WEAK DISCRETE MAXIMUM PRINCIPLE OF FINITE ELEMENT METHODS IN CONVEX POLYHEDRA

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Abstract. We prove that the Galerkin finite element solution u_h of the Laplace equation in a convex polyhedron Ω, with a quasi-uniform tetrahedral partition of the domain and with finite elements of polynomial degree $r \geqslant 1$, satisfies the following weak maximum principle:

$$\|u_h\|_{L^\infty(\Omega)} \leqslant C\,\|u_h\|_{L^\infty(\partial\Omega)}\,,$$

with a constant C independent of the mesh size h. By using this result, we show that the Ritz projection operator R_h is stable in L^∞ norm uniformly in h for $r \geq 2$, i.e.

$$\|R_h u\|_{L^\infty(\Omega)} \leqslant C\|u\|_{L^\infty(\Omega)}.$$

Thus we remove a logarithmic factor appearing in the previous results for convex polyhedral domains.

1. Introduction

Let S_h be a finite element space of Lagrange elements of degree $r \geqslant 1$ subject to a quasi-uniform tetrahedral partition $\mathfrak{T}$ of a convex polyhedron $\Omega \subset \mathbb{R}^3$, where h denotes the mesh size of the tetrahedral partition, and quasi-uniformity means that

$$\rho_\tau \geqslant ch \quad \forall\, \tau \in \mathfrak{T},$$

with ρ_τ denoting the radius of the largest ball inscribed in the tetrahedron $\tau \in \mathfrak{T}$.

Let $\mathring{S}_h$ be the subspace of S_h consisting of functions with zero boundary values. A function $u_h \in S_h$ is called a discrete harmonic if it satisfies

$$(\nabla u_h, \nabla \chi_h) = 0 \quad \forall\, \chi_h \in \mathring{S}_h. \tag{1.1}$$

In this article, we establish the following result, which we call weak maximum principle of finite element methods (for higher order equations it is often called Agmon–Miranda maximum principle).

Theorem 1.1. *A discrete harmonic function u_h satisfies the following estimate:*

$$\|u_h\|_{L^\infty(\Omega)} \leqslant C\,\|u_h\|_{L^\infty(\partial\Omega)}\,, \tag{1.2}$$

where the constant C is independent of the mesh size h.

As an application of the weak maximum principle, we show that the Ritz projection $R_h : H_0^1(\Omega) \to \mathring{S}_h$ defined by

$$(\nabla(u - R_h u), \nabla v_h) = 0 \quad \forall\, v_h \in \mathring{S}_h$$

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is stable in L^∞ norm for finite elements of degree $r \geqslant 2$, i.e.

$$\|R_h u\|_{L^\infty(\Omega)} \leqslant C\|u\|_{L^\infty(\Omega)} \quad \forall\, u \in H_0^1(\Omega) \cap L^\infty(\Omega).$$

Although this result is well-known for smooth domains [27, 29], for convex polyhedral domains the result was available only with an additional logarithmic factor [20, Theorem 12].

In the finite element literature, the "strict" discrete maximum principle

$$\|u_h\|_{L^\infty(\Omega)} \leqslant \|u_h\|_{L^\infty(\partial\Omega)}$$

i.e., with $C = 1$ in (1.2), has attracted a lot of attention; see [7, 8, 25, 31, 32], to mention a few. However, the sufficient conditions for the strict discrete maximum principle often put serious restrictions on the geometry of the mesh. For piecewise linear elements in two-dimensions, the strict discrete maximum principle generally requires the angles of the triangles to be less than $\pi/2$, or the sum of opposite angles of the triangles that share an edge to be less than π (for example, see [32, §5]), though these conditions are not necessary away from the boundary [10]. For quadratic elements in two dimensions, discrete maximum principe holds only for equilateral triangles [15]. The situation in three dimensions is more complicated [4, 18, 19, 33], essentially it is hard to guarantee the discrete maximum principe even for piecewise linear elements.

A different approach was taken in the work of Schatz [26], who proved that a weak maximum principle in the sense of (1.2) holds for a wide class of finite elements on general quasi-uniform triangulation of any two dimensional polygonal domain. The weak maximum principle was used to establish the stability of the Ritz projection in L^∞ and $W^{1,\infty}$ norms for two-dimensional polygons. Such L^∞- and $W^{1,\infty}$-stability results have a wide range of applications, for example to pointwise error estimates of finite element methods for parabolic problems [17, 21, 22], Stokes systems [3], nonlinear problems [11, 12, 23], obstacle problems [6], optimal control problems [1, 2], to name a few. As far as we know, [26] is the only paper that establishes weak maximum principle and L^∞ stability estimate (without the logarithmic factor) for the Ritz projection on nonsmooth domains.

In three dimensions the situation is less satisfactory. The stability of the Ritz projection in L^∞ and $W^{1,\infty}$ norms is available on smooth domains [27, 29] and convex polyhedral domains [14, 20]. However, on convex polyhedral domains in [20], the L^∞-stability constant depends logarithmically on the mesh size h, and it is not obvious how the logarithmic factor can be removed there. There are no results on the weak maximum principles in three dimensions even on smooth domains or convex polyhedra. The objective of this paper is to close this gap for convex polyhedral domains. In order to obtain the result, we have to modify the argument in [26] by extending the arguments to L^p norm for some $1 < p < 2$. This constitutes the main technical difficulty in the analysis of the paper. The mere adaptation of the L^2-norm based argument used in [26] for convex polyhedral domains, would yield a logarithmic factor. Unfortunately, the current analysis does not allow us to extend the results to nonconvex polyhedral domains or graded meshes. These would be the subject of future research.

The paper is organized as follows. In section 2 we state some preliminary results that we use later in our arguments. In section 3, we reduce the proof of the weak discrete maximum principle to a specific error estimate. Section 4 is devoted to the proof of this estimate, which constitutes the main technical part of the paper.

Finally, section 5, gives an application of the weak discrete maximum principle to show the stability of the Ritz projection in L^∞ norm uniform in h for higher order elements.

In the rest of this article, we denote by C a generic positive constant, which may be different at different occurrences but will be independent of the mesh size h.

2. Preliminary results

In this section, we present several well-known results that are used in our analysis. First result concerns global regularity of the weak solution $v \in H_0^1(\Omega)$ to the problem

$$(\nabla v, \nabla \chi) = (f, \chi) \quad \forall\, \chi \in H_0^1(\Omega). \tag{2.3}$$

On the general convex domains we naturally have the H^2 regularity (cf. [13]). However, on convex polyhedral domains, we have the following sharper $W^{2,p}(\Omega)$ regularity result (cf. [9, Corollary 3.12]).

Lemma 2.1. *Let Ω be a convex polyhedron. Then there exists a constant $p_0 > 2$ depending on Ω such that for any $1 < p < p_0$ and $f \in L^p(\Omega)$, the solution v of (2.3) is in $W^{2,p}(\Omega)$ and*

$$\|v\|_{W^{2,p}(\Omega)} \leqslant C\|f\|_{L^p(\Omega)}.$$

For any point $x^* \in \overline{\Omega}$ we denote $S_d(x^*) = \{x \in \Omega : |x - x^*| < d\}$. The following result, which is a version of the Poincare inequality, is an extension of Lemma 1.1 in [26], which was established in two dimensions for $p = 2$.

Lemma 2.2. *Let $1 < p < \infty$. If $\chi \in W_0^{1,p}(\Omega)$ and $x^* \in \partial\Omega$, then*

$$\|\chi\|_{L^p(S_{d_*}(x^*))} \leqslant C d_* \|\nabla \chi\|_{L^p(\Omega)}.$$

Proof. Similarly to [26, Lemma 1.1], we consider $\chi \in C_0^\infty(\Omega)$ and extend χ by zero outside Ω. By denoting $x^* = (x_1^*, x_2^*, x_3^*)$ and using the spherical coordinates centered at x^*, we define

$$\widetilde{\chi}(\rho, \varphi, \theta) = \chi(x_1^* + \rho\sin(\varphi)\cos(\theta), x_2^* + \rho\sin(\varphi)\sin(\theta), x_3^* + \rho\cos(\varphi)),$$

for $0 \leqslant \rho \leqslant d_*$, and some $\varphi \in [0, \pi]$ and $\theta \in [0, 2\pi]$. Since $\chi = 0$ on $\partial\Omega$, there exists $\theta^* \in [0, 2\pi]$ such that $\widetilde{\chi}(\rho, \varphi, \theta^*) = 0$. Therefore,

$$|\widetilde{\chi}(\rho, \varphi, \theta)| = \left| \int_{\theta^*}^{\theta} \partial_\theta \widetilde{\chi}(\rho, \varphi, \theta')\,\mathrm{d}\theta' \right| \leqslant \int_0^{2\pi} |\partial_\theta \widetilde{\chi}(\rho, \varphi, \theta')|\,\mathrm{d}\theta'.$$

From the chain rule, we have

$$\partial_\theta \widetilde{\chi}(\rho, \varphi, \theta) = -\partial_{x_1} \widetilde{\chi}(\rho, \varphi, \theta)\,\rho\sin(\varphi)\sin(\theta) + \partial_{x_2} \widetilde{\chi}(\rho, \varphi, \theta)\,\rho\sin(\varphi)\cos(\theta).$$

As a result, by Hölder's inequality, we obtain

$$|\widetilde{\chi}(\rho, \varphi, \theta)|^p \leqslant C \int_0^{2\pi} |\partial_\theta \widetilde{\chi}(\rho, \varphi, \theta')|^p\,\mathrm{d}\theta' \leqslant C \int_0^{2\pi} \rho^p |\nabla \widetilde{\chi}(\rho, \varphi, \theta')|^p\,\mathrm{d}\theta'.$$

Therefore,

$$\begin{aligned}
\int_{S_{d_*}(x^*)} |\chi(x)|^p\,\mathrm{d}x &= \int_0^{d_*} \int_0^{\pi} \int_0^{2\pi} |\widetilde{\chi}(\rho, \varphi, \theta)|^p \rho^2 \sin(\varphi)\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}\rho \\
&\leqslant C \int_0^{d_*} \int_0^{\pi} \int_0^{2\pi} \left(\int_0^{2\pi} \rho^p |\nabla \widetilde{\chi}(\rho, \varphi, \theta')|^p\,\mathrm{d}\theta' \right) \rho^2 \sin(\varphi)\,\mathrm{d}\theta\,\mathrm{d}\varphi\,\mathrm{d}\rho
\end{aligned}$$

$$\leqslant C d_*^p \int_0^{d_*} \int_0^{\pi} \int_0^{2\pi} |\nabla \widetilde{\chi}(\rho,\varphi,\theta')|^p \rho^2 \sin(\varphi)\,\mathrm{d}\theta'\,\mathrm{d}\varphi\,\mathrm{d}\rho$$
$$= C d_*^p \int_{S_{d_*}(x^*)} |\nabla \chi(x)|^p\,\mathrm{d}x.$$

This proves the desired result. □

The next result addresses the problem (2.3) when the source function f is supported in some part of Ω. It establishes the stability of the solution in $W^{1,p}$ norm and traces the dependence of the stability constant on the diameter of the support. The corresponding result in [26] is the equation (1.6) therein, which was established for $p = 2$ in two dimensions. In our situation we need it for larger range of p.

Lemma 2.3. *For any bounded Lipschitz domain Ω, there exist positive constants $\alpha \in (0, \frac{1}{2})$ and C (depending on Ω) such that for $\frac{3}{2} - \alpha \leqslant p \leqslant 3 + \alpha$ and $f \in L^p(\Omega)$ with* $\mathrm{supp}(f) \subset S_{d_*}(x_0)$ *and* $\mathrm{dist}(x_0, \partial\Omega) \leqslant d_*$*, the solution of* (2.3) *satisfies*

$$\|v\|_{W^{1,p}(\Omega)} \leqslant C d_* \|f\|_{L^p(\Omega)}.$$

Proof. If $\mathrm{dist}(x_0, \partial\Omega) \leqslant d_*$, then $S_{d_*}(x_0) \subset S_{2d_*}(\bar{x}_0)$ for some $\bar{x}_0 \in \partial\Omega$. For any $\chi \in W_0^{1,p'}(\Omega)$, there holds

$$\begin{aligned}|(\nabla v, \nabla \chi)| = |(f, \chi)| &\leqslant \|f\|_{L^p(S_{d_*}(x_0))} \|\chi\|_{L^{p'}(S_{d_*}(x_0))} \\ &\leqslant \|f\|_{L^p(S_{d_*}(x_0))} \|\chi\|_{L^{p'}(S_{2d_*}(\bar{x}_0))} \\ &\leqslant C d_* \|f\|_{L^p(\Omega)} \|\nabla \chi\|_{L^{p'}(\Omega)},\end{aligned}$$

where in the last step we have used Lemma 2.2.

For $\vec{w} \in C_0^\infty(\Omega)^3$, we let $\chi \in H^2(\Omega) \cap H_0^1(\Omega) \hookrightarrow W_0^{1,p'}(\Omega)$ (for $\frac{3}{2} - \alpha \leqslant p \leqslant 3 + \alpha$) be the solution of

$$\begin{cases} \Delta \chi = \nabla \cdot \vec{w} & \text{in } \Omega \\ \chi = 0 & \text{on } \partial\Omega. \end{cases}$$

The solution χ defined above satisfies

$$\nabla \cdot (\vec{w} - \nabla \chi) = 0,$$

and, according to [16, Theorem B], there exists a constant $\alpha \in (0, \frac{1}{2})$ such that

$$\|\nabla \chi\|_{L^{p'}(\Omega)} \leqslant C \|\vec{w}\|_{L^{p'}(\Omega)} \quad \text{for } \tfrac{3}{2} - \alpha \leqslant p \leqslant 3 + \alpha.$$

By using these properties, we have

$$|(\nabla v, \vec{w})| = |(\nabla v, \nabla \chi)| \leqslant C d_* \|f\|_{L^p(\Omega)} \|\nabla \chi\|_{L^{p'}(\Omega)} \leqslant C d_* \|f\|_{L^p(\Omega)} \|\vec{w}\|_{L^{p'}(\Omega)}.$$

Since $C_0^\infty(\Omega)^3$ is dense in $L^{p'}(\Omega)^3$ and the estimate above holds for all $\vec{w} \in C_0^\infty(\Omega)^3$, the duality pairing between $L^p(\Omega)^3$ and $L^{p'}(\Omega)^3$ implies the desired result. □

The next lemma concerns basic properties of harmonic functions on convex domains. The result is essentially the same as in [28, Lemma 8.3].

Lemma 2.4. *Let D and D_d be two subdomains satisfying $D \subset D_d \subset \Omega$, with*

$$D_d = \{x \in \Omega : dist(x, D) \leqslant d\},$$

where d is a positive constant. If $v \in H_0^1(\Omega)$ and v is harmonic on D_d, i.e.

$$(\nabla v, \nabla w) = 0, \quad \forall w \in H_0^1(D_d),$$

then the following estimates hold:

$$|v|_{H^2(D)} \leqslant Cd^{-1}\|v\|_{H^1(D_d)}, \tag{2.4a}$$

$$\|v\|_{H^1(D)} \leqslant Cd^{-1}\|v\|_{L_2(D_d)}. \tag{2.4b}$$

Finally, we need the best approximation property of the Ritz projection in $W^{1,p}$ norm. In [14], the best approximation property of the Ritz projection in $W^{1,\infty}$ norm was established on convex polyhedral domains. Together with the standard best approximation property in H^1 norm we obtain

$$\|v - R_h v\|_{W^{1,p}(\Omega)} \leqslant C \min_{\chi\in \mathring{S}_h} \|v-\chi\|_{W^{1,p}(\Omega)} \quad \forall\, v \in H_0^1(\Omega)\cap W^{1,p}(\Omega), \tag{2.5}$$

for any $2 \leqslant p \leqslant \infty$. Extension of the above result to $1 < p \leqslant \infty$ follows by duality (cf. [5, §8.5]). These can be summarized as below.

Lemma 2.5. *On a convex polyhedron Ω, the following estimate holds for any fixed $p \in (1,\infty]$:*

$$\|v - R_h v\|_{W^{1,p}(\Omega)} \leqslant Ch\|v\|_{W^{2,p}(\Omega)} \quad \forall\, v \in H_0^1(\Omega)\cap W^{2,p}(\Omega).$$

In sections 3–4, we would use several results from [24, 26, 27]. Some of these results were stated therein for sufficiently small mesh size h under certain hypothesis on the triangulation. Since, we concentrate on the Lagrange elements, all the hypotheses in [27] are trivially satisfied and we assume that our mesh size h is sufficiently small, say $h \leqslant h_0$ for some constant h_0, so these results hold.

3. Basic estimates

In this section, we derive some estimates we require to establish one of our key results, Theorem 1.1. This part of the argument up to (3.12) is analogous to the first part of the proof of [26, Theorem 1] up to equation (3.10). The dyadic decomposition part is also similar. The essential difference lies in the duality argument in section 4, after the equation (4.21).

In [27, Corollary 5.1], the following interior error estimate was established

$$\|u-u_h\|_{L^\infty(\Omega_1)} \leqslant Ch^l |\ln h|^{\bar r} |u|_{W^{l,\infty}(\Omega_2)} + Cd^{-3/q-p}\|u-u_h\|_{W^{-p,q}(\Omega_2)},$$

for $0 \leqslant l \leqslant r$, where $\bar r = 1$ for $r = 1$, $\bar r = 0$ for $r \geqslant 2$ and $\Omega_1 \subset\subset \Omega_2 \subset\subset \Omega$, with $\mathrm{dist}(\Omega_1, \partial\Omega_2) \geqslant d \geqslant kh$ and $\mathrm{dist}(\Omega_2, \partial\Omega) \geqslant d \geqslant kh$. Choosing $u = 0$, $p = 0$ and $q = 2$ in the above estimate, we obtain that there exists a constant C independent of h such that

$$\|u_h\|_{L^\infty(\Omega_1)} \leqslant Cd^{-\frac{3}{2}}\|u_h\|_{L^2(\Omega_2)}. \tag{3.1}$$

Let $x_0 \in \overline{\Omega}$ be a point satisfying

$$|u_h(x_0)| = \|u_h\|_{L^\infty(\Omega)} \quad \text{with} \quad d = \mathrm{dist}(x_0, \partial\Omega).$$

If $d \geqslant 2kh$ for $k \geqslant 1$, then we can choose $\Omega_1 = \{x_0\}$ and $\Omega_2 = S_{d/2}(x_0)$. In this case, the following interior L^∞ estimate holds (cf. [27, Corollary 5.1] and [26, Lemma 2.1 (ii)]):

$$|u_h(x_0)| \leqslant Cd^{-\frac{3}{2}}\|u_h\|_{L^2(S_d(x_0))}.$$

Otherwise, we have $d \leqslant 2kh$. In this case, the inverse inequality of finite element functions (cf. [5, Ch. 4.5]) implies

$$|u_h(x_0)| = \|u_h\|_{L^\infty(S_h(x_0))} \leqslant Ch^{-\frac{3}{2}}\|u_h\|_{L^2(S_h(x_0))}.$$

Hence, either for $d \geqslant 2kh$ or $d \leqslant 2kh$, the following estimate holds:

$$|u_h(x_0)| \leqslant C\rho^{-\frac{3}{2}}\|u_h\|_{L^2(S_\rho(x_0))}, \quad \text{with} \quad \rho = d + 2kh. \tag{3.2}$$

To estimate the term $\|u_h\|_{L^2(S_\rho(x_0))}$ on the right hand side of the inequality above, we use the following duality property:

$$\|u_h\|_{L^2(S_\rho(x_0))} = \sup_{\substack{\mathrm{supp}(\varphi)\subset S_\rho(x_0) \\ \|\varphi\|_{L^2(S_\rho(x_0))}\leqslant 1}} |(u_h, \varphi)|,$$

which implies the existence of a function $\varphi \in C_0^\infty(\Omega)$ with the following properties:

$$\mathrm{supp}(\varphi) \subset S_\rho(x_0), \quad \|\varphi\|_{L^2(S_\rho(x_0))} \leqslant 1 \tag{3.3}$$

and

$$\|u_h\|_{L^2(S_\rho(x_0))} \leqslant 2|(u_h, \varphi)|. \tag{3.4}$$

For this function φ, we define $v \in H_0^1(\Omega)$ to be the solution of

$$(\nabla v, \nabla \chi) = (\varphi, \chi) \quad \forall\, \chi \in H_0^1(\Omega), \tag{3.5}$$

and let $v_h \in \mathring{S}_h$ be the finite element solution of

$$(\nabla v_h, \nabla \chi_h) = (\varphi, \chi_h) \quad \forall\, \chi_h \in \mathring{S}_h.$$

Thus, v_h is the Ritz projection of v and satisfies

$$(\nabla(v - v_h), \nabla \chi_h) = 0 \quad \forall\, \chi_h \in \mathring{S}_h. \tag{3.6}$$

Let u be the solution of the problem (in weak form)

$$\begin{cases} (\nabla u, \nabla \chi) = 0 & \forall\, \chi \in H_0^1(\Omega), \\ u = u_h & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

Then the continuous maximum principle of (3.7) implies

$$\|u\|_{L^\infty(\Omega)} \leqslant \|u_h\|_{L^\infty(\partial\Omega)}. \tag{3.8}$$

Notice, that u_h is the Ritz projection of u, i.e.

$$\begin{cases} (\nabla(u - u_h), \nabla \chi_h) = 0 & \forall\, \chi_h \in \mathring{S}_h, \\ u - u_h = 0 & \text{on } \partial\Omega. \end{cases}$$

Therefore, we have

$$\begin{aligned} \|u_h\|_{L^2(S_\rho(x_0))} &\leqslant 2|(u_h, \varphi)| && \text{(here we used (3.4))} \\ &= 2|(u_h - u, \varphi) + (u, \varphi)| \\ &= 2|(\nabla(u_h - u), \nabla v) + (u, \varphi)| && \text{(here we used (3.5))} \\ &= 2|(\nabla u_h, \nabla v) + (u, \varphi)| && \text{(here we used (3.7))} \\ &\leqslant 2|(\nabla u_h, \nabla v)| + 2\|u\|_{L^\infty(\Omega)}\|\varphi\|_{L^1(\Omega)} \\ &\leqslant 2|(\nabla u_h, \nabla v)| + C\rho^{\frac{3}{2}}\|u_h\|_{L^\infty(\partial\Omega)}\|\varphi\|_{L^2(S_\rho(x_0))}, \end{aligned} \tag{3.9}$$

where we have used (3.8) and the Hölder inequality in deriving the last inequality.

To estimate $|(\nabla u_h, \nabla v)|$, we note that

$$\begin{aligned} (\nabla u_h, \nabla v) &= (\nabla u_h, \nabla(v - v_h)) && \text{(here we use (1.1) and } v_h \in \mathring{S}_h) \\ &= (\nabla(u_h - \chi_h), \nabla(v - v_h)) && \forall\, \chi_h \in \mathring{S}_h. \quad \text{(here we use (3.6))}. \end{aligned}$$

We simply choose χ_h to be equal to u_h at interior nodes and $\chi_h = 0$ on $\partial\Omega$; thus $u_h(x) - \chi_h(x)$ is zero when $\mathrm{dist}(x, \partial\Omega) \geqslant h$, and for any $r \geq 1$

$$\|u_h - \chi_h\|_{L^\infty(\Omega)} \leqslant C\|u_h\|_{L^\infty(\partial\Omega)}.$$

If we define

$$\Lambda_h = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \leqslant h\},$$

then using the inverse inequality,

$$\begin{aligned} |(\nabla u_h, \nabla v)| &\leqslant \|\nabla(u_h - \chi_h)\|_{L^\infty(\Lambda_h)}\|\nabla(v - v_h)\|_{L^1(\Lambda_h)} \\ &\leqslant Ch^{-1}\|u_h - \chi_h\|_{L^\infty(\Omega)}\|\nabla(v - v_h)\|_{L^1(\Lambda_h)} \\ &\leqslant Ch^{-1}\|u_h\|_{L^\infty(\partial\Omega)}\|\nabla(v - v_h)\|_{L^1(\Lambda_h)}. \end{aligned} \tag{3.10}$$

Then, substituting (3.9) and (3.10) into (3.2), we obtain

$$\|u_h\|_{L^\infty(\Omega)} \leqslant C\big(\rho^{-\frac{3}{2}}h^{-1}\|\nabla(v - v_h)\|_{L^1(\Lambda_h)} + 1\big)\|u_h\|_{L^\infty(\partial\Omega)}. \tag{3.11}$$

The proof of Theorem 1.1 will be completed if we establish

$$\rho^{-\frac{3}{2}}h^{-1}\|\nabla(v - v_h)\|_{L^1(\Lambda_h)} \leqslant C, \tag{3.12}$$

which will be accomplished in the next section.

4. Estimate of $\rho^{-\frac{3}{2}}h^{-1}\|\nabla(v - v_h)\|_{L^1(\Lambda_h)}$

Let $R_0 = \mathrm{diam}(\Omega)$ and $d_j = R_0 2^{-j}$ for $j = 0, 1, 2, \ldots$ We define a sequence of subdomains

$$A_j = \{x \in \Omega : d_{j+1} \leqslant |x - x_0| \leqslant d_j\}, \quad j = 0, 1, 2, \ldots$$

For each j we denote A_j^l to be a subdomain slightly larger than A_j, defined by

$$A_j^l = A_{j-l} \cup \cdots \cup A_j \cup A_{j+1} \cup \cdots \cup A_{j+l} \quad l < j,$$

$$A_j^l = A_0 \cup \cdots \cup A_j \cup A_{j+1} \cup \cdots \cup A_{j+l} \quad l \geqslant j, \quad l = 1, 2, 3, \ldots$$

Let $J = [\ln_2(R_0/2\kappa\rho)] + 1$, with $J = [\ln_2(R_0/2\kappa\rho)]$ denoting the greatest integer not exceeding $\ln_2(R_0/2\kappa\rho)$. The constant $\kappa \geq 1$ will be determined later; see (4.27). Then

$$\frac{1}{2}\kappa\rho \leqslant d_{J+1} \leqslant \kappa\rho$$

and

$$\mathrm{measure}(A_j \cap \Lambda_h) \leq Chd_j^2. \tag{4.13}$$

By using these subdomains defined above, we have

$$\begin{aligned} &\rho^{-\frac{3}{2}}h^{-1}\|\nabla(v - v_h)\|_{L^1(\Lambda_h)} \\ &\leqslant \rho^{-\frac{3}{2}}h^{-1}\left(\sum_{j=0}^{J}\|\nabla(v - v_h)\|_{L^1(\Lambda_h \cap A_j)} + \|\nabla(v - v_h)\|_{L^1(\Lambda_h \cap S_{\kappa\rho}(x_0))}\right) \\ &\leqslant C\rho^{-\frac{3}{2}}h^{-1}\sum_{j=0}^{J} h^{\frac{1}{2}}d_j\|\nabla(v - v_h)\|_{L^2(\Lambda_h \cap A_j)} \\ &\quad + C\kappa\rho^{-\frac{1}{2}}h^{-\frac{1}{2}}\|\nabla(v - v_h)\|_{L^2(\Lambda_h \cap S_{\kappa\rho}(x_0))}, \end{aligned} \tag{4.14}$$

where the Hölder inequality and (4.13) were used in deriving the last inequality.

Using global error estimate in H^1 norm, Lemma 2.1 with $p=2$ and (3.3), we obtain

$$\rho^{-\frac{1}{2}}h^{-\frac{1}{2}}\|\nabla(v-v_h)\|_{L^2(A_h\cap S_{\kappa\rho}(x_0))} \leqslant C\rho^{-\frac{1}{2}}h^{-\frac{1}{2}}h\|v\|_{H^2(\Omega)} \leqslant C\rho^{-\frac{1}{2}}h^{-\frac{1}{2}}h\|\varphi\|_{L^2(\Omega)} \leqslant C,$$

where we have used $\rho \geqslant h$ and $\|\varphi\|_{L^2(\Omega)} \leqslant 1$ in deriving the last inequality. Substituting the last inequality into (4.14) yields

$$\rho^{-\frac{3}{2}}h^{-1}\|\nabla(v-v_h)\|_{L^1(A_h)} \leqslant C\rho^{-\frac{3}{2}}h^{-\frac{1}{2}}\sum_{j=0}^{J} d_j\|\nabla(v-v_h)\|_{L^2(A_j)} + C\kappa. \tag{4.15}$$

Now, for $0 \leqslant j \leqslant J$, we use the following interior energy error estimate (proved in [24, Theorem 5.1], also see [26, Lemma 2.1 (i)]):

$$\|\nabla(v-v_h)\|_{L^2(A_j)} \leqslant C\|\nabla(v-I_hv)\|_{L^2(A_j^1)} + Cd_j^{-1}\|v-I_hv\|_{L^2(A_j^1)} + Cd_j^{-1}\|v-v_h\|_{L^2(A_j^1)}, \tag{4.16}$$

where I_h denotes the nodal interpolant. Using the approximation theory, we obtain

$$\begin{aligned}\|\nabla(v-v_h)\|_{L^2(A_j)} &\leqslant \left(Ch + Ch^2 d_j^{-1}\right)\|v\|_{H^2(A_j^1)} + Cd_j^{-1}\|v-v_h\|_{L^2(A_j^1)} \\ &\leqslant Chd_j^{\frac{1}{2}-\frac{3}{p}}\|v\|_{W^{1,p}(A_j^3)} + Cd_j^{-1}\|v-v_h\|_{L^2(A_j^1)} \quad \text{for } \tfrac{6}{5}<p<2,\end{aligned} \tag{4.17}$$

where we have used $d_j \geqslant h$ for $0 \leqslant j \leqslant J$ and the following inequality in deriving the last inequality:

$$\|v\|_{H^2(A_j^1)} \leqslant Cd_j^{\frac{1}{2}-\frac{3}{p}}\|v\|_{W^{1,p}(A_j^3)} \quad \text{for } \tfrac{6}{5}<p<2. \tag{4.18}$$

The inequality above follows from Lemma 2.4, the Hölder inequality and Sobolev embedding, i.e.,

$$\begin{aligned}\|v\|_{H^2(A_j^1)} &\leqslant Cd_j^{-2}\|v\|_{L^2(A_j^3)} \\ &\leqslant Cd_j^{-2+\frac{3}{2}-\frac{3}{q}}\|v\|_{L^q(A_j^3)} \quad \text{if } q>2 \\ &\leqslant Cd_j^{\frac{1}{2}-\frac{3}{p}}\|v\|_{W^{1,p}(A_j^3)} \quad \text{for } \tfrac{3}{q}=\tfrac{3}{p}-1 \text{ and } \tfrac{6}{5}<p<2 \text{ (so that } q>2).\end{aligned}$$

Here we need the constant $\kappa > 16$ to guarantee the condition $d_{J+4} > \rho$ for Lemma 2.4. This proves that (4.17) holds for $\frac{6}{5}<p<2$.

By applying Lemma 2.3 to (4.17) with $p=\frac{3}{2}$, we obtain

$$\begin{aligned}&\|\nabla(v-v_h)\|_{L^2(A_j)} \\ &\leqslant Chd_j^{-\frac{3}{2}}\rho\|\varphi\|_{L^{\frac{3}{2}}(S_\rho(x_0))} + Cd_j^{-1}\|v-v_h\|_{L^2(A_j^1)} \\ &\leqslant Chd_j^{-\frac{3}{2}}\rho^{\frac{3}{2}} + Cd_j^{-1}\|v-v_h\|_{L^2(A_j^1)},\end{aligned} \tag{4.19}$$

where the last inequality is due to the following Hölder inequality:

$$\|\varphi\|_{L^{\frac{3}{2}}(S_\rho(x_0))} \leqslant C\rho^{\frac{1}{2}}\|\varphi\|_{L^2(S_\rho(x_0))} \quad \text{with} \quad \|\varphi\|_{L^2(S_\rho(x_0))} \leqslant 1.$$

From (4.19) we see that

$$d_j\|\nabla(v-v_h)\|_{L^2(A_j)} \leqslant C\rho^{\frac{3}{2}}h^{\frac{1}{2}}\left(\frac{h}{d_j}\right)^{\frac{1}{2}} + C\|v-v_h\|_{L^2(A_j^1)}. \tag{4.20}$$

Then, substituting (4.20) into (4.15), we have

$$\rho^{-\frac{3}{2}}h^{-1}\|\nabla(v-v_h)\|_{L^1(\Lambda_h)} \leqslant C\sum_{j=0}^{J}\left(\frac{h}{d_j}\right)^{\frac{1}{2}} + C\rho^{-\frac{3}{2}}h^{-\frac{1}{2}}\sum_{j=0}^{J}\|v-v_h\|_{L^2(A_j^1)} + C\kappa \tag{4.21}$$

It remains to estimate $\sum_{j=0}^{J}\|v-v_h\|_{L^2(A_j^1)}$. To this end, we let χ be a smooth cut-off function satisfying

$$\chi = 1 \text{ on } A_j^1 \quad \text{and} \quad \chi = 0 \text{ outside } A_j^2.$$

Then

$$\begin{aligned}
\|v-v_h\|_{L^6(A_j^1)} &\leqslant \|\chi(v-v_h)\|_{L^6(\Omega)} \\
&\leqslant C\|\chi(v-v_h)\|_{H^1(\Omega)} \quad \text{(Sobolev embedding } H^1(\Omega)\hookrightarrow L^6(\Omega)) \\
&\leqslant C\|\nabla(v-v_h)\|_{L^2(A_j^2)} + Cd_j^{-1}\|v-v_h\|_{L^2(A_j^2)}.
\end{aligned} \tag{4.22}$$

By using (4.22) and the interpolation inequality (for $1<p<2$)

$$\|v-v_h\|_{L^2(A_j^1)} \leqslant \|v-v_h\|_{L^p(A_j^1)}^{1-\theta}\|v-v_h\|_{L^6(A_j^1)}^{\theta} \quad \text{with } \frac{1}{2} = \frac{1-\theta}{p}+\frac{\theta}{6}, \tag{4.23}$$

we obtain

$$\begin{aligned}
&\|v-v_h\|_{L^2(A_j^1)} \\
&\leqslant \|v-v_h\|_{L^p(A_j^1)}^{1-\theta}\big(C\|\nabla(v-v_h)\|_{L^2(A_j^2)} + Cd_j^{-1}\|v-v_h\|_{L^2(A_j^2)}\big)^{\theta} \\
&= (\varepsilon^{-\frac{\theta}{1-\theta}}\|v-v_h\|_{L^p(A_j^1)})^{1-\theta}\big(C\varepsilon\|\nabla(v-v_h)\|_{L^2(A_j^2)} + C\varepsilon d_j^{-1}\|v-v_h\|_{L^2(A_j^2)}\big)^{\theta} \\
&\leqslant C\varepsilon^{-\frac{\theta}{1-\theta}}\|v-v_h\|_{L^p(A_j^1)} + C\varepsilon\|\nabla(v-v_h)\|_{L^2(A_j^2)} + C\varepsilon d_j^{-1}\|v-v_h\|_{L^2(A_j^2)},
\end{aligned}$$

where ε can be an arbitrary positive number. By choosing $\varepsilon = d_j(\rho/d_j)^{\sigma}$ with $\sigma\in(0,1)$, we obtain

$$\|v-v_h\|_{L^2(A_j^1)} \leqslant C\left(\frac{\rho}{d_j}\right)^{-\frac{\theta\sigma}{1-\theta}} d_j^{-\frac{\theta}{1-\theta}}\|v-v_h\|_{L^p(A_j^1)} \tag{4.24}$$

$$+\left(\frac{\rho}{d_j}\right)^{\sigma}\big(Cd_j\|\nabla(v-v_h)\|_{L^2(A_j^2)} + C\|v-v_h\|_{L^2(A_j^2)}\big). \tag{4.25}$$

Hence,

$$\begin{aligned}
&\rho^{-\frac{3}{2}}h^{-\frac{1}{2}}\sum_{j=0}^{J}\|v-v_h\|_{L^2(A_j^1)} \\
&\leqslant C\rho^{-\frac{3}{2}}h^{-\frac{1}{2}}\sum_{j=0}^{J}\left(\frac{\rho}{d_j}\right)^{-\frac{\theta\sigma}{1-\theta}} d_j^{-\frac{\theta}{1-\theta}}\|v-v_h\|_{L^p(A_j^1)} \\
&\quad + C\rho^{-\frac{3}{2}}h^{-\frac{1}{2}}\sum_{j=0}^{J}\left(\frac{\rho}{d_j}\right)^{\sigma}\big(d_j\|\nabla(v-v_h)\|_{L^2(A_j^2)} + C\|v-v_h\|_{L^2(A_j^2)}\big)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\rho^{-\frac{3}{2}} h^{-\frac{1}{2}} \sum_{j=0}^{J} \left(\frac{\rho}{d_j}\right)^{-\frac{\theta\sigma}{1-\theta}} d_j^{-\frac{\theta}{1-\theta}} \|v - v_h\|_{L^p(A_j^1)} \\
&\quad + C \sum_{j=0}^{J} \left(\frac{\rho}{d_j}\right)^{\sigma} \left(\frac{h}{d_j}\right)^{\frac{1}{2}} \\
&\quad + C\rho^{-\frac{3}{2}} h^{-\frac{1}{2}} \sum_{j=0}^{J} \left(\frac{\rho}{d_j}\right)^{\sigma} \|v - v_h\|_{L^2(A_j^3)}, \qquad (4.26)
\end{aligned}
$$

where we have used (4.20) in deriving the last inequality. Note that

$$
\begin{aligned}
&\sum_{j=0}^{J} \left(\frac{\rho}{d_j}\right)^{\sigma} \|v - v_h\|_{L^2(A_j^3)} \\
&\leqslant C\left(\frac{\rho}{d_J}\right)^{\sigma} \|v - v_h\|_{L^2(S_{\kappa\rho}(x_0))} + 3 \sum_{j=0}^{J} \left(\frac{\rho}{d_j}\right)^{\sigma} \|v - v_h\|_{L^2(A_j^1)}.
\end{aligned}
$$

Combining the last two estimates, we obtain

$$
\begin{aligned}
&\rho^{-\frac{3}{2}} h^{-\frac{1}{2}} \sum_{j=0}^{J} \|v - v_h\|_{L^2(A_j^1)} \\
&\leqslant C\rho^{-\frac{3}{2}} h^{-\frac{1}{2}} \sum_{j=0}^{J} \left(\frac{\rho}{d_j}\right)^{-\frac{\theta\sigma}{1-\theta}} d_j^{-\frac{\theta}{1-\theta}} \|v - v_h\|_{L^p(A_j^1)} \\
&\quad + C \sum_{j=0}^{J} \left(\frac{\rho}{d_j}\right)^{\sigma} \left(\frac{h}{d_j}\right)^{\frac{1}{2}} \\
&\quad + C\rho^{-\frac{3}{2}} h^{-\frac{1}{2}} \left(\frac{\rho}{d_J}\right)^{\sigma} \|v - v_h\|_{L^2(S_{\kappa\rho}(x_0))} + C\rho^{-\frac{3}{2}} h^{-\frac{1}{2}} \sum_{j=0}^{J} \left(\frac{\rho}{d_j}\right)^{\sigma} \|v - v_h\|_{L^2(A_j^1)}.
\end{aligned}
$$

For given $\sigma \in (0,1)$, if $d_J \geqslant \kappa\rho$ for sufficiently large constant κ, then the last term can be absorbed by the left side. Hence, we have

$$
\begin{aligned}
\sum_{j=0}^{J} \rho^{-\frac{3}{2}} h^{-\frac{1}{2}} \|v - v_h\|_{L^2(A_j^1)} &\leqslant \sum_{j=0}^{J} C\rho^{-\frac{3}{2}} h^{-\frac{1}{2}} \left(\frac{\rho}{d_j}\right)^{-\frac{\theta\sigma}{1-\theta}} d_j^{-\frac{\theta}{1-\theta}} \|v - v_h\|_{L^p(A_j^1)} \\
&\quad + C \sum_{j=0}^{J} \left(\frac{\rho}{d_j}\right)^{\sigma} \left(\frac{h}{d_j}\right)^{\frac{1}{2}} \\
&\quad + C\rho^{-\frac{3}{2}} h^{-\frac{1}{2}} \left(\frac{\rho}{d_J}\right)^{\sigma} \|v - v_h\|_{L^2(S_{\kappa\rho}(x_0))}. \qquad (4.27)
\end{aligned}
$$

It remains to estimate $\|v - v_h\|_{L^p(A_j^1)}$ and $\|v - v_h\|_{L^2(S_{\kappa\rho}(x_0))}$. To this end, we let $\psi \in C_0^\infty(A_j^1)$ be a function satisfying

$$
\|v - v_h\|_{L^p(A_j^1)} \leqslant 2(v - v_h, \psi) \quad \text{and} \quad \|\psi\|_{L^q(A_j^1)} \leqslant 1, \quad \text{with} \quad \frac{1}{p} + \frac{1}{q} = 1. \qquad (4.28)
$$

Let $w \in H_0^1(\Omega)$ be the solution of

$$\begin{cases} -\Delta w = \psi & \text{in } \Omega, \\ \quad w = 0 & \text{on } \partial\Omega, \end{cases}$$

Then using Lemma 2.5 and Lemma 2.1, we obtain

$$\begin{aligned}
(v - v_h, \psi) &= (\nabla(v - v_h), \nabla w) \\
&= (\nabla(v - v_h), \nabla(w - I_h w)) \\
&\leqslant \|\nabla(v - v_h)\|_{L^p(\Omega)} \|\nabla(w - I_h w)\|_{L^q(\Omega)} \\
&\leqslant Ch^2 \|v\|_{W^{2,p}(\Omega)} \|w\|_{W^{2,q}(\Omega)} \\
&\leqslant Ch^2 \|\varphi\|_{L^p(\Omega)} \|\psi\|_{L^q(\Omega)} \\
&\leqslant Ch^2 \|\varphi\|_{L^p(S_\rho(x_0))} \|\psi\|_{L^q(A_j^1)} \\
&\leqslant Ch^2 \rho^{\frac{3}{p}-\frac{3}{2}} \|\varphi\|_{L^2(S_\rho(x_0))} \|\psi\|_{L^q(A_j^1)} \\
&\leqslant Ch^2 \rho^{\frac{3}{p}-\frac{3}{2}},
\end{aligned}$$

where we have used $\|\varphi\|_{L^2(S_\rho(x_0))} \leqslant 1$ and $\|\psi\|_{L^q(A_j^1)} \leqslant 1$ in deriving the last inequalities. This implies

$$\|v - v_h\|_{L^p(A_j^1)} \leqslant Ch^2 \rho^{\frac{3}{p}-\frac{3}{2}} \quad \text{and} \quad \|v - v_h\|_{L^2(S_{\kappa\rho}(x_0))} \leqslant Ch^2. \tag{4.29}$$

By substituting these estimates into (4.27), we obtain

$$\begin{aligned}
\sum_{j=0}^{J} \rho^{-\frac{3}{2}} h^{-\frac{1}{2}} \|v - v_h\|_{L^2(A_j^1)} &\leqslant \sum_{j=0}^{J} C \left(\frac{h}{\rho}\right)^{\frac{3}{2}} \left(\frac{\rho}{d_j}\right)^{\frac{3}{p}-\frac{3}{2}-\frac{\theta\sigma}{1-\theta}} d_j^{\frac{3}{p}-\frac{3}{2}-\frac{\theta}{1-\theta}} \\
&\quad + C \sum_{j=0}^{J} \left(\frac{\rho}{d_j}\right)^{\sigma} \left(\frac{h}{d_j}\right)^{\frac{1}{2}} + C.
\end{aligned} \tag{4.30}$$

Since $\frac{3}{p} - \frac{3}{2} - \frac{\theta}{1-\theta} = 0$ and $\sigma \in (0, 1)$, it follows that $\frac{3}{p} - \frac{3}{2} - \frac{\sigma\theta}{1-\theta} > 0$ and therefore

$$\begin{aligned}
\sum_{j=0}^{J} \rho^{-\frac{3}{2}} h^{-\frac{1}{2}} \|v - v_h\|_{L^2(A_j^1)} &\leqslant \sum_{j=0}^{J} C \left(\frac{h}{\rho}\right)^{\frac{3}{2}} \left(\frac{\rho}{d_j}\right)^{\frac{3}{p}-\frac{3}{2}-\frac{\theta\sigma}{1-\theta}} \\
&\quad + C \sum_{j=0}^{J} \left(\frac{\rho}{d_j}\right)^{\sigma} \left(\frac{h}{d_j}\right)^{\frac{1}{2}} + C \leqslant C.
\end{aligned} \tag{4.31}$$

Noting that, up to now, the constants κ and σ have been determined. Then, substituting (4.31) into (4.21), we obtain

$$\rho^{-\frac{3}{2}} h^{-1} \|\nabla(v - v_h)\|_{L^1(\Lambda_h)} \leqslant C. \tag{4.32}$$

This proves the desired result for sufficiently small mesh size $h \leqslant h_0$, as explained in the end of section 2.

For $h \geqslant h_0$, we denote by $\widetilde{g}_h \in S_h$ the finite element function satisfying $\widetilde{g}_h = u_h$ on $\partial\Omega$ and $\widetilde{g}_h = 0$ at the interior nodes of the domain Ω. Naturally,

$$\|\widetilde{g}_h\|_{L^\infty(\Omega)} \leqslant \|u_h\|_{L^\infty(\partial\Omega)}.$$

Since $\chi_h = u_h - \widetilde{g}_h \in \mathring{S}_h$, from (1.1), we have

$$(\nabla u_h, (\nabla(u_h - \widetilde{g}_h)) = 0$$

and as a result

$$\begin{aligned}\|\nabla(u_h-\widetilde g_h)\|_{L^2(\Omega)}^2 &= (\nabla(u_h-\widetilde g_h),\nabla(u_h-\widetilde g_h)) = -(\nabla\widetilde g_h,\nabla(u_h-\widetilde g_h))\\ &\leqslant C\|\nabla\widetilde g_h\|_{L^2(\Omega)}\|\nabla(u_h-\widetilde g_h)\|_{L^2(\Omega)}.\end{aligned}$$

Thus, using the inverse inequality and that $h \geqslant h_0$, we have

$$\begin{aligned}\|\nabla(u_h-\widetilde g_h)\|_{L^2(\Omega)} \leqslant C\|\nabla\widetilde g_h\|_{L^2(\Omega)} \leqslant Ch^{-1}\|\widetilde g_h\|_{L^2(\Omega)} &\leqslant Ch_0^{-1}\|\widetilde g_h\|_{L^\infty(\Omega)}\\ &\leqslant Ch_0^{-1}\|u_h\|_{L^\infty(\partial\Omega)}.\end{aligned}$$

By using the inverse inequality and the above estimate, we also have

$$\begin{aligned}\|u_h-\widetilde g_h\|_{L^\infty(\Omega)} &\leqslant Ch^{-\frac32}\|u_h-\widetilde g_h\|_{L^2(\Omega)}\\ &\leqslant Ch^{-\frac32}\|\nabla(u_h-\widetilde g_h)\|_{L^2(\Omega)}\\ &\leqslant Ch_0^{-\frac52}\|u_h\|_{L^\infty(\partial\Omega)}.\end{aligned}$$

By the triangle inequality, this proves

$$\|u_h\|_{L^\infty(\Omega)} \leqslant \|\widetilde g_h\| + \|u_h-\widetilde g_h\|_{L^\infty(\Omega)} \leqslant C\|u_h\|_{L^\infty(\partial\Omega)}$$

for $h \geqslant h_0$.

Combining the two cases $h \leqslant h_0$ and $h \geqslant h_0$, we obtain the desired result of Theorem 1.1.

5. Application to the Ritz projection

In this section, we adopt Schatz's argument to prove the maximum-norm stability of the Ritz projection. This argument uses the weak maximum principle established above to remove a logarithmic factor for finite elements of degree $r \geqslant 2$ in convex polyhedral domains under the following assumption:

(A) The tetrahedral partition of Ω can be extended to a larger convex domain $\widetilde\Omega$ quasi-uniformly, with $\Omega \subset\subset \widetilde\Omega$.

The logarithmic factor has been removed in previous articles only for $r \geq 2$ on smooth and two-dimensional polygonal domains.

For any function $u \in H_0^1(\Omega)$, we denote by $R_hu \in \mathring{S}_h$ the Ritz projection of u, defined by

$$(\nabla(u-R_hu),\nabla\chi_h) = 0 \quad \forall\,\chi_h \in \mathring{S}_h. \tag{5.33}$$

Theorem 5.1. *Under assumption* (A), *for finite elements of degree $r \geq 2$ the Ritz projection satisfies*

$$\|R_hu\|_{L^\infty(\Omega)} \leqslant C\|u\|_{L^\infty(\Omega)} \quad \forall\,u \in H_0^1(\Omega)\cap C(\overline{\Omega}). \tag{5.34}$$

Proof. Let $\widetilde u$ be the zero extension of u to the larger domain $\widetilde\Omega$. Let $\mathring{S}_h(\widetilde\Omega)$ be the finite element space subject to the tetrahedral partition of $\widetilde\Omega$ (with zero boundary values), and let $\widetilde u_h$ be the Ritz projection of $\widetilde u$ in the domain $\widetilde\Omega$, i.e.

$$\int_{\widetilde\Omega}\nabla(\widetilde u-\widetilde u_h)\cdot\nabla\chi_h\,\mathrm{d}x = 0 \quad \forall\,\chi_h \in \mathring{S}_h(\widetilde\Omega). \tag{5.35}$$

Since $u = \widetilde u$ on Ω, it follows that

$$\begin{aligned}\|u-u_h\|_{L^\infty(\Omega)} &= \|\widetilde u-u_h\|_{L^\infty(\Omega)} &(5.36)\\ &\leqslant \|\widetilde u-\widetilde u_h\|_{L^\infty(\Omega)} + \|\widetilde u_h-u_h\|_{L^\infty(\Omega)} &(5.37)\\ &:= E_1+E_2. &(5.38)\end{aligned}$$

By using [27, Theorem 5.1] (which requires $r \geq 2$ to remove a logarithmic factor and h sufficiently small, say $h \leqslant h_*$), we have

$$E_1 \leqslant C\|\widetilde{u} - I_h\widetilde{u}\|_{L^\infty(\Omega')} + C\|\widetilde{u} - \widetilde{u}_h\|_{L^2(\Omega')}, \tag{5.39}$$

where Ω' is some intermediate domain satisfying $\Omega \subset\subset \Omega' \subset\subset \widetilde{\Omega}$. Since the Lagrange interpolation operator I_h is stable in the L^∞ norm on $C(\overline{\Omega})$, it follows that

$$\|\widetilde{u} - I_h\widetilde{u}\|_{L^\infty(\Omega')} \leqslant C\|\widetilde{u}\|_{L^\infty(\Omega)} = C\|u\|_{L^\infty(\Omega)}. \tag{5.40}$$

To estimate $\|\widetilde{u} - \widetilde{u}_h\|_{L^2(\Omega')}$, we use a duality argument. Thus,

$$\|\widetilde{u} - \widetilde{u}_h\|_{L^2(\Omega')} \leqslant \|\widetilde{u} - \widetilde{u}_h\|_{L^2(\widetilde{\Omega})} = \sup_{\substack{\widetilde{\varphi} \in C_0^\infty(\widetilde{\Omega}) \\ \|\widetilde{\varphi}\|_{L^2(\widetilde{\Omega})} \leqslant 1}} \int_{\widetilde{\Omega}} (\widetilde{u} - \widetilde{u}_h)\, \widetilde{\varphi}\, \mathrm{d}x.$$

In particular, there exists a $\widetilde{\varphi} \in C_0^\infty(\widetilde{\Omega})$ satisfying

$$\|\widetilde{\varphi}\|_{L^2(\widetilde{\Omega})} \leqslant 1 \quad \text{and} \quad \|\widetilde{u} - \widetilde{u}_h\|_{L^2(\widetilde{\Omega})} \leqslant 2\int_{\widetilde{\Omega}} (\widetilde{u} - \widetilde{u}_h)\, \widetilde{\varphi}\, \mathrm{d}x. \tag{5.41}$$

For this $\widetilde{\varphi}$ we define $\widetilde{\psi} \in H_0^1(\Omega)$ to be the weak solution of

$$\begin{cases} -\Delta\widetilde{\psi} = \widetilde{\varphi} & \text{in } \widetilde{\Omega}, \\ \quad\ \widetilde{\psi} = 0 & \text{on } \partial\widetilde{\Omega}, \end{cases} \tag{5.42}$$

and denote by $\widetilde{\psi}_h \in \mathring{S}_h(\widetilde{\Omega})$ the Ritz projection of $\widetilde{\psi}$ in $\widetilde{\Omega}$, i.e.

$$\int_{\widetilde{\Omega}} \nabla(\widetilde{\psi} - \widetilde{\psi}_h) \cdot \nabla\widetilde{\chi}_h \, \mathrm{d}x = 0 \quad \forall\, \widetilde{\chi}_h \in \mathring{S}_h(\widetilde{\Omega}). \tag{5.43}$$

If we denote by $\widetilde{\mathfrak{T}}$ the set of tetrahedra in the partition of $\widetilde{\Omega}$, then testing (5.42) by $\widetilde{u} - \widetilde{u}_h$ yields

(5.44)

$$\begin{aligned}
\int_{\widetilde{\Omega}} (\widetilde{u} - \widetilde{u}_h)\, \widetilde{\varphi}\, \mathrm{d}x &= \int_{\widetilde{\Omega}} \nabla(\widetilde{u} - \widetilde{u}_h) \cdot \nabla\widetilde{\psi}\, \mathrm{d}x \qquad \text{(here we use integration by parts)} \\
&= \int_{\widetilde{\Omega}} \nabla(\widetilde{u} - \widetilde{u}_h) \cdot \nabla(\widetilde{\psi} - \widetilde{\psi}_h)\, \mathrm{d}x \quad \text{(here we use (5.35))} \\
&= \int_{\widetilde{\Omega}} \nabla\widetilde{u} \cdot \nabla(\widetilde{\psi} - \widetilde{\psi}_h)\, \mathrm{d}x \qquad\qquad \text{(here we use (5.43))} \\
&= \sum_{\tau \in \widetilde{\mathfrak{T}}} \int_\tau \nabla\widetilde{u} \cdot \nabla(\widetilde{\psi} - \widetilde{\psi}_h)\, \mathrm{d}x \\
&= -\sum_{\tau \in \widetilde{\mathfrak{T}}} \int_\tau \widetilde{u}\, \Delta(\widetilde{\psi} - \widetilde{\psi}_h)\, \mathrm{d}x + \int_{\partial\tau} \widetilde{u}\, \partial_n(\widetilde{\psi} - \widetilde{\psi}_h)\, \mathrm{d}s \\
&\leqslant C\|\widetilde{u}\|_{L^\infty(\widetilde{\Omega})} \sum_{\tau \in \widetilde{\mathfrak{T}}} \Big(\|\Delta(\widetilde{\psi} - \widetilde{\psi}_h)\|_{L^1(\tau)} + \|\partial_n(\widetilde{\psi} - \widetilde{\psi}_h)\|_{L^1(\partial\tau)} \Big) \\
&\leqslant C\|u\|_{L^\infty(\Omega)} \Big(h^{-1}\|\nabla(\widetilde{\psi} - \widetilde{\psi}_h)\|_{L^1(\widetilde{\Omega})} + \sum_{\tau \in \widetilde{\mathfrak{T}}} \|\widetilde{\psi} - \widetilde{\psi}_h\|_{W^{2,1}(\tau)} \Big),
\end{aligned}$$

where in the last step we have used $\|\widetilde{u}\|_{L^\infty(\widetilde{\Omega})} = \|u\|_{L^\infty(\Omega)}$ and the trace inequality

$$\|\partial_n(\widetilde{\psi} - \widetilde{\psi}_h)\|_{L^1(\partial\tau)} \leqslant Ch^{-1}\|\nabla(\widetilde{\psi} - \widetilde{\psi}_h)\|_{L^1(\tau)} + C\|\widetilde{\psi} - \widetilde{\psi}_h\|_{W^{2,1}(\tau)}.$$

By using a priori energy estimate and H^2 regularity, we have

$$\begin{aligned} \|\nabla(\widetilde{\psi} - \widetilde{\psi}_h)\|_{L^1(\widetilde{\Omega})} &\leqslant C\|\nabla(\widetilde{\psi} - \widetilde{\psi}_h)\|_{L^2(\widetilde{\Omega})} \\ &\leqslant C\|\nabla(\widetilde{\psi} - I_h\widetilde{\psi})\|_{L^2(\widetilde{\Omega})} \\ &\leqslant Ch\|\widetilde{\psi}\|_{H^2(\widetilde{\Omega})} \\ &\leqslant Ch\|\widetilde{\varphi}\|_{L^2(\widetilde{\Omega})} \leqslant Ch. \end{aligned} \tag{5.45}$$

Let $\tilde{I}_h$ be the Scott-Zhang interpolant. Then by the triangle and inverse inequalities, we have

$$\begin{aligned} &\sum_{\tau\in\widetilde{\mathfrak{T}}} \|\widetilde{\psi} - \widetilde{\psi}_h\|_{W^{2,1}(\tau)} \\ &\leqslant C\sum_{\tau\in\widetilde{\mathfrak{T}}} \Big(\|\widetilde{\psi} - \tilde{I}_h\widetilde{\psi}\|_{W^{2,1}(\tau)} + \|\tilde{I}_h\widetilde{\psi} - \widetilde{\psi}_h\|_{W^{2,1}(\tau)}\Big) \\ &\leqslant C\Big(\sum_{\tau\in\widetilde{\mathfrak{T}}} \|\widetilde{\psi} - \tilde{I}_h\widetilde{\psi}\|_{W^{2,1}(\tau)} + h^{-1}\|\tilde{I}_h\widetilde{\psi} - \widetilde{\psi}_h\|_{W^{1,1}(\widetilde{\Omega})}\Big) \\ &\leqslant C\Big(\sum_{\tau\in\widetilde{\mathfrak{T}}} \|\widetilde{\psi} - \tilde{I}_h\widetilde{\psi}\|_{W^{2,1}(\tau)} + h^{-1}\|\widetilde{\psi} - \tilde{I}_h\widetilde{\psi}\|_{W^{1,1}(\widetilde{\Omega})} + h^{-1}\|\widetilde{\psi} - \widetilde{\psi}_h\|_{W^{1,1}(\widetilde{\Omega})}\Big). \end{aligned}$$

Similarly as (5.45), we can prove the following estimate:

$$h^{-1}\|\widetilde{\psi} - \tilde{I}_h\widetilde{\psi}\|_{W^{1,1}(\widetilde{\Omega})} + h^{-1}\|\widetilde{\psi} - \widetilde{\psi}_h\|_{W^{1,1}(\widetilde{\Omega})} \leqslant C,$$

and by using the properties of $\tilde{I}_h$ (cf. [5, Theorem 4.8.3.8]),

$$\sum_{\tau\in\widetilde{\mathfrak{T}}} \|\widetilde{\psi} - \tilde{I}_h\widetilde{\psi}\|_{W^{2,1}(\tau)} \leqslant C\sum_{\tau\in\widetilde{\mathfrak{T}}} \|\widetilde{\psi}\|_{W^{2,1}(\tau)} \leqslant C\|\widetilde{\psi}\|_{H^2(\widetilde{\Omega})} \leqslant C\|\widetilde{\varphi}\|_{L^2(\widetilde{\Omega})} \leqslant C.$$

Now we substitute these estimates into (5.44). This yields

$$\|\widetilde{u} - \widetilde{u}_h\|_{L^2(\widetilde{\Omega})} \leqslant C\|u\|_{L^\infty(\Omega)}. \tag{5.46}$$

Then, by substituting (5.40) and (5.46) into (5.39), we obtain

$$E_1 \leqslant C\|u\|_{L^\infty(\Omega)}. \tag{5.47}$$

To estimate E_2, we use the fact that $\widetilde{u}_h - u_h$ is discrete harmonic in Ω, i.e.

$$\int_\Omega \nabla(\widetilde{u}_h - u_h)\cdot\nabla\chi_h \,\mathrm{d}x = \int_\Omega \nabla(\widetilde{u} - u)\cdot\nabla\chi_h \,\mathrm{d}x = 0 \quad \forall\, \chi_h \in \mathring{S}_h(\Omega).$$

Thus, by the weak discrete maximum principle proved in Theorem 1.1 and using the fact that $u_h = 0$ and $\widetilde{u} = 0$ on $\partial\Omega$, we have

$$
\begin{aligned}
E_2 &= \|\widetilde{u}_h - u_h\|_{L^\infty(\Omega)} \\
&\leqslant C\|\widetilde{u}_h - u_h\|_{L^\infty(\partial\Omega)} \\
&= C\|\widetilde{u}_h\|_{L^\infty(\partial\Omega)} && (\text{use } u_h = 0 \text{ on } \partial\Omega) \\
&= C\|\widetilde{u}_h - \widetilde{u}\|_{L^\infty(\partial\Omega)} && (\text{use } \widetilde{u} = 0 \text{ on } \partial\Omega) \\
&\leqslant C\|\widetilde{u}_h - \widetilde{u}\|_{L^\infty(\Omega)} \\
&= E_1,
\end{aligned}
\tag{5.48}
$$

which has already been estimated. Hence, substituting (5.47) and (5.48) into (5.36), we obtain

$$
\|u - u_h\|_{L^\infty(\Omega)} \leqslant C\|u\|_{L^\infty(\Omega)}. \tag{5.49}
$$

This completes the proof of Theorem 5.1 in the case $h \leqslant h_*$ for some positive constant h_*.

If $h \geqslant h_*$ then we pick up a point $x_0 \in \bar{\tau}_0$ (in some tetrahedron τ_0) satisfying $|u_h(x_0)| = \|u_h\|_{L^\infty(\Omega)}$. For such x_0 we define a regularized Green's function G as the solution of

$$
\begin{aligned}
-\Delta G(x) &= \tilde{\delta}(x), \quad && x \in \Omega, \\
G(x) &= 0, && x \in \partial\Omega,
\end{aligned}
\tag{5.50}
$$

where $\tilde{\delta} \in C^3(\overline{\Omega})$ is the regularized Delta function concentrated at x_0, satisfying $\mathrm{supp}(\tilde{\delta}) \subset \bar{\tau}_0$ and

$$
\begin{aligned}
&\int_\Omega \chi_h \widetilde{\delta}\, \mathrm{d}x = \chi_h(x_0), \quad \forall\, \chi_h \in \mathring{S}_h, \\
&\|\widetilde{\delta}\|_{W^{l,p}} \leq K h^{-l-3(1-1/p)} \quad \text{for } 1 \leq p \leq \infty, \ \ l = 0, 1, 2, 3.
\end{aligned}
$$

The construction of the function $\widetilde{\delta}$ can be found in [30, Lemma 2.2]. In particular, the construction of $\widetilde{\delta}$ can be done in any tetrahedron for the arbitrary mesh size h.

We define $G_h = R_h G \in \mathring{S}_h$, i.e.,

$$
(\nabla G_h, \nabla \chi_h) = (\tilde{\delta}, \chi_h) \quad \forall \chi_h \in \mathring{S}_h. \tag{5.51}
$$

The finite element function G_h defined by the equation above satisfies the following standard energy estimate:

$$
\|G_h\|_{H^1(\Omega)} \leqslant C\|\tilde{\delta}\|_{L^2(\Omega)}.
$$

Then using the Galerkin orthogonality, integration by parts, we obtain

$$
\begin{aligned}
u_h(x_0) = (\nabla u_h, \nabla G_h) = (\nabla u, \nabla G_h) &= \sum_{\tau \in \mathfrak{T}} \left[(u, \partial_n G_h)_{\partial\tau} + (u, -\Delta G_h)_\tau \right] \\
&\leqslant \|u\|_{L^\infty(\Omega)} \sum_{\tau \in \mathfrak{T}} \left(\|\partial_n G_h\|_{L^1(\partial\tau)} + \|\Delta G_h\|_{L^1(\tau)} \right).
\end{aligned}
\tag{5.52}
$$

Now, for $h \geq h_*$, using the trace and inverse inequality we have

$$
\begin{aligned}
\sum_{\tau \in \mathfrak{T}} \left[\|\partial_n G_h\|_{L^1(\partial\tau)} + \|\Delta G_h\|_{L^1(\tau)} \right] &\leqslant C h^{-1} \sum_{\tau \in \mathfrak{T}} \|\nabla G_h\|_{L^1(\tau)} \\
&\leqslant C h^{-1} \|G_h\|_{W^{1,1}(\Omega)}
\end{aligned}
$$

$$\leqslant Ch^{-1}\|G_h\|_{H^1(\Omega)}$$
$$\leqslant Ch^{-1}\|\tilde{\delta}\|_{L^2(\Omega)} \leqslant Ch_*^{-5/2},$$

since $\|\tilde{\delta}\|_{L^2(\Omega)} \leqslant Ch^{-3/2}$ and $h \geqslant h_*$.

Combining the two cases $h \leqslant h_*$ and $h \geqslant h_*$, we obtain the desired result of Theorem 5.1. □

6. Conclusion

In this article, we have proved the weak maximum principle of finite element method (Theorem 1.1). The main difference between the current proof and the proof in [26] for two-dimensional polygons is that we have used L^p estimates in place of some L^2 estimates in section 4, including (4.16), (4.18), (4.23), (4.24), (4.26), (4.28) and (4.29). As an application of the weak maximum principle of finite element methods, we have presented an L^∞-stability of Ritz projection (Theorem 5.1) by utilizing the argument in [26, Theorem 5.1].

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