1)$-dimensional vectors consisting of the nodal values of functions f and g, while M is a diagonal matrix consisting of the quadrature weights.

The extension of the discrete inner product to a three-dimensional rectangular domain $\Omega = (a_x, b_x)\times(a_y, b_y)\times(a_z, b_z)$ is similar by using the identity

$$I_h f = I_{h_x} I_{h_y} I_{h_z} f \quad \text{for } f \in C(\overline{\Omega}). \tag{2.1}$$

Since the weights for the Gauss–Lobatto quadrature are all non-negative, we can define the following discrete L^2 norm on the finite element space $\mathbf{S}_h^r$:

$$\|\mathbf{v}_h\|_{L_h^2} := \sqrt{(\mathbf{v}_h, \mathbf{v}_h)_h} \quad \text{for } \mathbf{v}_h \in \mathbf{S}_h^r,$$

which is also a semi-norm on $C(\overline{\Omega})$. Similarly, the norm $\|\cdot\|_{L_h^p}$ is defined by

$$\|\mathbf{v}_h\|_{L_h^p} := \Big(\int_\Omega I_h(|\mathbf{v}_h|^p)\mathrm{d}V\Big)^{\frac{1}{p}}$$

for $\mathbf{v}_h \in \mathbf{S}_h^r$ and $1 \leqslant p < \infty$. The next lemma gives the norm equivalence between $\|\cdot\|_{L_h^p}$ and $\|\cdot\|_{L^p}$.

Lemma 2.1. *Let $\mathbf{v}_h \in \mathbf{S}_h^r$ and $1 \leqslant p < \infty$. Then we have*

$$C_1\|\mathbf{v}_h\|_{L^p} \leqslant \|\mathbf{v}_h\|_{L_h^p} \leqslant C_2\|\mathbf{v}_h\|_{L^p} \quad \forall\, \mathbf{v}_h \in \mathbf{S}_h^r, \tag{2.2}$$

for some positive constants C_1 and C_2 that are independent of h.

The proof is straightforward and thus omitted (it can be proved based on the equivalence of norms in a finite-dimensional space and a scaling argument which transforms an element into a reference element).

2.2. The main result. Let $0 = t_0 < t_1 < \cdots < t_N = T$ be a uniform partition of the time interval $[0, T]$ with $t_n = n\tau$ and stepsize $\tau = T/N$.

The main theoretical result in this article is the following theorem on the convergence of the renormalized lumped mass method (1.7)–(1.8).

Theorem 2.2. *Let κ be any positive constant and assume that the solution of* (1.1)–(1.3) *is sufficiently smooth, i.e., with the following regularity:*

$$\begin{aligned}&\mathbf{m}_0 \in L^\infty(0,T;\mathbf{W}^{2,4}), \quad \mathbf{m} \in L^\infty(0,T;\mathbf{W}^{2,4}\cap\mathbf{H}^{2r}),\\ &\partial_t\mathbf{m} \in L^\infty(0,T;\mathbf{H}^{r+1}), \quad \partial_{tt}\mathbf{m} \in L^\infty(0,T;\mathbf{L}^2).\end{aligned}$$

Then there exists a positive constant τ_0 such that when $\kappa h^{r+1} \leqslant \tau \leqslant \tau_0$, the numerical scheme (1.7)–(1.8) *yields a unique solution $\mathbf{m}_h^n \in \mathbf{S}_h^r$ $(r \geqslant 1)$, $n = 1, \ldots, N$, with the following error bound:*

$$\max_{1\leqslant n\leqslant N}(\|\mathbf{m}_h^n - \mathbf{m}(\cdot,t_n)\|_{L^2} + \|\widetilde{\mathbf{m}}_h^n - \mathbf{m}(\cdot,t_n)\|_{L^2}) \leqslant C_\kappa(\tau + h^{r+1}), \tag{2.3}$$

where C_κ is a positive constant depending on κ (but independent of τ and h).

Remark 2.1. In computation, choosing the time stepsize τ much smaller than h^{r+1} would be a waste because then the spatial discretization error would dominate (thus further decreasing the stepsize would not make the error smaller). Hence, the time stepsize restriction $\tau \geqslant \kappa h^{r+1}$ is only a mild condition that does not affect practical computation.

Throughout, we denote by C a generic positive constant and by ε a small generic positive constant, independent of τ, h, and N, which could be different at different occurrences.

3. Proof of Theorem 2.2.

3.1. Preliminary results. In the proof of Theorem 2.2, the following several lemmas are used.

Lemma 3.1 (Error of the interpolation operator [14, Theorem 4.4.20]). *The Lagrange interpolation operator $I_h : C(\overline{\Omega}) \to \mathbf{S}_h^r$ satisfies the following error estimates:*

$$\|\mathbf{v} - I_h\mathbf{v}\|_{L^p(\Omega)} + h\|\mathbf{v} - I_h\mathbf{v}\|_{W^{1,p}} \leqslant Ch^{s+1}|\mathbf{v}|_{W^{s+1,p}}, \tag{3.1}$$

$$\|\mathbf{v} - I_h\mathbf{v}\|_{L^\infty(\Omega)} \leqslant Ch^{2-\frac{d}{2}}|\mathbf{v}|_{H^2}, \tag{3.2}$$

for $0 \leqslant s \leqslant r$ and $p > d/(s+1)$ (in this case $\mathbf{W}^{s+1,p} \hookrightarrow C(\overline{\Omega})$).

Lemma 3.2 (Error of the Ritz projection [14, Theorems 5.4.4 & 5.4.8]). *The Ritz projection $R_h : \mathbf{H}^1 \to \mathbf{S}_h^r$ defined by*

$$(\nabla(\mathbf{w} - R_h\mathbf{w}), \nabla\mathbf{v}_h) = 0 \tag{3.3}$$

for $\mathbf{v}_h \in \mathbf{S}_h^r$ with $\int_\Omega (\mathbf{w} - R_h\mathbf{w})\mathrm{d}V = \mathbf{0}$, satisfies the following error estimate:

$$\|\mathbf{w} - R_h\mathbf{w}\|_{L^2} + h\|\nabla(\mathbf{w} - R_h\mathbf{w})\|_{L^2} \leqslant Ch^{r+1}\|\mathbf{w}\|_{H^{r+1}} \quad \textit{for } \mathbf{w} \in \mathbf{H}^{r+1}. \tag{3.4}$$

Lemma 3.3 (Bramble–Hilbert Lemma [13]). *Let $a < b$ and $h = b - a$, and let F be a linear functional on $W^{k+1,p}(a,b)$ with $k \geqslant 1$ and $1 \leqslant p \leqslant \infty$. Assume that*

(i) *$|F(f)| \leqslant C_3\|f\|_{W^{k+1,p}(a,b)}$ for $f \in W^{k+1,p}(a,b)$, with some positive constant C_3 independent of h and f;*
(ii) *$F(f) = 0$ for all f that are polynomials of degree less than or equal to k.*

Then

$$|F(f)| \leqslant C_4h^{k+1}|f|_{W^{k+1,p}(a,b)} \quad \forall\, f \in W^{k+1,p}(a,b), \tag{3.5}$$

for some positive constant C_4 which is independent of h and f.

Lemma 3.4 (Inverse inequalities [14, Lemma 4.5.4 & Theorem 4.5.11]). *Let $1 \leqslant p, q \leqslant \infty$, $0 \leqslant m \leqslant l$, and assume that v_h is a function in some finite element space subject to the triangulation $\mathcal{K}$, with $v_h|_K \in W_p^l(K) \cap W_q^m(K)$ for $K \in \mathcal{K}$. Then*

$$\|v_h\|_{W_p^l(K)} \leqslant Ch^{m-l+n/p-n/q}\|v_h\|_{W_q^m(K)}, \tag{3.6}$$

$$\Big(\sum_{K\in\mathcal{K}} \|v_h\|_{W_p^l(K)}^p\Big)^{1/p} \leqslant Ch^{m-l+\min\{0,n/p-n/q\}}\Big(\sum_{K\in\mathcal{K}} \|v_h\|_{W_q^m(K)}^q\Big)^{1/q}, \tag{3.7}$$

where $\mathcal{K}$ is the set of triangles in the triangulation (as defined in Section 2.1).

Lemma 3.5. *For $\mathbf{v}_h \in \mathbf{S}_h^r$, the following discrete Sobolev interpolation and embedding inequalities hold:*

$$\|\mathbf{v}_h\|_{L^\infty} \leqslant C\|\mathbf{v}_h\|_{L^2}^{1-\frac{d}{4}}(\|\mathbf{v}_h\|_{L^2} + \|\Delta_h\mathbf{v}_h\|_{L^2})^{\frac{d}{4}}, \tag{3.8}$$

$$\|\nabla\mathbf{v}_h\|_{L^6} \leqslant C\|\Delta_h\mathbf{v}_h\|_{L^2}, \tag{3.9}$$

where the discrete Laplacian operator $\Delta_h : \mathbf{S}_h^r \to \mathbf{S}_h^r$ is defined via duality by

$$(\Delta_h\mathbf{v}_h, \mathbf{w}_h) = -(\nabla\mathbf{v}_h, \nabla\mathbf{w}_h) \quad \forall\, \mathbf{v}_h, \mathbf{w}_h \in \mathbf{S}_h^r. \tag{3.10}$$

The proof of Lemma 3.5 is presented in Appendix A.

Lemma 3.6. *Let $K \in \mathcal{K}$ and denote by $(\cdot,\cdot)_K$ the L^2 inner product on K. Let $\boldsymbol{V}_h$ be* *the space of vector-valued polynomials of some degree $\ell \geqslant 0$ on K. Then*

$$\big|(1, I_h\mathbf{f} - \mathbf{f})_K\big| \leqslant Ch^{2r} \sum_{i=1}^{d} \|\partial_i^{2r}\mathbf{f}\|_{L^1(K)} \quad \forall\, \mathbf{f} \in \boldsymbol{V}_h, \tag{3.11}$$

where C is a positive constant independent of h and $\mathbf{f}$ (but may depend on ℓ).

The proof of Lemma 3.6 is presented in Appendix B.

Lemma 3.7. *For $\mathbf{u} \in \mathbf{H}^{2r}$ and $\mathbf{v}_h \in \mathbf{S}_h^r$, the following superconvergence result holds for the Lagrange interpolation operator I_h:*

$$\big|\big(\nabla(\mathbf{u} - I_h\mathbf{u}), \nabla\mathbf{v}_h\big)\big| \leqslant Ch^{r+1}\|\mathbf{v}_h\|_{H^1}. \tag{3.12}$$

In the case of Dirichlet boundary condition, the superconvergence of tensor-product Q_r elements based on Gauss–Lobatto points in solving elliptic equations was established in [19, 27]. Here we need the superconvergence result of the Lagrange interpolation operator I_h in the sense of (3.12) (instead of the numerical solution of elliptic equations). A proof of this result is presented in Appendix C.

3.2. Estimates for the truncation error. Note that the exact solution of problem (1.1)–(1.3) satisfies the following equation

$$(\partial_t\mathbf{m}, \mathbf{v}_h) + (\nabla\mathbf{m}, \nabla\mathbf{v}_h) = (|\nabla\mathbf{m}|^2\mathbf{m}, \mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in \mathbf{S}_h^r,$$

which can be rewritten as

$$\begin{aligned} &\Big(\frac{I_h\mathbf{m}(t_n) - I_h\mathbf{m}(t_{n-1})}{\tau}, \mathbf{v}_h\Big)_h + (\nabla I_h\mathbf{m}(t_n), \nabla\mathbf{v}_h) \\ &= (|\nabla I_h\mathbf{m}(t_{n-1})|^2 I_h\mathbf{m}(t_n), \mathbf{v}_h) + \mathcal{E}(\mathbf{v}_h). \end{aligned} \tag{3.13}$$

Here, $\mathcal{E}(\mathbf{v}_h)$ denotes the truncation error, given by

$$\begin{aligned} \mathcal{E}(\mathbf{v}_h) &= \Big(\frac{I_h\mathbf{m}(t_n) - I_h\mathbf{m}(t_{n-1})}{\tau}, \mathbf{v}_h\Big)_h - \Big(\frac{I_h\mathbf{m}(t_n) - I_h\mathbf{m}(t_{n-1})}{\tau}, \mathbf{v}_h\Big) \\ &\quad + \Big(\frac{I_h\mathbf{m}(t_n) - I_h\mathbf{m}(t_{n-1})}{\tau}, \mathbf{v}_h\Big) - \Big(\partial_t\mathbf{m}(t_n), \mathbf{v}_h\Big) \\ &\quad + \Big(\nabla(I_h\mathbf{m}(t_n) - \mathbf{m}(t_n)), \nabla\mathbf{v}_h\Big) \\ &\quad + \Big(|\nabla\mathbf{m}(t_n)|^2\mathbf{m}(t_n) - |\nabla I_h\mathbf{m}(t_{n-1})|^2 I_h\mathbf{m}(t_n), \mathbf{v}_h\Big) \\ &=: \mathcal{E}_1(\mathbf{v}_h) + \mathcal{E}_2(\mathbf{v}_h) + \mathcal{E}_3(\mathbf{v}_h) + \mathcal{E}_4(\mathbf{v}_h). \end{aligned} \tag{3.14}$$

In the following, we present estimates for $\mathcal{E}_j(\mathbf{v}_h)$, $j = 1, 2, 3, 4$, respectively.

First, since finite element functions in $\mathbf{S}_h^r$ have at most rth-order nonzero partial derivative in each variable, by using the result of Lemma 3.6 we have

$$\begin{aligned} |\mathcal{E}_1(\mathbf{v}_h)| &\leqslant Ch^{2r} \sum_{K\in\mathcal{K}} \sum_{i=1}^{d} \Big\|\partial_i^{2r}\Big(I_h\frac{\mathbf{m}(t_n) - \mathbf{m}(t_{n-1})}{\tau} \cdot \mathbf{v}_h\Big)\Big\|_{L^1(K)} \\ &\leqslant Ch^{2r} \sum_{K\in\mathcal{K}} \Big\|I_h\frac{\mathbf{m}(t_n) - \mathbf{m}(t_{n-1})}{\tau}\Big\|_{H^r(K)} \|\mathbf{v}_h\|_{H^r(K)} \\ &\leqslant Ch^{2r} \sum_{K\in\mathcal{K}} \Big\|\frac{\mathbf{m}(t_n) - \mathbf{m}(t_{n-1})}{\tau}\Big\|_{H^{r+1}(K)} \|\mathbf{v}_h\|_{H^r(K)} \quad \text{(stability of } I_h \text{ in } H^{r+1}(K)) \\ &\leqslant Crh^{r+1} \sum_{K\in\mathcal{K}} \Big\|\frac{\mathbf{m}(t_n) - \mathbf{m}(t_{n-1})}{\tau}\Big\|_{H^{r+1}(K)} |\mathbf{v}_h\|_{H^1(K)} \\ &\leqslant Ch^{r+1}\|\mathbf{v}_h\|_{H^1}, \end{aligned} \tag{3.15}$$

where the inverse inequality (3.6) is used in the last to second inequality.

Second, by Lemma 3.1, it is easy to see that

$$|\mathcal{E}_2(\mathbf{v}_h)| \leqslant \Big|\Big(I_h\frac{\mathbf{m}(t_n) - \mathbf{m}(t_{n-1})}{\tau} - \frac{\mathbf{m}(t_n) - \mathbf{m}(t_{n-1})}{\tau}, \mathbf{v}_h\Big)\Big|$$

$$
\begin{aligned}
&+ \Big|\Big(\frac{\mathbf{m}(t_n)-\mathbf{m}(t_{n-1})}{\tau} - \partial_t \mathbf{m}(t_n), \mathbf{v}_h\Big)\Big| \\
&\leqslant Ch^{r+1}\|\mathbf{v}_h\|_{L^2} + C\tau\|\mathbf{v}_h\|_{L^2}.
\end{aligned}
$$

Third, it follows from Lemma 3.7 that

$$
|\mathcal{E}_3(\mathbf{v}_h)| \leqslant Ch^{r+1}\|\mathbf{v}_h\|_{H^1}.
$$

Finally, by using Lemma 3.1 again, we obtain

$$
\begin{aligned}
|\mathcal{E}_4(\mathbf{v}_h)| \leqslant\ & C\Big|\Big(|\nabla \mathbf{m}(t_n)|^2\mathbf{m}(t_n) - |\nabla I_h\mathbf{m}(t_n)|^2\mathbf{m}(t_n), \mathbf{v}_h\Big)\Big| \\
&+ C\Big|\Big(|\nabla I_h\mathbf{m}(t_n)|^2\mathbf{m}(t_n) - |\nabla I_h\mathbf{m}(t_{n-1})|^2\mathbf{m}(t_n), \mathbf{v}_h\Big)\Big| \\
&+ C\Big|\Big(|\nabla I_h\mathbf{m}(t_{n-1})|^2\mathbf{m}(t_n) - |\nabla I_h\mathbf{m}(t_{n-1})|^2 I_h\mathbf{m}(t_n), \mathbf{v}_h\Big)\Big|.
\end{aligned}
$$

The first term on the right-hand side can be estimated as follows:

$$
\begin{aligned}
&\Big(|\nabla \mathbf{m}(t_n)|^2\mathbf{m}(t_n) - |\nabla I_h\mathbf{m}(t_n)|^2\mathbf{m}(t_n), \mathbf{v}_h\Big) \\
&= \Big(\nabla \mathbf{m}(t_n)\cdot\nabla(\mathbf{m}(t_n) - I_h\mathbf{m}(t_n)), \mathbf{m}(t_n)\cdot\mathbf{v}_h\Big) \\
&\quad + \Big(\nabla I_h\mathbf{m}(t_n)\cdot\nabla(\mathbf{m}(t_n) - I_h\mathbf{m}(t_n)), \mathbf{m}(t_n)\cdot\mathbf{v}_h\Big) \\
&= 2\Big(\nabla \mathbf{m}(t_n)\cdot\nabla\big(\mathbf{m}(t_n) - I_h\mathbf{m}(t_n)\big), \mathbf{m}(t_n)\cdot\mathbf{v}_h\Big) \\
&\quad - \Big(\nabla\big(\mathbf{m}(t_n) - I_h\mathbf{m}(t_n)\big)\cdot\nabla\big(\mathbf{m}(t_n) - I_h\mathbf{m}(t_n)\big), \mathbf{m}(t_n)\cdot\mathbf{v}_h\Big) \\
&\leqslant -2\Big(\Delta\mathbf{m}(t_n)\cdot\big(\mathbf{m}(t_n) - I_h\mathbf{m}(t_n)\big), \mathbf{m}(t_n)\cdot\mathbf{v}_h\Big) \quad \text{(integration by parts is used)} \\
&\quad - 2\Big(\nabla\mathbf{m}(t_n)\big(\mathbf{m}(t_n) - I_h\mathbf{m}(t_n)\big), \nabla(\mathbf{m}(t_n)\cdot\mathbf{v}_h)\Big) \\
&\quad + \|\nabla(\mathbf{m}(t_n) - I_h\mathbf{m}(t_n))\|^2_{L^{\frac{12}{5}}}\|\mathbf{m}(t_n)\cdot\mathbf{v}_h\|_{L^6} \qquad \text{(Hölder's inequality is used)} \\
&\leqslant Ch^{r+1}\|\mathbf{v}_h\|_{H^1} + Ch^{r+1}\|\mathbf{v}_h\|_{H^1} + Ch^{r+1}\|\mathbf{v}_h\|_{H^1} \quad \text{(Lemma 3.1 is used)} \\
&\leqslant Ch^{r+1}\|\mathbf{v}_h\|_{H^1},
\end{aligned}
$$

where we have used the following estimate in the second to last inequality:

$$
\begin{aligned}
\|\nabla(\mathbf{m}(t_n) - I_h\mathbf{m}(t_n))\|^2_{L^{\frac{12}{5}}} &\leqslant \begin{cases} Ch^2\|\mathbf{m}(t_n)\|^2_{W^{2,\frac{12}{5}}} & \text{for } r=1, \\ Ch^{4r-4}\|\mathbf{m}(t_n)\|^2_{W^{2r-1,\frac{12}{5}}} & \text{for } r\geqslant 2, \end{cases} \\
&\leqslant \begin{cases} Ch^2\|\mathbf{m}(t_n)\|^2_{W^{2,4}} & \text{for } r=1, \\ Ch^{r+1}\|\mathbf{m}(t_n)\|^2_{H^{2r}} & \text{for } r\geqslant 2 \quad (4r-4\geqslant r+1). \end{cases}
\end{aligned}
$$

As a result, we have

$$
\begin{aligned}
|\mathcal{E}_4(\mathbf{v}_h)| &\leqslant Ch^{r+1}\|\mathbf{v}_h\|_{H^1} \\
&\quad + C\Big|\Big(|\nabla I_h\mathbf{m}(t_n) + \nabla I_h\mathbf{m}(t_{n-1})||\nabla I_h\mathbf{m}(t_n) - \nabla I_h\mathbf{m}(t_{n-1})|\mathbf{m}(t_n), \mathbf{v}_h\Big)\Big| \\
&\quad + C\Big|\Big(|\nabla I_h\mathbf{m}(t_{n-1})|^2\big(\mathbf{m}(t_n) - I_h\mathbf{m}(t_n)\big), \mathbf{v}_h\Big)\Big| \\
&\leqslant Ch^{r+1}\|\mathbf{v}_h\|_{H^1} + C\tau\|\mathbf{v}_h\|_{L^2} + Ch^{r+1}\|\mathbf{v}_h\|_{L^2}.
\end{aligned}
$$

By collecting the results above, we obtain the following estimate for the truncation error:

$$
|\mathcal{E}(\mathbf{v}_h)| \leqslant Ch^{r+1}\|\mathbf{v}_h\|_{H^1} + C\tau\|\mathbf{v}_h\|_{L^2}, \tag{3.16}
$$

and therefore, by using Young's inequality,

$$
|\mathcal{E}(\mathbf{v}_h)| \leqslant C\varepsilon^{-1}(\tau^2 + h^{2r+2}) + \varepsilon\|\nabla\mathbf{v}_h\|^2_{L^2} + C\|\mathbf{v}_h\|^2_{L^2}, \tag{3.17}
$$

with an arbitrary positive constant ε.

3.3. Error equations and an outline of the proof for Theorem 2.2. We define the following two types of error functions:

$$\mathbf{e}_h^n = I_h\mathbf{m}(t_n) - \mathbf{m}_h^n \quad \text{and} \quad \widetilde{\mathbf{e}}_h^n = I_h\mathbf{m}(t_n) - \widetilde{\mathbf{m}}_h^n,$$

for $n \geqslant 1$, with

$$\widetilde{\mathbf{e}}_h^0 = \mathbf{e}_h^0 = I_h\mathbf{m}^0 - \mathbf{m}_h^0 = \mathbf{0}. \tag{3.18}$$

By subtracting (1.7) from (3.13) we obtain the following equation for the error function $\widetilde{\mathbf{e}}_h^n$:

$$\Big(\frac{\widetilde{\mathbf{e}}_h^n - \mathbf{e}_h^{n-1}}{\tau}, \mathbf{v}_h\Big)_h + (\nabla\widetilde{\mathbf{e}}_h^n, \nabla\mathbf{v}_h) = \Big(|\nabla I_h\mathbf{m}(t_{n-1})|^2 I_h\mathbf{m}(t_n) - |\nabla\mathbf{m}_h^{n-1}|^2\widetilde{\mathbf{m}}_h^n, \mathbf{v}_h\Big) + \mathcal{E}(\mathbf{v}_h), \tag{3.19}$$

which holds for all $\mathbf{v}_h \in \mathbf{S}_h^r$ and $n = 1, 2, \dots, N$.

For the convenience of the reader, we present an outline of the proof for Theorem 2.2.

In order to bound the nonlinear terms arising from the error analysis, we will establish the following primary estimates by using mathematical induction:

$$\|\nabla\mathbf{e}_h^{n-1}\|_{L^4} \leqslant 1, \tag{3.20}$$

$$\|\widetilde{\mathbf{e}}_h^{n-1}\|_{L^\infty} \leqslant \frac{1}{4}, \tag{3.21}$$

$$\|\widetilde{\mathbf{e}}_h^{n-1}\|_{L^3} \leqslant \tau^{\frac{4}{7}}, \quad \|\widetilde{\mathbf{e}}_h^{n-1}\|_{L^6} \leqslant \tau^{\frac{2}{7}}. \tag{3.22}$$

For $n = 1$, (3.20)–(3.22) naturally hold as a result of (3.18). We assume that the numerical solution $\mathbf{m}_h^{n-1}$ is uniquely determined for $1 \leqslant n \leqslant k$, and (3.20)–(3.22) hold for $1 \leqslant n \leqslant k$. Then we shall prove that the linear system (1.7) is uniquely solvable for $n = k$ (thus $\mathbf{m}_h^k$ is uniquely determined) and (3.20)–(3.22) also hold for $n = k + 1$.

In Sections 3.4 and 3.5, we show that estimates (3.20)–(3.22) imply the following two lemmas, i.e., Lemmas 3.8 and 3.9, which give useful relations between the two types of error functions $\mathbf{e}_h^n$ and $\widetilde{\mathbf{e}}_h^n$.

Lemma 3.8. *If* (3.21) *holds for* $1 \leqslant n \leqslant k$, *then*

$$\|\mathbf{e}_h^{n-1}\|_{W^{1,p}} \leqslant C\|\widetilde{\mathbf{e}}_h^{n-1}\|_{W^{1,p}} + Ch \tag{3.23}$$

for $2 \leqslant p \leqslant 4$, *where* C *is a positive constant independent of* n, k, h, *and* τ.

Lemma 3.9. *For all* $n \geqslant 1$ *the following estimates hold*

$$\|\mathbf{e}_h^{n-1}\|_{L_h^p} \leqslant C\|\widetilde{\mathbf{e}}_h^{n-1}\|_{L_h^p}, \tag{3.24}$$

$$\|\mathbf{e}_h^{n-1}\|_{L^p} \leqslant C\|\widetilde{\mathbf{e}}_h^{n-1}\|_{L^p}, \tag{3.25}$$

for $1 \leqslant p < \infty$. *Furthermore, if* (3.22) *holds for* $1 \leqslant n \leqslant k$, *then*

$$\|\mathbf{e}_h^{n-1}\|_{L_h^2}^2 \leqslant \|\widetilde{\mathbf{e}}_h^{n-1}\|_{L_h^2}^2 + C\tau^{\frac{8}{7}}\|\widetilde{\mathbf{e}}_h^{n-1}\|_{H^1}^2, \tag{3.26}$$

where C *is a positive constant independent of* n, k, h, *and* τ.

By using the technical estimates (3.20)–(3.22) and Lemmas 3.8–3.9, we shall prove (3.20)–(3.22) for $n = k+1$ in Sections 3.7–3.9, and in the mean time prove the desired error bound given in Theorem 2.2.

3.4. Proof of Lemma 3.8. Since the exact solution satisfies $|\mathbf{m}(t_{n-1})| = 1$ everywhere in Ω, it follows that

$$|I_h\mathbf{m}(t_{n-1})| \leqslant |\mathbf{m}(t_{n-1})| + |\mathbf{m}(t_{n-1}) - I_h\mathbf{m}(t_{n-1})| \leqslant 1 + Ch\|\mathbf{m}(t_{n-1})\|_{W^{1,\infty}},$$

$$|I_h\mathbf{m}(t_{n-1})| \geqslant |\mathbf{m}(t_{n-1})| - |\mathbf{m}(t_{n-1}) - I_h\mathbf{m}(t_{n-1})| \geqslant 1 - Ch\|\mathbf{m}(t_{n-1})\|_{W^{1,\infty}}.$$

The two inequalities above imply, for sufficiently small h,

$$\frac{3}{4} \leqslant |I_h\mathbf{m}(t_{n-1})| \leqslant \frac{5}{4} \ \text{ pointwise in } \ \Omega. \tag{3.27}$$

If (3.21) holds then

$$\frac{1}{2} \leqslant |\widetilde{\mathbf{m}}_h^{n-1}| \leqslant \frac{3}{2} \ \text{ pointwise in } \ \Omega. \tag{3.28}$$

When $|f| \sim |g| \sim 1$ pointwise in Ω, by the inequality (7.11) from [2] there holds

$$\Big|\frac{f}{|f|} - \frac{g}{|g|}\Big| = \Big|\frac{f(|g|-|f|) - |f|(g-f)}{|f||g|}\Big| \leqslant 2\frac{|f-g|}{|g|} \leqslant C|f-g|, \tag{3.29}$$

and similarly,

$$\Big|\nabla\frac{f}{|f|} - \nabla\frac{g}{|g|}\Big| \leqslant C|\nabla g||f-g| + C|\nabla(f-g)|. \tag{3.30}$$

Then, by using the triangle inequality and (3.27)–(3.30), we have

$$\begin{aligned}
\|\mathbf{e}_h^{n-1}\|_{W^{1,p}} &= \Big\|I_h\Big(\frac{\widetilde{\mathbf{m}}_h^{n-1}}{|\widetilde{\mathbf{m}}_h^{n-1}|}\Big) - I_h\mathbf{m}(t_{n-1})\Big\|_{W^{1,p}} \\
&\leqslant \Big\|I_h(\frac{\widetilde{\mathbf{m}}_h^{n-1}}{|\widetilde{\mathbf{m}}_h^{n-1}|}) - \frac{\widetilde{\mathbf{m}}_h^{n-1}}{|\widetilde{\mathbf{m}}_h^{n-1}|}\Big\|_{W^{1,p}} + \Big\|\frac{\widetilde{\mathbf{m}}_h^{n-1}}{|\widetilde{\mathbf{m}}_h^{n-1}|} - \frac{I_h\mathbf{m}(t_{n-1})}{|I_h\mathbf{m}(t_{n-1})|}\Big\|_{W^{1,p}} \\
&\quad + \Big\|\frac{I_h\mathbf{m}(t_{n-1})}{|I_h\mathbf{m}(t_{n-1})|} - \frac{\mathbf{m}(t_{n-1})}{|\mathbf{m}(t_{n-1})|}\Big\|_{W^{1,p}} + \Big\|\mathbf{m}(t_{n-1}) - I_h\mathbf{m}(t_{n-1})\Big\|_{W^{1,p}} \\
&\leqslant \Big\|I_h(\frac{\widetilde{\mathbf{m}}_h^{n-1}}{|\widetilde{\mathbf{m}}_h^{n-1}|}) - \frac{\widetilde{\mathbf{m}}_h^{n-1}}{|\widetilde{\mathbf{m}}_h^{n-1}|}\Big\|_{W^{1,p}} + C\|\widetilde{\mathbf{e}}_h^{n-1}\|_{W^{1,p}} \\
&\quad + Ch\|\mathbf{m}(t_{n-1})\|_{W^{2,p}} + Ch\|\mathbf{m}(t_{n-1})\|_{W^{2,p}}.
\end{aligned} \tag{3.31}$$

It remains to estimate the first term on the right-hand side of (3.31). By Lemma 3.1, we derive on each small element K

$$\begin{aligned}
&\Big\|I_h(\frac{\widetilde{\mathbf{m}}_h^{n-1}}{|\widetilde{\mathbf{m}}_h^{n-1}|}) - \frac{\widetilde{\mathbf{m}}_h^{n-1}}{|\widetilde{\mathbf{m}}_h^{n-1}|}\Big\|_{W^{1,p}(K)} \leqslant Ch\Big\|\frac{\widetilde{\mathbf{m}}_h^{n-1}}{|\widetilde{\mathbf{m}}_h^{n-1}|}\Big\|_{W^{2,p}(K)} \leqslant Ch^{1+\frac{d}{p}}\Big\|\frac{\widetilde{\mathbf{m}}_h^{n-1}}{|\widetilde{\mathbf{m}}_h^{n-1}|}\Big\|_{W^{2,\infty}(K)} \\
&\leqslant Ch^{1+\frac{d}{p}}\Big(\|\widetilde{\mathbf{m}}_h^{n-1}\|_{W^{2,\infty}(K)} + \big\||\nabla\widetilde{\mathbf{m}}_h^{n-1}||\nabla\widetilde{\mathbf{m}}_h^{n-1}|\big\|_{L^\infty(K)}\Big) \quad \text{(here (3.28) is used)} \\
&\leqslant Ch^{1+\frac{d}{p}}\Big(\|\widetilde{\mathbf{e}}_h^{n-1}\|_{W^{2,\infty}(K)} + \|I_h\mathbf{m}(t_{n-1})\|_{W^{2,\infty}(K)} + \|\nabla\widetilde{\mathbf{e}}_h^{n-1}\|_{L^\infty(K)}\|\nabla\widetilde{\mathbf{e}}_h^{n-1}\|_{L^\infty(K)} \\
&\qquad + \|\nabla\widetilde{\mathbf{e}}_h^{n-1}\|_{L^\infty(K)}\|\nabla I_h\mathbf{m}(t_{n-1})\|_{L^\infty(K)} + \|\nabla I_h\mathbf{m}_{n-1}\|_{L^\infty(K)}^2\Big) \\
&\qquad\qquad \text{(here we replace } \widetilde{\mathbf{m}}_h^{n-1} \text{ by } I_h\mathbf{m}(t_{n-1}) - \widetilde{\mathbf{e}}_h^{n-1}) \\
&\leqslant Ch^{\frac{d}{p}}\Big(\|\widetilde{\mathbf{e}}_h^{n-1}\|_{W^{1,\infty}(K)} + \|\nabla\widetilde{\mathbf{e}}_h^{n-1}\|_{L^\infty(K)}\|\widetilde{\mathbf{e}}_h^{n-1}\|_{L^\infty(K)} + h\|\nabla\widetilde{\mathbf{e}}_h^{n-1}\|_{L^\infty(K)}\Big) \\
&\qquad\qquad \text{(here we use the inverse inequality (3.6))} \\
&\quad + Ch^{1+\frac{d}{p}}(\|I_h\mathbf{m}(t_{n-1})\|_{W^{2,\infty}(K)} + \|I_h\mathbf{m}(t_{n-1})\|_{W^{1,\infty}(K)}^2) \\
&\leqslant C\|\widetilde{\mathbf{e}}_h^{n-1}\|_{W^{1,p}(K)} + Ch(\|I_h\mathbf{m}(t_{n-1})\|_{W^{2,p}(K)} + \|I_h\mathbf{m}(t_{n-1})\|_{W^{1,p}(K)}) \quad \text{(inverse inequality)} \\
&\leqslant C\|\widetilde{\mathbf{e}}_h^{n-1}\|_{W^{1,p}(K)} + Ch\|\mathbf{m}(t_{n-1})\|_{W^{2,p}(K)},
\end{aligned}$$

where the inverse inequality (3.6) and (3.21) are used again in the last inequality. Consequently, by summing up the inequality above for all $K \in \mathcal{K}$, we have

$$\Big\|I_h\Big(\frac{\widetilde{\mathbf{m}}_h^{n-1}}{|\widetilde{\mathbf{m}}_h^{n-1}|}\Big) - \frac{\widetilde{\mathbf{m}}_h^{n-1}}{|\widetilde{\mathbf{m}}_h^{n-1}|}\Big\|_{W^{1,p}} \leqslant C\|\widetilde{\mathbf{e}}_h^{n-1}\|_{W^{1,p}} + Ch.$$

Substituting the above estimate into (3.31) yields the desired result of Lemma 3.8. ∎

3.5. Proof of Lemma 3.9. For the estimate (3.24), it suffices to show

$$|I_h\mathbf{m}(t_{n-1}) - \mathbf{m}_h^{n-1}| \leqslant C|I_h\mathbf{m}(t_{n-1}) - \widetilde{\mathbf{m}}_h^{n-1}| \quad \text{at all nodes.} \tag{3.32}$$

In fact, at each node we have $|I_h\mathbf{m}(t_{n-1})| = |\mathbf{m}_h^{n-1}| = 1$ because

$$I_h\mathbf{m}(t_{n-1}) = \mathbf{m}(t_{n-1}) \quad \text{and} \quad \mathbf{m}_h^{n-1} = I_h\Big(\frac{\widetilde{\mathbf{m}}_h^{n-1}}{|\widetilde{\mathbf{m}}_h^{n-1}|}\Big) = \frac{\widetilde{\mathbf{m}}_h^{n-1}}{|\widetilde{\mathbf{m}}_h^{n-1}|}.$$

At each node, we denote by $\widehat{\mathbf{m}}_h^{n-1}$ the projection of $\mathbf{m}(t_{n-1}) = I_h\mathbf{m}(t_{n-1})$ onto the straight line passing through the origin and $\mathbf{m}_h^{n-1}$, and denote by α the angle between the two unit vectors $\mathbf{m}_h^{n-1}$ and $\mathbf{m}(t_{n-1})$; then

$$\alpha \leqslant C\sin\alpha = C|I_h\mathbf{m}(t_{n-1}) - \widehat{\mathbf{m}}_h^{n-1}| \leqslant C|I_h\mathbf{m}(t_{n-1}) - \widetilde{\mathbf{m}}_h^{n-1}| \tag{3.33}$$

at each node. Since $|I_h\mathbf{m}(t_{n-1}) - \mathbf{m}_h^{n-1}|$ is the chord with respect to angle α, it follows that

$$\alpha \sim |I_h\mathbf{m}(t_{n-1}) - \mathbf{m}_h^{n-1}|. \tag{3.34}$$

Substituting this into the left-hand side of (3.33) yields (3.32), which further implies (3.24). Moreover, employing the norm equivalence given in Lemma 2.1, we obtain (3.25).

In the sequel, we denote by θ the angle between the vectors $\mathbf{m}_h^{n-1} - \mathbf{m}(t_{n-1})$ and $\widehat{\mathbf{m}}_h^{n-1} - \mathbf{m}(t_{n-1})$. Then at each node, it holds

$$\begin{aligned} |\mathbf{m}(t_{n-1}) - \mathbf{m}_h^{n-1}| - |\mathbf{m}(t_{n-1}) - \widetilde{\mathbf{m}}_h^{n-1}| &\leqslant |\mathbf{m}(t_{n-1}) - \mathbf{m}_h^{n-1}| - |\mathbf{m}(t_{n-1}) - \widehat{\mathbf{m}}_h^{n-1}| \\ &= |\mathbf{m}(t_{n-1}) - \mathbf{m}_h^{n-1}|(1-\cos\theta), \end{aligned} \tag{3.35}$$

where

$$1 - \cos\theta = 2\sin^2(\theta/2) \leqslant C\theta^2.$$

By a simple computation, we have $\theta = \frac{\alpha}{2} \sim |\mathbf{m}(t_{n-1}) - \mathbf{m}_h^{n-1}|$, where (3.34) is used. Substituting this estimate of θ into (3.35) and using (3.32), we obtain

$$|\mathbf{e}_h^{n-1}| \leqslant |\widetilde{\mathbf{e}}_h^{n-1}| + C|\mathbf{e}_h^{n-1}|^3 \leqslant |\widetilde{\mathbf{e}}_h^{n-1}| + C|\widetilde{\mathbf{e}}_h^{n-1}|^3,$$

which further implies

$$|\mathbf{e}_h^{n-1}|^2 \leqslant |\widetilde{\mathbf{e}}_h^{n-1}|^2 + C|\widetilde{\mathbf{e}}_h^{n-1}||\widetilde{\mathbf{e}}_h^{n-1}|^3 + C|\widetilde{\mathbf{e}}_h^{n-1}|^6,$$

at each node. This yields that

$$\begin{aligned} \|\mathbf{e}_h^{n-1}\|_{L_h^2}^2 &\le \|\widetilde{\mathbf{e}}_h^{n-1}\|_{L_h^2}^2 + C\|\widetilde{\mathbf{e}}_h^{n-1}\|_{L_h^3}\|\widetilde{\mathbf{e}}_h^{n-1}\|_{L_h^3}\|\widetilde{\mathbf{e}}_h^{n-1}\|_{L_h^6}^2 + C\|\widetilde{\mathbf{e}}_h^{n-1}\|_{L_h^6}^6 \\ &\leqslant \|\widetilde{\mathbf{e}}_h^{n-1}\|_{L_h^2}^2 + C\|\widetilde{\mathbf{e}}_h^{n-1}\|_{L^3}\|\widetilde{\mathbf{e}}_h^{n-1}\|_{L^3}\|\widetilde{\mathbf{e}}_h^{n-1}\|_{L^6}^2 + C\|\widetilde{\mathbf{e}}_h^{n-1}\|_{L^6}^6 \\ &\le \|\widetilde{\mathbf{e}}_h^{n-1}\|_{L_h^2}^2 + \tau^{\frac{8}{7}}\|\widetilde{\mathbf{e}}_h^{n-1}\|_{H^1}^2, \end{aligned}$$

where the Sobolev embedding $H^1 \hookrightarrow L^6$ and (3.22) are used in deriving the last inequality. This completes the proof of Lemma 3.9. ∎

3.6. Solvability of the linear system. The linear system (1.7) has a unique solution if and only if the corresponding homogeneous linear system

$$\left(\frac{\widetilde{\mathbf{m}}_h^n}{\tau}, \mathbf{v}_h\right)_h + (\nabla\widetilde{\mathbf{m}}_h^n, \nabla\mathbf{v}_h) = (|\nabla\mathbf{m}_h^{n-1}|^2\widetilde{\mathbf{m}}_h^n, \mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in \mathbf{S}_h^r, \tag{3.36}$$

has only the zero solution. Since the induction assumption (3.20) implies

$$\|\nabla\mathbf{m}_h^{n-1}\|_{L^4} \leqslant \|\nabla I_h\mathbf{m}(t_{n-1})\|_{L^4} + \|\nabla\mathbf{e}_h^{n-1}\|_{L^4} \leqslant C + 1, \tag{3.37}$$

for $1 \leqslant n \leqslant k$, substituting $\mathbf{v}_h = \widetilde{\mathbf{m}}_h^n$ into the homogeneous linear system (3.36) yields

$$\frac{\|\widetilde{\mathbf{m}}_h^n\|_{L_h^2}^2}{\tau} + \|\nabla\widetilde{\mathbf{m}}_h^n\|_{L^2}^2 = (|\nabla\mathbf{m}_h^{n-1}|^2\widetilde{\mathbf{m}}_h^n, \widetilde{\mathbf{m}}_h^n) \leqslant \|\nabla\mathbf{m}_h^{n-1}\|_{L^4}^2\|\widetilde{\mathbf{m}}_h^n\|_{L^4}^2 \leqslant C\|\widetilde{\mathbf{m}}_h^n\|_{L^2}^{\frac{1}{2}}\|\widetilde{\mathbf{m}}_h^n\|_{L^6}^{\frac{3}{2}},$$

where (3.37) and the Sobolev interpolation inequality (cf. [1, p. 135, Theorem 5.2]) are used. By using the Sobolev embedding $H^1 \hookrightarrow L^6$ and Young's inequality, we further obtain

$$\begin{aligned} \frac{\|\widetilde{\mathbf{m}}_h^n\|_{L_h^2}^2}{\tau} + \|\nabla\widetilde{\mathbf{m}}_h^n\|_{L^2}^2 \leqslant C\|\widetilde{\mathbf{m}}_h^n\|_{L^2}^{\frac{1}{2}}\|\widetilde{\mathbf{m}}_h^n\|_{H^1}^{\frac{3}{2}} &\leqslant C\|\widetilde{\mathbf{m}}_h^n\|_{L^2}^2 + \frac{1}{2}\|\widetilde{\mathbf{m}}_h^n\|_{H^1}^2 \\ &\leqslant C\|\widetilde{\mathbf{m}}_h^n\|_{L^2}^2 + \frac{1}{2}\|\nabla\widetilde{\mathbf{m}}_h^n\|_{L^2}^2. \end{aligned}$$

For sufficiently small τ, the two terms on the right-hand side can both be absorbed by the left-hand side. In this case, we obtain

$$\|\widetilde{\mathbf{m}}_h^n\|_{L_h^2} = 0.$$

This shows that the homogeneous linear system (3.36) has only the zero solution (when τ is smaller than some constant). This proves the unique solvability of the original linear system (1.7).

3.7. Error estimates. Taking $\mathbf{v}_h = \widetilde{\mathbf{e}}_h^n$ in the error equation (3.19) yields

$$\begin{aligned}
\frac{\|\widetilde{\mathbf{e}}_h^n\|_{L_h^2}^2 - \|\mathbf{e}_h^{n-1}\|_{L_h^2}^2}{2\tau} + \|\nabla\widetilde{\mathbf{e}}_h^n\|_{L^2}^2 &\leqslant \Big(|\nabla I_h\mathbf{m}(t_{n-1})|^2 I_h\mathbf{m}(t_n) - |\nabla\mathbf{m}_h^{n-1}|^2\widetilde{\mathbf{m}}_h^n, \widetilde{\mathbf{e}}_h^n\Big) + \mathcal{E}(\widetilde{\mathbf{e}}_h^n)\\
&= \Big(|\nabla I_h\mathbf{m}(t_{n-1})|^2 I_h\mathbf{m}(t_n) - |\nabla I_h\mathbf{m}(t_{n-1})|^2\widetilde{\mathbf{m}}_h^n, \widetilde{\mathbf{e}}_h^n\Big)\\
&\quad + \Big((|\nabla I_h\mathbf{m}(t_{n-1})|^2 - |\nabla\mathbf{m}_h^{n-1}|^2)\widetilde{\mathbf{m}}_h^n, \widetilde{\mathbf{e}}_h^n\Big) + \mathcal{E}(\widetilde{\mathbf{e}}_h^n)\\
&=: \mathcal{E}_5(\widetilde{\mathbf{e}}_h^n) + \mathcal{E}_6(\widetilde{\mathbf{e}}_h^n) + \mathcal{E}(\widetilde{\mathbf{e}}_h^n), \qquad (3.38)
\end{aligned}$$

for $1 \leqslant n \leqslant k$. In the following, we present detailed estimates of the right-hand side of (3.38).

The estimate (3.17) implies that

$$|\mathcal{E}(\widetilde{\mathbf{e}}_h^n)| \leqslant C(\tau^2 + h^{2r+2}) + \varepsilon\|\nabla\widetilde{\mathbf{e}}_h^n\|_{L^2}^2 + C\|\widetilde{\mathbf{e}}_h^n\|_{L^2}^2.$$

It is easy to see that

$$|\mathcal{E}_5(\widetilde{\mathbf{e}}_h^n)| \leqslant C\|\nabla I_h\mathbf{m}(t_{n-1})\|_{L^\infty}^2\|\widetilde{\mathbf{e}}_h^n\|_{L^2}^2 \leqslant C\|\widetilde{\mathbf{e}}_h^n\|_{L^2}^2.$$

Replacing $\widetilde{\mathbf{m}}_h^n$ by $I_h\mathbf{m}(t_n) - \widetilde{\mathbf{e}}_h^n$ in the expression of $\mathcal{E}_6(\widetilde{\mathbf{e}}_h^n)$ and using (3.24)–(3.25), we obtain

$$\begin{aligned}
&|\mathcal{E}_6(\widetilde{\mathbf{e}}_h^n)|\\
&\leqslant \Big|\Big(|\nabla I_h\mathbf{m}(t_{n-1})|^2 - |\nabla\mathbf{m}_h^{n-1}|^2, |\widetilde{\mathbf{e}}_h^n|^2\Big)\Big| + \Big|\Big((|\nabla I_h\mathbf{m}(t_{n-1})|^2 - |\nabla\mathbf{m}_h^{n-1}|^2)I_h\mathbf{m}(t_n), \widetilde{\mathbf{e}}_h^n\Big)\Big|\\
&= \Big|\Big(|\nabla I_h\mathbf{m}(t_{n-1})|^2 - |\nabla\mathbf{m}_h^{n-1}|^2, |\widetilde{\mathbf{e}}_h^n|^2\Big)\Big|\\
&\quad + \Big|\Big((2\nabla I_h\mathbf{m}(t_{n-1}) - \nabla\mathbf{e}_h^{n-1})\cdot\nabla\mathbf{e}_h^{n-1}, I_h\mathbf{m}(t_n)\cdot\widetilde{\mathbf{e}}_h^n\Big)\Big|.
\end{aligned}$$

The last term on the right-hand side can be estimated as follows:

$$\begin{aligned}
&\Big((2\nabla I_h\mathbf{m}(t_{n-1}) - \nabla\mathbf{e}_h^{n-1})\cdot\nabla\mathbf{e}_h^{n-1}, I_h\mathbf{m}(t_n)\cdot\widetilde{\mathbf{e}}_h^n\Big)\\
&= \Big(2\big(\nabla I_h\mathbf{m}(t_{n-1}) - \nabla\mathbf{m}(t_{n-1})\big)\cdot\nabla\mathbf{e}_h^{n-1}, I_h\mathbf{m}(t_n)\cdot\widetilde{\mathbf{e}}_h^n\Big)\\
&\quad + \Big(2\nabla\mathbf{m}(t_{n-1})\cdot\nabla\mathbf{e}_h^{n-1}, I_h\mathbf{m}(t_n)\cdot\widetilde{\mathbf{e}}_h^n\Big)\\
&\quad - \Big(\nabla\mathbf{e}_h^{n-1}\cdot\nabla\mathbf{e}_h^{n-1}, I_h\mathbf{m}(t_n)\cdot\widetilde{\mathbf{e}}_h^n\Big)\\
&\leqslant Ch\|\mathbf{m}(t_{n-1})\|_{W^{2,3}}\|\nabla\mathbf{e}_h^{n-1}\|_{L^2}\|\widetilde{\mathbf{e}}_h^n\|_{L^6}\\
&\quad - \Big[\Big(2\Delta\mathbf{m}(t_{n-1})\cdot\mathbf{e}_h^{n-1}, I_h\mathbf{m}(t_n)\cdot\widetilde{\mathbf{e}}_h^n\Big) + \Big(2\nabla\mathbf{m}(t_{n-1})\mathbf{e}_h^{n-1}, \nabla(I_h\mathbf{m}(t_n)\cdot\widetilde{\mathbf{e}}_h^n)\Big)\Big]\\
&\quad + C\|\mathbf{e}_h^{n-1}\|_{H^1}\|\nabla\mathbf{e}_h^{n-1}\|_{L^4}\|\widetilde{\mathbf{e}}_h^n\|_{L^4}\\
&\leqslant C\|\mathbf{e}_h^{n-1}\|_{L^2}\|\widetilde{\mathbf{e}}_h^n\|_{L^6} \qquad \text{(here the inverse inequality (3.7) is used)}\\
&\quad + C(\|\mathbf{e}_h^{n-1}\|_{L^4}\|\widetilde{\mathbf{e}}_h^n\|_{L^2} + \|\mathbf{e}_h^{n-1}\|_{L^4}\|\nabla\widetilde{\mathbf{e}}_h^n\|_{L^2})\\
&\quad + C(\|\widetilde{\mathbf{e}}_h^{n-1}\|_{H^1} + h)\|\nabla\mathbf{e}_h^{n-1}\|_{L^4}\|\widetilde{\mathbf{e}}_h^n\|_{L^4} \qquad \text{(here we use Lemma 3.8 with } p=2)\\
&\leqslant C\big(\|\mathbf{e}_h^{n-1}\|_{L^2}\|\widetilde{\mathbf{e}}_h^n\|_{L^6} + \|\mathbf{e}_h^{n-1}\|_{L^4}\|\nabla\widetilde{\mathbf{e}}_h^n\|_{L^2} + \|\widetilde{\mathbf{e}}_h^{n-1}\|_{H^1}\|\widetilde{\mathbf{e}}_h^n\|_{L^4} + \|\mathbf{e}_h^{n-1}\|_{L^4}\|\widetilde{\mathbf{e}}_h^n\|_{L^4}\big),
\end{aligned}$$

where (3.20) and the inverse inequality (3.7) are used in the last inequality. As a result, we have

$$\begin{aligned}
&|\mathcal{E}_6(\widetilde{\mathbf{e}}_h^n)|\\
&\leqslant C\|\widetilde{\mathbf{e}}_h^n\|_{L^2}^2 + C\|\nabla\mathbf{m}_h^{n-1}\|_{L^4}^2\|\widetilde{\mathbf{e}}_h^n\|_{L^4}^2\\
&\quad + C\Big(\|\mathbf{e}_h^{n-1}\|_{L^2}\|\widetilde{\mathbf{e}}_h^n\|_{L^6} + \|\mathbf{e}_h^{n-1}\|_{L^4}\|\nabla\widetilde{\mathbf{e}}_h^n\|_{L^2} + \|\widetilde{\mathbf{e}}_h^{n-1}\|_{H^1}\|\widetilde{\mathbf{e}}_h^n\|_{L^4} + \|\mathbf{e}_h^{n-1}\|_{L^4}\|\widetilde{\mathbf{e}}_h^n\|_{L^4}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\|\widetilde{\mathbf{e}}_h^n\|_{L^2}^2 + C\|\widetilde{\mathbf{e}}_h^n\|_{L^4}^2 \quad \text{(here (3.20) is used, which implies } \|\nabla \mathbf{m}_h^{n-1}\|_{L^4} \leqslant C\text{)}\\
&\quad + C\Big(\|\widetilde{\mathbf{e}}_h^{n-1}\|_{L^2}\|\widetilde{\mathbf{e}}_h^n\|_{L^6} + \|\widetilde{\mathbf{e}}_h^{n-1}\|_{L^4}\|\nabla\widetilde{\mathbf{e}}_h^n\|_{L^2} + \|\widetilde{\mathbf{e}}_h^{n-1}\|_{H^1}\|\widetilde{\mathbf{e}}_h^n\|_{L^4} + \|\widetilde{\mathbf{e}}_h^{n-1}\|_{L^4}\|\widetilde{\mathbf{e}}_h^n\|_{L^4}\Big)\\
&\leqslant C\|\widetilde{\mathbf{e}}_h^n\|_{L^2}^2 + C\|\widetilde{\mathbf{e}}_h^n\|_{L^2}^{\frac12}\|\widetilde{\mathbf{e}}_h^n\|_{L^6}^{\frac32}\\
&\quad + C\Big(\|\widetilde{\mathbf{e}}_h^{n-1}\|_{L^2}\|\widetilde{\mathbf{e}}_h^n\|_{L^6} + \|\widetilde{\mathbf{e}}_h^{n-1}\|_{L^2}^{\frac14}\|\widetilde{\mathbf{e}}_h^{n-1}\|_{L^6}^{\frac34}\|\nabla\widetilde{\mathbf{e}}_h^n\|_{L^2} + \|\widetilde{\mathbf{e}}_h^{n-1}\|_{H^1}\|\widetilde{\mathbf{e}}_h^n\|_{L^2}^{\frac14}\|\widetilde{\mathbf{e}}_h^n\|_{L^6}^{\frac34}\\
&\quad + \|\widetilde{\mathbf{e}}_h^{n-1}\|_{L^2}^{\frac14}\|\widetilde{\mathbf{e}}_h^{n-1}\|_{L^6}^{\frac34}\|\widetilde{\mathbf{e}}_h^n\|_{L^2}^{\frac14}\|\widetilde{\mathbf{e}}_h^n\|_{L^6}^{\frac34}\Big)\\
&\leqslant C(\|\widetilde{\mathbf{e}}_h^n\|_{L^2}^2 + \|\widetilde{\mathbf{e}}_h^{n-1}\|_{L^2}^2) + \varepsilon(\|\nabla\widetilde{\mathbf{e}}_h^n\|_{L^2}^2 + \|\nabla\widetilde{\mathbf{e}}_h^{n-1}\|_{L^2}^2),
\end{aligned}
$$

where ε can be arbitrarily small at the expense of enlarging the constant C.

Substituting the above estimates into (3.38) leads to

$$
\begin{aligned}
&\frac{\|\widetilde{\mathbf{e}}_h^n\|_{L_h^2}^2 - \|\mathbf{e}_h^{n-1}\|_{L_h^2}^2}{2\tau} + \|\nabla\widetilde{\mathbf{e}}_h^n\|_{L^2}^2\\
&\leqslant C(\tau^2 + h^{2r+2}) + \varepsilon(\|\nabla\widetilde{\mathbf{e}}_h^n\|_{L^2}^2 + \|\nabla\widetilde{\mathbf{e}}_h^{n-1}\|_{L^2}^2) + C(\|\widetilde{\mathbf{e}}_h^n\|_{L^2}^2 + \|\widetilde{\mathbf{e}}_h^{n-1}\|_{L^2}^2) \qquad (3.39)
\end{aligned}
$$

for $1 \leqslant n \leqslant k$. Then we substitute (3.26) into (3.39) and get

$$
\begin{aligned}
&\frac{\|\widetilde{\mathbf{e}}_h^n\|_{L_h^2}^2 - \|\widetilde{\mathbf{e}}_h^{n-1}\|_{L_h^2}^2}{2\tau} + \|\nabla\widetilde{\mathbf{e}}_h^n\|_{L^2}^2\\
&\leqslant C(\tau^2 + h^{2r+2}) + \varepsilon\|\nabla\widetilde{\mathbf{e}}_h^n\|_{L^2}^2 + C(\varepsilon + \tau^{\frac17})\|\nabla\widetilde{\mathbf{e}}_h^{n-1}\|_{L^2}^2 + C(\|\widetilde{\mathbf{e}}_h^n\|_{L^2}^2 + \|\widetilde{\mathbf{e}}_h^{n-1}\|_{L^2}^2).
\end{aligned}
$$

Summing up the above inequality yields

$$
\|\widetilde{\mathbf{e}}_h^n\|_{L_h^2}^2 + 2\tau\sum_{j=1}^n \|\nabla\widetilde{\mathbf{e}}_h^j\|_{L^2}^2 \leqslant C(\tau^2 + h^{2r+2}) + C(\varepsilon + \tau^{\frac17})\tau\sum_{j=1}^n \|\nabla\widetilde{\mathbf{e}}_h^j\|_{L^2}^2 + C\tau\sum_{j=1}^n \|\widetilde{\mathbf{e}}_h^j\|_{L^2}^2,
$$

for $1 \leqslant n \leqslant k$. Then, by choosing sufficiently small ε and τ, the second term on the right-hand side can be absorbed by the left-hand side. By applying the discrete Gronwall inequality, we obtain

$$
\max_{1\leqslant n\leqslant k}\|\widetilde{\mathbf{e}}_h^n\|_{L_h^2}^2 + \tau\sum_{j=1}^k \|\nabla\widetilde{\mathbf{e}}_h^j\|_{L^2}^2 \leqslant C(\tau + h^{r+1})^2. \qquad (3.40)
$$

Meanwhile, noting (3.24) and Lemma 2.1, it follows that

$$
\max_{1\leqslant n\leqslant k}\|\mathbf{e}_h^n\|_{L^2} + \max_{1\leqslant n\leqslant k}\|\widetilde{\mathbf{e}}_h^n\|_{L^2} \leqslant C(\tau + h^{r+1}). \qquad (3.41)
$$

This, together with Lemma 3.1, proves the desired error bound in (2.3) for $1 \leqslant n \leqslant k$.

It remains to complete the mathematical induction by proving (3.20)–(3.22) for $n = k+1$.

3.8. Proof of (3.20)–(3.21) **for** $n = k+1$**.** In this subsection, we prove (3.20)–(3.21) for $n = k+1$ in the two cases

$$
\tau \leqslant h^{1.875} \quad \text{and} \quad \tau \geqslant h^{1.875},
$$

respectively. The proof of (3.22) for $n = k+1$ is given in the next subsection.

If $\tau \leqslant h^{1.875}$, by using (3.41) and the inverse inequality (3.7), we have

$$
\|\nabla\widetilde{\mathbf{e}}_h^k\|_{L^4} \leqslant Ch^{-1-\frac{d}{4}}\|\widetilde{\mathbf{e}}_h^k\|_{L^2} \leqslant Ch^{-1-\frac{d}{4}}(\tau + h^{r+1}) \leqslant Ch^{\frac18}, \qquad (3.42)
$$

$$
\|\widetilde{\mathbf{e}}_h^k\|_{L^\infty} \leqslant Ch^{-\frac{d}{2}}\|\widetilde{\mathbf{e}}_h^k\|_{L^2} \leqslant Ch^{-\frac{d}{2}}(\tau + h^{r+1}) \leqslant Ch^{\frac14}. \qquad (3.43)
$$

If $\tau \geqslant h^{1.875}$, then we rewrite (3.19) as

$$
\begin{aligned}
(\Delta_h\widetilde{\mathbf{e}}_h^k, \mathbf{v}_h) &= \Big(\frac{\widetilde{\mathbf{e}}_h^k - \mathbf{e}_h^{k-1}}{\tau}, \mathbf{v}_h\Big)_h - \Big(|\nabla I_h\mathbf{m}(t_{k-1})|^2 I_h\mathbf{m}(t_k) - |\nabla\mathbf{m}_h^{k-1}|^2\widetilde{\mathbf{m}}_h^k, \mathbf{v}_h\Big) - \mathcal{E}(\mathbf{v}_h)\\
&=: \mathcal{G}_1 + \mathcal{G}_2 + \mathcal{G}_3 \quad \forall\, \mathbf{v}_h \in \mathbf{S}_h^r. \qquad (3.44)
\end{aligned}
$$

By using (3.40)–(3.41), it is easy to see that

$$
\begin{aligned}
|\mathcal{G}_1| \leqslant \Big\| \frac{\widetilde{\mathbf{e}}_h^k - \mathbf{e}_h^{k-1}}{\tau} \Big\|_{L_h^2} \|\mathbf{v}_h\|_{L_h^2} \leqslant C \Big\| \frac{\widetilde{\mathbf{e}}_h^k - \mathbf{e}_h^{k-1}}{\tau} \Big\|_{L^2} \|\mathbf{v}_h\|_{L^2} &\leqslant C\tau^{-1}(\tau + h^{r+1}) \|\mathbf{v}_h\|_{L^2} \\
&\leqslant C\tau^{-1}(\tau + \tau^{\frac{r+1}{1.875}}) \|\mathbf{v}_h\|_{L^2} \\
&\leqslant C\|\mathbf{v}_h\|_{L^2} \quad (\text{since } r \geqslant 1).
\end{aligned}
$$

By using (3.20), we have

$$
\begin{aligned}
|\mathcal{G}_2| &\leqslant \Big\| |\nabla I_h \mathbf{m}(t_{k-1})|^2 I_h \mathbf{m}(t_k) - |\nabla \mathbf{m}_h^{k-1}|^2 \widetilde{\mathbf{m}}_h^k \Big\|_{L^2} \|\mathbf{v}_h\|_{L^2} \\
&\leqslant \Big\| |\nabla I_h \mathbf{m}(t_{k-1}) + \nabla \mathbf{m}_h^{k-1}|\, |\nabla \mathbf{e}_h^{k-1}| \widetilde{\mathbf{m}}_h^k \Big\|_{L^2} \|\mathbf{v}_h\|_{L^2} + \Big\| |\nabla I_h \mathbf{m}(t_{k-1})|^2 \widetilde{\mathbf{e}}_h^k \Big\|_{L^2} \|\mathbf{v}_h\|_{L^2} \\
&\leqslant \Big\| \big(2|\nabla I_h \mathbf{m}(t_{k-1})| + |\nabla \mathbf{e}_h^{k-1}|\big) |\nabla \mathbf{e}_h^{k-1}| I_h \mathbf{m}(t_k) \Big\|_{L^2} \|\mathbf{v}_h\|_{L^2} \\
&\quad + \Big\| \big(2|\nabla I_h \mathbf{m}(t_{k-1})| + |\nabla \mathbf{e}_h^{k-1}|\big) |\nabla \mathbf{e}_h^{k-1}| \widetilde{\mathbf{e}}_h^k \Big\|_{L^2} \|\mathbf{v}_h\|_{L^2} + \Big\| |\nabla I_h \mathbf{m}(t_{k-1})|^2 \widetilde{\mathbf{e}}_h^k \Big\|_{L^2} \|\mathbf{v}_h\|_{L^2} \\
&\leqslant C\Big(\|\nabla I_h \mathbf{m}(t_{k-1})\|_{L^4} \|\nabla \mathbf{e}_h^{k-1}\|_{L^4} + \|\nabla \mathbf{e}_h^{k-1}\|_{L^4}^2 \Big) \|I_h \mathbf{m}(t_k)\|_{L^\infty} \|\mathbf{v}_h\|_{L^2} \\
&\quad + C\Big(\|\nabla I_h \mathbf{m}(t_{k-1})\|_{L^4} \|\nabla \mathbf{e}_h^{k-1}\|_{L^4} + \|\nabla \mathbf{e}_h^{k-1}\|_{L^4}^2 \Big) \|\widetilde{\mathbf{e}}_h^k\|_{L^\infty} \|\mathbf{v}_h\|_{L^2} \\
&\quad + C\|\nabla I_h \mathbf{m}(t_{k-1})\|_{L^4}^2 \|\widetilde{\mathbf{e}}_h^k\|_{L^\infty} \|\mathbf{v}_h\|_{L^2} \\
&\leqslant (C + C\|\widetilde{\mathbf{e}}_h^k\|_{L^\infty}) \|\mathbf{v}_h\|_{L^2}.
\end{aligned}
$$

By using (3.16) and the inverse inequality (3.7), we have

$$
|\mathcal{G}_3| \leqslant C(\tau + h^{r+1}) \|\mathbf{v}_h\|_{L^2} \leqslant C\|\mathbf{v}_h\|_{L^2}.
$$

Substituting the above estimates of $\mathcal{G}_j$, $j = 1, 2, 3$, into (3.44) yields

$$
\begin{aligned}
\|\Delta_h \widetilde{\mathbf{e}}_h^k\|_{L^2} &= \sup_{\mathbf{v}_h \in \mathbf{S}_h^r, \mathbf{v}_h \neq \mathbf{0}} \frac{|(\Delta_h \widetilde{\mathbf{e}}_h^k, \mathbf{v}_h)|}{\|\mathbf{v}_h\|_{L^2}} \\
&\leqslant C + C\|\widetilde{\mathbf{e}}_h^k\|_{L^\infty} \\
&\leqslant C + C\|\widetilde{\mathbf{e}}_h^k\|_{L^2}^{1-\frac{d}{4}} (\|\widetilde{\mathbf{e}}_h^k\|_{L^2} + \|\Delta_h \widetilde{\mathbf{e}}_h^k\|_{L^2})^{\frac{d}{4}} \quad (\text{here (3.8) is used}) \\
&\leqslant C + C\|\widetilde{\mathbf{e}}_h^k\|_{L^2} + \varepsilon \|\Delta_h \widetilde{\mathbf{e}}_h^k\|_{L^2},
\end{aligned}
$$

where ε can be arbitrarily small at the expense of enlarging the constant C. By choosing a sufficiently small ε, the last term on the right-hand side of the above inequality can be absorbed by the left-hand side. As a result, we obtain

$$
\|\Delta_h \widetilde{\mathbf{e}}_h^k\|_{L^2} \leqslant C, \tag{3.45}
$$

which further implies that

$$
\|\widetilde{\mathbf{e}}_h^k\|_{L^\infty} \leqslant C\|\widetilde{\mathbf{e}}_h^k\|_{L^2}^{1-\frac{d}{4}} (\|\widetilde{\mathbf{e}}_h^k\|_{L^2} + \|\Delta_h \widetilde{\mathbf{e}}_h^k\|_{L^2})^{\frac{d}{4}} \leqslant C(\tau + h^{r+1})^{\frac{1}{4}}. \tag{3.46}
$$

Since

$$
\|\nabla \widetilde{\mathbf{e}}_h^k\|_{L^2} = (\nabla \widetilde{\mathbf{e}}_h^k, \nabla \widetilde{\mathbf{e}}_h^k)^{\frac{1}{2}} = (-\Delta_h \widetilde{\mathbf{e}}_h^k, \widetilde{\mathbf{e}}_h^k)^{\frac{1}{2}} \leqslant \|\Delta_h \widetilde{\mathbf{e}}_h^k\|_{L^2}^{\frac{1}{2}} \|\widetilde{\mathbf{e}}_h^k\|_{L^2}^{\frac{1}{2}},
$$

it follows that (interpolating the L^4 norm by using the L^2 and L^6 norms)

$$
\|\nabla \widetilde{\mathbf{e}}_h^k\|_{L^4} \leqslant \|\nabla \widetilde{\mathbf{e}}_h^k\|_{L^2}^{\frac{1}{4}} \|\nabla \widetilde{\mathbf{e}}_h^k\|_{L^6}^{\frac{3}{4}} \leqslant C\|\widetilde{\mathbf{e}}_h^k\|_{L^2}^{\frac{1}{8}} \|\Delta_h \widetilde{\mathbf{e}}_h^k\|_{L^2}^{\frac{1}{8}+\frac{3}{4}} \leqslant C(\tau + h^{r+1})^{\frac{1}{8}}, \tag{3.47}
$$

where (3.9) is used in the second to last inequality, and (3.40) and (3.45) are used in the last inequality.

Combining (3.42)–(3.43) and (3.46)–(3.47) yields, in both cases $\tau \leqslant h^{1.875}$ and $\tau \geqslant h^{1.875}$,

$$
\|\nabla \widetilde{\mathbf{e}}_h^k\|_{L^4} \leqslant C(\tau + h)^{\frac{1}{8}}, \tag{3.48}
$$

$$
\|\widetilde{\mathbf{e}}_h^k\|_{L^\infty} \leqslant C(\tau + h)^{\frac{1}{4}}. \tag{3.49}
$$

For sufficiently small τ and h, (3.49) implies (3.21) for $n = k+1$. Then Lemma 3.8 implies that (3.23) also holds for $n = k+1$ (with $p = 4$ therein), which together with (3.48)–(3.49) further implies that

$$\|\nabla \mathbf{e}_h^k\|_{L^4} \leqslant C\|\widetilde{\mathbf{e}}_h^k\|_{W^{1,4}} + Ch \leqslant C(\tau+h)^{\frac{1}{8}}. \tag{3.50}$$

Hence, for sufficiently small τ and h, (3.49) and (3.50) lead to

$$\|\nabla \mathbf{e}_h^k\|_{L^4} \leqslant 1 \quad \text{and} \quad \|\widetilde{\mathbf{e}}_h^k\|_{L^\infty} \leqslant \frac{1}{4}. \tag{3.51}$$

This completes the proof of (3.20)–(3.21) for $n = k+1$.

3.9. Proof of (3.22) **for** $n = k+1$**.** On the one hand, by using the inverse inequality and the L^2 error bound in (3.41), we have

$$\|\widetilde{\mathbf{e}}_h^k\|_{L^3} \leqslant Ch^{-\frac{d}{6}}\|\widetilde{\mathbf{e}}_h^n\|_{L^2} \leqslant Ch^{-\frac{d}{6}}(\tau + h^{r+1}), \tag{3.52}$$

$$\|\widetilde{\mathbf{e}}_h^k\|_{L^6} \leqslant Ch^{-\frac{d}{3}}\|\widetilde{\mathbf{e}}_h^n\|_{L^2} \leqslant Ch^{-\frac{d}{3}}(\tau + h^{r+1}). \tag{3.53}$$

On the other hand, by the Sobolev interpolation inequality (cf. [1, p. 135, Theorem 5.2]) and (3.39), we have

$$\|\widetilde{\mathbf{e}}_h^k\|_{L^3} \leqslant C\|\widetilde{\mathbf{e}}_h^k\|_{H^1}^{\frac{d}{6}}\|\widetilde{\mathbf{e}}_h^k\|_{L^2}^{1-\frac{d}{6}} \leqslant C[\tau^{-\frac{1}{2}}(\tau+h^{r+1})]^{\frac{d}{6}}(\tau+h^{r+1})^{1-\frac{d}{6}} \leqslant C\tau^{-\frac{d}{12}}(\tau+h^{r+1}), \tag{3.54}$$

$$\|\widetilde{\mathbf{e}}_h^k\|_{L^6} \leqslant C\|\widetilde{\mathbf{e}}_h^k\|_{H^1}^{\frac{d}{3}}\|\widetilde{\mathbf{e}}_h^k\|_{L^2}^{1-\frac{d}{3}} \leqslant C[\tau^{-\frac{1}{2}}(\tau+h^{r+1})]^{\frac{d}{3}}(\tau+h^{r+1})^{1-\frac{d}{3}} \leqslant C\tau^{-\frac{d}{6}}(\tau+h^{r+1}). \tag{3.55}$$

Combining the estimates (3.52) and (3.54) yields

$$\begin{aligned}
\|\widetilde{\mathbf{e}}_h^k\|_{L^3} &\leqslant C\min\{h^{-\frac{d}{6}}, \tau^{-\frac{d}{12}}\}(\tau+h^{r+1}) \\
&\leqslant C\min\{h^{-\frac{d}{6}}, \tau^{-\frac{d}{12}}\}\,\tau + C\min\{h^{-\frac{d}{6}}, \tau^{-\frac{d}{12}}\}\,h^{r+1} \\
&\leqslant C(\tau^{1-\frac{d}{12}} + h^{(r+1)(1-\frac{d}{6(r+1)})}) \\
&\leqslant C(\tau^{\frac{3}{4}} + h^{(r+1)\frac{3}{4}}) \quad \text{(because } r+1 \geqslant 2 \text{ and } d \leqslant 3) \\
&\leqslant C\tau^{\frac{3}{4}} \quad (\text{when } \tau \geqslant \kappa h^{r+1}) \\
&\leqslant \tau^{\frac{4}{7}} \quad (\text{when } \tau \text{ sufficiently small}).
\end{aligned} \tag{3.56}$$

Similarly, combining the estimates (3.53) and (3.55) yields

$$\begin{aligned}
\|\widetilde{\mathbf{e}}_h^k\|_{L^6} &\leqslant C\min\{h^{-\frac{d}{3}}, \tau^{-\frac{d}{6}}\}(\tau+h^{r+1}) \\
&\leqslant C\min\{h^{-\frac{d}{3}}, \tau^{-\frac{d}{6}}\}\,\tau + C\min\{h^{-\frac{d}{3}}, \tau^{-\frac{d}{6}}\}\,h^{r+1} \\
&\leqslant C(\tau^{1-\frac{d}{6}} + h^{r+1(1-\frac{d}{3(r+1)})}) \\
&\leqslant C(\tau^{\frac{1}{2}} + h^{(r+1)\frac{1}{2}}) \quad \text{(because } r+1 \geqslant 2 \text{ and } d \leqslant 3) \\
&\leqslant C\tau^{\frac{1}{2}} \quad (\text{when } \tau \geqslant \kappa h^{r+1}) \\
&\leqslant \tau^{\frac{2}{7}} \quad (\text{when } \tau \text{ is sufficiently small}).
\end{aligned} \tag{3.57}$$

The estimates (3.56)–(3.57) imply (3.22) for $n = k+1$.

The mathematical induction on (3.20)–(3.22) is completed. As a result, the error bounds in (3.40)–(3.41) hold for $k = N$. Then, combining the error bound in (3.41) and Lemma 3.1, we obtain the desired error bound (2.3). This completes the proof of Theorem 2.2.

4. Numerical results. In this section, we present numerical results to support the theoretical result proved in Theorem 2.2, by illustrating the convergence rates of the renormalized lumped mass method.

We consider problem (1.1) on the two-dimensional domain $\Omega = [1/2, 3/2] \times [1/2, 3/2]$ and let $T = 0.5$. The initial value of the solution is chosen to be

$$\mathbf{m}^0 = \frac{1}{S}[\tilde{m}_1^0, \tilde{m}_2^0, \tilde{m}_3^0]^\top, \tag{4.1}$$

where

$$\tilde{m}_1^0(x,y) = \sin(\pi x)\cos(2\pi y) + 1,$$
$$\tilde{m}_2^0(x,y) = \cos(2\pi x)\cos(2\pi y) + 2,$$
$$\tilde{m}_3^0(x,y) = \sin(\pi y),$$

and $S(x,y) = \sqrt{\tilde{m}_1^0(x,y)^2 + \tilde{m}_2^0(x,y)^2 + \tilde{m}_3^0(x,y)^2}$. Clearly, the initial data (4.1) satisfies the pointwise constraint $|\mathbf{m}^0| = 1$ and the boundary condition (1.2). We solve problem (1.1) with the above initial condition by the proposed method (1.7)–(1.8) on both rectangular and triangular meshes.

Rectangular Mesh: With tensor-product Q_1 and Q_2 elements, the temporal discretization errors of the numerical solutions are presented in Tables 4.1–4.2 with different time stepsizes τ and mesh sizes h. The numerical results in Tables 4.1–4.2 indicate that the spatial discretization errors are sufficiently small (further decreasing the spatial mesh size does not affect the temporal discretization error) and can be neglected in observing the first-order convergence in time, which is consistent with the theoretical result proved in Theorem 2.2.

TABLE 4.1
Temporal discretization error $\|\mathbf{m}_{h,\tau}^N - \mathbf{m}_{h,\tau/2}^N\|_{L^2}$ *with* Q_1 *element*

$h\backslash\tau$	1/20	1/40	1/80	1/160	convergence rate
1/32	2.233e-3	8.433e-4	3.693e-4	1.784e-4	≈ 1.05
1/64	2.244e-3	8.441e-4	3.691e-4	1.784e-4	≈ 1.05
1/128	2.246e-3	8.443e-4	3.691e-4	1.784e-4	≈ 1.05

TABLE 4.2
Temporal discretization error $\|\mathbf{m}_{h,\tau}^N - \mathbf{m}_{h,\tau/2}^N\|_{L^2}$ *with* Q_2 *element*

$h\backslash\tau$	1/20	1/40	1/80	1/160	convergence rate
1/32	2.247e-3	8.443e-4	3.691e-4	1.784e-4	≈ 1.05
1/64	2.247e-3	8.444e-4	3.691e-4	1.784e-4	≈ 1.05
1/128	2.247e-3	8.444e-4	3.691e-4	1.784e-4	≈ 1.05

The spatial discretization errors of the numerical solutions with Q_r elements are presented in Tables 4.3–4.4 for $r = 1, 2$. The numerical results in Tables 4.3–4.4 indicate that the temporal discretization errors are sufficiently small (further decreasing the time stepsize does not essentially affect the spatial discretization error) and can be neglected in observing the $(r+1)$th-order convergence in space, which is consistent with the theoretical result proved in Theorem 2.2.

TABLE 4.3
Spatial discretization error $\|\mathbf{m}_{h,\tau}^N - \mathbf{m}_{h/2,\tau}^N\|_{L^2}$ *with* Q_1 *element*

$\tau\backslash h$	1/16	1/32	1/64	1/128	convergence rate
1/80	2.946e-5	7.413e-6	1.857e-6	4.644e-7	≈ 2.00
1/160	2.594e-5	6.477e-6	1.619e-6	4.048e-7	≈ 2.00
1/320	2.492e-5	6.194e-6	1.547e-6	3.865e-7	≈ 2.00

TABLE 4.4
Spatial discretization error $\|\mathbf{m}_{h,\tau}^N - \mathbf{m}_{h/2,\tau}^N\|_{L^2}$ *with* Q_2 *element*

$\tau \backslash h$	1/16	1/32	1/64	1/128	convergence rate
1/80	1.874e-7	1.827e-8	2.082e-9	2.535e-10	≈ 3.04
1/160	1.377e-7	1.497e-8	1.792e-9	2.214e-10	≈ 3.02
1/320	1.178e-7	1.357e-8	1.658e-9	2.060e-10	≈ 3.01

Triangular Mesh: The temporal and spatial discretization errors on triangular meshes are presented in Tables 4.5–4.7 and Tables 4.8–4.10, respectively, for $\widetilde{P}_r$ elements with $r = 1, 2, 3$ (the specific definitions are in [18]). First-order convergence in time and $(r+1)$th-order convergence in space are observed numerically. The spatial convergence is one order higher than our theoretical result for triangular mesh.

TABLE 4.5
Temporal discretization error $\|\mathbf{m}_{h,\tau}^N - \mathbf{m}_{h,\tau/2}^N\|_{L^2}$ *with* $\widetilde{P}_1$ *element*

$h \backslash \tau$	1/20	1/40	1/80	1/160	convergence rate
1/32	7.373e-4	3.259e-4	1.504e-4	7.190e-5	≈ 1.06
1/64	7.842e-4	3.523e-4	1.645e-4	7.920e-5	≈ 1.05
1/128	7.585e-4	3.377e-4	1.567e-4	7.517e-5	≈ 1.06

TABLE 4.6
Temporal discretization error $\|\mathbf{m}_{h,\tau}^N - \mathbf{m}_{h,\tau/2}^N\|_{L^2}$ *with* $\widetilde{P}_2$ *element*

$h \backslash \tau$	1/20	1/40	1/80	1/160	convergence rate
1/32	7.170e-4	3.148e-4	1.445e-4	6.882e-5	≈ 1.07
1/64	7.168e-4	3.147e-4	1.445e-4	6.880e-5	≈ 1.07
1/128	7.169e-4	3.148e-4	1.445e-4	6.881e-5	≈ 1.07

TABLE 4.7
Temporal discretization error $\|\mathbf{m}_{h,\tau}^N - \mathbf{m}_{h,\tau/2}^N\|_{L^2}$ *with* $\widetilde{P}_3$ *element*

$h \backslash \tau$	1/20	1/40	1/80	1/160	convergence rate
1/32	7.170e-4	3.148e-4	1.445e-4	6.883e-5	≈ 1.07
1/64	7.170e-4	3.148e-4	1.445e-4	6.883e-5	≈ 1.07
1/128	7.170e-4	3.148e-4	1.445e-4	6.883e-5	≈ 1.07

TABLE 4.8
Spatial discretization error $\|\mathbf{m}_{h,\tau}^N - \mathbf{m}_{h/2,\tau}^N\|_{L^2}$ *with* $\widetilde{P}_1$ *element*

$\tau \backslash h$	1/16	1/32	1/64	1/128	convergence rate
1/80	5.193e-5	1.299e-5	3.249e-6	8.123e-7	≈ 2.00
1/160	4.995e-5	1.248e-5	3.120e-6	7.798e-7	≈ 2.00
1/320	4.977e-5	1.242e-5	3.103e-6	7.757e-7	≈ 2.00

5. Conclusions.

We have proved the optimal-order convergence of a linearly implicit lumped mass method, with renormalization at the finite element nodes at every time level, for the equations describing heat flow of harmonic maps. The proof is based on a geometric relation (3.26) (that has not been previously used in the literature) between the errors of the auxiliary and renormalized numerical solutions. The error of the numerical solution is

Table 4.9
Spatial discretization error $\|\mathbf{m}_{h,\tau}^N - \mathbf{m}_{h/2,\tau}^N\|_{L^2}$ *with* $\widetilde{P}_2$ *element*

$\tau\backslash h$	1/8	1/16	1/32	1/64	convergence rate
1/80	1.718e-6	1.067e-7	1.334e-8	1.668e-9	≈ 3.00
1/160	1.699e-6	8.989e-8	1.109e-8	1.382e-9	≈ 3.00
1/320	1.734e-6	8.862e-8	1.073e-8	1.330e-9	≈ 3.01

Table 4.10
Spatial discretization error $\|\mathbf{m}_{h,\tau}^N - \mathbf{m}_{h/2,\tau}^N\|_{L^2}$ *with* $\widetilde{P}_3$ *element*

$\tau\backslash h$	1/8	1/16	1/32	1/64	convergence rate
1/80	1.465e-7	1.011e-9	5.835e-11	3.622e-12	≈ 4.00
1/160	1.208e-7	8.423e-10	4.833e-11	2.998e-12	≈ 4.01
1/320	1.099e-7	7.984e-10	4.637e-11	2.879e-12	≈ 4.00

shown to be $O(\tau + h^{r+1})$ when the tensor-product Q_r elements on rectangular mesh is used, where τ and h are the time stepsize and spatial mesh size, respectively. Since the geometric relation holds only for $\mathbf{e}_h^n = \mathbf{m}_h^n - I_h\mathbf{m}(t_n)$ (instead of $\mathbf{m}_h^n - R_h\mathbf{m}(t_n)$ where R_h is the Ritz projection operator), the optimal-order convergence in space is proved by utilizing the superconvergence result of the Lagrange interpolation operator in Lemma 3.7 (instead of using the Ritz projection operator R_h).

The error analysis in this paper can be extended to triangular mesh straightforwardly, by using the lumped mass FEM on triangular mesh constructed in [18] (with finite element space $\mathbf{S}_h^r = (\widetilde{V}_h^r)^d$, where $\widetilde{V}_h^r$ is defined in [18, Section 5]). In this case, the quadrature error bound in (3.15) should be replaced by the following result (cf. [18, Lemma 5.2 with $q = 1$ and $p = k - 1$]):

$$|\mathcal{E}_1(\mathbf{v}_h)| \leqslant Ch^r\|\mathbf{v}_h\|_{H^1}. \tag{5.1}$$

Moreover, since the superconvergence result in Lemma 3.7 does not hold for triangular mesh, the estimate (3.12) should be replaced by the following standard result:

$$\big|\big(\nabla(\mathbf{u} - I_h\mathbf{u}), \nabla\mathbf{v}_h\big)\big| \leqslant Ch^r\|\mathbf{v}_h\|_{H^1}. \tag{5.2}$$

With these changes, the error analysis in this article would yield the following error bound under the stepsize restriction $\tau \geqslant h^r$:

$$\max_{1\leqslant n\leqslant N}(\|\mathbf{m}_h^n - \mathbf{m}(\cdot,t_n)\|_{L^2} + \|\widetilde{\mathbf{m}}_h^n - \mathbf{m}(\cdot,t_n)\|_{L^2}) \leqslant C(\tau + h^r) \quad \text{for } r \geqslant 2, \tag{5.3}$$

where the condition $r \geqslant 2$ is required in (3.56)–(3.57) (in which h^{r+1} should be replaced by h^r). Hence, the result for triangular mesh is one-order lower than the result for rectangular mesh.

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Appendix A: Proof of Lemma 3.5. To prove (3.8), we define $\mathbf{v} \in \mathbf{H}^2$ to be the solution of the following problem

$$\begin{cases} \Delta\mathbf{v} = \Delta_h\mathbf{v}_h & \text{in } \Omega, \\ \partial_\nu\mathbf{v} = \mathbf{0} & \text{on } \partial\Omega, \end{cases} \tag{A.1}$$

for any given $\mathbf{v}_h \in \mathbf{S}_h^r$, with $\int_\Omega \mathbf{v}\mathrm{d}V = \mathbf{0}$. Then, we split the bound into three parts:

$$\|\mathbf{v}_h\|_{L^\infty} \leqslant \Big\|\mathbf{v}_h - \frac{1}{|\Omega|}\int_\Omega \mathbf{v}_h\mathrm{d}V - \mathbf{v}\Big\|_{L^\infty} + \|\mathbf{v}\|_{L^\infty} + \Big|\frac{1}{|\Omega|}\int_\Omega \mathbf{v}_h\mathrm{d}V\Big|. \tag{A.2}$$

In the following, we analyze the right-hand side of (A.2).

First, we consider the estimate of the second term on the right-hand side of (A.2). The standard H^2-regularity estimate of problem (A.1) implies

$$\|\mathbf{v}\|_{H^2} \leqslant C\|\Delta_h \mathbf{v}_h\|_{L^2}. \tag{A.3}$$

By the definition of Δ_h in (3.10), we have

$$(\Delta_h \mathbf{v}_h, \mathbf{w}_h) = -(\nabla \mathbf{v}_h, \nabla \mathbf{w}_h) \leqslant Ch^{-2}\|\mathbf{v}_h\|_{L^2}\|\mathbf{w}_h\|_{L^2}$$

for $\mathbf{w}_h \in \mathbf{S}_h^r$, which implies

$$\|\Delta_h \mathbf{v}_h\|_{L^2} \leqslant Ch^{-2}\|\mathbf{v}_h\|_{L^2}. \tag{A.4}$$

Then, assuming $\boldsymbol{\psi} \in \mathbf{H}^2$ to be the solution of the problem

$$\begin{cases} \Delta\boldsymbol{\psi} = \mathbf{v} & \text{in } \Omega, \\ \partial_\nu \boldsymbol{\psi} = \mathbf{0} & \text{on } \partial\Omega, \end{cases}$$

with $\int_\Omega \boldsymbol{\psi} \mathrm{d}V = \mathbf{0}$, it follows from (A.4) and integration by parts that

$$\begin{aligned} (\mathbf{v}, \mathbf{v}) &= (\Delta\psi, \mathbf{v}) = (\psi, \Delta\mathbf{v}) = (\psi, \Delta_h \mathbf{v}_h) \\ &= (R_h\psi, \Delta_h \mathbf{v}_h) + (\psi - R_h\psi, \Delta_h \mathbf{v}_h) \\ &\leqslant -(\nabla R_h\psi, \nabla \mathbf{v}_h) + \|\psi - R_h\psi\|_{L^2}\|\Delta_h \mathbf{v}_h\|_{L^2} \\ &\leqslant -(\nabla\psi, \nabla\mathbf{v}_h) + Ch^2\|\psi\|_{H^2}\|\Delta_h \mathbf{v}_h\|_{L^2} \quad \text{(here (A.4) is used)} \\ &\leqslant (\Delta\psi, \mathbf{v}_h) + C\|\psi\|_{H^2}\|\mathbf{v}_h\|_{L^2} \\ &\leqslant C\|\psi\|_{H^2}\|\mathbf{v}_h\|_{L^2} \\ &\leqslant C\|\mathbf{v}\|_{L^2}\|\mathbf{v}_h\|_{L^2}, \end{aligned}$$

where $R_h\psi$ denotes the Ritz projection defined in (3.3). As a result, we have

$$\|\mathbf{v}\|_{L^2} \leqslant C\|\mathbf{v}_h\|_{L^2}, \tag{A.5}$$

which together with the Sobolev interpolation inequality [1, Theorem 5.9] and (A.3) yields

$$\|\mathbf{v}\|_{L^\infty} \leqslant C\|\mathbf{v}\|_{L^2}^{1-\frac{d}{4}}\|\mathbf{v}\|_{H^2}^{\frac{d}{4}} \leqslant C\|\mathbf{v}_h\|_{L^2}^{1-\frac{d}{4}}\|\Delta_h \mathbf{v}_h\|_{L^2}^{\frac{d}{4}}. \tag{A.6}$$

Next, we consider the estimate of the first term on the right-hand side of (A.2). By (A.1), there holds

$$\begin{aligned} \big(\nabla(\mathbf{v}_h - \mathbf{v}), \nabla(\mathbf{v}_h - \mathbf{v})\big) = \big(\nabla(\mathbf{v}_h - \mathbf{v}), \nabla\mathbf{v}\big) &= \big(\nabla(\mathbf{v}_h - \mathbf{v}), \nabla(I_h\mathbf{v} - \mathbf{v})\big) \\ &\leqslant \|\nabla(\mathbf{v}_h - \mathbf{v})\|_{L^2}\|\nabla(I_h\mathbf{v} - \mathbf{v})\|_{L^2}, \end{aligned}$$

which leads to

$$\|\nabla(\mathbf{v}_h - \mathbf{v})\|_{L^2} \leqslant \|\nabla(I_h\mathbf{v} - \mathbf{v})\|_{L^2} \leqslant Ch\|\mathbf{v}\|_{H^2}. \tag{A.7}$$

Let $\mathbf{g} := \mathbf{v}_h - \frac{1}{|\Omega|}\int_\Omega \mathbf{v}_h \mathrm{d}V - \mathbf{v}$ so that $\int_\Omega \mathbf{g}\mathrm{d}V = 0$, and define $\boldsymbol{\varphi}_g$ to be the solution of

$$\begin{cases} \Delta\boldsymbol{\varphi}_g = \mathbf{g} & \text{in } \Omega, \\ \partial_\nu \boldsymbol{\varphi}_g = \mathbf{0} & \text{on } \partial\Omega, \end{cases}$$

with $\int_\Omega \boldsymbol{\varphi}_g \mathrm{d}V = \mathbf{0}$. Then, we have

$$\begin{aligned} \|\mathbf{g}\|_{L^2}^2 = -(\nabla\mathbf{g}, \nabla\boldsymbol{\varphi}_g) = -(\nabla(\mathbf{v}_h - \mathbf{v}), \nabla\boldsymbol{\varphi}_g) &= -(\nabla(\mathbf{v}_h - \mathbf{v}), \nabla(\boldsymbol{\varphi}_g - I_h\boldsymbol{\varphi}_g)) \\ &\leqslant \|\nabla(\mathbf{v}_h - \mathbf{v})\|_{L^2}\|\nabla(\boldsymbol{\varphi}_g - I_h\boldsymbol{\varphi}_g)\|_{L^2} \\ &\leqslant Ch^2\|\mathbf{v}\|_{H^2}\|\boldsymbol{\varphi}_g\|_{H^2} \\ &\leqslant Ch^2\|\mathbf{v}\|_{H^2}\|\mathbf{g}\|_{L^2}, \end{aligned}$$

which implies

$$\left\|\mathbf{v}_h - \frac{1}{|\Omega|}\int_\Omega \mathbf{v}_h \mathrm{d}V - \mathbf{v}\right\|_{L^2} \leqslant Ch^2\|\mathbf{v}\|_{H^2}.$$

Thus, we bound the first term on the right-hand side of (A.2) by

$$
\begin{aligned}
&\Big\|\mathbf{v}_h - \frac{1}{|\Omega|}\int_\Omega \mathbf{v}_h \mathrm{d}V - \mathbf{v}\Big\|_{L^\infty} \\
&\leqslant \Big\|\mathbf{v}_h - \frac{1}{|\Omega|}\int_\Omega \mathbf{v}_h \mathrm{d}V - I_h\mathbf{v}\Big\|_{L^\infty} + \|I_h\mathbf{v} - \mathbf{v}\|_{L^\infty} \\
&\leqslant Ch^{-\frac{d}{2}}\Big\|\mathbf{v}_h - \frac{1}{|\Omega|}\int_\Omega \mathbf{v}_h \mathrm{d}V - I_h\mathbf{v}\Big\|_{L^2} + Ch^{2-\frac{d}{2}}\|\mathbf{v}\|_{H^2} \quad \text{(here Lemma 3.1 is used)} \\
&\leqslant Ch^{-\frac{d}{2}}\Big\|\mathbf{v}_h - \frac{1}{|\Omega|}\int_\Omega \mathbf{v}_h \mathrm{d}V - \mathbf{v}\Big\|_{L^2} + Ch^{-\frac{d}{2}}\|\mathbf{v} - I_h\mathbf{v}\|_{L^2} + Ch^{2-\frac{d}{2}}\|\mathbf{v}\|_{H^2} \\
&\leqslant Ch^{2-\frac{d}{2}}\|\mathbf{v}\|_{H^2} \\
&\leqslant Ch^{2-\frac{d}{2}}\|\varDelta_h\mathbf{v}_h\|_{L^2}, \qquad \text{(A.8)}
\end{aligned}
$$

where (A.3) is used in the last inequality.

Substituting (A.6) and (A.8) into (A.2) results in

$$
\begin{aligned}
\|\mathbf{v}_h\|_{L^\infty} &\leqslant Ch^{2-\frac{d}{2}}\|\varDelta_h\mathbf{v}_h\|_{L^2}^{1-\frac{d}{4}}\|\varDelta_h\mathbf{v}_h\|_{L^2}^{\frac{d}{4}} + C\|\mathbf{v}_h\|_{L^2}^{1-\frac{d}{4}}\|\varDelta_h\mathbf{v}_h\|_{L^2}^{\frac{d}{4}} + \|\mathbf{v}_h\|_{L^2} \\
&\leqslant C\|\mathbf{v}_h\|_{L^2}^{1-\frac{d}{4}}\|\varDelta_h\mathbf{v}_h\|_{L^2}^{\frac{d}{4}} + \|\mathbf{v}_h\|_{L^2}^{1-\frac{d}{4}}\|\mathbf{v}_h\|_{L^2}^{\frac{d}{4}} \\
&\leqslant C\|\mathbf{v}_h\|_{L^2}^{1-\frac{d}{4}}(\|\mathbf{v}_h\|_{L^2} + \|\varDelta_h\mathbf{v}_h\|_{L^2})^{\frac{d}{4}},
\end{aligned}
$$

where we have used (A.4) in the second to last inequality.

Furthermore, by the inverse inequality (3.7) and (A.7), we get

$$
\begin{aligned}
\|\nabla\mathbf{v}_h\|_{L^6} &\leqslant \|\nabla(I_h\mathbf{v} - \mathbf{v}_h)\|_{L^6} + \|\nabla I_h\mathbf{v}\|_{L^6} \\
&\leqslant Ch^{-\frac{d}{3}}\|\nabla(I_h\mathbf{v} - \mathbf{v}_h)\|_{L^2} + C\|\mathbf{v}\|_{H^2} \\
&\leqslant Ch^{-\frac{d}{3}}\|\nabla(I_h\mathbf{v} - \mathbf{v})\|_{L^2} + Ch^{-\frac{d}{3}}\|\nabla(\mathbf{v} - \mathbf{v}_h)\|_{L^2} + C\|\mathbf{v}\|_{H^2} \\
&\leqslant Ch^{1-\frac{d}{3}}\|\mathbf{v}\|_{H^2} + C\|\mathbf{v}\|_{H^2} \\
&\leqslant C\|\varDelta_h\mathbf{v}_h\|_{L^2},
\end{aligned}
$$

where (A.3) is used in the last inequality. The proof of Lemma 3.5 is complete. ∎

Appendix B: Proof of Lemma 3.6. In the case $d = 1$ there holds

$$
\Big|(1, I_h f - f)_K\Big| \leqslant C\|f\|_{L^\infty(K)} \leqslant C\|f\|_{W^{2r,1}(K)} \quad \forall\, f \in W^{2r,1}(K).
$$

Hence, the functional $F : W^{2r,1}(K) \to \mathbb{R}$ defined by $F(f) := (1, I_h f - f)_K$ satisfies the condition (i) in Lemma 3.3. We further note that the $(r+1)$-point Gaussian–Lobatto quadrature is exact for polynomials of degree not larger than $2r-1$, i.e., $F(f) = 0$ for all f that are polynomials of degree less than or equal to $2r-1$. As a result, F satisfies the condition (ii) in Lemma 3.3 for $k = 2r-1$. As a result of Lemma 3.3, the following inequality holds:

$$
\big|(1, I_h f - f)_K\big| \leqslant Ch^{2r}\|\partial_x^{2r} f\|_{L^1(K)} \quad \forall\, f \in W^{2r,1}(K).
$$

This proves the desired inequality (3.11) in the case $d = 1$.

In the case $d = 3$, it follows from (2.1) that

$$
\begin{aligned}
&\big|(1, I_h\mathbf{f} - \mathbf{f})_K\big| \\
&= \big|(1, I_{h_x}I_{h_y}I_{h_z}\mathbf{f} - I_{h_y}I_{h_z}\mathbf{f})_K\big| + \big|(1, I_{h_y}I_{h_z}\mathbf{f} - I_{h_z}\mathbf{f})_K\big| + \big|(1, I_{h_z}\mathbf{f} - \mathbf{f})_K\big| \\
&\leqslant Ch^{2r}\|\partial_x^{2r}(I_{h_y}I_{h_z}\mathbf{f})\|_{L^1(K)} + Ch^{2r}\|\partial_y^{2r}(I_{h_z}\mathbf{f})\|_{L^1(K)} + Ch^{2r}\|\partial_z^{2r}\mathbf{f}\|_{L^1(K)} \\
&\leqslant Ch^{2r+d}\|\partial_x^{2r}(I_{h_y}I_{h_z}\mathbf{f})\|_{L^\infty(K)} + Ch^{2r+d}\|\partial_y^{2r}(I_{h_z}\mathbf{f})\|_{L^\infty(K)} + Ch^{2r+d}\|\partial_z^{2r}\mathbf{f}\|_{L^\infty(K)} \\
&= Ch^{2r+d}\|I_{h_y}I_{h_z}\partial_x^{2r}\mathbf{f}\|_{L^\infty(K)} + Ch^{2r+d}\|I_{h_z}\partial_y^{2r}\mathbf{f}\|_{L^\infty(K)} + Ch^{2r+d}\|\partial_z^{2r}\mathbf{f}\|_{L^\infty(K)} \\
&\leqslant Ch^{2r+d}\|\partial_x^{2r}\mathbf{f}\|_{L^\infty(K)} + Ch^{2r+d}\|\partial_y^{2r}\mathbf{f}\|_{L^\infty(K)} + Ch^{2r+d}\|\partial_z^{2r}\mathbf{f}\|_{L^\infty(K)}.
\end{aligned}
$$

If $\mathbf{f} \in \mathbf{V}_h$, then (3.11) follows from the inequality above and the inverse inequality (3.6). The proof for the case $d = 2$ is similar and thus omitted. □

Appendix C: Proof of Lemma 3.7. We first consider the case $d = 1$. Let $[a, b] := \cup_{j=1}^{J} \mathcal{I}_j$. Since $u - I_h u = 0$ at the two end points of each subinterval $\mathcal{I}_j$, by using integration by parts on each subinterval $\mathcal{I}_j$ we obtain

$$\begin{aligned}\Big|\Big(\partial_x(u - I_h u), \partial_x v_h\Big)\Big| &= \Big|\sum_{j=1}^{J}\Big(\partial_x(u - I_h u), \partial_x v_h\Big)_{\mathcal{I}_j}\Big| = \Big|\sum_{j=1}^{J}\Big(u - I_h u, \partial_{xx} v_h\Big)_{\mathcal{I}_j}\Big| \\ &= \Big|\sum_{j=1}^{J}\Big(u, \partial_{xx} v_h\Big)_{\mathcal{I}_j} - \Big(I_h u, \partial_{xx} v_h\Big)_{\mathcal{I}_j}\Big|.\end{aligned}$$

Due to the fact that $I_h u \cdot \partial_{xx} v_h$ on each subinterval $\mathcal{I}_j$ is a polynomial of degree not higher than $2r - 2$; hence $(I_h u, \partial_{xx} v_h)_{\mathcal{I}_j}$ equals the Gauss–Lobatto quadrature for $u \cdot \partial_{xx} v_h$ on $\mathcal{I}_j$. Employing Lemma 3.6, it follows that

$$\begin{aligned}\Big|\Big(\partial_x(u - I_h u), \partial_x v_h\Big)\Big| &\leqslant \sum_{j=1}^{J} C h^{2r}\|u\partial_{xx} v_h\|_{W^{2r,1}(\mathcal{I}_j)} \leqslant \sum_{j=1}^{J} C h^{2r}\|u\|_{H^{2r}(\mathcal{I}_j)}\|v_h\|_{H^r(\mathcal{I}_j)} \\ &\leqslant C h^{2r}\|u\|_{H^{2r}(a,b)}\|v_h\|_{H^r(a,b)} \\ &\leqslant C h^{r+1}\|v_h\|_{H^1(a,b)}.\end{aligned}$$

When $d > 1$, for example $d = 2$, we have

$$\begin{aligned}\Big|\Big(\nabla(\mathbf{u} - I_h\mathbf{u}), \nabla\mathbf{v}_h\Big)\Big| &\leqslant \Big|\Big(\partial_x(\mathbf{u} - I_h\mathbf{u}), \partial_x\mathbf{v}_h\Big)\Big| + \Big|\Big(\partial_y(\mathbf{u} - I_h\mathbf{u}), \partial_y\mathbf{v}_h\Big)\Big| \\ &\leqslant \int_{a_y}^{b_y} C h^{r+1}\|\mathbf{v}_h\|_{H^1_x}\mathrm{d}y + \int_{a_x}^{b_x} C h^{r+1}\|\mathbf{v}_h\|_{H^1_y}\mathrm{d}x,\end{aligned}$$

where

$$\begin{aligned}\int_{a_y}^{b_y} C h^{r+1}\|\mathbf{v}_h\|_{H^1_x}\mathrm{d}y &= C h^{r+1}\int_{a_y}^{b_y}\Big(\int_{a_x}^{b_x}(|\partial_x\mathbf{v}_h|^2 + |\mathbf{v}_h|^2)\mathrm{d}x\Big)^{\frac{1}{2}}\mathrm{d}y \\ &\leqslant C h^{r+1}\Big(\int_{a_y}^{b_y}\int_{a_x}^{b_x}(|\partial_x\mathbf{v}_h|^2 + |\mathbf{v}_h|^2)\mathrm{d}x\mathrm{d}y\Big)^{\frac{1}{2}} \leqslant C h^{r+1}\|\mathbf{v}_h\|_{H^1},\end{aligned}$$

and similarly

$$\int_{a_x}^{b_x} C h^{r+1}\|\mathbf{v}_h\|_{H^1_y}\mathrm{d}x \leqslant C h^{r+1}\|\mathbf{v}_h\|_{H^1}.$$

The above results yield (3.12) immediately. When $d = 3$, the estimate (3.12) can be also proved by similar analysis. The proof of Lemma 3.7 is complete. □

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