MAXIMUM-NORM STABILITY OF THE FINITE ELEMENT METHOD FOR THE NEUMANN PROBLEM IN NONCONVEX POLYGONS WITH LOCALLY REFINED MESH

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ABSTRACT. The Galerkin finite element solution u_h of the Possion equation $-\Delta u = f$ under the Neumann boundary condition in a possibly nonconvex polygon Ω, with a graded mesh locally refined at the corners of the domain, is shown to satisfy the following maximum-norm stability:

$$\|u_h\|_{L^\infty(\Omega)} \le C\ell_h \|u\|_{L^\infty(\Omega)},$$

where $\ell_h = \ln(2+1/h)$ for piecewise linear elements and $\ell_h = 1$ for higher-order elements. As a result of the maximum-norm stability, the following best approximation result holds:

$$\|u-u_h\|_{L^\infty(\Omega)} \le C\ell_h \|u - I_h u\|_{L^\infty(\Omega)},$$

where I_h denotes the Lagrange interpolation operator onto the finite element space. For a locally quasi-uniform triangulation sufficiently refined at the corners, the above best approximation property implies the following optimal-order error bound in the maximum norm:

$$\|u-u_h\|_{L^\infty(\Omega)} \le \begin{cases} C\ell_h h^{k+2-\frac{2}{p}} \|f\|_{W^{k,p}(\Omega)} & \text{if } r \ge k+1, \\ C\ell_h h^{k+1} \|f\|_{H^k(\Omega)} & \text{if } r = k, \end{cases}$$

where $r \ge 1$ is the degree of finite elements, k is any nonnegative integer no larger than r, and $p \in [2,\infty)$ can be arbitrarily large.

1 Introduction

This article concerns the maximum-norm stability of Galerkin finite element approximations to the Neumann boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \partial_n u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

under the condition $\int_\Omega f \mathrm{d}x = 0$ (for the existence of solution) with the normalization condition $\int_\Omega u \mathrm{d}x = 0$ (for the uniqueness of the solution), where Ω is a two-dimensional polygon. The Galerkin finite element solution of (1.1) is defined by the weak formulation:

$$(\nabla u_h, \nabla v_h) = (f, v_h) \quad \forall\, v_h \in S_h, \tag{1.2}$$

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with the normalization condition $\int_\Omega u_h \mathrm{d}x = 0$, where S_h is the Lagrange finite element subspace of $H^1(\Omega)$ consisting of all piecewise polynomials of degree $r \geq 1$ subject to a locally quasi-uniform triangulation.

It is well known that the finite element approximation to (1.1) is stable in the H^1 norm on a general polygon with general triangulation, i.e.,

$$\|u_h\|_{H^1(\Omega)} \leq C\|u\|_{H^1(\Omega)}, \tag{1.3}$$

where the constant C is independent of the solution u and the mesh size h. The result can be interpreted as the H^1 stability of the Ritz projection. Since the Ritz projection of $u - I_h u$ is $u_h - I_h u$, where I_h denotes the Lagrange interpolation operator onto the finite element space, replacing u by $u - I_h u$ in (1.3) yields the following best approximation property in the H^1 norm:

$$\|u - u_h\|_{H^1(\Omega)} \leq C\|u - I_h u\|_{H^1(\Omega)}. \tag{1.4}$$

The objective of this article is to establish the following analogous stability result in the L^∞ norm on a general polygon (possibly nonconvex) with locally refined triangulation at the corners:

$$\|u_h\|_{L^\infty(\Omega)} \leq C\ell_h\|u\|_{L^\infty(\Omega)}, \tag{1.5}$$

where

$$\ell_h = \begin{cases} \ln(2 + 1/h) & \text{if } r = 1, \\ 1 & \text{if } r \geq 2. \end{cases} \tag{1.6}$$

Such maximum-norm stability results have important applications in resolvent estimates of discretized elliptic operators [6, 28], discrete maximal L^p regularity of parabolic equations [13, 14, 23, 24], and pointwise error estimates of finite element solutions for elliptic, parabolic and optimal control problems [19, 21, 22, 33]. In particular, the maximum-norm stability result in (1.5) would completely reduce pointwise error estimation to interpolation errors, i.e.,

$$\|u - u_h\|_{L^\infty(\Omega)} \leq C\ell_h\|u - I_h u\|_{L^\infty(\Omega)}. \tag{1.7}$$

The maximum-norm stability result in (1.5) has been established for convex polygons and polyhedra with globally quasi-uniform mesh in [20], and for smooth domains in [18, 31]. It is known that the logarithmic factor $\ln(2+1/h)$ in the piecewise linear case $r = 1$ cannot be removed in general; see [11]. For the Dirichlet boundary condition, the maximum-norm stability has been established in [29] for nonconvex polygons by utilizing a weak maximum principle of finite element methods under globally quasi-uniform mesh. However, the argument using weak maximum principle of finite element methods cannot be extended to the Neumann problem in nonconvex polygons, or Dirichlet/Neumann problems in nonconvex polyhedra, or locally refined mesh. Whether the maximum-norm stability (1.5) can hold, under either globally quasi-uniform mesh or locally refined mesh, is still an open question for the Neumann problem in nonconvex polygons/polyhedra and the Dirichlet problem in nonconvex polyhedra (except for the special case of piecewise linear finite elements with non-obtuse quasi-uniform tetrahedral mesh [12]).

In contrast to the maximum-norm stability result in (1.5), the almost optimal-order error estimate

$$\|u - u_h\|_{L^\infty(\Omega)} \leq Ch^{r+1-\epsilon} \quad (\epsilon > 0 \text{ is any fixed number}) \tag{1.8}$$

was shown in [30] for sufficiently smooth f and general polygons, with triangulations locally refined at the corners, i.e.,

$$\hbar(x) \sim \min_j |x - z_j|^{1-\gamma_j} h \tag{1.9}$$

where $\hbar(x)$ denotes the mesh size at point x, and z_j denotes the jth corner of the polygon Ω. It is assumed that the local refinement parameter $\gamma_j \in (0, 1]$ corresponding to the corner z_j satisfies the condition $\gamma_j < \beta_j/r$, where $\beta_j = \pi/\theta_j$ and θ_j is the interior angle at the corner z_j. The convergence order for piecewise linear finite elements was improved in [2] for the Dirichlet problem with explicit dependence on a Hölder norm of f, i.e.,

$$\|u - u_h\|_{L^\infty(\Omega)} \le Ch^2 \ln(2 + 1/h)\|f\|_{C^\sigma(\overline{\Omega})} \tag{1.10}$$

under the condition $\gamma_j < \beta_j/2$ and $\sigma > 0$. More recently, an optimal-order error estimate

$$\|u - u_h\|_{L^\infty(\Omega)} \le Ch^2 \ln(2 + 1/h)\|u\|_{W^{2,\infty}_\sigma}, \tag{1.11}$$

was shown in [1] under the condition $\gamma_j < \beta_j/2$, for the Neumann problem with piecewise linear finite elements, with explicit dependence on some weighted $W^{2,\infty}$ norm of the solution u.

The $W^{1,\infty}$ stability of finite element approximations was shown for convex polygons and polyhedra under mildly graded meshes in [10], i.e.,

$$\|u_h\|_{W^{1,\infty}(\Omega)} \le C\|u\|_{W^{1,\infty}(\Omega)}. \tag{1.12}$$

The $W^{1,p}$ stability

$$\|u_h\|_{W^{1,p}(\Omega)} \le C\|u\|_{W^{1,p}(\Omega)} \tag{1.13}$$

was shown in [25] for convex polygons with mesh satisfying (1.9). Such $W^{1,\infty}$ stability estimates were also established for the Stokes equation in convex polyhedra [17]. The extension of these results to nonconvex polygons or polyhedra still remains open, similarly as the L^∞ stability result in (1.5).

In this article, we prove the maximum-norm stability (1.5) for general polygons, with finite elements of arbitrary degree and locally refined mesh satisfying (1.9); see Theorem 2.1. The local refinement parameter γ_j is only required to satisfy $\gamma_j \in (0, \beta_j) \cap (0, 1]$, which is weaker than the condition $\gamma_j < \beta_j/2$ required to obtain the maximum-norm error estimates in the literature. Some new techniques are developed to prove such maximum-norm stability results in nonconvex polygons and with graded mesh. In particular, in the literature of maximum-norm stability and error estimates for finite element methods, people often use a "dyadic decomposition" corresponding to a point x_0 where u_h attains its maximum, i.e.,

$$\Omega = \bigcup_{j=0}^{J} \Omega_j \quad \text{with} \quad \Omega_j = \{x \in \Omega : \rho_{j+1} \le |x - x_0| < \rho_j\} \text{ and } \rho_j = 2^{-j}\text{diameter}(\Omega),$$

and reduce the problem to some technical estimates on the subdomains Ω_j, in order to derive estimates of $|u_h(x_0)|$ or $|u_h(x_0) - u(x_0)|$. In this article, we introduce a "double dyadic decomposition" corresponding to both x_0 and a corner z_0, as described in Section 4.1. In this way, different estimates can be obtained on the subdomains closer to x_0 and the subdomains closer to z_0, respectively. Therefore, such a double dyadic decomposition is convenient for analysis of the maximum-norm stability of finite element methods with graded mesh locally refined at a corner.

As a consequence of (1.5) and a local regularity result to be established in this article, we also obtain the following maximum-norm error estimates:

$$\|u-u_h\|_{L^\infty(\Omega)} \le \begin{cases} C\ell_h h^{k+2-\frac{2}{p}}\|f\|_{W^{k,p}(\Omega)} & \text{if } r\ge k+1 \text{ and } \gamma_j\in\left(0,\frac{\min(1,\beta_j)}{k+2-2/p}\right] \text{ at corners,}\\ C\ell_h h^{k+1}\|f\|_{H^k(\Omega)} & \text{if } r=k \text{ and } \gamma_j\in\left(0,\frac{\min(1,\beta_j)}{k+1}\right] \text{ at corners,}\end{cases} \tag{1.14}$$

where k is any nonnegative integer and $p\in[2,\infty)$ can be arbitrarily large; see Corollary 2.1. In particular, if f is sufficiently smooth compared with the degree of finite elements (in the case $r=k$), then the order of convergence is optimal with respect to the degree of finite elements (up to a factor ℓ_h); if f is not sufficiently smooth compared with the degree of finite elements (in the case $r\ge k+1$), then the order of convergence is optimal with respect to the regularity of f.

The rest of this article is organized as follows. In Section 2 we present the notation, assumptions and main theorems. In Section 3 we present local H^{1+s}, $W^{2,p}$ and $H^{2+\alpha}$ estimates of Green's function in nonconvex polygons. These results are used in Section 4 to prove the maximum-norm stability of the Ritz projection. The proof of is presented in Section 5. Some technical estimates are presented in Appendices A–C. Throughout this article, we denote by C a generic positive constant, which may be different at different occurrences but will be independent of the mesh size h.

2 Main results

2.1 Triangulation locally refined at the corners

Let Ω be a nonconvex polygon, with vertices z_j, $j=0,\dots,m-1$, oriented counter clockwise, and denote by Γ_j the edge between the vertices z_j and z_{j+1}, with $z_m=z_0$. Let θ_j be the interior angle of the polygon Ω at the vertex z_j, and define $\beta_j:=\pi/\theta_j$. We assume that the domain Ω is triangulated with the following properties.

(1) Local quasi-uniformity: The ratio between the radius of circumcircle and the radius of inscribed circle of each triangle is bounded, and the ratios between the diameters of adjacent triangles are bounded.

(2) Local refinement at the corners: Let h denote the mesh size of the triangulation (maximal diameter of the triangles). Let $h_{*,j}\sim h^{1/\gamma_j}$ for some constant $\gamma_j\in(0,\beta_j)\cap(0,1]$, represent the diameter (up to a constant multiple) of triangles near the corner z_j, and let $\hbar(x)$ denote the maximal diameter of triangles which contain x. We assume that $\hbar(x)$ is equivalent to h away from (when x is outside a neighborhood of) the corners and satisfies the following conditions near the corners z_j, $j=0,\dots,m-1$:

$$\hbar(x)\sim|x-z_j|^{1-\gamma_j}h, \qquad \text{if } |x-z_j|>2h_{*,j}, \tag{2.1a}$$

$$\hbar(x)\sim h_{*,j}, \qquad \text{if } |x-z_j|\le 2h_{*,j}. \tag{2.1b}$$

Hence, the mesh is locally refined at the corners z_j, $j=0,\dots,m-1$, and is quasi-uniform away from the corners. In particular, the mesh size near the corner z_j is $h_{*,j}^{1-\gamma_j}h\sim h_{*,j}$, with $h_{*,j}\sim h^{1/\gamma_j}$.

If we denote by N the number of degrees of freedom in the triangulation above, then the following inequality can be shown:

$$N \le Ch^{-2}. \tag{2.2}$$

Namely, the number of degrees of freedom in the above locally refined triangulation is equivalent to the number of degrees of freedom in a quasi-uniform triangulation with mesh size h.

Let $\mathcal{T}_h$ denote the set of triangles in the triangulation of the domain Ω, and let S_h be the finite element space of degree $r \ge 1$ subject to the triangulation, i.e.,

$$S_h = \{v_h \in H^1(\Omega) : v_h|_\tau \text{ is a polynomial of degree } r \text{ for all } \tau \in \mathcal{T}_h\}.$$

2.2 Main results

Theorem 2.1 *Let Ω be a polygon which is triangulated as described in Section 2.1. Then the Ritz projection $R_h : H^1(\Omega) \to S_h$ defined by*

$$(\nabla(u - R_h u), \nabla v_h) = 0, \quad \forall\, v_h \in S_h, \tag{2.3}$$

with the normalization condition $\int_\Omega R_h u \mathrm{d}x = \int_\Omega u \mathrm{d}x$, satisfies the following stability estimate:

$$\|R_h u\|_{L^\infty(\Omega)} \le C\ell_h \|u\|_{L^\infty(\Omega)} \quad \forall\, u \in C(\overline{\Omega}) \cap H^1(\Omega), \tag{2.4}$$

where ℓ_h is defined in (1.6).

Remark 2.1 Since $C(\overline{\Omega}) \cap H^1(\Omega)$ is dense in $C(\overline{\Omega})$, the stability inequality (2.4) implies that the Ritz projection has an extension $R_h : C(\overline{\Omega}) \to S_h$. The maximum-norm stability of the Ritz projection in Theorem 2.1 immediately implies (1.5) for the solutions of (1.1)–(1.2).

The L^∞ stability of the Ritz projection in Theorem 2.1 immediately implies that the solutions of (1.1)–(1.2) have the following property:

$$\|I_h u - u_h\|_{L^\infty(\Omega)} = \|R_h(I_h u - u)\|_{L^\infty(\Omega)} \le C\ell_h \|I_h u - u\|_{L^\infty(\Omega)}. \tag{2.5}$$

and therefore

$$\|u - u_h\|_{L^\infty(\Omega)} \le C\ell_h \|u - I_h u\|_{L^\infty(\Omega)}. \tag{2.6}$$

The inequality above is called the best approximation property in maximum norm. By using this best approximation property (2.6) and the regularity result in Lemma 5.1, we can prove the following maximum-norm error estimate, which is optimal with respect to regularity of f.

Corollary 2.1 *Let $f \in W^{k,p}(\Omega)$, where k is a nonnegative integer and $p \ge 2$ is a real number such that $(k,p) \ne (0,2)$ and $(1 - \frac{1}{p})\frac{2\theta_j}{\pi}$ is not an integer for $j = 0, 1, \dots, m-1$. Then the solutions of* (1.1)–(1.2) *satisfy the error bound in* (1.14).

The rest part of this paper is devoted to the proof of Theorem 2.1 and Corollary 2.1. For simplicity, in the proof of Theorem 2.1 we assume that there is only one reentrant corner at z_0 with $\theta_0 \in (\pi, 2\pi)$, with $\theta_j \in (0, \pi)$ for $j = 1, \dots, m-1$, and assume that the mesh is locally refined only at the reentrant corner z_0 with a parameter $\gamma = \gamma_0$. The proof would be similar if there are multiple reentrant corners or the mesh is refined at multiple corners.

3 Local estimates of Green's function in nonconvex polygons

In this section we present local $W^{2,p}$ and $H^{2+\alpha}$ estimates for the solution of the Neumann problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ \partial_n u = g & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

Note that the compatibility condition

$$\int_\Omega f \mathrm{d}x = \int_{\partial\Omega} g \mathrm{d}s \tag{3.2}$$

is automatically satisfied once u is a solution of (3.1). Conversely, this compatibility condition also guarantees the existence and uniqueness of solutions to (3.1) under the normalization condition $\int_\Omega u \mathrm{d}x = 0$ (for uniqueness).

Throughout this paper, we denote by s a number satisfying the following condition (unless otherwise specified):

$$s \in \Big(\frac{1}{2}, \beta\Big) \quad \text{with} \quad \beta = \frac{\pi}{\theta_0} \in \Big(\frac{1}{2}, 1\Big). \tag{3.3}$$

We denote by $H^{s-1}(\Omega)$ the dual space of $H^{1-s}(\Omega)$. Then $L^{p_s}(\Omega) \hookrightarrow H^{s-1}(\Omega)$ for $p_s = 2/(2-s)$.

3.1 H^{1+s} estimates in a polygon

Lemma 3.1 (Existence of lifted functions in $H^{s+1}(\Omega)$) *Let $\phi \in H^{s+\frac{1}{2}}_{\text{piecewise}}(\partial\Omega)$ and $g \in H^{s-\frac{1}{2}}_{\text{piecewise}}(\partial\Omega)$. There exists a lifted function $w \in H^{s+1}(\Omega)$ satisfying*

$$w = \phi \in H^{s+\frac{1}{2}}(\partial\Omega) \quad \textit{and} \quad \partial_n w = g \in H^{s-\frac{1}{2}}_{\text{piecewise}}(\partial\Omega) \quad \textit{on } \partial\Omega,$$

if and only if the following condition holds:

$$\phi \textit{ is continuous at the corners } z_j,\ j = 0, 1, \ldots, m-1. \tag{3.4}$$

In this case, the lifted function w satisfies the following estimate:

$$\|w\|_{H^{s+1}(\Omega)} \le C\Big(\|\phi\|_{H^{s+\frac{1}{2}}_{\text{piecewise}}(\partial\Omega)} + \|g\|_{H^{s-\frac{1}{2}}_{\text{piecewise}}(\partial\Omega)}\Big). \tag{3.5}$$

Proof. Condition (3.4) with $s \in (\frac{1}{2}, 1)$ is exactly the condition (5.3) in [3, Theorem 5.2] in the case $n = 0$ and $m = 2$. As a result, the existence of the lifted function and its boundedness in $H^{s+1}(\Omega)$ follow from [3, Theorem 5.2 and Corollary 5.3]. ∎

The following regularity result can be proved by using Lemma 3.1.

Lemma 3.2 *Let s be any number satisfying (3.3). For any given $f \in H^{s-1}(\Omega)$ and $g \in H^{s-\frac{1}{2}}_{\text{piecewise}}(\partial\Omega)$, the solution of (3.1) is in $H^{s+1}(\Omega)$, and*

$$\|u\|_{H^{s+1}(\Omega)} \le C(\|f\|_{H^{s-1}(\Omega)} + \|g\|_{H^{s-\frac{1}{2}}_{\text{piecewise}}(\partial\Omega)}). \tag{3.6}$$

Proof. Let $\phi = 0$. Then condition (3.4) is fulfilled, and Lemma 3.1 implies that there exists a function $w \in H^{s+1}(\Omega)$ satisfying

$$\partial_n w = g \text{ on } \partial\Omega \quad \text{and} \quad \|w\|_{H^{s+1}(\Omega)} \le C\|g\|_{H^{s-\frac{1}{2}}_{\text{piecewise}}(\partial\Omega)}. \tag{3.7}$$

If u is the solution of (3.1) then $u - w$ is the solution of

$$\begin{cases} -\Delta(u-w) = f + \Delta w & \text{in } \Omega, \\ \partial_n(u-w) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.8}$$

with the homogeneous Neumann boundary condition. Since $w \in H^{s+1}(\Omega)$, it follows that $f + \Delta w \in H^{s-1}(\Omega)$. In [8, (23.3)] it is shown that, when $0 < s < \beta$ as shown in (3.3), the solution to the Neumann problem (3.8) has the following regularity result:

$$\begin{aligned} \|u-w\|_{H^{s+1}(\Omega)} &\le C\|f+\Delta w\|_{H^{s-1}(\Omega)} \\ &\le C\|f\|_{H^{s-1}(\Omega)} + C\|w\|_{H^{s+1}(\Omega)} \\ &\le C\|f\|_{H^{s-1}(\Omega)} + C\|g\|_{H^{s-\frac{1}{2}}_{\text{piecewise}}(\partial\Omega)}. \end{aligned} \tag{3.9}$$

The two estimates in (3.7) and (3.9) imply the desired result in Lemma 3.2. ∎

For a subdomain $D \subset \Omega$ we define the fractional-order Sobolev space on D by

$$\|v\|_{H^{\alpha+k}(D)} = \inf_{\tilde v} \|\tilde v\|_{H^{\alpha+k}(\Omega)} \quad \text{when } \alpha \in (0,1) \text{ and } k \text{ is a nonnegative integer,} \tag{3.10}$$

where the infimum extends over all possible extensions $\tilde v \in H^{\alpha+k}(\Omega)$ such that $\tilde v = v$ on D. The definition in (3.10) is equivalent to the usual definition of Sobolev spaces when D is a fixed Lipschitz domain (see [32, p. 181, Theorem 5]), but is more convenient for analysis when the subdomain D is nonsmooth and not fixed. By using the regularity result in Lemma 3.2, we prove the following local H^{s+1} estimate.

Lemma 3.3 *Let $D = B_d(z) \cap \Omega$ and $D' = B_{2d}(z) \cap \Omega$ be subdomains of Ω, where $z \in \Omega$ and $0 < d <$ diameter(Ω), and let ω be a smooth cut-off function satisfying*

$$\omega(x) \equiv 1, \qquad x \in B_d(z) \tag{3.11a}$$

$$\omega(x) \equiv 0, \qquad x \in \mathbb{R}^2 \backslash B_{3d/2}(z) \tag{3.11b}$$

$$|\nabla^k \omega| \le C_k d^{-k}, \qquad k = 1, 2, \dots \tag{3.11c}$$

Then for any given $f \in L^{p_s}(\Omega)$ and $g = 0$, with $\int_\Omega f \mathrm{d}x = 0$ and $p_s := 2/(2-s)$, the solution of (3.1) *satisfies*

$$\|u - u_D\|_{H^{s+1}(D)} \le C\|f\|_{L^{p_s}(D')} + Cd^{-s}(\|u\|_{L^{2,\infty}(\Omega)} + \|\nabla u\|_{L^{2,\infty}(\Omega)}), \tag{3.12}$$

where u_D is some constant depending on both u and the subdomain D, satisfying $|u_D| \le Cd^{-2}\|u\|_{L^1(\Omega)}$, and $\|\cdot\|_{L^{p,\infty}(\Omega)}$ denotes the weak L^p norm defined by

$$\|w\|_{L^{p,\infty}(\Omega)} := \sqrt{\sup_{\lambda>0} \lambda |\{x \in \Omega : |w(x)|^p \ge \lambda\}|}, \tag{3.13}$$

where $|\{x \in \Omega : |w(x)|^p \ge \lambda\}|$ denotes the measure of the set $\{x \in \Omega : |w(x)|^p \ge \lambda\}$.

Proof. Since $p_s = 2/(2-s)$, the following two Sobolev embedding results hold and will be used frequently:

$$L^{p_s}(\Omega) \hookrightarrow H^{s-1}(\Omega) \quad \text{and} \quad W^{1,p_s}(\Omega) \hookrightarrow H^s(\Omega). \tag{3.14}$$

Let $L^{p,q}(\Omega)$ be the Lorentz space (see [15, §1.4]), and let $W^{1,p,q}(\Omega)$ be the space of functions w such that

$$\|w\|_{W^{1,p,q}(\Omega)} := \left(\|u\|^p_{L^{p,q}(\Omega)} + \|\nabla u\|^p_{L^{p,q}(\Omega)}\right)^{\frac{1}{p}} < \infty.$$

In the case $q = \infty$, the $L^{p,q}(\Omega)$ norm is equivalent to the definition in (3.13).

Let $E : L^1(\Omega) \to L^1(\mathbb{R}^2)$ be Stein's extension operator as described in [32, p. 181, Theorem 5], which is bounded from $W^{k,p}(\Omega)$ to $W^{k,p}(\mathbb{R}^2)$ for $1 \le p \le \infty$ and $k \ge 0$. According to [26, Example 7], the real interpolation space between $W^{1,p_1}(\Omega)$ and $W^{1,p_2}(\Omega)$ is

$$(W^{1,p_1}(\Omega), W^{1,p_1}(\Omega))_{\theta,q} = W^{1,p,q}(\Omega) \quad \text{with } \frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2} \text{ and } 1 \le q \le \infty.$$

By choosing $q = \infty$ and using the real interpolation result, we obtain that Stein's extension operator is bounded from $W^{1,p,\infty}(\Omega)$ to $W^{1,p,\infty}(\mathbb{R}^2)$ for $1 < p < \infty$. We denote $\bar{u} = Eu$ so that $\bar{u} = u$ in Ω and

$$\|\bar{u}\|_{L^{2,\infty}(\mathbb{R}^2)} + \|\nabla\bar{u}\|_{L^{2,\infty}(\mathbb{R}^2)} \le C(\|u\|_{L^{2,\infty}(\Omega)} + \|\nabla u\|_{L^{2,\infty}(\Omega)}). \tag{3.15}$$

Since $f \in L^{p_s}(\Omega) \hookrightarrow H^{s-1}(\Omega)$, with $\int_\Omega f \mathrm{d}x = 0$ and $g = 0$, it follows that (3.1) has a unique solution in $H^1(\Omega)$ and therefore the right-hand side of (3.15) is bounded ($L^{2,\infty}$ norm is weaker than L^2 norm).

Let $\bar{u}_d$ be the average of $\bar{u}$ on $B_{2d}(z)$. Then $\tilde{u} := \omega(\bar{u} - \bar{u}_d)$ is the solution of

$$\begin{cases} -\Delta\tilde{u} = \tilde{f}, & \text{in } \Omega, \\ \partial_n\tilde{u} = \tilde{g}\cdot n & \text{on } \partial\Omega, \end{cases} \tag{3.16}$$

where

$$\tilde{f} = f\omega - 2\nabla\bar{u}\cdot\nabla\omega - (\bar{u} - \bar{u}_d)\Delta\omega, \tag{3.17}$$

$$\tilde{g} = (\bar{u} - \bar{u}_d)\nabla\omega \quad \text{and} \quad \tilde{g}\cdot n = (\bar{u} - \bar{u}_d)\nabla\omega\cdot n. \tag{3.18}$$

Since $f \in L^{p_s}(\Omega) \hookrightarrow H^{s-1}(\Omega)$ and $\nabla\bar{u} \in L^2(\Omega) \hookrightarrow H^{s-1}(\Omega)$, it follows that $\tilde{f} \in H^{s-1}(\Omega)$. Since $\bar{u} \in H^1(\Omega) \hookrightarrow W^{1,p_s}(\Omega) \hookrightarrow H^s(\Omega)$, it follows that

$$\tilde{g} \in W^{1,p_s}(\Omega) \hookrightarrow H^s(\Omega) \quad \text{and} \quad \tilde{g}\cdot n = (\bar{u} - \bar{u}_d)\nabla\omega\cdot n \in H^{s-\frac{1}{2}}_{\text{piecewise}}(\partial\Omega).$$

Then Lemma 3.2 implies that

$$\begin{aligned} \|\tilde{u}\|_{H^{s+1}(\Omega)} &\le C(\|\tilde{f}\|_{H^{s-1}(\Omega)} + \|\tilde{g}\cdot n\|_{H^{s-\frac{1}{2}}_{\text{piecewise}}(\partial\Omega)}) \\ &\le C(\|\tilde{f}\|_{H^{s-1}(\Omega)} + \|\tilde{g}\|_{H^s(\Omega)}) \\ &\le C(\|\tilde{f}\|_{H^{s-1}(\Omega)} + \|\tilde{g}\|_{W^{1,p_s}(\Omega)}) \\ &\le C\|\omega f\|_{L^{p_s}(\Omega)} + C\|\nabla\bar{u}\cdot\nabla\omega\|_{L^{p_s}(\Omega)} \\ &\quad + C\|(\bar{u} - \bar{u}_d)\Delta\omega\|_{L^{p_s}(\Omega)} + C\|(\bar{u} - \bar{u}_d)\nabla\omega\|_{W^{1,p_s}(\Omega)}. \end{aligned} \tag{3.19}$$

By using (3.11c) to estimate $\nabla\omega$ and $\Delta\omega$ on the right-hand side of (3.19), we have

$$\begin{aligned} \|\tilde{u}\|_{H^{s+1}(\Omega)} &\le C\|\omega f\|_{L^{p_s}(\Omega)} + Cd^{-2}\|\bar{u} - \bar{u}_d\|_{L^{p_s}(B_{2d}(z))} + Cd^{-1}\|\nabla\bar{u}\|_{L^{p_s}(B_{2d}(z))} \\ &\le C\|f\|_{L^{p_s}(D')} + Cd^{-1}\|\nabla\bar{u}\|_{L^{p_s}(B_{2d}(z))} \qquad \text{(by Poincaré's inequality)} \\ &\le C\|f\|_{L^{p_s}(D')} + Cd^{-2+\frac{2}{p_s}}\|\nabla\bar{u}\|_{L^{2,\infty}(\mathbb{R}^2)} \qquad \text{([15, Exercise 1.1.15])} \\ &\le C\|f\|_{L^{p_s}(D')} + Cd^{-2+\frac{2}{p_s}}(\|u\|_{L^{2,\infty}(\Omega)} + \|\nabla u\|_{L^{2,\infty}(\Omega)}). \end{aligned} \tag{3.20}$$

Since $p_s = \frac{2}{2-s}$ implies $-2 + \frac{2}{p_s} = -s$, and $\tilde{u} = \omega(\bar{u} - \bar{u}_d)$ is an extension of $u - \bar{u}_d$ from D to Ω, the last inequality implies (3.12) in view of the definition in (3.10), with $u_D = \bar{u}_d$, i.e., the average of $\bar{u} = Eu$ in $B_{2d}(z)$. Therefore,

$$|u_D| = \frac{1}{|B_{2d}(z)|}\left|\int_{B_{2d}(z)} Eu\mathrm{d}x\right| \le Cd^{-2}\int_{B_{2d}(z)} |Eu|\mathrm{d}x \le Cd^{-2}\|u\|_{L^1(\Omega)},$$

where the boundedness of the extension operator $E : L^1(\Omega) \to L^1(\mathbb{R}^2)$ is used. ∎

3.2 A priori $W^{2,p}$ and $H^{2+\alpha}$ estimates in a polygon

It is well known that in a nonconvex polygon, $f \in L^p(\Omega)$ and $g \in W^{1-1/p}_{\text{piecewise}}(\partial\Omega)$ with $p > 1$ may not imply $u \in W^{2,p}(\Omega)$ for the solution of (3.1). However, for a solution u which is a priori in $W^{2,p}(\Omega)$, we still have the following $W^{2,p}$ estimates.

Lemma 3.4 (A priori $W^{2,p}$ estimates) *Let $u \in W^{2,p}(\Omega)$, with $p > 1$, be a solution of* (3.1)*, and assume that the following conditions are satisfied:*

$$2 - \frac{2}{p} \ \text{ and } \ \left(1 - \frac{1}{p}\right)\frac{2\theta_j}{\pi} \ \text{ are not integers for } j = 0, 1, \dots, m-1. \tag{3.21}$$

Then

$$|u|_{W^{1,p}(\Omega)} + |u|_{W^{2,p}(\Omega)} \le C(\|f\|_{L^p(\Omega)} + \|g\|_{W^{1-1/p,p}_{\text{piecewise}}(\partial\Omega)}), \tag{3.22}$$

where

$$W^{1-1/p,p}_{\text{piecewise}}(\partial\Omega) = \{q \in L^p(\partial\Omega) : \ q \in W^{1-1/p,p}(\Gamma_j),\ j = 0, 1, \dots, m-1\}. \tag{3.23}$$

In particular, if $u \in H^2(\Omega)$ is a solution of (3.1) *with $g = 0$, then*

$$|u|_{H^1(\Omega)} + |u|_{H^2(\Omega)} \le C\|f\|_{L^2(\Omega)}. \tag{3.24}$$

Proof. From [16, Corollary 4.4.4.14] we know that for the given $f \in L^p(\Omega)$ and $g \in W^{1-1/p}_{\text{piecewise}}(\partial\Omega)$ satisfying the compatibility condition $\int_\Omega f \mathrm{d}x = \int_{\partial\Omega} g \mathrm{d}s$ (which must be true if $u \in W^{2,p}(\Omega)$ is the solution of (3.1)), there exist some constants $c_{j,n}$, $n = 1, \dots, K_j$ and $j = 0, \dots, m-1$, such that

$$u - \sum_{j=0}^{m-1} \sum_{n=1}^{K_j} c_{j,n} S_{j,n} \in W^{2,p}(\Omega), \tag{3.25}$$

where $S_{j,n}$, $n = 1, \dots, K_j$, are some weakly singular functions (independent of f and g) not in $W^{2,p}(\Omega)$, but $\Delta S_{j,n} \in L^p(\Omega)$ and $\partial_n S_{j,n} = 0$ on $\partial\Omega$. The number of such singular terms depend only on Ω and p. In fact, we have

$$S_{j,n}(x) = \phi(|x - z_j|)|x - z_j|^{\frac{n\pi}{\theta_j}} \cos\Big(\frac{n\pi}{\theta_j}\Theta_j(x)\Big), \tag{3.26}$$

where $\Theta_j(x)$ denotes the angle between the two vectors $x - z_j$ and $z_{j+1} - z_j$, and $\phi : \mathbb{R} \to \mathbb{R}$ is some smooth cut-off function such that $\phi(|x - z_j|) = 0$ when x is outside a small neighborhood of the corner z_j, and K_j is the largest integer such that $K_j < \big(1 - \frac{1}{p}\big)\frac{2\theta_j}{\pi}$,

Let X be the Banach space spanned by $W^{2,p}(\Omega)$ and $S_{j,n}$, with $n = 1, \dots, K_j$ and $j = 0, 1, \dots, m-1$, and define $X_0 = \{v \in X : \int_\Omega v \mathrm{d}x = 0\}$. Let $Y = \{(f, g) \in L^p(\Omega) \times W^{1-1/p,p}_{\text{piecewise}}(\partial\Omega) : \int_\Omega f \mathrm{d}x = \int_{\partial\Omega} g \mathrm{d}s\}$. Then the above-mentioned regularity result implies that the operator

$$(\Delta, \partial_n) : X_0 \to Y$$

is one-to-one, bounded and onto. Therefore, there exists a bounded right inverse of the above operator. This implies that

$$\sum_{j=0}^{m-1}\sum_{n=1}^{K_j}|c_{j,n}| + \left\| u - c - \sum_{j=0}^{m-1}\sum_{n=1}^{K_j} c_{j,n}S_{j,n}\right\|_{W^{2,p}(\Omega)} \le C(\|f\|_{L^p(\Omega)} + \|g\|_{W^{1-1/p}_{\text{piecewise}}(\partial\Omega)}) \quad (3.27)$$

for some constant $c = \frac{1}{|\Omega|}\int_\Omega u\mathrm{d}x$. If u is a priori in $W^{2,p}(\Omega)$ then $c_{j,n} = 0$ for $n = 1, \ldots, K_j$ and $j = 0, \ldots, m-1$. In this case, the above inequality implies (3.22).

In the case $p = 2$ and $g = 0$, it is shown in [16, Theorem 4.3.1.4] that any solution $u \in H^2(\Omega)$ of (3.1) with $g = 0$ satisfies the following estimate:

$$\|u\|_{H^2(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}).$$

Let $u_\Omega = \frac{1}{|\Omega|}\int_\Omega u\mathrm{d}x$ be the average of u over Ω. Then $u - u_\Omega$ is also a solution of (3.1) with $g = 0$. Replacing u by $u - u_\Omega$ in the inequality above yields

$$\|u - u_\Omega\|_{H^2(\Omega)} \le C(\|f\|_{L^2(\Omega)} + \|u - u_\Omega\|_{L^2(\Omega)}) \le C(\|f\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\Omega)}).$$

By substituting the standard energy estimate $\|\nabla u\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}$ into the inequality above, we obtain (3.24). ∎

Similar as Lemma 3.4, for a solution which is a priori in $H^{2+\alpha}(\Omega)$, the standard $H^{2+\alpha}$ estimates still hold. The proof of this result requires using the existence of a function $w \in H^{2+\alpha}(\Omega)$ satisfying $\partial_n w = g$, for any $g \in H^{\alpha+\frac{1}{2}}_{\text{piecewise}}(\partial\Omega)$. This is guaranteed by the following lemma.

Lemma 3.5 (Existence of lifted functions in $H^{2+\alpha}(\Omega)$) *Let $\alpha \in (0,1)$, and let $\phi \in H^{\frac{3}{2}+\alpha}_{\text{piecewise}}(\partial\Omega)$ and $g \in H^{\frac{1}{2}+\alpha}_{\text{piecewise}}(\partial\Omega)$. There exists a lifted function $w \in H^{2+\alpha}(\Omega)$ satisfying*

$$w = \phi \quad \text{and} \quad \partial_n w = g \quad \text{on } \partial\Omega,$$

if and only if the following conditions hold:

$$\phi \text{ and } (\partial_\tau\phi)\tau + gn \text{ are both continuous at the corners } z_j,\ j = 0, 1, \ldots, m-1, \quad (3.28)$$

where τ and n denote the unit tangential and normal vectors on the boundary $\partial\Omega$, respectively. In this case, the lifted function w satisfies the following estimate:

$$\|w\|_{H^{2+\alpha}(\Omega)} \le C\Big(\|\phi\|_{H^{\frac{3}{2}+\alpha}_{\text{piecewise}}(\partial\Omega)} + \|g\|_{H^{\frac{1}{2}+\alpha}_{\text{piecewise}}(\partial\Omega)}\Big). \quad (3.29)$$

Proof. Condition (3.4) is exactly the condition (5.3) in [3, Theorem 5.2] in the case $n = 0, 1$ and $m = 2$ therein. As a result, the existence of the lifted function and its boundedness in $H^{2+\alpha}(\Omega)$ follow from [3, Theorem 5.2 and Corollary 5.3]. ∎

Lemma 3.6 (A priori $H^{2+\alpha}$ estimate) *Let $u \in H^{2+\alpha}(\Omega)$ be a solution of (3.1), with*

$$\alpha \in (0,1) \quad \text{and} \quad \frac{(1+\alpha)\theta_j}{\pi} \text{ is not an integer for } j = 0, 1, \ldots, m-1. \quad (3.30)$$

Then there exist constants c and C such that

$$\|u - c\|_{H^{2+\alpha}(\Omega)} \le C(\|f\|_{H^\alpha(\Omega)} + \|g\|_{H^{\frac{1}{2}+\alpha}_{\text{piecewise}}(\partial\Omega)}),$$

where

$$H^{\frac{1}{2}+\alpha}_{\rm piecewise}(\partial\Omega) = \{q\in L^2(\partial\Omega):\ q\in H^{\frac{1}{2}+\alpha}(\Gamma_j),\ j=0,1,\dots,m-1\},$$

$|c|\le C\|u\|_{L^1(\Omega)}$ *and the constant* C *is independent of* u*,* f *and* g*.*

Proof. We define ϕ to be a cubic polynomial on each side Γ_j, $j=0,1,\dots,m-1$, such that

$$\phi=0,\quad \partial_{\tau_+}\phi=\frac{g_- - g_+ n_+\cdot n_-}{\tau_+\cdot n_-}\quad\text{and}\quad \partial_{\tau_-}\phi=\frac{g_+ - g_- n_-\cdot n_+}{\tau_-\cdot n_+},$$

at every corner z_j from both sides of the corner, where τ_- and n_- denote the tangential and normal vectors on the left side of the corner, τ_+ and n_+ denote the normal vectors on the right side of the corner, and g_- and g_+ denote the values of g on the left and right sides of the corner. Then ϕ and g satisfy the conditions in (3.28). In fact, the above expressions of $\partial_{\tau_+}\phi$ and $\partial_{\tau_-}\phi$ at a corner can be solved from the following two equations:

$$\begin{aligned}
&[(\partial_{\tau_+}\phi_+)\tau_+ + g_+n_+]\cdot n_- = [(\partial_{\tau_-}\phi_-)\tau_- + g_-n_-]\cdot n_- = g_-\,,\\
&[(\partial_{\tau_-}\phi_-)\tau_- + g_-n_-]\cdot n_+ = [(\partial_{\tau_+}\phi_+)\tau_+ + g_+n_+]\cdot n_+ = g_+\,.
\end{aligned}$$

Therefore, Lemma 3.5 implies that there exists a lifted function $w\in H^{\alpha+2}(\Omega)$ satisfying

$$\partial_n w = g\in H^{\alpha+\frac{1}{2}}_{\rm piecewise}(\partial\Omega)\ \text{ on } \partial\Omega\quad\text{and}\quad \|w\|_{H^{2+\alpha}(\Omega)}\le C\|g\|_{H^{\frac{1}{2}+\alpha}_{\rm piecewise}(\partial\Omega)}. \tag{3.31}$$

If u is the solution of (3.1) then $u-w$ is the solution of

$$\begin{cases} -\Delta(u-w) = f+\Delta w & \text{in } \Omega,\\ \partial_n(u-w)=0 & \text{on } \partial\Omega. \end{cases} \tag{3.32}$$

with the homogeneous Neumann boundary condition. Since $w\in H^{\alpha+2}(\Omega)$, it follows that $f+\Delta w\in H^\alpha(\Omega)$. In [8, (5.11) and p. 210] (also see [7, page 24]) it is shown that the solution of the PDE problem (3.32) with $f+\Delta w\in H^\alpha(\Omega)$ can be written as a singular part plus a regular part (similarly as in the proof of Lemma 3.4), i.e.,

$$u-w-\sum_{j=0}^{m-1}\sum_{n=1}^{K_j}c_{j,n}S_{j,n}\in H^{2+\alpha}(\Omega)\quad\text{with}\quad S_{j,n}(x)=|x-z_j|^{\frac{n\pi}{\theta_j}}\cos\Big(\frac{n\pi}{\theta_j}\Theta_j(x)\Big), \tag{3.33}$$

where K_j is the largest integer such that $K_j<\Big(1-\dfrac{1}{p}\Big)\dfrac{2\theta_j}{\pi}$. Moreover, the following estimate holds (similarly as in the proof of Lemma 3.4):

$$\begin{aligned}
\sum_{j=0}^{m-1}\sum_{n=1}^{K_j}|c_{j,n}| + \Big\|u-w-c_0-\sum_{j=0}^{m-1}\sum_{n=1}^{K_j}c_{j,n}S_{j,n}\Big\|_{H^{2+\alpha}(\Omega)} &\le C\|f+\Delta w\|_{H^\alpha(\Omega)}\\
&\le C\|f\|_{H^\alpha(\Omega)}+\|w\|_{H^{2+\alpha}(\Omega)}.
\end{aligned}$$

where $c_0=\frac{1}{|\Omega|}\int_\Omega(u-w)\mathrm{d}x =: c+c_1$, where $c=\frac{1}{|\Omega|}\int_\Omega u\mathrm{d}x$ and $c_1=\frac{1}{|\Omega|}\int_\Omega w\mathrm{d}x$. By using the triangle inequality and (3.31), the inequality above implies that

$$\begin{aligned}
\sum_{j=0}^{m-1}\sum_{n=1}^{K_j}|c_{j,n}| + \Big\|u-c-\sum_{j=0}^{m-1}\sum_{n=1}^{K_j}c_{j,n}S_{j,n}\Big\|_{H^{2+\alpha}(\Omega)} &\le C\|f\|_{H^\alpha(\Omega)}+C\|w\|_{H^{2+\alpha}(\Omega)}+Cc_1\\
&\le C(\|f\|_{H^\alpha(\Omega)}+\|g\|_{H^{\frac{1}{2}+\alpha}_{\rm piecewise}(\partial\Omega)}).
\end{aligned}$$

If u is a priori in $H^{2+\alpha}(\Omega)$ then $c_{j,n} = 0$ for $n = 1, \ldots, K_j$ and $j = 0, \ldots, m-1$. In this case, the above inequality implies the desired result in Lemma 3.6. ∎

3.3 $W^{2,p}$ and $H^{2+\alpha}$ estimates away from the reentrant corner

As mentioned at the end of Section 2, we assume that there is only one reentrant corner at z_0 with $\theta_0 \in (\pi, 2\pi)$, with $\theta_j \in (0, \pi)$ for $j = 1, \ldots, m-1$. We define

$$\alpha_0 := \min\left(\frac{2\pi}{\theta_0} - 1\,,\ \min_{1\le j\le m-1}\left(\frac{\pi}{\theta_j}\right) - 1\right) \in (0,1), \tag{3.34}$$

$$p_0 := \frac{2}{1-\alpha_0}. \tag{3.35}$$

Then any $p \in (2, p_0)$ and $\alpha \in (0, \alpha_0)$ satisfy the conditions in (3.21) and (3.30). Moreover, we have the following qualitative regularity results away from the reentrant corner as a result of the decomposition in (3.25) and (3.33).

Lemma 3.7 For any $0 < d < \text{diameter}(\Omega)$, the solution of (3.1) has the following properties:

(1) If $f \in L^p(\Omega)$ and $g \in W^{1-1/p,p}_{\text{piecewise}}(\partial\Omega)$, then $u \in W^{2,p}(\Omega\backslash\overline{B_d(z_0)})$;

(2) If $f \in H^\alpha(\Omega)$ and $g \in H^{1/2+\alpha}_{\text{piecewise}}(\partial\Omega)$, then $u \in H^{2+\alpha}(\Omega\backslash\overline{B_d(z_0)})$.

For the Neumann problem (3.1) with $g = 0$, by using Lemma 3.4 and 3.6 we have the following local quantitative estimates away from the reentrant corner.

Lemma 3.8 (Local $W^{2,p}$ and $H^{2+\alpha}$ estimates away from the reentrant corner) *Let $p \in [2, p_0)$ and $\alpha \in (0, \alpha_0)$, and let u be a solution of (3.1) with $g = 0$. Let $D = B_d(z) \cap \Omega$ and $D' = B_{d+d_\star}(z) \cap \Omega$, with $z \in \Omega$, be subdomains of Ω such that*

$$d \le K d_\star \quad \text{and} \quad d + d_\star < |z - z_0|.$$

Then

$$|u|_{H^1(D)} \le C_K(d_\star\|f\|_{L^2(D')} + \|\nabla u\|_{L^{2,\infty}(D')}), \qquad \text{if } f \in L^2(D'), \tag{3.36a}$$

$$|u|_{W^{1,p}(D)} + |u|_{W^{2,p}(D)} \le C_K\left(\|f\|_{L^p(D')} + d_\star^{-2+2/p}\|\nabla u\|_{L^{2,\infty}(D')}\right), \quad \text{if } f \in L^p(D'). \tag{3.36b}$$

Moreover, there exists $\beta \in (0,1)$ such that D can be covered by a bounded number of smaller subdomains $D_j = B_{\beta d/2}(\zeta_j) \cap \Omega$ (with some points $\zeta_j \in D$), $j = 1, \ldots, J_\beta$, such that

$$\|u - u_{D_j'}\|_{H^{2+\alpha}(D_j')} \le C_K d_\star^{-\alpha}\left(\|f\|_{L^2(D')} + d_\star\|\nabla f\|_{L^2(D')} + d_\star^{-1}\|\nabla u\|_{L^{2,\infty}(D')}\right), \quad \text{if } f \in H^1(D'), \tag{3.37}$$

where $D_j' = B_{\beta d}(\zeta_j) \cap \Omega$ and $u_{D_j'}$ are some constants depending on both u and D_j, satisfying $|u_{D_j'}| \le C d_\star^{-2}\|u\|_{L^1(D')}$.

Remark 3.1 In Section 4, we will decompose the domain into some subdomains and cover each subdomain by a finite number of balls $B_d(z)$ (with several different z in the subdomain). Then we apply the estimates on each $B_{d_j/8}(z)$ to obtain Lemma 4.2.

In the estimation of the Green function and the regularized Green function (see Lemmas 3.9, 4.1 and 4.2), and in the proof of Theorem 2.1, we have to frequently use some $H^{2+\alpha}$ estimates of the Green function. Since we cannot directly prove such $H^{2+\alpha}$ estimates on the subdomain $D = B_d(z) \cap \Omega$ (which may intersects two adjacent sides of Ω and therefore nonconvex), we have to cover D by some smaller convex subdomains $D_j = B_{\beta d/2}(\zeta_j) \cap \Omega$ which intersect at most one edge of Ω (and therefore convex) and use the estimates on these convex subdomains D_j. This is the motivation of dividing D into subdomains D_j, $j = 1, \ldots, J_\beta$.

Proof. Without loss of generality, we can assume that f is qualitatively smooth enough, provided our quantitative estimates presented below are independent of the assumed extra smoothness of f.

Let $\zeta \in D$ be any fixed point and consider $D_\zeta := B_{\beta d}(\zeta) \cap \Omega$ and $D'_\zeta := B_{\beta d + \beta d_\star}(\zeta) \cap \Omega$, with $\beta = \frac{1}{2(1+K)} \sin(2\pi - \theta_0)$. Then we have $\beta d + \beta d_\star < \frac{1}{2}|\zeta - z_0| \sin(2\pi - \theta_0)$, which guarantees that the disk $B_{\beta d + \beta d_\star}(\zeta)$ can intersects at most one side of the wedge at corner z_0. As a result, the subdomain D'_ζ is convex and $|D'_\zeta| \geq d_\star^2/C$. In this case, the following Poincaré's inequality holds (cf. [4, Theorems 1.1–1.2]):

$$\|u - u_{D'_\zeta}\|_{L^q(D'_\zeta)} \leq C d_\star \|\nabla u\|_{L^q(D'_\zeta)} \quad \forall\, 1 \leq q \leq \infty, \tag{3.38}$$

where $u_{D'_\zeta}$ denotes the average of u on D'_ζ.

Since $D = B_d(z) \cap \Omega$ can be covered by a finite number of subdomains of the type $B_{\beta d/2}(\zeta) \cap \Omega$ with $\zeta \in D$ (the number depends only on β, independent of z, d, $d_\star$), we only need to prove the following estimates in $D_\zeta = B_{\beta d}(\zeta) \cap \Omega$:

$$|u|_{H^1(D_\zeta)} \leq C_K \big(d_\star \|f\|_{L^2(D'_\zeta)} + \|\nabla u\|_{L^{2,\infty}(D'_\zeta)}\big), \tag{3.39a}$$

$$|u|_{W^{1,p}(D_\zeta)} + |u|_{W^{2,p}(D_\zeta)} \leq C_K \big(\|f\|_{L^p(D'_\zeta)} + d_\star^{-2+2/p} \|\nabla u\|_{L^{2,\infty}(D'_\zeta)}\big) \quad \text{for } p \in [2, p_0), \tag{3.39b}$$

$$\|u - c\|_{H^{2+\alpha}(D_\zeta)} \leq C_K d_\star^{-\alpha} \big(\|f\|_{L^2(D'_\zeta)} + d_\star \|\nabla f\|_{L^2(D'_\zeta)} + d_\star^{-1} \|\nabla u\|_{L^{2,\infty}(D'_\zeta)}\big), \tag{3.39c}$$

where c is some constant depending on u and D_ζ, satisfying that $|c| \leq C d_\star^{-2} \|u\|_{L^1(D'_\zeta)}$. Then (3.36)–(3.37) follow from (3.39).

To prove (3.39a)–(3.39c), we introduce a convex subdomain $\widetilde{D}'_\zeta := B_{\beta d + \beta d_\star/2}(\zeta) \cap \Omega$, which satisfies $D_\zeta \subset \widetilde{D}'_\zeta \subset D'_\zeta$, and define a smooth cut-off function $\omega(x)$ such that

$$\omega(x) \equiv 1, \quad x \in B_{\beta d}(\zeta) \tag{3.40a}$$

$$\omega(x) \equiv 0, \quad x \in \mathbb{R}^2 \backslash B_{\beta d + \beta d_\star/2}(\zeta) \tag{3.40b}$$

$$|\nabla^k \omega| \leq C_k d_\star^{-k}, \quad k = 1, 2, \ldots \tag{3.40c}$$

In view of Lemma 3.7, we have $\tilde{u} = \omega(u - u_{\widetilde{D}'_\zeta}) \in W^{2,p}(\Omega)$, and it is the solution of the equation

$$\begin{cases} -\Delta \tilde{u} = \tilde{f} & \text{in } \Omega, \\ \partial_n \tilde{u} = \tilde{g} \cdot n & \text{on } \partial\Omega, \end{cases} \tag{3.41}$$

where

$$\tilde{f} = f\omega - 2\nabla u \cdot \nabla \omega - (u - u_{\widetilde{D}'_\zeta})\Delta\omega \quad \text{and} \quad \tilde{g} = (u - u_{\widetilde{D}'_\zeta})\nabla\omega. \tag{3.42}$$

For $p \in (2, p_0)$, we choose $q \in (1, 2)$ satisfying $1 = 2/q - 2/p$ so that $W^{1,q}(\Omega) \hookrightarrow L^p(\Omega)$ and $W^{2,q}(\Omega) \hookrightarrow W^{1,p}(\Omega)$. Since $1 < q \le p < p_0$, it follows that q also satisfies the condition (3.21), as explained below (3.35). Then Lemma 3.4 implies that

$$\begin{aligned}
\|\nabla\tilde u\|_{W^{1,q}(\Omega)} &\le C\|\tilde f\|_{L^q(\Omega)} + C\|\tilde g\cdot n\|_{W^{1-1/q,q}_{\text{piecewise}}(\partial\Omega)} \\
&\le C\|\tilde f\|_{L^q(\Omega)} + C\|\tilde g\|_{W^{1,q}(\Omega)} \\
&\le C(\|f\|_{L^q(\widetilde D'_\zeta)} + d_\star^{-1}\|\nabla u\|_{L^q(\widetilde D'_\zeta)} + d_\star^{-2}\|u - u_{\widetilde D'_\zeta}\|_{L^q(\widetilde D'_\zeta)}) \\
&\le C(\|f\|_{L^q(\widetilde D'_\zeta)} + d_\star^{-1}\|\nabla u\|_{L^q(\widetilde D'_\zeta)}) \quad \text{(the Poincaré inequality (3.38) is used)} \\
&\le C(d_\star^{-1+2/q}\|f\|_{L^2(\widetilde D'_\zeta)} + d_\star^{-2+2/q}\|\nabla u\|_{L^{2,\infty}(\widetilde D'_\zeta)})
\end{aligned} \tag{3.43}$$

and so

$$\begin{aligned}
\|\nabla\tilde u\|_{L^p(\Omega)} &\le C\|\nabla\tilde u\|_{W^{1,q}(\Omega)} \\
&\le C(d_\star^{-1+2/q}\|f\|_{L^2(\widetilde D'_\zeta)} + d_\star^{-2+2/q}\|\nabla u\|_{L^{2,\infty}(\widetilde D'_\zeta)}) \\
&\le C(d_\star^{2/p}\|f\|_{L^2(\widetilde D'_\zeta)} + d_\star^{-1+2/p}\|\nabla u\|_{L^{2,\infty}(\widetilde D'_\zeta)}),
\end{aligned} \tag{3.44}$$

where $1 = 2/q - 2/p$ is used in the last inequality. Since $\omega = 1$ on D_ζ and $\tilde u = \omega(u - u_{\widetilde D'_\zeta})$, the last inequality implies that

$$\|\nabla u\|_{L^p(D_\zeta)} \le C(d_\star^{2/p}\|f\|_{L^2(\widetilde D'_\zeta)} + d_\star^{-1+2/p}\|\nabla u\|_{L^{2,\infty}(\widetilde D'_\zeta)}). \tag{3.45}$$

By using Hölder's inequality, we can further derive the following two inequalities:

$$\|\nabla u\|_{L^2(D_\zeta)} \le d_\star^{1-2/p}\|\nabla u\|_{L^p(D_\zeta)} \le C(d_\star\|f\|_{L^2(\widetilde D'_\zeta)} + \|\nabla u\|_{L^{2,\infty}(\widetilde D'_\zeta)}), \tag{3.46}$$

$$\|\nabla u\|_{L^p(D_\zeta)} \le C(d_\star\|f\|_{L^p(\widetilde D'_\zeta)} + d_\star^{-1+2/p}\|\nabla u\|_{L^{2,\infty}(\widetilde D'_\zeta)}). \tag{3.47}$$

This proves (3.39a).

Similarly as (3.46)–(3.47), replacing D_ζ and $\widetilde D'_\zeta$ by $\widetilde D'_\zeta$ and D'_ζ, respectively, we also have the following estimates:

$$\|\nabla u\|_{L^2(\widetilde D'_\zeta)} \le C(d_\star\|f\|_{L^2(D'_\zeta)} + \|\nabla u\|_{L^{2,\infty}(D'_\zeta)}), \tag{3.48}$$

$$\|\nabla u\|_{L^p(\widetilde D'_\zeta)} \le C(d_\star\|f\|_{L^p(D'_\zeta)} + d_\star^{-1+2/p}\|\nabla u\|_{L^{2,\infty}(D'_\zeta)}). \tag{3.49}$$

The last inequality and Lemma 3.4 imply

$$\begin{aligned}
|\tilde u|_{W^{1,p}(\Omega)} + |\tilde u|_{W^{2,p}(\Omega)} &\le C\|\tilde f\|_{L^p(\Omega)} + C\|\tilde g\cdot n\|_{W^{1-1/p,p}_{\text{piecewise}}(\partial\Omega)} \\
&\le C\|\tilde f\|_{L^p(\Omega)} + C\|\tilde g\|_{W^{1,p}(\Omega)} \\
&\le C(\|f\|_{L^p(\widetilde D'_\zeta)} + d_\star^{-1}\|\nabla u\|_{L^p(\widetilde D'_\zeta)} + d_\star^{-2}\|u - u_{\widetilde D'_\zeta}\|_{L^p(\widetilde D'_\zeta)}) \\
&\le C(\|f\|_{L^p(\widetilde D'_\zeta)} + d_\star^{-1}\|\nabla u\|_{L^p(\widetilde D'_\zeta)}) \\
&\le C\left(\|f\|_{L^p(\widetilde D'_\zeta)} + d_\star^{-1}(d_\star\|f\|_{L^p(D'_\zeta)} + d_\star^{-1+2/p}\|\nabla u\|_{L^{2,\infty}(D'_\zeta)})\right) \\
&\le C(\|f\|_{L^p(D'_\zeta)} + d_\star^{-2+2/p}\|\nabla u\|_{L^{2,\infty}(D'_\zeta)}).
\end{aligned}$$

This proves (3.39b) in the case $p \in (2, p_0)$.

For $p=2$ we have $\tilde u=\tilde u_1+\tilde u_2$, where $\tilde u_1$ and $\tilde u_2$ are solutions of

$$\begin{cases} -\Delta \tilde u_1 = \tilde f & \text{in } \Omega, \\ \partial_n \tilde u_1 = 0 & \text{on } \partial\Omega, \end{cases} \quad \text{and} \quad \begin{cases} -\Delta \tilde u_2 = 0 & \text{in } \Omega, \\ \partial_n \tilde u_2 = \tilde g\cdot n & \text{on } \partial\Omega. \end{cases}$$

By applying (3.24) and (3.22) to $\tilde u_1$ and $\tilde u_2$, respectively, we obtain

$$\begin{aligned} |\tilde u_1|_{H^1(\Omega)} + |\tilde u_1|_{H^2(\Omega)} &\le C\|\tilde f\|_{L^2(\Omega)} \\ &\le C(\|f\|_{L^2(\widetilde D'_\zeta)} + d_\star^{-1}\|\nabla u\|_{L^2(\widetilde D'_\zeta)} + d_\star^{-2}\|u-u_{\widetilde D'_\zeta}\|_{L^2(\widetilde D'_\zeta)}) \\ &\le C(\|f\|_{L^2(\widetilde D'_\zeta)} + d_\star^{-1}\|\nabla u\|_{L^2(\widetilde D'_\zeta)}) \end{aligned}$$

and

$$\begin{aligned} |\tilde u_2|_{W^{1,p}(\Omega)} + |\tilde u_2|_{W^{2,p}(\Omega)} &\le C\|\tilde g\cdot n\|_{W^{1-1/p,p}_{\text{piecewise}}(\partial\Omega)} \\ &\le C\|\tilde g\|_{W^{1,p}(\Omega)} \\ &\le Cd_\star^{-1}\|\nabla u\|_{L^p(\widetilde D'_\zeta)} \\ &\le C(d_\star^{-1+\frac2p}\|f\|_{L^2(D'_\zeta)} + d_\star^{-2+\frac2p}\|\nabla u\|_{L^{2,\infty}(D'_\zeta)}) \end{aligned}$$

where (3.45) is used in the last inequality. By using Hölder's inequality, we have

$$\begin{aligned} |\tilde u_2|_{H^1(\Omega)} + |\tilde u_2|_{H^2(\Omega)} &\le Cd_\star^{1-\frac2p}(|\tilde u_2|_{W^{1,p}(\Omega)} + |\tilde u_2|_{W^{2,p}(\Omega)}) \\ &\le C(\|f\|_{L^2(D'_\zeta)} + d_\star^{-1}\|\nabla u\|_{L^{2,\infty}(D'_\zeta)}). \end{aligned}$$

Combining the estimates of $\tilde u_1$ and $\tilde u_2$, we obtain

$$|\tilde u|_{H^1(\Omega)} + |\tilde u|_{H^2(\Omega)} \le C(\|f\|_{L^2(D'_\zeta)} + d_\star^{-1}\|\nabla u\|_{L^{2,\infty}(D'_\zeta)}). \tag{3.50}$$

This proves (3.39b) in the case $p=2$.

Next, we prove (3.39c). In view of the qualitative regularity results in Lemma 3.7, we have $\tilde u=\omega(u-u_{\widetilde D'_\zeta})\in H^{2+\alpha}(\Omega)$, which is the solution of (3.41). For the given $\alpha\in(0,\alpha_0)$, we choose $p=2/(1-\alpha)$ so that $H^\alpha(\Omega)\hookrightarrow L^p(\Omega)$. Let $\widetilde\omega$ be a smooth cut-off function such that

$$\widetilde\omega(x)\equiv 1, \quad x\in B_{\beta d+\beta d_\star/2}(\zeta) \tag{3.51a}$$

$$\widetilde\omega(x)\equiv 0, \quad x\in \mathbb{R}^2\backslash B_{\beta d+\beta d_\star}(\zeta) \tag{3.51b}$$

$$|\nabla^k\widetilde\omega|\le C_k d_\star^{-k}, \quad k=1,2,\dots \tag{3.51c}$$

so that $\widetilde\omega=1$ on $\widetilde D'_\zeta$ and $\widetilde\omega=0$ outside D'_ζ.

Since

$$\begin{aligned} &\|f\widetilde\omega\|_{L^2(\Omega)} \le C\|f\|_{L^2(D'_\zeta)} \\ &\|f\widetilde\omega\|_{H^1(\Omega)} \le Cd_\star^{-1}\|f\|_{L^2(D'_\zeta)} + C\|\nabla f\|_{L^2(D'_\zeta)}, \end{aligned}$$

it follows from the interpolation and Young's inequalities that

$$\begin{aligned} \|f\widetilde\omega\|_{H^\alpha(\Omega)} \le \|f\widetilde\omega\|_{L^2(\Omega)}^{1-\alpha}\|f\widetilde\omega\|_{H^1(\Omega)}^{\alpha} &\le Cd_\star^{-\alpha}\|f\|_{L^2(D'_\zeta)} + C\|f\|_{L^2(D'_\zeta)}^{1-\alpha}\|\nabla f\|_{L^2(D'_\zeta)}^{\alpha} \\ &\le Cd_\star^{-\alpha}\|f\|_{L^2(D'_\zeta)} + Cd_\star^{1-\alpha}\|\nabla f\|_{L^2(D'_\zeta)}. \end{aligned} \tag{3.52}$$

Similarly, for the smooth cut-off function ω defined in (3.40), the following result holds:

$$\begin{aligned}\|f\omega\|_{H^\alpha(\Omega)} \le \|f\omega\|_{L^2(\Omega)}^{1-\alpha}\|f\omega\|_{H^1(\Omega)}^{\alpha} &\le Cd_\star^{-\alpha}\|f\|_{L^2(\widetilde{D}'_\zeta)} + C\|f\|_{L^2(\widetilde{D}'_\zeta)}^{1-\alpha}\|\nabla f\|_{L^2(\widetilde{D}'_\zeta)}^{\alpha}\\ &\le Cd_\star^{-\alpha}\|f\|_{L^2(\widetilde{D}'_\zeta)} + Cd_\star^{1-\alpha}\|\nabla f\|_{L^2(\widetilde{D}'_\zeta)}.\end{aligned} \tag{3.53}$$

By an obvious change of domain from D'_ζ to $\widetilde{D}'_\zeta$, (3.39b) implies

$$\begin{aligned}|\tilde{u}|_{W^{1,p}(\Omega)} + |\tilde{u}|_{W^{2,p}(\Omega)} &\le C(\|f\|_{L^p(\widetilde{D}'_\zeta)} + d_\star^{-2+2/p}\|\nabla u\|_{L^{2,\infty}(\widetilde{D}'_\zeta)})\\ &\le C(\|\widetilde{\omega}f\|_{L^p(\Omega)} + d_\star^{-2+2/p}\|\nabla u\|_{L^{2,\infty}(\widetilde{D}'_\zeta)})\\ &\le C(\|\widetilde{\omega}f\|_{H^\alpha(\Omega)} + d_\star^{-2+2/p}\|\nabla u\|_{L^{2,\infty}(\widetilde{D}'_\zeta)})\\ &\le C(d_\star^{-\alpha}\|f\|_{L^2(D'_\zeta)} + d_\star^{1-\alpha}\|\nabla f\|_{L^2(D'_\zeta)} + d_\star^{-2+2/p}\|\nabla u\|_{L^{2,\infty}(D'_\zeta)}).\end{aligned} \tag{3.54}$$

which reduces to

$$|u|_{W^{1,p}(D_\zeta)} + |u|_{W^{2,p}(D_\zeta)} \le C(d_\star^{-\alpha}\|f\|_{L^2(D'_\zeta)} + d_\star^{1-\alpha}\|\nabla f\|_{L^2(D'_\zeta)} + d_\star^{-2+2/p}\|\nabla u\|_{L^{2,\infty}(D'_\zeta)}).$$

Again, by an obvious change of domain from D_ζ to $\widetilde{D}'_\zeta$, we can rewrite the last inequality as

$$|u|_{W^{1,p}(\widetilde{D}'_\zeta)} + |u|_{W^{2,p}(\widetilde{D}'_\zeta)} \le C(d_\star^{-\alpha}\|f\|_{L^2(D'_\zeta)} + d_\star^{1-\alpha}\|\nabla f\|_{L^2(D'_\zeta)} + d_\star^{-2+2/p}\|\nabla u\|_{L^{2,\infty}(D'_\zeta)}).$$

Since $1-2/p=\alpha$, by using Hölder's inequality we get

$$\begin{aligned}|u|_{H^1(\widetilde{D}'_\zeta)} + |u|_{H^2(\widetilde{D}'_\zeta)} &\le Cd_\star^{1-2/p}(|u|_{W^{1,p}(\widetilde{D}'_\zeta)} + |u|_{W^{2,p}(\widetilde{D}'_\zeta)})\\ &\le C(\|f\|_{L^2(D'_\zeta)} + d_\star\|\nabla f\|_{L^2(D'_\zeta)} + d_\star^{-1}\|\nabla u\|_{L^{2,\infty}(D'_\zeta)})\end{aligned} \tag{3.55}$$

In order to obtain $H^{2+\alpha}$ estimate for $\tilde{u}$, we first estimate $\|\tilde{f}\|_{H^\alpha(\Omega)}$ below, where $\tilde{f}$ is defined in (3.42). By the properties of ω and the Hölder inequality, we have

$$\|(u-u_{\widetilde{D}'_\zeta})\Delta\omega\|_{L^2(\Omega)} \le Cd_\star^{-2}\|u-u_{\widetilde{D}'_\zeta}\|_{L^2(\widetilde{D}'_\zeta)} \le Cd_\star^{-1}\|\nabla u\|_{L^2(\widetilde{D}'_\zeta)},$$

$$\begin{aligned}\|(u-u_{\widetilde{D}'_\zeta})\Delta\omega\|_{H^1(\Omega)} &\le \|(u-u_{\widetilde{D}'_\zeta})\Delta\omega\|_{L^2(\Omega)} + \|\nabla u\Delta\omega\|_{L^2(\Omega)} + \|(u-u_{\widetilde{D}'_\zeta})\nabla\Delta\omega\|_{L^2(\Omega)}\\ &\le Cd_\star^{-2}\|u-u_{\widetilde{D}'_\zeta}\|_{L^2(\widetilde{D}'_\zeta)} + Cd_\star^{-2}\|\nabla u\|_{L^2(\widetilde{D}'_\zeta)} + Cd_\star^{-3}\|u-u_{\widetilde{D}'_\zeta}\|_{L^2(\widetilde{D}'_\zeta)}\\ &\le Cd_\star^{-2}\|\nabla u\|_{L^2(\widetilde{D}'_\zeta)}.\end{aligned}$$

By the interpolation inequality between $L^2(\Omega)$ and $H^1(\Omega)$, we have

$$\begin{aligned}\|(u-u_{\widetilde{D}'_\zeta})\Delta\omega\|_{H^\alpha(\Omega)} &\le \|(u-u_{\widetilde{D}'_\zeta})\Delta\omega\|_{L^2(\Omega)}^{1-\alpha}\|(u-u_{\widetilde{D}'_\zeta})\Delta\omega\|_{H^1(\Omega)}^{\alpha}\\ &\le Cd_\star^{-1-\alpha}\|\nabla u\|_{L^2(\widetilde{D}'_\zeta)}.\end{aligned} \tag{3.56}$$

Similarly, we have

$$\|\nabla u\cdot\nabla\omega\|_{L^2(\Omega)} \le Cd_\star^{-1}\|\nabla u\|_{L^2(\widetilde{D}'_\zeta)},$$

$$\|\nabla u\cdot\nabla\omega\|_{H^1(\Omega)} \le Cd_\star^{-2}\|\nabla u\|_{L^2(\widetilde{D}'_\zeta)} + Cd_\star^{-1}(|u|_{H^1(\widetilde{D}'_\zeta)} + |u|_{H^2(\widetilde{D}'_\zeta)})$$

$$\leq C d_\star^{-2}\|\nabla u\|_{L^2(\widetilde{D}'_\zeta)} + C d_\star^{-1}|u|_{H^2(\widetilde{D}'_\zeta)},$$

which imply the following result by the Sobolev interpolation inequality and Young's inequality:

$$\begin{aligned}
\|\nabla u\cdot\nabla\omega\|_{H^\alpha(\Omega)} &\leq \|\nabla u\cdot\nabla\omega\|_{L^2(\Omega)}^{1-\alpha}\|\nabla u\cdot\nabla\omega\|_{H^1(\Omega)}^{\alpha}\\
&\leq C d_\star^{-(1-\alpha)}\|\nabla u\|_{L^2(\widetilde{D}'_\zeta)}^{1-\alpha}(C d_\star^{-2}\|\nabla u\|_{L^2(\widetilde{D}'_\zeta)} + C d_\star^{-1}|u|_{H^2(\widetilde{D}'_\zeta)})^\alpha\\
&\leq C d_\star^{-(1-\alpha)}\|\nabla u\|_{L^2(\widetilde{D}'_\zeta)}^{1-\alpha}(C d_\star^{-2}\|\nabla u\|_{L^2(\widetilde{D}'_\zeta)}^\alpha + C d_\star^{-1}|u|_{H^2(\widetilde{D}'_\zeta)}^\alpha)\\
&\leq C d_\star^{-1-\alpha}\|\nabla u\|_{L^2(\widetilde{D}'_\zeta)} + C d_\star^{-1}\|\nabla u\|_{L^2(\widetilde{D}'_\zeta)}^{1-\alpha}|u|_{H^2(\widetilde{D}'_\zeta)}^\alpha. \qquad (3.57)
\end{aligned}$$

The estimates in (3.53) and (3.56)–(3.57) imply that

$$\begin{aligned}
\|\tilde{f}\|_{H^\alpha(\Omega)} \leq& C d_\star^{-\alpha}\|f\|_{L^2(\widetilde{D}_\zeta)} + C d_\star^{1-\alpha}\|\nabla f\|_{L^2(\widetilde{D}'_\zeta)}\\
&+ C d_\star^{-1-\alpha}\|\nabla u\|_{L^2(\widetilde{D}'_\zeta)} + C d_\star^{-1}\|\nabla u\|_{L^2(\widetilde{D}'_\zeta)}^{1-\alpha}|u|_{H^2(\widetilde{D}'_\zeta)}^\alpha. \qquad (3.58)
\end{aligned}$$

By applying Lemma 3.6 to equation (3.41) and using (3.58), we obtain

$$\begin{aligned}
&\|\tilde{u}-c_1\|_{H^{2+\alpha}(\Omega)}\\
&\leq C\|\tilde{f}\|_{H^\alpha(\Omega)} + C\|\tilde{g}\cdot n\|_{H^{1/2+\alpha}_{\text{piecewise}}(\partial\Omega)}\\
&\leq C\|\tilde{f}\|_{H^\alpha(\Omega)} + C\|\tilde{g}\|_{H^{1+\alpha}(\Omega)}\\
&\leq C d_\star^{-\alpha}\|f\|_{L^2(\widetilde{D}'_\zeta)} + C d_\star^{1-\alpha}\|\nabla f\|_{L^2(\widetilde{D}'_\zeta)}\\
&\quad + C d_\star^{-1-\alpha}\|\nabla u\|_{L^2(\widetilde{D}'_\zeta)} + C d_\star^{-1}\|\nabla u\|_{L^2(\widetilde{D}'_\zeta)}^{1-\alpha}|u|_{H^2(\widetilde{D}'_\zeta)}^\alpha\\
&\leq C d_\star^{-\alpha}\|f\|_{L^2(\widetilde{D}'_\zeta)} + C d_\star^{1-\alpha}\|\nabla f\|_{L^2(\widetilde{D}'_\zeta)} + C d_\star^{-1-\alpha}\|\nabla u\|_{L^2(\widetilde{D}'_\zeta)} + C d_\star^{-\alpha}|u|_{H^2(\widetilde{D}'_\zeta)},
\end{aligned}$$

where c_1 is some constant satisfying $|c_1| \leq C\int_\Omega |\tilde{u}|\mathrm{d}x \leq C\int_{\widetilde{D}'_j}|u|\mathrm{d}x$. Then, substituting (3.48) and (3.55) into the inequality above, we have

$$\|u-u_{\widetilde{D}'_\zeta}-c_1\|_{H^{2+\alpha}(D_\zeta)} \leq C_K\left(d_\star^{-\alpha}\|f\|_{L^2(D'_\zeta)} + d_\star^{1-\alpha}\|\nabla f\|_{L^2(D'_\zeta)} + d_\star^{-1-\alpha}\|\nabla u\|_{L^{2,\infty}(D'_\zeta)}\right).$$

where the constant $c := u_{\widetilde{D}'_\zeta} + c_1$ satisfies $|c| \leq C d_j^{-2}\int_{\widetilde{D}'_j}|u|\mathrm{d}x$. This proves (3.39c).

The proof of Lemma 3.8 is complete. ∎

3.4 Local estimates of the Green function

Let $\Gamma(x,y)$ be the Green function of the Neumann problem, defined by

$$\begin{cases} -\Delta\Gamma(\cdot,y) = \delta_y - \frac{1}{|\Omega|} & \text{in } \Omega,\\ \partial_n\Gamma(\cdot,y) = 0 & \text{on } \partial\Omega, \end{cases} \qquad (3.59)$$

where $\delta_y(x)$ is the delta function satisfying $\int_\Omega \delta_y(x)\phi(x)\mathrm{d}x = \phi(y)$ for any $\phi\in C(\overline{\Omega})$. For uniqueness of the Green function, we impose the normalization condition $\int_\Omega \Gamma(x,y)\,dx = 0$. the Green function satisfies the basic weak L^2 estimate [27, Theorem 1.3]:

$$\|\nabla\Gamma(\cdot,y)\|_{L^{2,\infty}(\Omega)} \leq C. \qquad (3.60)$$

Since $\int_\Omega \Gamma(x,y)\,dx = 0$, the above estimate also implies (via the Sobolev embedding inequality)

$$\|\Gamma(\cdot,y)\|_{L^q(\Omega)} \le C\|\nabla\Gamma(\cdot,y)\|_{L^{2,\infty}(\Omega)} \le C \quad \text{for } 1<q<\infty. \tag{3.61}$$

We will need the following local estimates of the Green function in the next section.

Lemma 3.9 (Local estimates of the Green function) *Let $p\in(2,p_0)$ and $\alpha\in(0,\alpha_0)$, and let $D = B_d(z)\cap\Omega$ with*

$$d \le K d_\star \qquad \text{and} \qquad d+d_\star < \min(|z-z_0|,|z-y|).$$

Then the Green function $\Gamma(x,y)$ satisfies

$$\|\Gamma(\cdot,y)\|_{H^1(D)} \le C_K, \tag{3.62a}$$

$$\|\Gamma(\cdot,y)\|_{H^2(D)} \le C_K d_\star^{-1}, \tag{3.62b}$$

$$\|\Gamma(\cdot,y)\|_{W^{2,p}(D)} \le C_K d_\star^{-2+2/p}. \tag{3.62c}$$

Moveover, there exists $\beta\in(0,1)$ such that D can be covered by a bounded number of subdomains $D_j = B_{\beta d/2}(\zeta_j)\cap\Omega$, $j=1,\dots,J_\beta$, with

$$\|\Gamma(\cdot,y) - c_{D_j}(y)\|_{H^{2+\alpha}(D_j')} \le C_K d_\star^{-1-\alpha}, \tag{3.63}$$

where $D_j' = B_{\beta d}(\zeta_j)\cap\Omega$ and $c_{D_j}(y)$ is some constant depending on D_j and y, satisfying $|c_{D_j}(y)| \le C d_\star^{-2}$.

If $d+d_\star \le K d_\#$ and $d+d_\star+d_\# < |z-y|$, then the following improved estimates hold:

$$\|\Gamma(\cdot,y)\|_{H^2(D)} \le C_K d_\star^{s-1} d_\#^{-s}, \tag{3.64a}$$

$$\|\Gamma(\cdot,y)\|_{W^{2,p}(D)} \le C_K d_\star^{-2+\frac{2}{p}+s} d_\#^{-s}. \tag{3.64b}$$

Proof. The condition $d+d_\star < \min(|z-z_0|,|z-y|)$ guarantees that the subdomain $D' = B_{d+d_\star}(z)\cap\Omega$ is away from the reentrant corner and the singularity point y. As a result, the solution $\Gamma(\cdot,y)$ of equation (3.59) has $W^{2,p}$ and $H^{2+\alpha}$ regularity in the subdomain D'.

By applying Lemma 3.8 and using the basic estimate (3.60), we have

$$\begin{aligned}
&|\Gamma(\cdot,y)|_{H^1(D)} \le C_K\big(d_\star\|1/|\Omega|\|_{L^2(D')} + \|\nabla\Gamma(\cdot,y)\|_{L^{2,\infty}(D')}\big) \le C_K,\\
&|\Gamma(\cdot,y)|_{W^{1,p}(D)} + |\Gamma(\cdot,y)|_{W^{2,p}(D)} \le C_K\big(\|\tfrac{1}{|\Omega|}\|_{L^p(D')} + d_\star^{-2+2/p}\|\nabla\Gamma(\cdot,y)\|_{L^{2,\infty}(D')}\big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \le C_K d_\star^{-2+2/p},\\
&|\Gamma(\cdot,y)|_{H^1(D)} + |\Gamma(\cdot,y)|_{H^2(D)} \le C_K d_\star^{1-2/p}(|\Gamma(\cdot,y)|_{W^{1,p}(D)} + |\Gamma(\cdot,y)|_{W^{2,p}(D)}) \le C_K d_\star^{-1}.
\end{aligned}$$

Moreover, according to Lemma 3.8, D can be covered by a bounded number of subdomains $D_j = B_{\beta d/2}(\zeta_j)\cap\Omega$, $j=1,\dots,J_\beta$, with

$$\|\Gamma(\cdot,y) - c_{D_j}\|_{H^{2+\alpha}(D_j')} \le C_K\big(d_\star^{-\alpha}\|1/|\Omega|\|_{L^2(D')} + d_\star^{-1-\alpha}\|\nabla\Gamma(\cdot,y)\|_{L^{2,\infty}(D')}\big) \le C d_\star^{-1-\alpha}.$$

where $c_{D_j}(y)$ is some constant which satisfy $|c_{D_j}(y)| \le C d_\star^{-2}\|\Gamma(\cdot,y)\|_{L^1(D_j')} \le C d_\star^{-2}$. The above semi-norm estimates and the L^q-norm estimate in (3.61) together imply the desired results in (3.62)–(3.63).

Let $D'' = B_{d+d_\star+d_\#}(z)\cap\Omega$. Let ω be a smooth cut-off function such that

$$\omega(x) \equiv 1, \quad x\in B_{d+d_\star}(z) \tag{3.65a}$$

$$\omega(x) \equiv 0, \quad x\in \Omega\backslash B_{d+d_\star+d_\#}(z) \tag{3.65b}$$

$$|\nabla^k \omega| \le C_k d_{\#}^{-k}, \quad k = 1, 2, \dots \tag{3.65c}$$

Hence, $\omega = 1$ on D' and $\omega = 0$ outside D''. Then applying Lemma 3.8 yields

$$\begin{aligned}
&|\Gamma(\cdot, y)|_{W^{1,p}(D)} + |\Gamma(\cdot, y)|_{W^{2,p}(D)} \\
&\le C_K \big(\|1/|\Omega|\|_{L^p(D')} + d_\star^{-2+\frac{2}{p}} \|\nabla\Gamma(\cdot, y)\|_{L^{2,\infty}(D')} \big) \\
&\le C_K \big(C + d_\star^{-2+\frac{2}{p}} d_\star^{1-\frac{2}{q}} \|\nabla\Gamma(\cdot, y)\|_{L^q(D')} \big) \quad \text{(Hölder inequality with } q > 2 \text{ satisfying } s = 1 - \tfrac{2}{q}) \\
&\le C_K \big(C + d_\star^{-2+\frac{2}{p}+s} \|\omega(\Gamma(\cdot, y) - c)\|_{H^{s+1}(\Omega)} \big) \qquad \text{(with } H^{s+1}(\Omega) \hookrightarrow W^{1,q}(\Omega)) \\
&\le C_K \Big[C + d_\star^{-2+\frac{2}{p}+s} \Big(\|1/|\Omega|\|_{L^{p_s}(D'')} + d_{\#}^{-s} (\|\Gamma(\cdot, y)\|_{L^{2,\infty}(\Omega)} + \|\nabla\Gamma(\cdot, y)\|_{L^{2,\infty}(\Omega)}) \Big) \Big] \\
&\le C_K d_\star^{-2+\frac{2}{p}+s} d_{\#}^{-s},
\end{aligned}$$

where c can be an arbitrary constant in the third to last inequality, and we have used (3.20) in estimating $\|\omega(\Gamma(\cdot, y) - c)\|_{H^{s+1}(\Omega)}$ with $u = \Gamma(\cdot, y)$, $c = \bar{u}_d$, $f = 1/|\Omega|$ and $p_s = 2/(2 - s)$. By using Hölder's inequality, we further derive the following result:

$$|\Gamma(\cdot, y)|_{H^1(D)} + |\Gamma(\cdot, y)|_{H^2(D)} \le C d_\star^{1-\frac{2}{p}} (|\Gamma(\cdot, y)|_{W^{1,p}(D)} + |\Gamma(\cdot, y)|_{W^{2,p}(D)}) \le C_K d_\star^{s-1} d_{\#}^{-s}.$$

The two semi-norm estimates above and the L^q-norm (3.61) together imply the desired results in (3.64). The proof of Lemma 3.9 is complete. ∎

4 Proof of Theorem 2.1

We only need to prove the following result for any given point $x_0 \in \Omega$ in the interior of some triangle τ_0:

$$|R_h u(x_0)| \le C\|u\|_{L^\infty(\Omega)} \quad \text{for any given } x_0 \in \Omega.$$

We first focus on the case $|x_0 - z_0| > 16\kappa h_*$, where $\kappa \ge 1$ is a parameter to be determined later. From now on, we keep the generic positive constant C independent of κ until it is determined, and keep C independent of x_0. The case $|x_0 - z_0| \le 16\kappa h_*$ will be discussed after the parameter κ is determined.

4.1 Double dyadic decomposition of the domain

We decompose the domain Ω into disjoint subsets

$$\Omega = O_* \bigcup \cup_{j=0}^{J_*} O_j \bigcup \widetilde{O}_* \bigcup \cup_{j=0}^{J_{x_0}} \widetilde{O}_j \bigcup \cup_{j=0}^{J+1} \Omega_j, \tag{4.1}$$

where

$$O_* := \{x \in \Omega : |x - z_0| < d_{J_*+1}\}, \tag{4.2a}$$

$$O_j := \{x \in \Omega : d_{j+1} \le |x - z_0| < d_j\}, \qquad j = 0, 1, \dots, J_*, \tag{4.2b}$$

$$\widetilde{O}_* := \{x \in \Omega : |x - x_0| < d_{J_{x_0}+1}\}, \tag{4.2c}$$

$$\widetilde{O}_j := \{x \in \Omega : d_{j+1} \le |x - x_0| < d_j\}, \qquad j = 0, 1, \dots, J_{x_0}, \tag{4.2d}$$

$$\Omega_j := \{x \in \Omega : \rho_{j+1} \le |x - z_0| < \rho_j\}, \qquad j = 0, 1, \dots, J, \tag{4.2e}$$

$$\Omega_{J+1} := \{x \in \Omega : |x - x_0| \ge d_0,\ d_0 \le |x - z_0| < \rho_{J+1}\}, \tag{4.2f}$$

with $d_j = 2^{-j-2}|x_0 - z_0|$, $J_* = \left[\log_2\left(\frac{|x_0-z_0|}{16\kappa h_*}\right)\right]$ and $J_{x_0} = \left[\log_2\left(\frac{|x_0-z_0|}{16\kappa^{\gamma\sigma}\hbar(x_0)}\right)\right]$ for some $\sigma \in (0,1)$, and $\rho_j = 2^{-j}\mathrm{diameter}(\Omega)$ and $J = \left[\log_2\left(\frac{\mathrm{diameter}(\Omega)}{16|x_0-z_0|}\right)\right]$, so that

$$2\kappa h_* \le d_{J_*+1} \le 4\kappa h_*, \tag{4.3a}$$
$$2\kappa^{\gamma\sigma}\hbar(x_0) \le d_{J_{x_0}+1} \le 4\kappa^{\gamma\sigma}\hbar(x_0), \tag{4.3b}$$
$$8|x_0 - z_0| \le \rho_{J+1} \le 16|x_0 - z_0|, \tag{4.3c}$$
$$\mathrm{dist}(O_j, \widetilde{O}_i) \sim |x_0 - z_0|, \tag{4.3d}$$
$$\mathrm{dist}(O_j, \Omega_i) \sim \mathrm{dist}(\widetilde{O}_k, \Omega_i) \sim \rho_i, \tag{4.3e}$$

where $\mathrm{dist}(O_j, \widetilde{O}_i)$ denotes the distance between the two sets O_j and $\widetilde{O}_i$. Moreover, we have

$$K^{-1}h_* \le \hbar(x) \le 2K\kappa^{1-\gamma}h_*, \qquad \forall\, x \in O_*, \tag{4.4a}$$
$$\hbar(x) \sim d_j^{1-\gamma}h \qquad \forall\, x \in O_j, \tag{4.4b}$$
$$\hbar(x) \sim \hbar(x_0) \qquad \forall\, x \in \widetilde{O}_j \cup \widetilde{O}_*, \tag{4.4c}$$
$$\hbar(x) \sim \rho_j^{1-\gamma}h \qquad \forall\, x \in \Omega_j, \tag{4.4d}$$

for some positive constant K (independent of κ). We denote by h_j the mesh size in O_j and $\mathtt{h}_j$ the mesh size in Ω_j.

Let

$$O'_j = O_{j-1/2} \cup O_j \cup O_{j+1/2}, \tag{4.5}$$
$$\widetilde{O}'_j = \widetilde{O}_{j-1/2} \cup \widetilde{O}_j \cup \widetilde{O}_{j+1/2}, \tag{4.6}$$
$$\Omega'_j = \Omega_{j-1/2} \cup \Omega_j \cup \Omega_{j+1/2}, \tag{4.7}$$
$$\tag{4.8}$$

with

$$O_{j-1/2} := \{x \in \Omega : d_{j+1/2} \le |x - z_0| < d_{j-1/2}\}, \tag{4.9a}$$
$$\widetilde{O}_{j-1/2} := \{x \in \Omega : d_{j+1/2} \le |x - x_0| < d_{j-1/2}\}, \tag{4.9b}$$
$$\Omega_{j-1/2} := \{x \in \Omega : \rho_{j+1/2} \le |x - z_0| < \rho_{j-1/2}\}, \tag{4.9c}$$
$$O_{-1} = \widetilde{O}_{-1} := \Omega_{J+1}, \tag{4.9d}$$
$$\Omega_{J+2} := O_0 \cup \widetilde{O}_0, \tag{4.9e}$$
$$\tag{4.9f}$$

and

$$\mathcal{O}_{z_0} := \{O_j : 0 \le j \le J_*\}, \qquad \mathcal{O}'_{z_0} := \mathcal{O}_{z_0} \cup \{O_*\}, \tag{4.10a}$$
$$\mathcal{O}_{x_0} := \{\widetilde{O}_j : 0 \le j \le J_{x_0}\}, \qquad \mathcal{O}'_{x_0} := \mathcal{O}_{x_0} \cup \{\widetilde{O}_*\}, \tag{4.10b}$$
$$\mathcal{O} := \{\Omega_j : 0 \le j \le J+1\}, \qquad \mathcal{O}' := \mathcal{O} \cup \{\Omega_{J+2}\}. \tag{4.10c}$$

Then we have $\Omega = O_* \bigcup \widetilde{O}_* \bigcup \big(\cup_{O_j \in \mathcal{O}_{z_0}} O_j\big) \bigcup \big(\cup_{\widetilde{O}_j \in \mathcal{O}_{x_0}} \widetilde{O}_j\big) \bigcup \big(\cup_{\Omega_j \in \mathcal{O}} \Omega_j\big)$.

Remark 4.1 In the case $|x_0 - z_0| > 16\kappa h_* \sim 16\kappa h^{1/\gamma}$ we have $|x_0 - z_0|^\gamma > 16^\gamma \kappa^\gamma h$ and

$$\frac{|x_0 - z_0|}{\kappa^{\gamma\sigma}\hbar(x_0)} \sim \frac{|x_0 - z_0|}{\kappa^{\gamma\sigma}|x_0 - z_0|^{1-\gamma}h} \sim \frac{|x_0 - z_0|^\gamma}{\kappa^{\gamma\sigma}h} \geq \frac{\kappa^{\gamma(1-\sigma)}}{C}.$$

Hence, for the fixed $\sigma \in (0,1)$, we can choose κ sufficiently large to make sure that $|x_0 - z_0| \geq 16\kappa^{\gamma\sigma}\hbar(x_0)$.

Remark 4.2 The double dyadic decomposition $O_j := \{x \in \Omega : d_{j+1} \leq |x - z_0| < d_j\}$ and $\widetilde{O}_j := \{x \in \Omega : d_{j+1} \leq |x - x_0| < d_j\}$ are defined for $d_j = 2^{-j-2}|x_0 - z_0|$, $j = 1, \ldots, J_*$ and therefore with radius d_j smaller than $|x_0 - z_0|$. They reduce to the single dyadic decomposition $\Omega_j := \{x \in \Omega : \rho_{j+1} \leq |x - z_0| < \rho_j\}$ when the radius exceeds $|x_0 - z_0|$. We use ρ_j to denote the radius when it is bigger than $|x_0 - z_0|$.

4.2 Regularized Green's function

Recall that τ_0 is the triangle which contains x_0. We denote by $\tilde{\delta}_{x_0} \in C^3(\tau_0)$ a regularized Delta function which has the following properties:

$$\tilde{\delta}_{x_0} \text{ is compactly supported in } \tau_0, \tag{4.11a}$$

$$(\tilde{\delta}_{x_0}, v_h)_{\tau_0} = v_h(x_0), \quad \forall\, v_h \in S_h, \tag{4.11b}$$

$$\int_\Omega \tilde{\delta}_{x_0}(x)\, dx = 1, \tag{4.11c}$$

$$\|\tilde{\delta}_{x_0}\|_{W^{l,p}(\Omega)} \leq C\hbar(x_0)^{-l-2(1-\frac{1}{p})} \quad \text{for } 1 \leq p \leq \infty,\ \ l = 0,1,2,3. \tag{4.11d}$$

It is known that such a smoothed Delta function exists; see [34, Lemma 2.2].

The regularized Green's function $G(x, x_0)$ is defined by using the regularized Delta function, as the solution of

$$\begin{cases} -\Delta G(\cdot, x_0) = \tilde{\delta}_{x_0}(\cdot) - \frac{1}{|\Omega|} & \text{in } \Omega, \\ \partial_n G(\cdot, x_0) = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.12}$$

Since $\int_\Omega (\tilde{\delta}(x) - \frac{1}{|\Omega|})\mathrm{d}x = 0$, the equation above admits a unique solution up to a constant. The discrete Green's function $G_h(\cdot, x_0) \in S_h$ is defined as the finite element solution of the problem

$$(\nabla G_h(\cdot, x_0), \nabla v_h) = v_h(x_0) - \frac{1}{|\Omega|}\int_\Omega v_h(x)\mathrm{d}x, \quad \forall\, v_h \in S_h, \tag{4.13}$$

which is also well defined up to a constant. For uniqueness, we further impose the condition

$$\int_\Omega G(x, x_0)\mathrm{d}x = \int_\Omega G_h(x, x_0)\mathrm{d}x = 0.$$

Similarly as the local estimates of Green's function in Lemma 3.9, the following local estimates of the regularized Green's function hold.

Lemma 4.1 (Local estimates of the regularized Green's function) *Let $p \in (2, p_0)$ and $\alpha \in (0, \alpha_0)$. Let $D = B_d(z) \cap \Omega$ and assume that*

$$d \leq K d_\star \quad \text{and} \quad \overline{B_{d+d_\star}(z)} \cap \{z_0\} = \overline{B_{d+d_\star}(z)} \cap \mathrm{supp}(\tilde{\delta}_{x_0}) = \emptyset.$$

Then the regularized Green's function $G(x, x_0)$ satisfies the following estimates:

$$\|G(\cdot, x_0)\|_{H^1(D)} \leq C_K, \tag{4.14a}$$

$$\|G(\cdot,x_0)\|_{H^2(D)} \le C_K d_\star^{-1}, \tag{4.14b}$$

$$\|G(\cdot,x_0)\|_{W^{2,p}(D)} \le C_K d_\star^{-2+2/p}. \tag{4.14c}$$

Moveover, there exists $\beta \in (0,1)$ such that D can be covered by a bounded number of subdomains $D_j = B_{\beta d/2}(\zeta_j) \cap \Omega$, $j = 1,\ldots,J_\beta$, with $\zeta_j \in D$, and

$$\|G(\cdot,x_0) - c_{D_j}\|_{H^{2+\alpha}(D_j')} \le C_K d_\star^{-1-\alpha}, \tag{4.15}$$

where $D_j' = B_{\beta d}(\zeta_j) \cap \Omega$ and c_{D_j} is some constant depending on G and D_j.

If $d + d_\star \le K d_\#$ and $d + d_\star + d_\# < \mathrm{dist}(z, \mathrm{supp}(\tilde\delta_{x_0}))$, then the following estimates hold:

$$\|G(\cdot,x_0)\|_{H^2(D)} \le C_K d_\star^{s-1} d_\#^{-s}, \tag{4.16}$$

$$\|G(\cdot,x_0)\|_{W^{2,p}(D)} \le C_K d_\star^{-2+\frac{2}{p}+s} d_\#^{-s}. \tag{4.17}$$

Proof. By representing $G(x,x_0)$ in terms of the continuous Green function, i.e.

$$G(x,x_0) = \int_\Omega \Gamma(x,\xi)\tilde\delta_{x_0}(\xi)\,\mathrm{d}\xi,$$

we see that $|x-\xi| \ge d_\star$ when $x \in D$ and $\xi \in \mathrm{supp}(\tilde\delta_{x_0})$. Therefore, the following estimate holds as a result of Lemma 3.9:

$$\begin{aligned}\|G(\cdot,x_0)\|_{W^{2,p}(D)} &\le \int_\Omega \|\Gamma(\cdot,\xi)\|_{W^{2,p}(D)}|\tilde\delta_{x_0}(\xi)|\,\mathrm{d}\xi \\ &\le \int_\Omega C d_\star^{-2+2/p}|\tilde\delta_{x_0}(\xi)|\,\mathrm{d}\xi \le C d_\star^{-2+2/p}.\end{aligned}$$

The other estimates in (4.14) and (4.16)–(4.17) can be proved in the same way.

The local $H^{2+\alpha}$ estimate in (4.15) needs to be proved in a slightly different way, by considering the following expression:

$$G(x,x_0) - c = \int_\Omega (\Gamma(x,y) - c_{D_j}(y))\tilde\delta_{x_0}(y)\mathrm{d}y$$

where $c = \int_\Omega c_{D_j}(y)\tilde\delta_{x_0}(y)\mathrm{d}y$ satisfies $|c| \le \int_\Omega C d_\star^{-2}\tilde\delta_{x_0}(y)\mathrm{d}y \le C d_\star^{-2}$. By using (3.63) we obtain

$$\begin{aligned}\|G(\cdot,x_0) - c\|_{H^{2+\alpha}(D_j')} &\le \int_\Omega \|\Gamma(\cdot,y) - c_{D_j}(y)\|_{H^{2+\alpha}(D_j')}|\tilde\delta_{x_0}(y)|\,\mathrm{d}y \\ &\le \int_\Omega C d_\star^{-1-\alpha}|\tilde\delta_{x_0}(y)|\,\mathrm{d}y \le C d_\star^{-1-\alpha}.\end{aligned}$$

∎

Each O_j can be covered by a finite number of balls $B_{d_j/8}(z)$ with $z \in O_j$, and the number of balls are independent of d_j, where each ball $B_{d_{j+2}}(z)$ satisfies the condition of Lemma 4.1 with $d = d_\star = d_j/8$. Similarly, $\widetilde O_j$ and Ω_j can also be covered by balls of radius $d_j/8$ and $\rho_j/8$, respectively. Hence, Lemma 4.1 immediately implies the following results.

Lemma 4.2 (Local estimates in O_j, $\widetilde O_j$ and Ω_j) *Let $p \in (2,p_0)$ and $\alpha \in (0,\alpha_0)$. Then we have*

$$\|G(\cdot,x_0)\|_{H^1(O_j)} \le C, \tag{4.18a}$$

$$\|G(\cdot,x_0)\|_{H^2(O_j)} \le C d_j^{-1}, \tag{4.18b}$$

$$\|G(\cdot,x_0)\|_{W^{2,p}(O_j)} \le C d_j^{-2+2/p}, \tag{4.18c}$$

$$\|G(\cdot,x_0)\|_{H^1(\widetilde{O}_j)} \le C, \tag{4.18d}$$

$$\|G(\cdot,x_0)\|_{H^2(\widetilde{O}_j)} \le C d_j^{-1}, \tag{4.18e}$$

$$\|G(\cdot,x_0)\|_{W^{2,p}(\widetilde{O}_j)} \le C d_j^{-2+2/p}, \tag{4.18f}$$

$$\|G(\cdot,x_0)\|_{H^1(\Omega_j)} \le C, \tag{4.18g}$$

$$\|G(\cdot,x_0)\|_{H^2(\Omega_j)} \le C \rho_j^{-1}, \tag{4.18h}$$

$$\|G(\cdot,x_0)\|_{W^{2,p}(\Omega_j)} \le C \rho_j^{-2+2/p}. \tag{4.18i}$$

Moveover, there exists $\beta \in (0,1)$ *such that* O_j *can be covered by a bounded number of subdomains* $D_{j,i} = B_{\beta d_j/2}(\zeta_{j,i}) \cap \Omega$, $i = 1, \ldots, J_\beta$, *with* $\zeta_{j,i} \in O_j$, *and*

$$\|G(\cdot,x_0) - c_{D_{j,i}}\|_{H^{2+\alpha}(D'_{j,i})} \le C d_j^{-1-\alpha}, \tag{4.19}$$

where $D'_{j,i} = B_{\beta d_j}(\zeta_{j,i}) \cap \Omega$ *and* c_{D_j} *is some constant depending on* G *and* $D_{j,i}$.

Similarly, there exists $\beta \in (0,1)$ *such that* $\widetilde{O}_j$ *can be covered by a bounded number of subdomains* $\widetilde{D}_{j,i} = B_{\beta d_j/2}(\zeta_{j,i}) \cap \Omega$, $i = 1, \ldots, J_\beta$, *with* $\zeta_{j,i} \in \widetilde{O}_j$, *and*

$$\|G(\cdot,x_0) - c_{\widetilde{D}_{j,i}}\|_{H^{2+\alpha}(\widetilde{D}'_j)} \le C d_j^{-1-\alpha}, \tag{4.20}$$

where $\widetilde{D}'_{j,i} = B_{\beta d_j}(\zeta_{j,i}) \cap \Omega$ *and* $c_{\widetilde{D}_{j,i}}$ *is some constant depending on* G *and* $\widetilde{D}_{j,i}$.

There exists $\beta \in (0,1)$ *such that* Ω_j *can be covered by a bounded number of subdomains* $\hat{D}_{j,i} = B_{\beta\rho_j/2}(\zeta_{j,i}) \cap \Omega$, $i = 1, \ldots, J_\beta$, *with* $\zeta_{j,i} \in \Omega_j$, *and*

$$\|G(\cdot,x_0) - c_{\hat{D}_{j,i}}\|_{H^{2+\alpha}(\hat{D}'_{j,i})} \le C \rho_j^{-1-\alpha}, \tag{4.21}$$

where $\hat{D}'_{j,i} = B_{\beta\rho_j/}(\zeta_{j,i}) \cap \Omega$ *and* $c_{\hat{D}_{j,i}}$ *is some constant depending on* G *and* $\hat{D}_{j,i}$.

If $\Omega_j \in \mathcal{O}$ *and* $\widetilde{O}_i \in \mathcal{O}_{x_0} \backslash \{\widetilde{O}_0\}$, *then*

$$\sup_{y \in \Omega_j} \|G(\cdot,y)\|_{H^2(\widetilde{O}'_i)} \le C|x_0 - z_0|^{s-1} \rho_j^{-s}. \tag{4.22}$$

Proof. (4.18) is a consequence of (4.14); (4.19)–(4.21) follow from (4.15); (4.22) is a consequence of (4.16). ∎

Since G_h is piecewisely defined in the elements and therefore not in $H^2(\Omega)$, we denote by $\nabla^2_{\mathcal{T}} G_h$ the elementwise second-order derivative (Hessian) of G_h. For the simplicity of notation, we denote

$$\|\nabla^2_{\mathcal{T}}(G_h - G)\|_{L^1(D)} := \sum_{\tau \cap D \neq \emptyset} \|\nabla^2(G_h - G)\|_{L^1(\tau)}$$

and

$$\|G(\cdot,x_0)\|_{H^{2+\alpha}_*(O_j)} := \sum_i \|G(\cdot,x_0) - c_{D_{j,i}}\|_{H^{2+\alpha}(D'_{j,i})}, \tag{4.23}$$

$$\|G(\cdot,x_0)\|_{H^{2+\alpha}_*(\widetilde{O}_j)} := \sum_i \|G(\cdot,x_0) - \widetilde{c}_{D_{j,i}}\|_{H^{2+\alpha}(\widetilde{D}'_{j,i})}, \tag{4.24}$$

$$\|G(\cdot,x_0)\|_{H^{2+\alpha}_*(\Omega_j)} := \sum_i \|G(\cdot,x_0) - \hat{c}_{D_{j,i}}\|_{H^{2+\alpha}(\hat{D}'_{j,i})}, \tag{4.25}$$

where the number of terms in the three summations above is bounded (independent of h), as mentioned in Lemma 4.2. Then we have the following estimates:

$$\begin{aligned}
\|\nabla[G(\cdot,x_0)-I_hG(\cdot,x_0)]\|_{L^2(O_j)} &\le \sum_i \|\nabla[G(\cdot,x_0)-I_hG(\cdot,x_0)]\|_{L^2(D_{j,i})} \\
&= \sum_i \|\nabla[(G(\cdot,x_0)-c_{D_{j,i}})-I_h(G(\cdot,x_0)-c_{D_{j,i}})]\|_{L^2(D_{j,i})} \\
&\le \sum_i Ch^{1+\alpha}\|G(\cdot,x_0)-c_{D_{j,i}}\|_{H^{2+\alpha}(D'_{j,i})} \quad (\text{if } r\ge 2) \\
&= Ch^{1+\alpha}\|G(\cdot,x_0)\|_{H^{2+\alpha}_*(O_j)}, \qquad (4.26)
\end{aligned}$$

$$\begin{aligned}
\|\nabla^2_{\mathcal{T}}[G(\cdot,x_0)-I_hG(\cdot,x_0)]\|_{L^2(O_j)} &\le \sum_i \|\nabla^2_{\mathcal{T}}[G(\cdot,x_0)-I_hG(\cdot,x_0)]\|_{L^2(D_{j,i})} \\
&= \sum_i \|\nabla^2_{\mathcal{T}}[(G(\cdot,x_0)-c_{D_{j,i}})-I_h(G(\cdot,x_0)-c_{D_{j,i}})]\|_{L^2(D_{j,i})} \\
&\le \sum_i Ch^{\alpha}\|G(\cdot,x_0)-c_{D_{j,i}}\|_{H^{2+\alpha}(D'_{j,i})} \quad (\text{if } r\ge 2) \\
&= Ch^{\alpha}\|G(\cdot,x_0)\|_{H^{2+\alpha}_*(O_j)}, \qquad (4.27)
\end{aligned}$$

and similarly, the following estimates hold for $r\ge 2$:

$$\|\nabla[G(\cdot,x_0)-I_hG(\cdot,x_0)]\|_{L^2(\widetilde{O}_j)} \le Ch^{1+\alpha}\|G(\cdot,x_0)\|_{H^{2+\alpha}_*(\widetilde{O}_j)}, \qquad (4.28)$$

$$\|\nabla^2_{\mathcal{T}}[G(\cdot,x_0)-I_hG(\cdot,x_0)]\|_{L^2(\widetilde{O}_j)} \le Ch^{\alpha}\|G(\cdot,x_0)\|_{H^{2+\alpha}_*(\widetilde{O}_j)}, \qquad (4.29)$$

$$\|\nabla[G(\cdot,x_0)-I_hG(\cdot,x_0)]\|_{L^2(\Omega_j)} \le Ch^{1+\alpha}\|G(\cdot,x_0)\|_{H^{2+\alpha}_*(\Omega_j)}, \qquad (4.30)$$

$$\|\nabla^2_{\mathcal{T}}[G(\cdot,x_0)-I_hG(\cdot,x_0)]\|_{L^2(\Omega_j)} \le Ch^{\alpha}\|G(\cdot,x_0)\|_{H^{2+\alpha}_*(\Omega_j)}. \qquad (4.31)$$

For $r=1$ the estimates in (4.26)–(4.31) should be replaced by the following standard estimates:

$$\begin{aligned}
\|\nabla[G(\cdot,x_0)-I_hG(\cdot,x_0)]\|_{L^2(O_j)} &\le Ch\|G(\cdot,x_0)\|_{H^2(O'_j)}, \\
\|\nabla^2_{\mathcal{T}}[G(\cdot,x_0)-I_hG(\cdot,x_0)]\|_{L^2(O_j)} &\le C\|G(\cdot,x_0)\|_{H^2(O'_j)}, \\
\|\nabla[G(\cdot,x_0)-I_hG(\cdot,x_0)]\|_{L^2(\widetilde{O}_j)} &\le Ch\|G(\cdot,x_0)\|_{H^2(\widetilde{O}'_j)}, \\
\|\nabla^2_{\mathcal{T}}[G(\cdot,x_0)-I_hG(\cdot,x_0)]\|_{L^2(\widetilde{O}_j)} &\le C\|G(\cdot,x_0)\|_{H^2(\widetilde{O}'_j)}, \\
\|\nabla[G(\cdot,x_0)-I_hG(\cdot,x_0)]\|_{L^2(\Omega_j)} &\le Ch\|G(\cdot,x_0)\|_{H^2(\Omega'_j)}, \\
\|\nabla^2_{\mathcal{T}}[G(\cdot,x_0)-I_hG(\cdot,x_0)]\|_{L^2(\Omega_j)} &\le C\|G(\cdot,x_0)\|_{H^2(\Omega'_j)}.
\end{aligned}$$

4.3 Reduction to the estimation of $\|\nabla^2_{\mathcal{T}}(G_h-G)\|_{L^1(\Omega)}+\|\hbar^{-1}\nabla(G_h-G)\|_{L^1(\Omega)}$

The standard Lagrange interpolation operator $I_h\colon C(\overline{\Omega})\to S_h$ has the following approximation properties (cf. [5, Theorem 3.1.5])

$$\|u-I_hu\|_{L^q(D)} \le Ch^{2+2(\frac{1}{q}-\frac{1}{p})}|u|_{W^{2,p}(D')}, \quad \text{for } 1<p\le q\le\infty, \qquad (4.32)$$

where $D \subset D' \subset \Omega$ can be any subdomains such that $\{\tau : \tau \cap D \neq \emptyset\} \subset D'$.

Then we have

$$\begin{aligned}
&\left|R_h u(x_0) - I_h u(x_0) - \frac{1}{|\Omega|}\int_\Omega (R_h u - I_h u)\mathrm{d}x\right| \\
&= |(\nabla G_h, \nabla(R_h u - I_h u))| \\
&= |(\nabla G_h, \nabla(u - I_h u))| \\
&= |(\nabla(G_h - G), \nabla(u - I_h u)) + (\nabla G, \nabla(u - I_h u))| \\
&= \left|(\nabla(G_h - G), \nabla(u - I_h u)) + (\tilde{\delta}_{x_0}, u) - I_h u(x_0) - \frac{1}{|\Omega|}\int_\Omega (u - I_h u)\mathrm{d}x\right| \\
&= \Bigg|\sum_{\tau\in\mathcal{T}_h} (-\Delta(G_h - G), u - I_h u)_\tau + \sum_{e\in\mathcal{E}_h} ([\partial_n(G_h - G)], u - I_h u)_e \\
&\qquad + (\tilde{\delta}_{x_0}, u) - I_h u(x_0) - \frac{1}{|\Omega|}\int_\Omega (u - I_h u)\mathrm{d}x\Bigg| \\
&\le \Bigg(\sum_{\tau\in\mathcal{T}_h} \|\Delta(G_h - G)\|_{L^1(\tau)} + \sum_{e\in\mathcal{E}_h} \|[\partial_n(G_h - G)]\|_{L^1(e)} + C\Bigg)\|u\|_{L^\infty(\Omega)} \\
&\le C\big(\|\nabla^2_{\mathcal{T}}(G_h - G)\|_{L^1(\Omega)} + \|\hbar^{-1}\nabla(G_h - G)\|_{L^1(\Omega)} + 1\big)\|u\|_{L^\infty(\Omega)}. \qquad (4.33)
\end{aligned}$$

where we have used the following trace inequality in the last inequality of (4.33) (e is an edge of τ):

$$\|\nabla(G_h - G)\|_{L^1(e)} \le C\left(\|\hbar^{-1}\nabla(G_h - G)\|_{L^1(\tau)} + \|\nabla^2_{\mathcal{T}}(G_h - G)\|_{L^1(\tau)}\right). \qquad (4.34)$$

It remains to prove

$$\|\nabla^2_{\mathcal{T}}(G_h - G)\|_{L^1(\Omega)} + \|\hbar^{-1}\nabla(G_h - G)\|_{L^1(\Omega)} \le C\ell_h, \qquad (4.35)$$

where ℓ_h is defined in (1.6). Once the above inequality is proved, (4.33) would reduce to

$$\begin{aligned}
|R_h u(x_0) - I_h u(x_0)| &\le C\ell_h\|u\|_{L^\infty(\Omega)} + \left|\frac{1}{|\Omega|}\int_\Omega (R_h u - I_h u)\mathrm{d}x\right| \\
&= C\ell_h\|u\|_{L^\infty(\Omega)} + \left|\frac{1}{|\Omega|}\int_\Omega (u - I_h u)\mathrm{d}x\right| \\
&\le C\ell_h\|u\|_{L^\infty(\Omega)},
\end{aligned}$$

where we have used the normalization condition $\int_\Omega R_h u\mathrm{d}x = \int_\Omega u\mathrm{d}x$ for the Ritz projeciton. By using the triangle inequality, we obtain from the inequality above

$$|R_h u(x_0)| \le C\ell_h\|u\|_{L^\infty(\Omega)} + |I_h u(x_0)| \le C\ell_h\|u\|_{L^\infty(\Omega)}. \qquad (4.36)$$

Since the constant C is independent of x_0, the inequality above implies (2.4) and therefore complete the proof of Theorem 2.1.

It remains to prove the key estimate (4.35), which is presented in the next two subsections.

4.4 Reduction to the estimation of $\|\hbar^{-1}\nabla(G_h - G)\|_{L^1(\Omega)}$

By using the inverse inequality, we have

$$
\begin{aligned}
&\|\nabla^2_{\mathcal{T}}(G_h - G)\|_{L^1(\Omega)} + \|\hbar^{-1}\nabla(G_h - G)\|_{L^1(\Omega)} \\
&\le \|\nabla^2_{\mathcal{T}}(G_h - I_hG)\|_{L^1(\Omega)} + \|\nabla^2_{\mathcal{T}}(I_hG - G)\|_{L^1(\Omega)} + \|\hbar^{-1}\nabla(G_h - G)\|_{L^1(\Omega)} \\
&\le C\|\hbar^{-1}\nabla(G_h - I_hG)\|_{L^1(\Omega)} + \|\nabla^2_{\mathcal{T}}(I_hG - G)\|_{L^1(\Omega)} + \|\hbar^{-1}\nabla(G_h - G)\|_{L^1(\Omega)} \\
&\le C\|\hbar^{-1}\nabla(G_h - G)\|_{L^1(\Omega)} + C\|\hbar^{-1}\nabla(G - I_hG)\|_{L^1(\Omega)} + \|\nabla^2_{\mathcal{T}}(G - I_hG)\|_{L^1(\Omega)}.
\end{aligned}
\tag{4.37}
$$

The last two terms on the right-hand side are estimated in the following lemma.

Lemma 4.3 There exists a constant C, independent of h, such that

$$
\|\hbar^{-1}\nabla(G - I_hG)\|_{L^1(\Omega)} + \|\nabla^2_{\mathcal{T}}(G - I_hG)\|_{L^1(\Omega)} \le C\ell_h. \tag{4.38}
$$

Proof. By using the decomposition (4.1), we have

$$
\begin{aligned}
&\|\nabla^2_{\mathcal{T}}(G - I_hG)\|_{L^1(\Omega)} + \|\hbar^{-1}\nabla(G - I_hG)\|_{L^1(\Omega)} \\
&\le \|\nabla^2_{\mathcal{T}}(G - I_hG)\|_{L^1(O_*)} + \|\hbar^{-1}\nabla(G - I_hG)\|_{L^1(O_*)} \\
&\quad + \sum_{O_j \in \mathcal{O}_{z_0}} \|\nabla^2_{\mathcal{T}}(G - I_hG)\|_{L^1(O_j)} + \sum_{O_j \in \mathcal{O}_{z_0}} \|\hbar^{-1}\nabla(G - I_hG)\|_{L^1(O_j)} \\
&\quad + \|\nabla^2_{\mathcal{T}}(G - I_hG)\|_{L^1(\widetilde{O}_*)} + \|\hbar^{-1}\nabla(G - I_hG)\|_{L^1(\widetilde{O}_*)} \\
&\quad + \sum_{\widetilde{O}_j \in \mathcal{O}_{x_0}} \|\nabla^2_{\mathcal{T}}(G - I_hG)\|_{L^1(\widetilde{O}_j)} + \sum_{\widetilde{O}_j \in \mathcal{O}_{x_0}} \|\hbar^{-1}\nabla(G - I_hG)\|_{L^1(\widetilde{O}_j)} \\
&\quad + \sum_{\Omega_j \in \mathcal{O}} \|\nabla^2_{\mathcal{T}}(G - I_hG)\|_{L^1(\Omega_j)} + \sum_{\Omega_j \in \mathcal{O}} \|\hbar^{-1}\nabla(G - I_hG)\|_{L^1(\Omega_j)}.
\end{aligned}
$$

To estimate the integrals on O_*, we use the following result: for sufficiently small $q \in (1,2)$ the $W^{2,q}$ estimate $\|G\|_{W^{2,q}(\Omega)} \le C\|\tilde{\delta} - 1/|\Omega|\|_{L^q(\Omega)}$ holds; see [16]. By applying this result and the H^{s+1} estimate in Lemma 3.2, we obtain

$$
\begin{aligned}
&\|\hbar^{-1}\nabla(G - I_hG)\|_{L^1(O_*)} + \|\nabla^2_{\mathcal{T}}(G - I_hG)\|_{L^1(O_*)} \\
&\le Ch_*^{-1}d_{J_*}\|\nabla(G - I_hG)\|_{L^2(O_*)} + Cd_{J_*}^{2/q'}\|\nabla^2_{\mathcal{T}}(G - I_hG)\|_{L^q(O_*)} \\
&\le C\kappa(\kappa^{1-\gamma}h_*)^s\|G\|_{H^{s+1}(\Omega)} + C\kappa^{2/q'}h_*^{2/q'}\|G\|_{W^{2,q}(\Omega)} \\
&\le C\kappa(\kappa^{1-\gamma}h_*)^s\|\tilde{\delta}_{x_0} - 1/|\Omega|\|_{H^{s-1}(\Omega)} + C\kappa^{2/q'}h_*^{2/q'}\|\tilde{\delta}_{x_0} - 1/|\Omega|\|_{L^q(\Omega)} \\
&\le C\kappa(\kappa^{1-\gamma}h_*)^s\|\tilde{\delta}_{x_0} - 1/|\Omega|\|_{L^{p_s}(\Omega)} + C\kappa^{2/q'}h_*^{2/q'}\|\tilde{\delta}_{x_0} - 1/|\Omega|\|_{L^q(\Omega)} \\
&\le C\kappa(\kappa^{1-\gamma}h_*)^s\hbar(x_0)^{-2+2/p_s} + C\kappa^{2/q'}h_*^{2-2/q}\hbar(x_0)^{-2+2/q} \quad \text{(here (4.11d) is used)} \\
&\le C\kappa + C\kappa^{2\gamma/q'},
\end{aligned}
\tag{4.39}
$$

where we have chosen $p_s := 2/(2-s)$, which satisfies $L^{p_s}(\Omega) \hookrightarrow H^{s-1}(\Omega)$ and $-2 + 2/p_s = -s$, and we have also used the following relation in the derivation of the last inequality:

$$
\hbar(x_0) \sim |x_0 - z_0|^{1-\gamma}h \ge C(\kappa h_*)^{1-\gamma}h_*^{\gamma} \sim C\kappa^{1-\gamma}h_*.
$$

Similarly, the following estimate holds:

$$\begin{aligned}
&\|\hbar^{-1}\nabla(G-I_hG)\|_{L^1(\widetilde O_*)}+\|\nabla^2_{\mathcal T}(G-I_hG)\|_{L^1(\widetilde O_*)}\\
&\le C\hbar(x_0)^{-1}d_{J_{x_0}}\|\nabla(G-I_hG)\|_{L^2(\widetilde O_*)}+Cd_{J_{x_0}}^{2/q'}\|\nabla^2_{\mathcal T}(G-I_hG)\|_{L^q(\widetilde O_*)}\\
&\le C\kappa^{\gamma\sigma}\hbar(x_0)^s\|G\|_{H^{s+1}(\Omega)}+C(\kappa^{\gamma\sigma}\hbar(x_0))^{2/q'}\|G\|_{W^{2,q}(\Omega)}\\
&\le C\kappa^{\gamma\sigma}\hbar(x_0)^s\|\tilde\delta_{x_0}-1/|\Omega|\|_{H^{s-1}(\Omega)}+C(\kappa^{\gamma\sigma}\hbar(x_0))^{2/q'}\|\tilde\delta_{x_0}-1/|\Omega|\|_{L^q(\Omega)}\\
&\le C\kappa^{\gamma\sigma}\hbar(x_0)^s\|\tilde\delta_{x_0}-1/|\Omega|\|_{L^{p_s}(\Omega)}+C(\kappa^{\gamma\sigma}\hbar(x_0))^{2/q'}\|\tilde\delta_{x_0}-1/|\Omega|\|_{L^q(\Omega)}\\
&\le C\kappa^{\gamma\sigma}\hbar(x_0)^s\hbar(x_0)^{-2+2/p_s}+C\kappa^{\gamma\sigma2/q'}\hbar(x_0)^{2-2/q}\hbar(x_0)^{-2+2/q}\quad\text{(here (4.11d) is used)}\\
&\le C(\kappa^{\gamma\sigma}+\kappa^{\gamma\sigma2/q'})\\
&\le C\kappa^{\gamma\sigma}.
\end{aligned}\tag{4.40}$$

We estimate the integrals on O_j below by using Lemma 4.2:

$$\begin{aligned}
&\|\hbar^{-1}\nabla(G-I_hG)\|_{L^1(O_j)}+\|\nabla^2_{\mathcal T}(G-I_hG)\|_{L^1(O_j)}\\
&\le C\left(d_jh_j^{-1}\|\nabla(G-I_hG)\|_{L^2(O_j)}+d_j\|\nabla^2_{\mathcal T}(G-I_hG)\|_{L^2(O_j)}\right)\\
&\le\begin{cases}Cd_j\|G\|_{H^2(O_j)} & \text{if } r=1,\\ Cd_jh_j^\alpha\|G\|_{H^{2+\alpha}_*(O_j)} & \text{if } r\ge 2,\end{cases}\quad\text{(here the notation in (4.26)–(4.31) is used)}\\
&\le\begin{cases}C & \text{if } r=1,\\ Ch_j^\alpha d_j^{-\alpha} & \text{if } r\ge 2.\end{cases}
\end{aligned}\tag{4.41}$$

Since $h\sim h_*^\gamma$, $d_{J_*+1}\ge 2\kappa h_*$ and $\kappa\ge 1$, it follows that

$$\sum_j h_j^\alpha d_j^{-\alpha}\sim\sum_j(d_j^{1-\gamma}h)^\alpha d_j^{-\alpha}=\sum_j h^\alpha d_j^{-\gamma\alpha}\le Ch_*^{\gamma\alpha}(2\kappa h_*)^{-\gamma\alpha}\le C.$$

By using the inequality above in the case $r\ge 2$, and using the inequality $J_*\le C\ln(2+1/h)$ in the case $r=1$, we obtain

$$\begin{aligned}
&\sum_{O_j\in\mathcal O_{z_0}}\|\hbar^{-1}\nabla(G-I_hG)\|_{L^1(O_j)}+\sum_{O_j\in\mathcal O_{z_0}}\|\nabla^2_{\mathcal T}(G-I_hG)\|_{L^1(O_j)}\\
&\le\begin{cases}C\ln(2+1/h) & \text{if } r=1,\\ C & \text{if } r\ge 2,\end{cases}\\
&=C\ell_h.
\end{aligned}\tag{4.42}$$

In the same way, one can prove that

$$\begin{aligned}
&\sum_{\widetilde O_j\in\mathcal O_{x_0}}\|\hbar^{-1}\nabla(G-I_hG)\|_{L^1(\widetilde O_j)}+\sum_{\widetilde O_j\in\mathcal O_{x_0}}\|\nabla^2_{\mathcal T}(G-I_hG)\|_{L^1(\widetilde O_j)}\\
&\quad+\sum_{\Omega_j\in\mathcal O}\|\hbar^{-1}\nabla(G-I_hG)\|_{L^1(\Omega_j)}+\sum_{\Omega_j\in\mathcal O}\|\nabla^2_{\mathcal T}(G-I_hG)\|_{L^1(\Omega_j)}\le C\ell_h.
\end{aligned}\tag{4.43}$$

Summing up (4.39)–(4.40) and (4.42)–(4.43), we obtain the desired result (4.38). ∎

Then, by substituting Lemma 4.3 into (4.37), we obtain

$$\|\nabla^2_{\mathcal T}(G_h-G)\|_{L^1(\Omega)}+\|\hbar^{-1}\nabla(G_h-G)\|_{L^1(\Omega)}\le C\|\hbar^{-1}\nabla(G_h-G)\|_{L^1(\Omega)}+C\ell_h.\tag{4.44}$$

Now it remains to estimate $\|\hbar^{-1}\nabla(G_h - G)\|_{L^1(\Omega)}$.

4.5 Estimation of $\|\hbar^{-1}\nabla(G_h - G)\|_{L^1(\Omega)}$

We consider the decomposition

$$\begin{aligned}
\|\hbar^{-1}\nabla(G_h - G)\|_{L^1(\Omega)} &\le Ch_*^{-1}\|\nabla(G-G_h)\|_{L^1(O_*)} + C\hbar(x_0)^{-1}\|\nabla(G-G_h)\|_{L^1(\widetilde{O}_*)} \\
&\quad + C\sum_{O_j\in\mathcal{O}_{z_0}} h_j^{-1}\|\nabla(G-G_h)\|_{L^1(O_j)} \\
&\quad + C\sum_{\widetilde{O}_j\in\mathcal{O}_{x_0}} \hbar(x_0)^{-1}\|\nabla(G-G_h)\|_{L^1(\widetilde{O}_j)} \\
&\quad + C\sum_{\Omega_j\in\mathcal{O}} \mathtt{h}_j^{-1}\|\nabla(G-G_h)\|_{L^1(\Omega_j)} \\
&\le C\kappa\|\nabla(G-G_h)\|_{L^2(O_*)} + C\kappa\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_*)} \\
&\quad + C\sum_{O_j\in\mathcal{O}_{z_0}} d_j h_j^{-1}\|\nabla(G-G_h)\|_{L^2(O_j)} \\
&\quad + C\sum_{\widetilde{O}_j\in\mathcal{O}_{x_0}} d_j\hbar(x_0)^{-1}\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_j)} \\
&\quad + C\sum_{\Omega_j\in\mathcal{O}} \rho_j\mathtt{h}_j^{-1}\|\nabla(G-G_h)\|_{L^2(\Omega_j)}.
\end{aligned} \tag{4.45}$$

Again, we use the notation $p_s = 2/(2-s)$, with $L^{p_s}(\Omega)\hookrightarrow H^{s-1}(\Omega)$. Then we have

$$\begin{aligned}
&\|\nabla(G-G_h)\|_{L^2(O_*)} + \|\nabla(G-G_h)\|_{L^2(\widetilde{O}_*)} \\
&\le 2\|\nabla(G-G_h)\|_{L^2(\Omega)} \\
&\le 2\|\nabla(G-I_hG)\|_{L^2(\Omega)} \\
&\le 2\|\nabla(G-I_hG)\|_{L^2(O_*)} + 2\|\nabla(G-I_hG)\|_{L^2(\widetilde{O}_*)} \\
&\quad + 2\sum_{O_j\in\mathcal{O}_{z_0}}\|\nabla(G-I_hG)\|_{L^2(O_j)} + 2\sum_{\widetilde{O}_j\in\mathcal{O}_{x_0}}\|\nabla(G-I_hG)\|_{L^2(\widetilde{O}_j)} \\
&\quad + 2\sum_{\Omega_j\in\mathcal{O}}\|\nabla(G-I_hG)\|_{L^2(\Omega_j)} \\
&\le C(\kappa^{1-\gamma}h_*)^s\|G\|_{H^{s+1}(O_*')} + C\hbar(x_0)^s\|G\|_{H^{s+1}(\widetilde{O}_*')} \\
&\quad + C\sum_{O_j\in\mathcal{O}_{z_0}} h_j\|G\|_{H^2(O_j')} + C\sum_{\widetilde{O}_j\in\mathcal{O}_{x_0}}\hbar(x_0)\|G\|_{H^2(\widetilde{O}_j')} + C\sum_{\Omega_j\in\mathcal{O}}\mathtt{h}_j\|G\|_{H^2(\Omega_j')} \\
&\le C(\kappa^{1-\gamma}h_*)^s\|\tilde\delta_{x_0} - 1/|\Omega|\|_{H^{s-1}(\Omega)} + C\hbar(x_0)^s\|\tilde\delta_{x_0} - 1/|\Omega|\|_{H^{s-1}(\Omega)} \\
&\quad + C\sum_{O_j\in\mathcal{O}_*} h_j d_j^{-1} + C\sum_{\widetilde{O}_j\in\mathcal{O}_{x_0}}\hbar(x_0)d_j^{-1} + C\sum_{\Omega_j\in\mathcal{O}}\mathtt{h}_j\rho_j^{-1} \quad \text{(Lemma 4.2 is used here)} \\
&\le C(\kappa^{1-\gamma}h_*)^s\|\tilde\delta_{x_0} - 1/|\Omega|\|_{L^{p_s}(\Omega)} + C\hbar(x_0)^s\|\tilde\delta_{x_0} - 1/|\Omega|\|_{L^{p_s}(\Omega)} + C \\
&\le C(\kappa^{1-\gamma}h_*)^s\hbar(x_0)^{-2+2/p_s} + C\hbar(x_0)^s\hbar(x_0)^{-2+2/p_s} + C \quad \text{(here (4.11d) is used)} \\
&\le C.
\end{aligned} \tag{4.46}$$

The estimates (4.45)–(4.46) imply

$$\|\hbar^{-1}\nabla(G_h - G)\|_{L^1(\Omega)} \le C + C\mathcal{M}, \tag{4.47}$$

where

$$\begin{aligned}\mathcal{M} := &\sum_{O_j\in\mathcal{O}_{z_0}} d_j h_j^{-1}\|\nabla(G-G_h)\|_{L^2(O_j)} \\ &+ \sum_{\widetilde O_j\in\mathcal{O}_{x_0}} d_j \hbar(x_0)^{-1}\|\nabla(G-G_h)\|_{L^2(\widetilde O_j)} \\ &+ \sum_{\Omega_j\in\mathcal{O}} \rho_j \mathtt{h}_j^{-1}\|\nabla(G-G_h)\|_{L^2(\Omega_j)}.\end{aligned} \tag{4.48}$$

To estimate $\mathcal{M}$, we need to use the following local energy error estimate (cf. [9, Theorem 3.4]), which holds in general polygons.

Lemma 4.4 *For any $\xi \in \Omega$, let L_0 and L_1 be two concentric annuli such that $\{x \in \mathbb{R}^2 :$ $\mathrm{dist}(x, L_0) < d\} \subset L_1$,* and consider the subdomains $D = L_0 \cap \Omega$ and $D' = L_1 \cap \Omega$ of Ω. Moreover, we assume that $\hbar(x) < d$ and $\hbar(x) \sim \hbar(y)$ for all $x, y \in D'$. Then any function $u \in W^{1,1}(\Omega) \cap H^1(D')$ satisfies

$$\|u - R_h u\|_{H^1(D)} \le C\left(\|u - I_h u\|_{H^1(D')} + d^{-1}\|u - I_h u\|_{L^2(D')} + d^{-1}\|u - R_h u\|_{L^2(D')}\right).$$

Since $G_h = R_h G$, we can apply Lemma 4.4 with $u = G(\cdot, x_0)$ and use the local regularity estimates in Lemma 4.2. Then we obtain

$$\begin{aligned} & d_j h_j^{-1}\|\nabla(G-G_h)\|_{L^2(O_j)} \\ &\le C\left(d_j h_j^{-1}\|\nabla(G-I_hG)\|_{L^2(O_j')} + h_j^{-1}\|G-I_hG\|_{L^2(O_j')}\right) + Ch_j^{-1}\|G-G_h\|_{L^2(O_j')} \\ &\le \begin{cases} C\left(d_j\|G\|_{H^2(O_j'')} + h_j\|G\|_{H^2(O_j'')}\right) + Ch_j^{-1}\|G-G_h\|_{L^2(O_j')} & \text{if } r=1, \\ C\left(d_j h_j^{\alpha}\|G\|_{H_*^{2+\alpha}(O_j'')} + h_j^{1+\alpha}\|G\|_{H_*^{2+\alpha}(O_j'')}\right) & \\ \qquad\qquad +Ch_j^{-1}\|G-G_h\|_{L^2(O_j')} & \text{if } r\ge 2, \end{cases} \\ &\le \begin{cases} C + Ch_j^{-1}\|G-G_h\|_{L^2(O_j')} & \text{if } r=1, \\ C\left(d_j^{-\alpha}h_j^{\alpha} + d_j^{-1-\alpha}h_j^{1+\alpha}\right) + Ch_j^{-1}\|G-G_h\|_{L^2(O_j')} & \text{if } r\ge 2. \end{cases}\end{aligned} \tag{4.49}$$

Since $J_* \le C\ln(2+1/h)$ and $\sum_{O_j\in\mathcal{O}_{z_0}} d_j^{-\alpha}h_j^{\alpha} \le C$, it follows that

$$\sum_{O_j\in\mathcal{O}_{z_0}} d_j h_j^{-1}\|\nabla(G-G_h)\|_{L^2(O_j)} \le C\ell_h + C\sum_{O_j\in\mathcal{O}_{z_0}} h_j^{-1}\|G-G_h\|_{L^2(O_j')} \tag{4.50}$$

In the same way, one can prove that

$$\sum_{\widetilde O_j\in\mathcal{O}_{x_0}} d_j\hbar(x_0)^{-1}\|\nabla(G-G_h)\|_{L^2(\widetilde O_j)} \le C\ell_h + C\sum_{\widetilde O_j\in\mathcal{O}_{x_0}} \hbar(x_0)^{-1}\|G-G_h\|_{L^2(\widetilde O_j')}, \tag{4.51}$$

$$\sum_{\Omega_j\in\mathcal{O}} \rho_j\mathtt{h}_j^{-1}\|\nabla(G-G_h)\|_{L^2(\Omega_j)} \le C\ell_h + C\sum_{\Omega_j\in\mathcal{O}} \mathtt{h}_j^{-1}\|G-G_h\|_{L^2(\Omega_j')}. \tag{4.52}$$

Hence, by summing up (4.50)–(4.52), we have

$$\begin{aligned}\mathcal{M} &\le C\ell_h + C\sum_{O_j\in\mathcal{O}_{z_0}} h_j^{-1}\|G-G_h\|_{L^2(O_j')} \\ &\quad + C\sum_{\widetilde{O}_j\in\mathcal{O}_{x_0}} \hbar(x_0)^{-1}\|G-G_h\|_{L^2(\widetilde{O}_j')} + C\sum_{\Omega_j\in\mathcal{O}} \mathtt{h}_j^{-1}\|G-G_h\|_{L^2(\Omega_j')} \\ &\le C\ell_h + C\sum_{O_j\in\mathcal{O}_{z_0}'} h_j^{-1}\|G-G_h\|_{L^2(O_j)} \\ &\quad + C\sum_{\widetilde{O}_j\in\mathcal{O}_{x_0}'} \hbar(x_0)^{-1}\|G-G_h\|_{L^2(\widetilde{O}_j)} + C\sum_{\Omega_j\in\mathcal{O}} \mathtt{h}_j^{-1}\|G-G_h\|_{L^2(\Omega_j)},\end{aligned} \tag{4.53}$$

where we have used (4.46) in the last inequality.

The following three technical estimates can be proved for some $\sigma\in(\gamma,\beta)$, and their proofs are presented in Appendix:

$$\sum_{O_j\in\mathcal{O}_{z_0}'} h_j^{-1}\|G-G_h\|_{L^2(O_j)} \le C\kappa^{(1-\sigma)\gamma} + C\kappa^{-\gamma\sigma}\mathcal{M}, \tag{4.54}$$

$$\sum_{\widetilde{O}_j\in\mathcal{O}_{x_0}'} \hbar(x_0)^{-1}\|G-G_h\|_{L^2(\widetilde{O}_j)} \le C\kappa^{\gamma(1-\sigma)} + C\kappa^{-\gamma\sigma^2}\mathcal{M}, \tag{4.55}$$

$$\sum_{\Omega_j\in\mathcal{O}} \mathtt{h}_j^{-1}\|G-G_h\|_{L^2(\Omega_j)} \le C\kappa^{\gamma(1-\sigma)} + C\kappa^{-\gamma\sigma}\mathcal{M}. \tag{4.56}$$

Substituting (4.54)–(4.56) into (4.53), we have

$$\mathcal{M} \le C\ell_h + C\kappa^{\gamma(1-\sigma)} + C\kappa^{-\gamma\sigma^2}\mathcal{M}. \tag{4.57}$$

Then, by choosing κ sufficiently large, the last term of the above inequality would be absorbed by its left-hand side, and therefore we obtain

$$\mathcal{M} \le C\ell_h. \tag{4.58}$$

By substituting the above result into (4.47) and using (4.44), we obtain the key estimate (4.35). This completes the proof of Theorem 2.1 in the case $|x_0-z_0|>16\kappa h_*$.

4.6 The case $|x_0-z_0|\le 16\kappa h_*$

Note that κ is a fixed constant already determined below (4.57). In the case $|x_0-z_0|\le 16\kappa h_*$, we decompose the domain Ω into disjoint subsets

$$\Omega = \bigcup_{j=0}^{J+1}\Omega_j, \tag{4.59}$$

where

$$\Omega_j := \{x\in\Omega : \rho_{j+1}\le |x-z_0| < \rho_j\}, \qquad j=0,1,\dots,J, \tag{4.60a}$$

$$\Omega_{J+1} := \{x\in\Omega : |x-z_0|<\rho_{J+1}\}, \tag{4.60b}$$

with $\rho_j = 2^{-j}\mathrm{diameter}(\Omega)$ and $J=\left[\log_2\left(\frac{\mathrm{diameter}(\Omega)}{16\kappa h_*\lambda}\right)\right]$, so that

$$8\kappa h_*\lambda \le \rho_{J+1} \le 16\kappa h_*\lambda, \tag{4.61a}$$

where λ is a constant to be determined later (like the constant κ in the previous subsections).

The rest proof is similar as the proof for the case $|x_0 - z_0| > 16\kappa h_*$, except that the decomposition (4.1) is replaced by the simpler one (4.59). In particular, inequality (4.33) still holds, i.e.,

$$\begin{aligned}&\left|R_h u(x_0) - I_h u(x_0) - \frac{1}{|\Omega|}\int_\Omega (R_h u - I_h u)\mathrm{d}x\right| \\ &\leq C\big(\|\nabla^2_{\mathcal{T}}(G_h - G)\|_{L^1(\Omega)} + \|\hbar^{-1}\nabla(G_h - G)\|_{L^1(\Omega)} + 1\big)\|u\|_{L^\infty(\Omega)},\end{aligned} \tag{4.62}$$

and (4.47)–(4.48) would be replaced by

$$\|\nabla^2_{\mathcal{T}}(G_h - G)\|_{L^1(\Omega)} + \|\hbar^{-1}\nabla(G_h - G)\|_{L^1(\Omega)} \leq C\ell_h + C\mathcal{M}, \tag{4.63}$$

with

$$\mathcal{M} := \sum_{j=0}^{J} \rho_j h_j^{-1}\|\nabla(G - G_h)\|_{L^2(\Omega_j)}. \tag{4.64}$$

The similar estimates as in the previous subsections would yield the following estimate (similarly as (4.57)):

$$\mathcal{M} \leq C\ell_h + C\lambda^{\gamma(1-\sigma)} + C\lambda^{-\gamma\sigma}\mathcal{M}. \tag{4.65}$$

By choosing sufficiently large λ, the last term of (4.65) can be absorbed by its left-hand side and therefore,

$$\mathcal{M} \leq C\ell_h. \tag{4.66}$$

Substituting (4.63) and (4.66) into (4.62) would yield the desired result, i.e.,

$$|R_h u(x_0)| \leq C\ell_h\|u\|_{L^\infty(\Omega)}. \tag{4.67}$$

This completes the proof of Theorem 2.1. ∎

5 Proof of Corollary 2.1

In this section, we prove Corollary 2.1 by applying the result of Theorem 2.1 and assuming that the triangulation satisfies the general conditions described in Section 2.1.

We first prove the following local $W^{k+2,p}$ regularity for the solutions of the Poisson equation and then use this result to prove Corollary 2.1.

Lemma 5.1 (Local $W^{k+2,p}$ estimates) *Let k and p be nonnegative integer and real number, respectively, such that $p \geq 2$, $(k,p) \neq (0,2)$, and $(1-\frac{1}{p})\frac{2\theta_j}{\pi}$ are not integers for $j = 0,1,\dots,m-1$. Let $f \in W^{k,p}(\Omega)$ (satisfying the compatibility condition $\int_\Omega f\mathrm{d}x = 0$) and let u be the unique solution of (1.1), and let $d > 0$ be small enough so that* $\mathrm{dist}(\Omega\cap B_{3d}(z_i), z_j) \geq C$ *when $i \neq j$ and $i,j = 0,1,\dots,m-1$. Then*

$$\|u\|_{W^{k+2,p}(\Omega\cap B_{2d}(z_j)\backslash B_d(z_j))} \leq Cd^{-k-1+\frac{2}{p}-\frac{2}{q_j}}\|f\|_{W^{k,p}(\Omega)}, \tag{5.1}$$

where $q_j = 2/(1-\beta_j)$ if $\beta_j < 1$ and $q_j = \infty$ if $\beta_j > 1$ and $k \geq 1$.

Proof. Since we have assumed that $k \geq 0$, $p \geq 2$ and $(k,p) \neq (0,2)$, there are two cases: (1) If $k = 0$ then $p > 2$; (2) If $k \geq 1$ then $W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for some $q > 2$. In either case, $f \in W^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$ for some $q > 2$. We can choose such a fixed $q > 2$ such that condition (3.21) is satisfied. Then from (3.25)–(3.27) (or [16, Corollary 4.4.4.11]) we know there exist some constants $c_{j,n}$, $n = 1, \dots, K_j$ and $j = 0, \dots, m-1$, such that

$$u - \sum_{j=0}^{m-1} \sum_{n=1}^{K_j} c_{j,n} S_{j,n} \in W^{2,q}(\Omega),$$

where the expression of $S_{j,n}$ in (3.26) implies that

$$\|\nabla S_{j,n}\|_{L^{q_j,\infty}(\Omega)} \le C \quad \text{with } q_j = 2/(1-\beta_j) \text{ if } \beta_j = \tfrac{\pi}{\omega_j} < 1, \text{ and } q_j = \infty \text{ if } \beta_j > 1.$$

Moreover, as explained in the text above (3.27), there exists $c \in \mathbb{R}$ such that

$$\sum_{j=0}^{m-1} \sum_{n=1}^{K_j} |c_{j,n}| + \left\| u - c - \sum_{j=0}^{m-1} \sum_{n=1}^{K_j} c_{j,n} S_{j,n} \right\|_{W^{2,q}(\Omega)} \le C\|f\|_{L^q(\Omega)} \le C\|f\|_{W^{k,p}(\Omega)}.$$

Since $q > 2$, it follows that $W^{2,q}(\Omega) \hookrightarrow W^{1,\infty}(\Omega) \hookrightarrow W^{1,q_j}(\Omega)$. As a result, the two inequalities above and the triangle inequality imply that

$$\|\nabla u\|_{L^{q_j,\infty}(\Omega \cap B_{2d}(z_j))} \le C\|f\|_{W^{k,p}(\Omega)} \quad \text{for } j = 0, 1, \dots, m-1.$$

This is the basic estimate to be used in the following proof.

For any fixed $\beta \in (0,1)$, the circular region $\Omega \cap B_{2d}(z_0) \backslash B_d(z_0)$ can be covered by a bounded number of disks of radius βd (the number depends on β). We shall present estimates of the solution in each of these disks. To this end, for $\zeta \in \Omega \cap B_{2d}(z_j) \backslash B_d(z_j)$ and $k \ge 0$, we denote by ω_k a smooth cut-off function such that

$$\omega_k(x) \equiv 1, \qquad \text{in } B_{\beta d/2^{k+3}}(\zeta) \tag{5.2a}$$

$$\omega_k(x) \equiv 0, \qquad \text{outside } B_{\beta d/2^{k+2}}(\zeta) \tag{5.2b}$$

$$|\nabla^l \omega_k| \le C_l d^{-l}, \qquad l = 1, 2, \dots \tag{5.2c}$$

and let c be an arbitrary constant. Since u is the solution of (3.1), it follows that $\tilde{u} = \omega_k(u-c)$ is the solution of

$$\begin{cases} -\Delta \tilde{u} = \tilde{f} & \text{in } \Omega, \\ \partial_n \tilde{u} = \tilde{g} \cdot n & \text{on } \partial\Omega, \end{cases} \tag{5.3}$$

with

$$\tilde{f} = f\omega_k - 2\nabla u \cdot \nabla \omega_k - (u-c)\Delta\omega_k, \tag{5.4}$$

$$\tilde{g} = (u-c)\nabla\omega_k. \tag{5.5}$$

Note that the functions $\tilde{u}$, $\tilde{f}$ and $\tilde{g}$ are all supported on $B_{\beta d/2^{k+2}}(\zeta)$. If $B_{\beta d/2}(\zeta) \cap \partial\Omega = \emptyset$ then the equation in (5.3) actually holds on $\mathbb{R}^2$. Then the $W^{k+2,p}$ estimates of $\tilde{u}$ can be obtained similarly as (but simpler than) the following argument for the more complicated case that $B_{\beta d/2}(\zeta) \cap \partial\Omega \neq \emptyset$.

Without loss of generality, we focus on the case $B_{\beta d/2}(\zeta) \cap \partial\Omega \neq \emptyset$ and, by choosing β small enough we can make sure that $B_{\beta d}(\zeta)$ does not intersect other sides of Ω. Via a rotation we can assume that one side of $\partial\Omega \cap B_{\beta d/2}(\zeta)$ is contained in $\mathbb{R}_+ \times \{0\}$. Since $\tilde{u}$ is supported in $B_{\beta d/2}(\zeta)$, it follows that (5.3) holds in the upper half plane, i.e.,

$$\begin{cases} -\Delta \tilde{u} = \tilde{f} & \text{in } \mathbb{R} \times \mathbb{R}_+, \\ \partial_n \tilde{u} = \tilde{g} \cdot n & \text{on } \mathbb{R} \times \{0\}, \end{cases} \tag{5.6}$$

where $\tilde{g}(x_1, 0) = 0$ for $x_1 \leq 0$. Let $B_{\beta d/2}(\zeta)_+ = B_{\beta d/2}(\zeta) \cap (\mathbb{R} \times \mathbb{R}_+)$. By applying the $W^{k+1,p}$ estimates on the upper half plane we obtain

$$
\begin{aligned}
&|u|_{W^{k+2,p}(\mathbb{R}\times\mathbb{R}_+)} \\
&\leq C\|\tilde{f}\|_{W^{k,p}(\mathbb{R}\times\mathbb{R}_+)} + C\|\tilde{g}\cdot n\|_{W^{k+1-\frac{1}{p},p}(\mathbb{R}\times\{0\})} \\
&\leq C\|\tilde{f}\|_{W^{k,p}(\mathbb{R}\times\mathbb{R}_+)} + C\|\tilde{g}\|_{W^{k+1,p}(\mathbb{R}\times\mathbb{R}_+)} \\
&\leq C\sum_{j=0}^{k} d^{-j}\|f\|_{W^{k-j,p}(B_{\beta d/2^{k+2}}(\zeta)_+)} + C\sum_{j=0}^{k} d^{-j-1}\|u\|_{W^{k+1-j,p}(B_{\beta d/2^{k+2}}(\zeta)_+)} \\
&\quad + C\sum_{j=0}^{k} d^{-j-2}\|u-c\|_{W^{k-j,p}(B_{\beta d/2^{k+2}}(\zeta)_+)} + C\sum_{j=0}^{k+1} d^{-j-1}\|u-c\|_{W^{k+1-j,p}(B_{\beta d/2^{k+2}}(\zeta)_+)},
\end{aligned}
$$

where we have substituted the expressions (5.4)–(5.5) in deriving the last inequality. By choosing c to be the average of u on $B_{d/2^{k+2}}(\zeta)$, we have

$$
|u-c|_{L^p(B_{\beta d/2^{k+2}}(\zeta)_+)} \leq Cd|u|_{W^{1,p}(B_{\beta d/2^{k+2}}(\zeta)_+)}.
$$

Since $w_k = 1$ on $B_{\beta d/2^{k+3}}(\zeta)$, it follows that

$$
\begin{aligned}
&|u|_{W^{k+2,p}(B_{\beta d/2^{k+3}}(\zeta)\cap\Omega)} \\
&\leq C\sum_{j=0}^{k} d^{-j}\|f\|_{W^{k-j,p}(B_{\beta d/2^{k+2}}(\zeta)\cap\Omega)} + C\sum_{j=0}^{k} d^{-j-1}\|u\|_{W^{k+1-j,p}(B_{\beta d/2^{k+2}}(\zeta)\cap\Omega)}.
\end{aligned} \tag{5.7}
$$

Since the right-hand side contains strictly lower-order norms of u than the left-hand side, by iterating the inequality with respect to k we can obtain

$$
|u|_{W^{k+2,p}(B_{\beta d/2^{k+3}}(\zeta)\cap\Omega)} \leq C\sum_{j=0}^{k} d^{-j}\|f\|_{W^{k-j,p}(B_{\beta d/4}(\zeta)\cap\Omega)} + Cd^{-k-1}\|\nabla u\|_{L^p(B_{\beta d/4}(\zeta)\cap\Omega)}.
$$

Then, substituting the L^p estimate for ∇u in (3.49) into the above inequality, we have

$$
\begin{aligned}
&|u|_{W^{k+2,p}(B_{\beta d/2^{k+3}}(\zeta)\cap\Omega)} \\
&\leq C\sum_{j=0}^{k} d^{-j}\|f\|_{W^{k-j,p}(B_{\beta d/2}(\zeta)\cap\Omega)} + Cd^{-k-2+\frac{2}{p}}\|\nabla u\|_{L^2(B_{\beta d/2}(\zeta)\cap\Omega)} \\
&\leq C\sum_{j=0}^{k} d^{-j}\|f\|_{W^{k-j,p}(B_{\beta d/2}(\zeta)\cap\Omega)} + Cd^{-k-1+\frac{2}{p}-\frac{2}{q}}\|\nabla u\|_{L^{q,\infty}(B_{\beta d/2}(\zeta)\cap\Omega)},
\end{aligned} \tag{5.8}
$$

where the last inequality follows from using Hölder's inequality with the weak L^q norm, for arbitrary $q \in [2, \infty]$. The proof of above inequality relies on the iteration of (5.7). In fact, we can construct many intermediate disks between $B_{\beta d}(\zeta)$ and $B_{\beta d/2}(\zeta)$ and apply similar iterations as (5.7). In this way, we would obtain a similar inequality as (5.8), but with $B_{\beta d/2^{k+3}}(\zeta)$ replaced by $B_{\beta d/4}(\zeta)$ on the left-hand side, i.e.,

$$
|u|_{W^{k+2,p}(B_{\beta d/4}(\zeta)\cap\Omega)} \leq C\sum_{j=0}^{k} d^{-j}\|f\|_{W^{k-j,p}(B_{\beta d/2}(\zeta)\cap\Omega)} + Cd^{-k-1+\frac{2}{p}-\frac{2}{q}}\|\nabla u\|_{L^{q,\infty}(B_{\beta d/2}(\zeta)\cap\Omega)}, \tag{5.9}
$$

Since the circular region $\Omega \cap B_{2d}(z_j)\backslash B_d(z_j)$ can be covered by a bounded number of balls of radius βd, by summing up the $W^{k+2,p}$ estimates over these balls, we obtain

$$\begin{aligned}\|u\|_{W^{k+2,p}(\Omega\cap B_{2d}(z_j)\backslash B_d(z_j))} &\le C\sum_{j=0}^{k} d^{-j}\|f\|_{W^{k-j,p}(\Omega)} + Cd^{-k-1+\frac{2}{p}-\frac{2}{q_j}}\|\nabla u\|_{L^{q_j,\infty}(\Omega\cap B_{3d}(z_j))}\\ &\le Cd^{-k-1+\frac{2}{p}-\frac{2}{q_j}}\|f\|_{W^{k,p}(\Omega)}. \end{aligned} \tag{5.10}$$

This proves the desired result of Lemma 5.1. ∎

Let d be a sufficiently small constant such that $\mathrm{dist}(z_i, z_j) \ge 4d$ for any two different vertices z_i and z_j of the polygon Ω. For any $j = 0, 1, \ldots, m-1$, we denote $D_{j,i} = \Omega \cap B_{2d_i}(z_j)\backslash B_{d_i}(z_j)$, with $d_i = 2^{-i-2}d$ for $i = 0, 1, \ldots, I_j$, where I_j is determined by $2^{-I_j}d \sim \kappa h_{*,j}$, where κ can be chosen to be large enough so that $d_i \ge C\kappa^{\gamma_j}h_{j,i}$ is bigger than twice of the mesh size in $B_{d_i}(z_j)$.

Let $\Omega_0 = \{x \in \Omega : \mathrm{dist}(x, z_j) \ge d/4 \text{ for } j = 0, 1, \ldots, m-1\}$. We denote by $h_{j,i}$ the mesh size in $D_{j,i}$. According to the mesh size choice in (2.1), we have

$$h_{j,i} = d_i^{1-\gamma_j}h.$$

Then

$$\Omega = \Omega_0 \cup \bigcup_{j=0}^{m-1} \cup_{i=0}^{I_j} D_{i,j}.$$

Theorem 1.2 implies that the finite element solution given by (1.2) satisfies the following error estimate:

$$\begin{aligned}\|u-u_h\|_{L^\infty(\Omega)} &= \|u - I_hu - R_h(u-I_hu)\|_{L^\infty(\Omega)}\\ &\le C\ell_h\|u-I_hu\|_{L^\infty(\Omega)}\\ &\le C\ell_h \max_{0\le j\le m-1}\max_{1\le i\le I_j}\|u-I_hu\|_{L^\infty(\Omega\cap B_{2d_i}(z_j)\backslash B_{d_i}(z_j))} + C\ell_h\|u-I_hu\|_{L^\infty(\Omega_0)}. \end{aligned} \tag{5.11}$$

We consider two cases separately.

Case 1: $r \ge k+1$. In this case, we have

$$\begin{aligned}\|u-I_hu\|_{L^\infty(\Omega\cap B_{2d_i}(z_j)\backslash B_{d_i}(z_j))} &\le Ch_{j,i}^{k+2-\frac{2}{p}}\|u\|_{W^{k+2,p}(\Omega\cap B_{4d_i}(z_j)\backslash B_{d_i/2}(z_j))}\\ &\le Ch_{j,i}^{k+2-\frac{2}{p}}d_i^{-k-1+\frac{2}{p}-\frac{2}{q_j}}\|f\|_{W^{k,p}(\Omega)}\\ &\le Cd_i^{(1-\gamma_j)(k+2-\frac{2}{p})}h^{k+2-\frac{2}{p}}d_i^{-k-1+\frac{2}{p}-\frac{2}{q_j}}\|f\|_{W^{k,p}(\Omega)}\\ &\le Ch^{k+2-\frac{2}{p}}d_i^{(1-\gamma_j)(k+2-\frac{2}{p})-k-1+\frac{2}{p}-\frac{2}{q_j}}\|f\|_{W^{k,p}(\Omega)}\\ &\le Ch^{k+2-\frac{2}{p}}d_i^{-\gamma_j(k+2-\frac{2}{p})+1-\frac{2}{q_j}}\|f\|_{W^{k,p}(\Omega)}. \end{aligned} \tag{5.12}$$

By choosing $-\gamma_j(k+2-\frac{2}{p})+1-\frac{2}{q_j} \ge 0$, or equivalently $\gamma_j \le \frac{1-2/q_j}{k+2-2/p} = \frac{\min(1,\beta_j)}{k+2-2/p}$ (as $q_j = \infty$ when $\beta_j > 1$), we obtain

$$\|u-I_hu\|_{L^\infty(\Omega\cap B_{2d_i}(z_j)\backslash B_{d_i}(z_j))} \le Ch^{k+2-\frac{2}{p}}\|f\|_{W^{k,p}(\Omega)}. \tag{5.13}$$

Since the mesh size in Ω_0 is $O(h)$, it follows that

$$\|u - I_h u\|_{L^\infty(\Omega_0)} \le Ch^{k+2-\frac{2}{p}} \|u\|_{W^{k+2,p}(\Omega_0')} \le Ch^{k+2-\frac{2}{p}} \|f\|_{W^{k,p}(\Omega)}, \tag{5.14}$$

where $\Omega_0' = \{x \in \Omega : \mathrm{dist}(x, z_j) \ge d/8\} \supset \Omega_0$. By substituting the two estimates above into (5.11), we obtain

$$\|u - u_h\|_{L^\infty(\Omega)} \le C\ell_h h^{k+2-\frac{2}{p}} \|f\|_{W^{k,p}(\Omega)}, \tag{5.15}$$

This proves the desired result of Corollary 2.1 in the case $r \ge k+1$.

Case 2: $r = k \ge 1$. In this case, we choose $p = 2$ in Lemma 5.1 and replace (5.12) by the following estimate:

$$\begin{aligned}
\|u - I_h u\|_{L^\infty(\Omega\cap B_{2d_i}(z_j)\setminus B_{d_i}(z_j))} &\le Ch_{j,i}^{r+1} \|u\|_{H^{r+2}(\Omega\cap B_{4d_i}(z_j)\setminus B_{d_i/2}(z_j))} \\
&\le Ch_{j,i}^{r+1} d_i^{-r-\frac{2}{q_j}} \|f\|_{H^r(\Omega)} \\
&\le Ch^{r+1} d_i^{(1-\gamma_j)(r+1)-r-\frac{2}{q_j}} \|f\|_{H^r(\Omega)}.
\end{aligned} \tag{5.16}$$

By choosing $(1-\gamma_j)(r+1) - r - \frac{2}{q_j} \ge 0$, or equivalently $\gamma_j \le \frac{1-2/q_j}{r+1} = \frac{\min(1,\beta_j)}{r+1}$ (as $q_j = \infty$ when $\beta_j > 1$), we obtain

$$\|u - I_h u\|_{L^\infty(\Omega\cap B_{2d_i}(z_j)\setminus B_{d_i}(z_j))} \le Ch^{r+1} \|f\|_{H^r(\Omega)}. \tag{5.17}$$

Since the mesh size in Ω_0 is $O(h)$, it follows that

$$\|u - I_h u\|_{L^\infty(\Omega_0)} \le Ch^{r+1} \|u\|_{H^{r+1}(\Omega_0')} \le Ch^{r+1} \|f\|_{H^r(\Omega)}, \tag{5.18}$$

where $\Omega_0' = \{x \in \Omega : \mathrm{dist}(x, z_j) \ge d/8\} \supset \Omega_0$. By substituting the two estimates above into (5.11), we obtain

$$\|u - u_h\|_{L^\infty(\Omega)} \le C\ell_h h^{r+1} \|f\|_{H^r(\Omega)}, \tag{5.19}$$

This proves the desired result of Corollary 2.1 in the case $r = k$. ∎

6 Conclusions

We have proved the maximum-norm stability of finite element solutions to the Poisson equation with the Neumann boundary condition in a polygon which is possibly nonconvex. The use of graded mesh, with triangulations locally refined at the reentrant corners, is essential to the proof. With the maximum-norm stability result, the error estimation in the L^∞ norm can be reduced to an interpolation error estimate. By analyzing the interpolation error, the error estimate is derived in terms of the smoothness $f \in W^{k,p}(\Omega)$ of the right-hand side, with $2 \le p < \infty$. By using norms in Lorentz spaces $L^{p,q}(\Omega)$ (instead of the usual Lebesgue spaces) for the singular functions, it is possible to choose the grading parameter in the limit.

The analysis in this article may be extended to the Dirichlet boundary condition and more general elliptic equations with variable diffusion coefficients in two-dimensional polygonal domains. The maximum-norm stability of finite element solutions in three-dimensional nonconvex polyhedral domains still remains open.

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Appendix A: Proof of (4.54)

For $O_j \in \mathcal{O}'_{z_0}$, we estimate $\|G - G_h\|_{L^2(O_j)}$ via the duality

$$\|G - G_h\|_{L^2(O_j)} = \sup_{\substack{\psi \in C_0^\infty(O_j) \\ \|\psi\|_{L^2(O_j)} \le 1}} (G - G_h, \psi).$$

For any given $\psi \in C_0^\infty(O_j)$ satisfying $\|\psi\|_{L^2(O_j)} \le 1$, we define w as the solution of

$$\begin{cases} -\Delta w = \psi - \overline{\psi} & \text{in } \Omega, \\ \partial_n w = 0 & \text{on } \partial\Omega, \end{cases} \tag{A.1}$$

where $\overline{\psi} := \frac{1}{|\Omega|}\int_\Omega \psi(x)dx$ is the average of ψ in the domain Ω. The solution w exists and is unique under the condition $\int_\Omega w(x)\mathrm{d}x = 0$. It is known that the solution of (A.1) satisfies that

$$\|w\|_{H^{s+1}(\Omega)} \le C\|\psi\|_{H^{s-1}(\Omega)} \quad \text{for } s = 0 \text{ and } s \in \left(\frac{1}{2}, \beta\right);$$

see Lemma 3.2 with $g = 0$. By the complex interpolation, we have

$$\|w\|_{H^{s+1}(\Omega)} \le C\|\psi\|_{H^{s-1}(\Omega)} \quad \text{for } s \in [0, \beta). \tag{A.2}$$

Here we choose $s > \gamma$ to be sufficiently close to γ so that $(1-\gamma)(1+s) < 1$, and choose $\sigma = s$ in the definition of J_{x_0} below (4.2). Inequality (A.2) will be used in Appendices A, B and C.

By using the Sobolev embedding $L^{p_s}(\Omega) \hookrightarrow H^{s-1}(\Omega)$ with $p_s = 2/(2-s)$ and Hölder's inequality, we have

$$\|\psi\|_{H^{s-1}(\Omega)} \le C\|\psi\|_{L^{p_s}(\Omega)} \le C|O_j|^{\frac{1}{p_s}-\frac{1}{2}}\|\psi\|_{L^2(\Omega)} \le Cd_j^{\frac{2}{p_s}-1}\|\psi\|_{L^2(\Omega)} = Cd_j^{1-s}. \tag{A.3}$$

This inequality will be frequently used below.

By using the normalization condition $\int_\Omega G_h \mathrm{d}x = \int_\Omega G \mathrm{d}x$, we have $(G - G_h, \overline{\psi}) = 0$ and therefore

$$
\begin{aligned}
(G - G_h, \psi) &= (G - G_h, \psi - \overline{\psi}) = (G - G_h, -\Delta w) = (\nabla(G - G_h), \nabla w) \\
&= (\nabla(G - G_h), \nabla(w - I_h w)) \\
&= (\nabla(G - G_h), \nabla(w - I_h w))_{O_*} \\
&\quad + (\nabla(G - G_h), \nabla(w - I_h w))_{\widetilde{O}_*} \\
&\quad + \sum_{O_i \in \mathcal{O}_{z_0} \cup \{O_{-1}\}} (\nabla(G - G_h), \nabla(w - I_h w))_{O_i} \\
&\quad + \sum_{\widetilde{O}_i \in \mathcal{O}_{x_0}} (\nabla(G - G_h), \nabla(w - I_h w))_{\widetilde{O}_i} \\
&\quad + \sum_{\Omega_i \in \mathcal{O} \setminus \{\Omega_{J+1}\}} (\nabla(G - G_h), \nabla(w - I_h w))_{\Omega_i} \\
&=: \sum_{j=1}^{5} \mathcal{E}_j.
\end{aligned}
\tag{A.4}
$$

We estimate $\mathcal{E}_j$, $j = 1, \ldots, 5$, separately.

The first term on the right-hand side of (A.4) can be estimated by

$$
\begin{aligned}
|\mathcal{E}_1| &= |(\nabla(G - G_h), \nabla(w - I_h w))_{O_*}| \\
&\leq \|\nabla(G - G_h)\|_{L^2(O_*)} \|\nabla(w - I_h w)\|_{L^2(O_*)} \\
&\leq C(\kappa^{1-\gamma} h_*)^s \|\nabla(G - G_h)\|_{L^2(O_*)} \|w\|_{H^{s+1}(\Omega)} \\
&\leq C(\kappa^{1-\gamma} h_*)^s \|\nabla(G - G_h)\|_{L^2(O_*)} \|\psi\|_{H^{s-1}(\Omega)} \\
&\leq C(\kappa^{1-\gamma} h_*)^s d_j^{1-s} \|\nabla(G - G_h)\|_{L^2(O_*)} \\
&\leq C(\kappa^{1-\gamma} h_*)^s d_j^{1-s},
\end{aligned}
\tag{A.5}
$$

where we have used (A.3) in the second to last inequality, and (4.46) in the last inequality.

The second term on the right-hand side of (A.4) can be estimated by

$$
\begin{aligned}
|\mathcal{E}_2| &= |(\nabla(G - G_h), \nabla(w - I_h w))_{\widetilde{O}_*}| \\
&\leq C\hbar(x_0) \|w\|_{H^2(\widetilde{O}_*)} \|\nabla(G - G_h)\|_{L^2(\widetilde{O}_*)} \\
&\leq C\hbar(x_0) \|w\|_{H^2(\widetilde{O}_*)},
\end{aligned}
\tag{A.6}
$$

where we have used (4.46). Note that $\|w\|_{H^2(\widetilde{O}_*)}$ can be estimated by using Green's formula, i.e.,

$$
w(x) = \int_\Omega \Gamma(x, y) \psi(y) \mathrm{d}y = \int_{O_j} \Gamma(x, y) \psi(y) \mathrm{d}y.
$$

Since $\psi \in C_0^\infty(O_j)$ and $\|\psi\|_{L^2(O_j)} \leq 1$, it follows that

$$
\begin{aligned}
\|w\|_{H^2(\widetilde{O}_*)} &\leq \sup_{y \in O_j} \|\Gamma(\cdot, y)\|_{H^2(\widetilde{O}_*)} \|\psi\|_{L^1(O_j)} \\
&\leq C \mathrm{dist}(\widetilde{O}_*, O_j)^{-1} |O_j|^{\frac{1}{2}} \|\psi\|_{L^2(O_j)} \\
&\leq C |x_0 - z_0|^{-1} d_j.
\end{aligned}
$$

Then, substituting this result into (A.6), we have

$$|\mathcal{E}_2| \leq C\hbar(x_0)|x_0 - z_0|^{-1} d_j. \tag{A.7}$$

For $O_i \in \mathcal{O}_{z_0} \cup \{O_{-1}\}$ we have

$$\begin{aligned} |\mathcal{E}_3| = |(\nabla(G - G_h), \nabla(w - I_h w))_{O_i}| &\leq C h_i^s \|w\|_{H^{s+1}(\Omega)} \|\nabla(G - G_h)\|_{L^2(O_i)} \\ &\leq C h_i^s \|\psi\|_{H^{s-1}(\Omega)} \|\nabla(G - G_h)\|_{L^2(O_i)} \\ &\leq C h_i^s \|\psi\|_{L^{p_s}(\Omega)} \|\nabla(G - G_h)\|_{L^2(O_i)} \\ &\leq C h_i^s d_j^{\frac{2}{p_s}-1} \|\psi\|_{L^2(\Omega)} \|\nabla(G - G_h)\|_{L^2(O_i)} \\ &\leq C h_i^s d_j^{1-s} \|\nabla(G - G_h)\|_{L^2(O_i)}. \end{aligned} \tag{A.8}$$

For $\widetilde{O}_i \in \mathcal{O}_{x_0}$ there holds

$$\begin{aligned} \|w\|_{H^2(\widetilde{O}_i')} &\leq \sup_{y \in O_j} \|G(\cdot, y)\|_{H^2(\widetilde{O}_i')} \|\psi\|_{L^1(O_j)} \\ &\leq C|x_0 - z_0|^{-1} |O_j|^{\frac{1}{2}} \|\psi\|_{L^2(O_j)} \\ &\leq C|x_0 - z_0|^{-1} d_j, \end{aligned}$$

which implies that

$$\begin{aligned} |\mathcal{E}_4| = |(\nabla(G - G_h), \nabla(w - I_h w))_{\widetilde{O}_i}| &\leq C\hbar(x_0) \|w\|_{H^2(\widetilde{O}_i')} \|\nabla(G - G_h)\|_{L^2(\widetilde{O}_i)} \\ &\leq C\hbar(x_0)|x_0 - z_0|^{-1} d_j \|\nabla(G - G_h)\|_{L^2(\widetilde{O}_i)}. \end{aligned} \tag{A.9}$$

For $\Omega_i \in \mathcal{O}\backslash\{\Omega_{J+1}\}$ there holds

$$\begin{aligned} \|w\|_{H^2(\Omega_i')} &\leq \sup_{y \in O_j} \|G(\cdot, y)\|_{H^2(\Omega_i')} \|\psi\|_{L^1(O_j)} \\ &\leq C\rho_i^{-1} |O_j|^{\frac{1}{2}} \|\psi\|_{L^2(O_j')} \\ &\leq C\rho_i^{-1} d_j, \end{aligned}$$

which implies that

$$\begin{aligned} |\mathcal{E}_5| = |(\nabla(G - G_h), \nabla(w - I_h w))_{\Omega_i}| &\leq C\mathtt{h}_i \|w\|_{H^2(\Omega_i')} \|\nabla(G - G_h)\|_{L^2(\Omega_i)} \\ &\leq C\mathtt{h}_i \rho_i^{-1} d_j \|\nabla(G - G_h)\|_{L^2(\Omega_i)}. \end{aligned} \tag{A.10}$$

Overall, substituting the estimates of $|\mathcal{E}_j|$, $j = 1, \ldots, 5$, into (A.4), we obtain (via the duality argument)

$$\begin{aligned} \|G - G_h\|_{L^2(O_j)} \leq \quad & C(\kappa^{1-\gamma} h_*)^s d_j^{1-s} + C\hbar(x_0)|x_0 - z_0|^{-1} d_j \\ & + C \sum_{O_i \in \mathcal{O}_{z_0} \cup \{O_{-1}\}} (d_j^{1-s} h_i^{s+1} d_i^{-1}) d_i h_i^{-1} \|\nabla(G - G_h)\|_{L^2(O_i)} \\ & + C \sum_{\widetilde{O}_i \in \mathcal{O}_{x_0}} (d_j \hbar(x_0)^2 |x_0 - z_0|^{-1} d_i^{-1}) d_i \hbar(x_0)^{-1} \|\nabla(G - G_h)\|_{L^2(\widetilde{O}_i)} \\ & + C \sum_{\Omega_i \in \mathcal{O}\backslash\{\Omega_{J+1}\}} (d_j \mathtt{h}_i^2 \rho_i^{-2}) \rho_i \mathtt{h}_i^{-1} \|\nabla(G - G_h)\|_{L^2(\Omega_i)}, \end{aligned} \tag{A.11}$$

and so

$$
\begin{aligned}
&\sum_{O_j\in\mathcal{O}'_{z_0}} h_j^{-1}\|G-G_h\|_{L^2(O_j)}\\
&\le C\sum_{O_j\in\mathcal{O}'_{z_0}}((\kappa^{1-\gamma}h_*)^s d_j^{1-s}h_j^{-1}+\hbar(x_0)|x_0-z_0|^{-1}d_jh_j^{-1})\\
&\quad + C\sum_{O_i\in\mathcal{O}_{z_0}\cup\{O_{-1}\}}\sum_{O_j\in\mathcal{O}'_{z_0}}(d_j^{1-s}h_j^{-1}h_i^{s+1}d_i^{-1})d_ih_i^{-1}\|\nabla(G-G_h)\|_{L^2(O_i)}\\
&\quad + C\sum_{\widetilde{O}_i\in\mathcal{O}_{x_0}}\sum_{O_j\in\mathcal{O}'_{z_0}}(d_jh_j^{-1}\hbar(x_0)^2d_i^{-1}|x_0-z_0|^{-1})d_i\hbar(x_0)^{-1}\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_i)}\\
&\quad + C\sum_{\Omega_i\in\mathcal{O}\setminus\{\Omega_{J+1}\}}\sum_{O_j\in\mathcal{O}'_{z_0}}(d_jh_j^{-1}\mathtt{h}_i^2\rho_i^{-2})\rho_i\mathtt{h}_i^{-1}\|\nabla(G-G_h)\|_{L^2(\Omega_i)}\\
&=: L_1+L_2+L_3+L_4.
\end{aligned}
\tag{A.12}
$$

Since $\gamma<s<\beta$, as shown in (3.3), we have

$$
\begin{aligned}
L_1 &= C\sum_{O_j\in\mathcal{O}'_{z_0}}((\kappa^{1-\gamma}h_*)^s d_j^{1-s}h_j^{-1}+\hbar(x_0)|x_0-z_0|^{-1}d_jh_j^{-1})\\
&\le C\sum_{O_j\in\mathcal{O}'_{z_0}}(\kappa^{1-\gamma}h_*)^s d_j^{1-s}(hd_j^{1-\gamma})^{-1}\\
&\quad + C\sum_{O_j\in\mathcal{O}'_{z_0}}\hbar(x_0)|x_0-z_0|^{-1}d_j(hd_j^{1-\gamma})^{-1}\\
&\le C\sum_{O_j\in\mathcal{O}'_{z_0}}((\kappa^{1-\gamma}h_*)^s h_*^{-\gamma}d_j^{-(s-\gamma)}\\
&\quad + C\sum_{O_j\in\mathcal{O}'_{z_0}}(h|x_0-z_0|^{1-\gamma})|x_0-z_0|^{-1}h_*^{-\gamma}d_j^{\gamma}\\
&\le C((\kappa^{1-\gamma}h_*)^s h_*^{-\gamma}(\kappa h_*)^{-(s-\gamma)}\\
&\quad + Ch_*^{\gamma}|x_0-z_0|^{1-\gamma}|x_0-z_0|^{-1}h_*^{-\gamma}|x_0-z_0|^{\gamma}\\
&\le C\kappa^{(1-s)\gamma}+C
\end{aligned}
$$

By using the definition of $\mathcal{M}$ in (4.48), we have

$$
\begin{aligned}
L_2 &\le C\mathcal{M}\max_{O_i\in\mathcal{O}_{z_0}\cup\{O_{-1}\}}\sum_{O_j\in\mathcal{O}'_{z_0}}\left(d_j^{1-s}h_j^{-1}h_i^{s+1}d_i^{-1}\right)\\
&\le C\mathcal{M}\max_{O_i\in\mathcal{O}_{z_0}\cup\{O_{-1}\}}\sum_{O_j\in\mathcal{O}'_{z_0}} d_j^{1-s}(hd_j^{1-\gamma})^{-1}(hd_i^{1-\gamma})^{s+1}d_i^{-1}\\
&\le C\mathcal{M}\max_{O_i\in\mathcal{O}_{z_0}\cup\{O_{-1}\}}\sum_{O_j\in\mathcal{O}'_{z_0}} h^s d_j^{-(s-\gamma)}d_i^{-[1-(1-\gamma)(s+1)]}\\
&\le C\mathcal{M}h_*^{\gamma s}(\kappa h_*)^{-(s-\gamma)}(\kappa h_*)^{-[1-(1-\gamma)(s+1)]}\quad\text{(note that } 1-(1-\gamma)(s+1)>0)\\
&\le C\kappa^{-\gamma s}\mathcal{M},
\end{aligned}
$$

$$
\begin{aligned}
L_3 &\le C\mathcal{M} \max_{\widetilde{O}_i\in\mathcal{O}_{x_0}} \sum_{O_j\in\mathcal{O}'_{z_0}} \left(d_j h_j^{-1}\hbar(x_0)^2 d_i^{-1}|x_0-z_0|^{-1}\right)\\
&\le C\mathcal{M} \sum_{O_j\in\mathcal{O}'_{z_0}} d_j(hd_j^{1-\gamma})^{-1}\hbar(x_0)^2(\kappa^{\gamma s}\hbar(x_0))^{-1}|x_0-z_0|^{-1}\\
&\le C\mathcal{M} \sum_{O_j\in\mathcal{O}'_{z_0}} h^{-1}d_j^{\gamma}\kappa^{-\gamma s}\hbar(x_0)|x_0-z_0|^{-1}\\
&\le C\mathcal{M}h^{-1}|x_0-z_0|^{\gamma}\kappa^{-\gamma s}(h|x_0-z_0|^{1-\gamma})|x_0-z_0|^{-1}\\
&\le C\kappa^{-\gamma s}\mathcal{M},
\end{aligned}
$$

$$
\begin{aligned}
L_4 &\le C\mathcal{M} \max_{\Omega_i\in\mathcal{O}\backslash\{\Omega_{J+1}\}} \sum_{O_j\in\mathcal{O}'_{z_0}} (d_j h_j^{-1}\mathtt{h}_i^2\rho_i^{-2})\\
&\le C\mathcal{M} \max_{\Omega_i\in\mathcal{O}\backslash\{\Omega_{J+1}\}} \sum_{O_j\in\mathcal{O}'_{z_0}} d_j(hd_j^{1-\gamma})^{-1}(h\rho_i^{1-\gamma})^2\rho_i^{-2}\\
&\le C\mathcal{M} \max_{\Omega_i\in\mathcal{O}\backslash\{\Omega_{J+1}\}} \sum_{O_j\in\mathcal{O}'_{z_0}} hd_j^{\gamma}\rho_i^{-2\gamma}\\
&\le C\mathcal{M} \max_{\Omega_i\in\mathcal{O}\backslash\{\Omega_{J+1}\}} h_*^{\gamma}|x_0-z_0|^{\gamma}|x_0-z_0|^{-2\gamma}\\
&\le C\mathcal{M} \max_{\Omega_i\in\mathcal{O}\backslash\{\Omega_{J+1}\}} h_*^{\gamma}(\kappa h_*)^{-\gamma}\\
&\le C\kappa^{-\gamma}\mathcal{M}.
\end{aligned}
$$

Substituting the estimates of L_1, L_2, L_3 and L_4 into (A.12) and using $\kappa \ge 1$, we obtain

$$
\sum_{O_j\in\mathcal{O}'_{z_0}} h_j^{-1}\|G-G_h\|_{L^2(O_j)} \le C\kappa^{(1-s)\gamma} + C\kappa^{-\gamma s}\mathcal{M}. \tag{A.13}
$$

This completes the proof of (4.54).

Appendix B: Proof of (4.55)

The proof of (4.55) is similar as the proof of (4.54) in Appendix A. The main difference here is that we focus on the subdomains $\widetilde{O}_j$ which are closer to the singular point x_0 than the reentrant corner z_0 (in Appendix A we focus on the subdomains O_j which are closer to the reentrant corner z_0). For the convenience of readers, we include the complete proof here.

For $\widetilde{O}_j \in \mathcal{O}'_{x_0}$, we estimate $\|G-G_h\|_{L^2(\widetilde{O}_j)}$ via the duality

$$
\|G-G_h\|_{L^2(\widetilde{O}_j)} = \sup_{\substack{\psi\in C_0^\infty(\widetilde{O}_j)\\ \|\psi\|_{L^2(\widetilde{O}_j)}\le 1}} (G-G_h,\psi).
$$

For any given $\psi \in C_0^\infty(\widetilde{O}_j)$ satisfying $\|\psi\|_{L^2(\widetilde{O}_j)} \le 1$, we define w as the solution of

$$
\begin{cases} -\Delta w = \psi - \overline{\psi} & \text{in } \Omega,\\ \partial_n w = 0 & \text{on } \partial\Omega, \end{cases} \tag{B.1}
$$

where $\overline{\psi} := \frac{1}{|\Omega|}\int_\Omega \psi(x)dx$ is the average of ψ in the domain Ω. The solution w exists and is unique under the condition $\int_\Omega w(x)\mathrm{d}x = 0$. Moreover, inequality (A.3) holds similarly here.

By using the normalization condition $\int_\Omega G_h \mathrm{d}x = \int_\Omega G\mathrm{d}x$, we have $(G - G_h, \overline{\psi}) = 0$ and therefore

$$
\begin{aligned}
(G - G_h, \psi) &= (G - G_h, \psi - \overline{\psi}) = (G - G_h, -\Delta w) = (\nabla(G - G_h), \nabla w) \\
&= (\nabla(G - G_h), \nabla(w - I_h w)) \\
&= (\nabla(G - G_h), \nabla(w - I_h w))_{O_*} \\
&\quad + \sum_{O_i \in \mathcal{O}_{z_0}} (\nabla(G - G_h), \nabla(w - I_h w))_{O_i} \\
&\quad + \sum_{\Omega_i \in \mathcal{O}\setminus\{\Omega_{J+1}\}} (\nabla(G - G_h), \nabla(w - I_h w))_{\Omega_i} \\
&\quad + (\nabla(G - G_h), \nabla(w - I_h w))_{\widetilde{O}_*} \\
&\quad + \sum_{\widetilde{O}_i \in \mathcal{O}_{x_0} \cup \{\widetilde{O}_{-1}\}} (\nabla(G - G_h), \nabla(w - I_h w))_{\widetilde{O}_i} \\
&=: \sum_{j=1}^{5} \widetilde{\mathcal{E}}_j.
\end{aligned}
\tag{B.2}
$$

The term $|\widetilde{\mathcal{E}}_1|$ can be estimated in the same way as (A.5), i.e.

$$
|\widetilde{\mathcal{E}}_1| = |(\nabla(G - G_h), \nabla(w - I_h w))_{O_*}| \le C(\kappa^{1-\gamma} h_*)^s d_j^{1-s}. \tag{B.3}
$$

The second and third terms on the right-hand side of (B.2) can be estimated by

$$
\begin{aligned}
|\widetilde{\mathcal{E}}_2| = |(\nabla(G - G_h), \nabla(w - I_h w))_{O_i}| &\le C h_i^s \|w\|_{H^{s+1}(O_i')} \|\nabla(G - G_h)\|_{L^2(O_i)} \\
&\le C h_i^s \|\psi\|_{H^{s-1}(\Omega)} \|\nabla(G - G_h)\|_{L^2(O_i)} \\
&\le C h_i^s d_j^{1-s} \|\nabla(G - G_h)\|_{L^2(O_i)} \\
&\le C(d_j^{1-s} h_i^{s+1} d_i^{-1}) d_i h_i^{-1} \|\nabla(G - G_h)\|_{L^2(O_i)}.
\end{aligned}
\tag{B.4}
$$

and

$$
\begin{aligned}
|\widetilde{\mathcal{E}}_3| = |(\nabla(G - G_h), \nabla(w - I_h w))_{\Omega_i}| &\le C \mathtt{h}_i^s \|w\|_{H^{s+1}(\Omega_i')} \|\nabla(G - G_h)\|_{L^2(\Omega_i)} \\
&\le C \mathtt{h}_i^s \|\psi\|_{H^{s-1}(\Omega)} \|\nabla(G - G_h)\|_{L^2(\Omega_i)} \\
&\le C \mathtt{h}_i^s d_j^{1-s} \|\nabla(G - G_h)\|_{L^2(\Omega_i)} \\
&\le C(d_j^{1-s} \mathtt{h}_i^{s+1} \rho_i^{-1}) \rho_i \mathtt{h}_i^{-1} \|\nabla(G - G_h)\|_{L^2(\Omega_i)}.
\end{aligned}
\tag{B.5}
$$

To estimate $|\widetilde{\mathcal{E}}_4|$, we consider the two different cases below.

Case (1): If $j = J_{x_0}$ or $j = J_{x_0} + 1$, then $\widetilde{O}_j' \cap \widetilde{O}_*' \neq \emptyset$ and $d_j \le C\kappa\hbar(x_0)$. In this case, by applying (3.36b) with $p = 2$ we have

$$
\begin{aligned}
\|w\|_{H^2(\widetilde{O}_*')} &\le C\left(\|\psi - \bar{\psi}\|_{L^2(\widetilde{O}_*'')} + (\kappa^{\gamma s}\hbar(x_0))^{-1} \|\nabla w\|_{L^2(\widetilde{O}_*'')}\right) \\
&\le C\left(\|\psi\|_{L^2(\widetilde{O}_*'')} + (\kappa^{\gamma s}\hbar(x_0))^{-1} \|\nabla w\|_{L^2(\widetilde{O}_*'')}\right),
\end{aligned}
$$

and so

$$
\begin{aligned}
|\widetilde{\mathcal{E}}_4| &= |(\nabla(G-G_h),\nabla(w-I_hw))_{\widetilde{O}_*}| \\
&\le C\hbar(x_0)\|w\|_{H^2(\widetilde{O}_*')}\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_*)} \\
&\le C\hbar(x_0)\left(\|\psi\|_{L^2(\widetilde{O}_*'')}+(\kappa^{\gamma s}\hbar(x_0))^{-1}\|\nabla w\|_{L^2(\widetilde{O}_*'')}\right)\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_*)} \\
&\le C\hbar(x_0)\left(\|\psi\|_{L^2(\widetilde{O}_*'')}+(\kappa^{\gamma s}\hbar(x_0))^{s-1}\|\nabla w\|_{L^{2/(1-s)}(\widetilde{O}_*'')}\right)\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_*)} \\
&\le C\hbar(x_0)\left(\|\psi\|_{L^2(\widetilde{O}_*'')}+(\kappa^{\gamma s}\hbar(x_0))^{s-1}\|w\|_{H^{s+1}(\Omega)}\right)\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_*)} \\
&\le C\hbar(x_0)\left(1+(\kappa^{\gamma s}\hbar(x_0))^{s-1}\|\psi\|_{H^{s-1}(\Omega)}\right)\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_*)} \\
&\le C\hbar(x_0)\left(1+(\kappa^{\gamma s}\hbar(x_0))^{s-1}(\kappa^{\gamma s}\hbar(x_0))^{1-s}\right)\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_*)} \\
&\le C\hbar(x_0),
\end{aligned}
\tag{B.6}
$$

where we have used Hölder's inequality in deriving the third inequality of (B.6), and used (4.46) in deriving the last inequality.

Case (2): If $0\le j<J_{x_0}$, then $\widetilde{O}_j'\cap\widetilde{O}_*'=\emptyset$ and for $p\in(2,p_0)$ as in Lemma 4.2 we have

$$
\begin{aligned}
|\widetilde{\mathcal{E}}_4| &= |(\nabla(G-G_h),\nabla(w-I_hw))_{\widetilde{O}_*}| \\
&\le C\hbar(x_0)\|w\|_{W^{2,p}(\widetilde{O}_*')}\|\nabla(G-G_h)\|_{L^{p'}(\widetilde{O}_*)} \\
&\le C\hbar(x_0)|\widetilde{O}_*|^{\frac{1}{p'}-\frac{1}{2}}\|w\|_{W^{2,p}(\widetilde{O}_*')}\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_*)} \\
&\le C\hbar(x_0)(\kappa^{\gamma s}\hbar(x_0))^{\frac{2}{p'}-1}\|w\|_{W^{2,p}(\widetilde{O}_*')},
\end{aligned}
$$

where we have used (4.46) in the last inequality. Then, by applying the local $W^{2,p}$ estimate in Lemma 4.2 to the expression $w(x)=\int_{\widetilde{O}_j}\Gamma(x,y)\psi(y)\mathrm{d}y$ and using Hölder's inequality, we have

$$
\|w\|_{W^{2,p}(\widetilde{O}_*')}\le\sup_{y\in\widetilde{O}_j}\|\Gamma(\cdot,y)\|_{W^{2,p}(\widetilde{O}_*')}\|\psi\|_{L^1(\widetilde{O}_j)}\le Cd_j^{-2+2/p}d_j\|\psi\|_{L^2(\widetilde{O}_j)}\le Cd_j^{2/p-1},
$$

which implies that

$$
|\widetilde{\mathcal{E}}_4| = |(\nabla(G-G_h),\nabla(w-I_hw))_{\widetilde{O}_*}|\le C\kappa^{\gamma s(\frac{2}{p'}-1)}\hbar(x_0)^{\frac{2}{p'}}d_j^{2/p-1}. \tag{B.7}
$$

Similarly, we also consider the following two different cases in the estimation of $|\widetilde{\mathcal{E}}_5|$.

Case (1'): If $|j-i|\le 2$, then $\widetilde{O}_j'\cap\widetilde{O}_i'\neq\emptyset$. In this case, $d_j\sim d_i$ and $h_i\sim h_j\sim\hbar(x_0)$, and we have

$$
\begin{aligned}
|\widetilde{\mathcal{E}}_5| &= |(\nabla(G-G_h),\nabla(w-I_hw))_{\widetilde{O}_i}| \\
&\le C\hbar(x_0)^s\|w\|_{H^{s+1}(\Omega)}\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_i)} \\
&\le C\hbar(x_0)^s\|\psi\|_{H^{s-1}(\Omega)}\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_i)} \\
&\le C(\hbar(x_0)^{s+1}d_i^{-s})d_ih(x_0)^{-1}\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_i)}.
\end{aligned}
\tag{B.8}
$$

Case (2'): If $|j-i|\ge 3$, then $\widetilde{O}_j'\cap\widetilde{O}_i'=\emptyset$. In this case, $h_i\sim h_j\sim\hbar(x_0)$, and for $p\in(2,p_0)$ as in Lemma 4.2 we have

$$
\begin{aligned}
|\widetilde{\mathcal{E}}_5| &= |(\nabla(G-G_h),\nabla(w-I_hw))_{\widetilde{O}_i}| \\
&\le C\hbar(x_0)\|w\|_{W^{2,p}(\widetilde{O}_i')}\|\nabla(G-G_h)\|_{L^{p'}(\widetilde{O}_i)}
\end{aligned}
$$

$$\leq C\hbar(x_0)\|w\|_{W^{2,p}(\widetilde{O}_i')} d_i^{\frac{2}{p'}-1}\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_i)}$$
$$\leq C\Big(\hbar(x_0)^2 d_i^{\frac{2}{p'}-2}\|w\|_{W^{2,p}(\widetilde{O}_i')}\Big) d_i\hbar(x_0)^{-1}\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_i)}.$$

By using the local $W^{2,p}$ estimate in Lemma 4.2, we have

$$\begin{aligned}\|w\|_{W^{2,p}(\widetilde{O}_i')} &\leq \sup_{y\in\widetilde{O}_j}\|\Gamma(\cdot,y)\|_{W^{2,p}(\widetilde{O}_i')}\|\psi\|_{L^1(\widetilde{O}_j)}\\ &\leq C\max(d_i,d_j)^{-2+\frac{2}{p}} d_j\|\psi\|_{L^2(\widetilde{O}_j)}\\ &\leq C\max(d_i,d_j)^{-\frac{2}{p'}} d_j,\end{aligned}$$

which implies that

$$|\widetilde{\mathcal{E}}_5| \leq C\Big(\hbar(x_0)^2 d_i^{\frac{2}{p'}-2}\max(d_i,d_j)^{-\frac{2}{p'}} d_j\Big) d_i\hbar(x_0)^{-1}\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_i)}. \tag{B.9}$$

Substituting the estimates of $|\widetilde{\mathcal{E}}_j|,\ j=1,\dots,5$ into (B.2), we obtain (via the duality argument)

$$\begin{aligned}&\|(G-G_h\|_{L^2(\widetilde{O}_j)}\\ &\leq C(\kappa^{1-\gamma}h_*)^s d_j^{1-s}\\ &\quad+\sum_{O_i\in\mathcal{O}_{z_0}} C(h_i^{s+1}d_i^{-1}d_j^{1-s})d_i h_i^{-1}\|\nabla(G-G_h)\|_{L^2(O_i)}\\ &\quad+\sum_{\Omega_i\in\mathcal{O}} C(\mathtt{h}_i^{s+1}\rho_i^{-1}d_j^{1-s})\rho_i\mathtt{h}_i^{-1}\|\nabla(G-G_h)\|_{L^2(\Omega_i)}\\ &\quad+C\hbar(x_0)(\delta_{j,J_{x_0}}+\delta_{j,J_{x_0}+1})+C\kappa^{\gamma s(\frac{2}{p'}-1)}\hbar(x_0)^{\frac{2}{p'}}d_j^{1-\frac{2}{p'}}(1-\delta_{j,J_{x_0}}-\delta_{j,J_{x_0}+1})\\ &\quad+C\sum_{\widetilde{O}_i\in\mathcal{O}_{x_0}}(\hbar(x_0)^{s+1}d_i^{-s})d_i h_i^{-1}\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_i)}(\delta_{i,j}+\delta_{i,j-1}+\delta_{i,j+1}+\delta_{i,j-2}+\delta_{i,j+2})\\ &\quad+\sum_{|i-j|\geq 3}\Big(\hbar(x_0)^2 d_i^{-\frac{2}{p}}\max(d_i,d_j)^{-\frac{2}{p'}}d_j\Big)d_i\hbar(x_0)^{-1}\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_i)}.\end{aligned} \tag{B.10}$$

Hence, we have

$$
\begin{aligned}
&\sum_{\widetilde{O}_j\in\mathcal{O}'_{x_0}} \hbar(x_0)^{-1}\|G-G_h\|_{L^2(\widetilde{O}_j)} \\
&\le C\sum_{\widetilde{O}_j\in\mathcal{O}'_{x_0}} \hbar(x_0)^{-1}(\kappa^{1-\gamma}h_*)^s d_j^{1-s} \\
&\quad + C\sum_{O_i\in\mathcal{O}_{z_0}}\sum_{\widetilde{O}_j\in\mathcal{O}'_{x_0}} (\hbar(x_0)^{-1}h_i^{s+1}d_i^{-1}d_j^{1-s})d_i h_i^{-1}\|\nabla(G-G_h)\|_{L^2(O_i)} \\
&\quad + C\sum_{\Omega_i\in\mathcal{O}}\sum_{\widetilde{O}_j\in\mathcal{O}'_{x_0}} (\hbar(x_0)^{-1}\mathtt{h}_i^{s+1}\rho_i^{-1}d_j^{1-s})\rho_i \mathtt{h}_i^{-1}\|\nabla(G-G_h)\|_{L^2(\Omega_i)} \\
&\quad + C + C\sum_{\widetilde{O}_j\in\mathcal{O}'_{x_0}} \kappa^{\gamma s(\frac{2}{p'}-1)}\hbar(x_0)^{\frac{2}{p'}-1}d_j^{1-\frac{2}{p'}} \\
&\quad + C\sum_{\widetilde{O}_j\in\mathcal{O}_{x_0}} (\hbar(x_0)^s d_j^{-s})d_j\hbar(x_0)^{-1}\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_j)} \\
&\quad + C\sum_{\widetilde{O}_i\in\mathcal{O}_{x_0}}\sum_{\widetilde{O}_j\in\mathcal{O}'_{x_0}} \left(\frac{\hbar(x_0)d_j}{d_i^{\frac{2}{p}}\max(d_i,d_j)^{\frac{2}{p'}}}\right) d_i\hbar(x_0)^{-1}\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_i)} \\
&=: E_1+E_2+E_3+E_4+E_5+E_6.
\end{aligned}
\tag{B.11}
$$

Since $\gamma < s < \beta$, as shown in (3.3), we have

$$
\begin{aligned}
E_1 &= C\sum_{\widetilde{O}_j\in\mathcal{O}'_{x_0}} \hbar(x_0)^{-1}(\kappa^{1-\gamma}h_*)^s d_j^{1-s} \\
&\le C(h|x_0-z_0|^{1-\gamma})^{-1}(\kappa^{1-\gamma}h_*)^s|x_0-z_0|^{1-s} \\
&\le Ch^{-1}(\kappa^{1-\gamma}h_*)^s|x_0-z_0|^{-(s-\gamma)} \\
&\le Ch_*^{-\gamma}(\kappa^{1-\gamma}h_*)^s(\kappa h_*)^{-(s-\gamma)} \\
&\le C\kappa^{\gamma(1-s)},
\end{aligned}
$$

$$
\begin{aligned}
E_2 &\le C\mathcal{M}\max_{O_i\in\mathcal{O}_{z_0}}\sum_{\widetilde{O}_j\in\mathcal{O}'_{x_0}} \left(\hbar(x_0)^{-1}h_i^{s+1}d_i^{-1}d_j^{1-s}\right) \\
&\le C\mathcal{M}\max_{O_i\in\mathcal{O}_{z_0}} \left((h|x_0-z_0|^{1-\gamma})^{-1}(hd_i^{1-\gamma})^{s+1}d_i^{-1}|x_0-z_0|^{1-s}\right) \\
&\le C\mathcal{M}\max_{O_i\in\mathcal{O}_{z_0}} \left(h^s|x_0-z_0|^{-(s-\gamma)}d_i^{-[1-(1-\gamma)(1+s)]}\right) \\
&\le C\mathcal{M}h_*^{\gamma s}(\kappa h_*)^{-(s-\gamma)}(\kappa h_*)^{-[1-(1-\gamma)(1+s)]} \\
&\le C\kappa^{-\gamma s}\mathcal{M},
\end{aligned}
$$

$$
E_3 \le C\mathcal{M}\max_{\Omega_i\in\mathcal{O}}\sum_{\widetilde{O}_j\in\mathcal{O}_{x_0}} \left(\hbar(x_0)^{-1}\mathtt{h}_i^{s+1}\rho_i^{-1}d_j^{1-s}\right)
$$

$$\le C\mathcal{M} \max_{\Omega_i\in\mathcal{O}} \Big((h|x_0-z_0|^{1-\gamma})^{-1}(h\rho_i^{1-\gamma})^{s+1}\rho_i^{-1}|x_0-z_0|^{1-s}\Big)$$
$$\le C\mathcal{M} \max_{\Omega_i\in\mathcal{O}} \Big(h^s|x_0-z_0|^{-(s-\gamma)}\rho_i^{-[1-(1-\gamma)(1+s)]}\Big)$$
$$\le C\mathcal{M}h_*^{\gamma s}(\kappa h_*)^{-(s-\gamma)}(\kappa h_*)^{-[1-(1-\gamma)(1+s)]}$$
$$\le C\kappa^{-\gamma s}\mathcal{M},$$

$$\begin{aligned}E_4 &\le C + C\sum_{\widetilde{O}_j\in\mathcal{O}'_{x_0}} \kappa^{\frac{2}{p'}-1}\hbar(x_0)^{\frac{2}{p'}-1}d_j^{-\left(\frac{2}{p'}-1\right)}\\ &\le C + C\kappa^{\gamma s(\frac{2}{p'}-1)}\hbar(x_0)^{\frac{2}{p'}-1}(\kappa^{\gamma s}\hbar(x_0))^{-\left(\frac{2}{p'}-1\right)}\\ &\le C,\end{aligned}$$

$$E_5 \le C\mathcal{M}\max_{\widetilde{O}_j\in\mathcal{O}'_{x_0}} \hbar(x_0)^s d_j^{-s} \le C\kappa^{-\gamma s^2}\mathcal{M},$$

$$\begin{aligned}E_6 &\le C\mathcal{M}\max_{\widetilde{O}_i\in\mathcal{O}_{x_0}}\sum_{\widetilde{O}_j\in\mathcal{O}_{x_0}}\left(\frac{\hbar(x_0)d_j}{d_i^{\frac{2}{p}}\max(d_i,d_j)^{\frac{2}{p'}}}\right)\\ &\le C\mathcal{M}\max_{\widetilde{O}_i\in\mathcal{O}_{x_0}}\left(\frac{\hbar(x_0)}{d_i}\right)\\ &\le C\kappa^{-\gamma s}\mathcal{M}.\end{aligned}$$

Substituting the estimates of E_k, $k=1,\dots,6$, into (B.11), we obtain

$$\sum_{\widetilde{O}_j\in\mathcal{O}_{x_0}} \hbar(x_0)^{-1}\|G-G_h\|_{L^2(O_j)} \le C\kappa^{\gamma(1-s)} + C\kappa^{-\gamma s^2}\mathcal{M}. \tag{B.12}$$

Appendix C: Proof of (4.56)

The proof of (4.55) is similar as the proof of (4.54) in Appendix A. The main difference here is that the subdomains Ω_j are away from both the reentrant corner z_0 and the singular point x_0, and therefore the analysis would become simpler. For the convenience of readers, we include the complete proof here.

For $\Omega_j\in\mathcal{O}$, we estimate $\|G-G_h\|_{L^2(\Omega_j)}$ via the duality

$$\|G-G_h\|_{L^2(\Omega_j)} = \sup_{\substack{\psi\in C_0^\infty(\Omega_j)\\ \|\psi\|_{L^2(\Omega_j)}\le 1}} (G-G_h,\psi).$$

For any given $\psi\in C_0^\infty(\Omega_j)$ satisfying $\|\psi\|_{L^2(\Omega_j)}\le 1$, we define w as the solution of

$$\begin{cases} -\Delta w = \psi - \overline{\psi} & \text{in } \Omega,\\ \partial_n w = 0 & \text{on } \partial\Omega,\end{cases} \tag{C.1}$$

where $\overline{\psi} := \frac{1}{|\Omega|}\int_\Omega \psi(x)dx$ is the average of ψ in the domain Ω. The solution w exists and is unique under the condition $\int_\Omega w(x)\mathrm{d}x = 0$. Moreover, inequality (A.3) holds similarly here.

By using the normalization condition $\int_\Omega G_h \mathrm{d}x = \int_\Omega G \mathrm{d}x$, we have $(G - G_h, \overline{\psi}) = 0$ and therefore

$$\begin{aligned}(G - G_h, \psi) &= (G - G_h, \psi - \overline{\psi}) = (G - G_h, -\Delta w) = (\nabla(G - G_h), \nabla w) \\ &= (\nabla(G - G_h), \nabla(w - I_h w)) \\ &= (\nabla(G - G_h), \nabla(w - I_h w))_{O_*} \\ &\quad + \sum_{O_i \in \mathcal{O}_{z_0} \setminus \{O_0\}} (\nabla(G - G_h), \nabla(w - I_h w))_{O_i} \\ &\quad + \sum_{\Omega_i \in \mathcal{O} \cup \{\Omega_{J+2}\}} (\nabla(G - G_h), \nabla(w - I_h w))_{\Omega_i} \\ &\quad + (\nabla(G - G_h), \nabla(w - I_h w))_{\widetilde{O}_*} \\ &\quad + \sum_{\widetilde{O}_i \in \mathcal{O}_{x_0} \setminus \{\widetilde{O}_0\}} (\nabla(G - G_h), \nabla(w - I_h w))_{\widetilde{O}_i} \\ &= \sum_{j=1}^{5} \mathcal{E}_j^*.\end{aligned} \tag{C.2}$$

The term $|\mathcal{E}_1^*|$ can be estimated in the same way as (A.5), i.e.

$$|\mathcal{E}_1^*| = |(\nabla(G - G_h), \nabla(w - I_h w))_{O_*}| \le C(\kappa^{1-\gamma} h_*)^s \rho_j^{1-s}. \tag{C.3}$$

$|\mathcal{E}_2^*|$ and $|\mathcal{E}_3^*|$ can be estimated by

$$\begin{aligned}|\mathcal{E}_2^*| = |(\nabla(G - G_h), \nabla(w - I_h w))_{O_i}| &\le C h_i^s \|w\|_{H^{s+1}(O_i')} \|\nabla(G - G_h)\|_{L^2(O_i)} \\ &\le C h_i^s \|\psi\|_{H^{s-1}(\Omega)} \|\nabla(G - G_h)\|_{L^2(O_i)} \\ &\le C h_i^s \rho_j^{1-s} \|\nabla(G - G_h)\|_{L^2(O_i)}\end{aligned} \tag{C.4}$$

and

$$\begin{aligned}|\mathcal{E}_3^*| = |(\nabla(G - G_h), \nabla(w - I_h w))_{\Omega_i}| &\le C \mathtt{h}_i^s \|w\|_{H^{s+1}(\Omega_i')} \|\nabla(G - G_h)\|_{L^2(\Omega_i)} \\ &\le C \mathtt{h}_i^s \|\psi\|_{H^{s-1}(\Omega)} \|\nabla(G - G_h)\|_{L^2(\Omega_i)} \\ &\le C \mathtt{h}_i^s \rho_j^{1-s} \|\nabla(G - G_h)\|_{L^2(\Omega_i)}.\end{aligned} \tag{C.5}$$

$|\mathcal{E}_4^*|$ can be estimated by

$$\begin{aligned}|\mathcal{E}_4^*| = |(\nabla(G - G_h), \nabla(w - I_h w))_{\widetilde{O}_*}| &\le C \hbar(x_0) \|w\|_{H^2(\widetilde{O}_*')} \|\nabla(G - G_h)\|_{L^2(\widetilde{O}_*)} \\ &\le C \hbar(x_0) \|w\|_{H^2(\widetilde{O}_*')}\end{aligned} \tag{C.6}$$

where we have used (4.46) in the last inequality. By using (4.22), we get

$$\begin{aligned}\|w\|_{H^2(\widetilde{O}_*')} &\le \sup_{y \in \Omega_j} \|G(\cdot, y)\|_{H^2(\widetilde{O}_*')} \|\psi\|_{L^1(\Omega_j)} \\ &\le C|x_0 - z_0|^{s-1} \rho_j^{-s} |\Omega_j|^{\frac{1}{2}} \|\psi\|_{L^2(\Omega_j)} \\ &\le C|x_0 - z_0|^{s-1} \rho_j^{1-s},\end{aligned}$$

which implies

$$|\mathcal{E}_4^*| \le C|x_0 - z_0|^{s-1} \hbar(x_0) \rho_j^{1-s}. \tag{C.7}$$

By using (4.22) we can estimate $|\mathcal{E}_5^*|$ similarly as $|\mathcal{E}_4^*|$, i.e.,

$$\begin{aligned} |\mathcal{E}_5^*| &= |(\nabla(G-G_h), \nabla(w - I_h w))_{\widetilde{O}_i}| \\ &\le C(|x_0 - z_0|^{s-1}\hbar(x_0)^2 d_i^{-1}\rho_j^{1-s} d_i \hbar(x_0)^{-1}\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_i)}. \end{aligned} \tag{C.8}$$

Then, substituting the estimates of $|\mathcal{E}_j^*|$ into (C.2), we obtain (via the duality argument)

$$\begin{aligned} \|G-G_h\|_{L^2(\Omega_j)} &\le C(\kappa^{1-\gamma}h_*)^s\rho_j^{1-s} \\ &\quad + C\sum_{O_i\in\mathcal{O}_{z_0}}(h_i^{s+1}d_i^{-1}\rho_j^{1-s})d_i h_i^{-1}\|\nabla(G-G_h)\|_{L^2(O_i)} \\ &\quad + C\sum_{\Omega_i\in\mathcal{O}\cup\{\Omega_{J+2}\}}(\mathtt{h}_i^{s+1}\rho_i^{-1}\rho_j^{1-s})\rho_i\mathtt{h}_i^{-1}\|\nabla(G-G_h)\|_{L^2(\Omega_i)} \\ &\quad + C|x_0-z_0|^{s-1}\hbar(x_0)\rho_j^{1-s} \\ &\quad + C\sum_{\widetilde{O}_i\in\mathcal{O}_{x_0}\setminus\{\widetilde{O}_0\}}\frac{\hbar(x_0)^2\rho_j^{1-s}}{|x_0-z_0|^{1-s}d_i}d_i\hbar(x_0)^{-1}\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_i)}. \end{aligned} \tag{C.9}$$

As a result, we have

$$\begin{aligned} \sum_{\Omega_j\in\mathcal{O}}\mathtt{h}_j^{-1}\|G-G_h\|_{L^2(\Omega_j)} &\le C\sum_{\Omega_j\in\mathcal{O}}(\kappa^{1-\gamma}h_*)^s\rho_j^{1-s}\mathtt{h}_j^{-1} \\ &\quad + C\sum_{O_i\in\mathcal{O}_{z_0}}\sum_{\Omega_j\in\mathcal{O}}(h_i^{s+1}d_i^{-1}\rho_j^{1-s}\mathtt{h}_j^{-1})d_i h_i^{-1}\|\nabla(G-G_h)\|_{L^2(O_i)} \\ &\quad + C\sum_{\Omega_i\in\mathcal{O}\cup\{\Omega_{J+2}\}}\sum_{\Omega_j\in\mathcal{O}}(\mathtt{h}_i^{s+1}\rho_i^{-1}\rho_j^{1-s}\mathtt{h}_j^{-1})\rho_i\mathtt{h}_i^{-1}\|\nabla(G-G_h)\|_{L^2(\Omega_i)} \\ &\quad + C\sum_{\Omega_j\in\mathcal{O}}|x_0-z_0|^{s-1}\hbar(x_0)\rho_j^{1-s}\mathtt{h}_j^{-1} \\ &\quad + C\sum_{\widetilde{O}_i\in\mathcal{O}_{x_0}}\sum_{\Omega_j\in\mathcal{O}}\frac{\hbar(x_0)^2\rho_j^{1-s}}{|x_0-z_0|^{1-s}d_i\mathtt{h}_j}d_i\hbar(x_0)^{-1}\|\nabla(G-G_h)\|_{L^2(\widetilde{O}_i)} \\ &=: F_1+F_2+F_3+F_4+F_5, \end{aligned} \tag{C.10}$$

where

$$\begin{aligned} F_1 &= C\sum_{\Omega_j\in\mathcal{O}}(\kappa^{1-\gamma}h_*)^s\rho_j^{1-s}\mathtt{h}_j^{-1} \\ &\le C\sum_{\Omega_j\in\mathcal{O}}(\kappa^{1-\gamma}h_*)^s\rho_j^{1-s}(h\rho_j^{1-\gamma})^{-1} \\ &\le C(\kappa^{1-\gamma}h_*)^s h^{-1}\sum_{\Omega_j\in\mathcal{O}}\rho_j^{-(s-\gamma)} \\ &\le C(\kappa^{1-\gamma}h_*)^s h^{-1}|x_0-z_0|^{-(s-\gamma)} \\ &\le C(\kappa^{1-\gamma}h_*)^s h^{-1}(\kappa h_*)^{-(s-\gamma)} \\ &\le C\kappa^{\gamma(1-s)}, \end{aligned}$$

$$
\begin{aligned}
F_2 &\le C \max_{O_i\in\mathcal{O}_{z_0}} \sum_{\Omega_j\in\mathcal{O}} (h_i^{s+1} d_i^{-1}\rho_j^{1-s}\mathtt{h}_j^{-1})\,\mathcal{M}\\
&\le C \max_{O_i\in\mathcal{O}_{z_0}} \sum_{\Omega_j\in\mathcal{O}} (hd_i^{1-\gamma})^{s+1} d_i^{-1}\rho_j^{1-s}(h\rho_j^{1-\gamma})^{-1}\mathcal{M}\\
&\le C \max_{O_i\in\mathcal{O}_{z_0}} \sum_{\Omega_j\in\mathcal{O}} h^s d_i^{(1-\gamma)(s+1)-1}\rho_j^{-(s-\gamma)}\mathcal{M}\\
&\le C \max_{O_i\in\mathcal{O}_{z_0}} h^s d_i^{-[\gamma(s+1)-s]}|x_0-z_0|^{-(s-\gamma)}\mathcal{M}\\
&\le C h_*^{\gamma s}(\kappa h_*)^{-[\gamma(s+1)-s]}(\kappa h_*)^{-(s-\gamma)}\mathcal{M}\\
&\le C\kappa^{-\gamma s}\mathcal{M},
\end{aligned}
$$

$$
\begin{aligned}
F_3 &\le C \max_{\Omega_i\in\mathcal{O}\cup\{\Omega_{J+2}\}} \sum_{\Omega_j\in\mathcal{O}} (\mathtt{h}_i^{s+1}\rho_i^{-1}\rho_j^{1-s}\mathtt{h}_j^{-1})\mathcal{M}\\
&\le C \max_{\Omega_i\in\mathcal{O}\cup\{\Omega_{J+2}\}} \sum_{\Omega_j\in\mathcal{O}} (h\rho_i^{1-\gamma})^{s+1}\rho_i^{-1}\rho_j^{1-s}(h\rho_j^{1-\gamma})^{-1}\mathcal{M}\\
&\le C \max_{\Omega_i\in\mathcal{O}\cup\{\Omega_{J+2}\}} \sum_{\Omega_j\in\mathcal{O}} h^s\rho_i^{(1-\gamma)(s+1)-1}\rho_j^{-(s-\gamma)}\mathcal{M}\\
&\le C \max_{\Omega_i\in\mathcal{O}\cup\{\Omega_{J+2}\}} h^s\rho_i^{-[\gamma(s+1)-s]}|x_0-z_0|^{-(s-\gamma)}\mathcal{M}\\
&\le C h_*^{\gamma s}(\kappa h_*)^{-[\gamma(s+1)-s]}(\kappa h_*)^{-(s-\gamma)}\mathcal{M}\\
&\le C\kappa^{-\gamma s}\mathcal{M},
\end{aligned}
$$

$$
\begin{aligned}
F_4 &\le C \sum_{\Omega_j\in\mathcal{O}} |x_0-z_0|^{s-1}\hbar(x_0)\rho_j^{1-s}\mathtt{h}_j^{-1}\\
&\le C \sum_{\Omega_j\in\mathcal{O}} |x_0-z_0|^{s-1}(h|x_0-z_0|^{1-\gamma})\rho_j^{1-s}(h\rho_j^{1-\gamma})^{-1}\\
&\le C \sum_{\Omega_j\in\mathcal{O}} |x_0-z_0|^{s-\gamma}\rho_j^{-(s-\gamma)}\\
&\le C,
\end{aligned}
$$

$$
\begin{aligned}
F_5 &\le C \max_{\widetilde{O}_i\in\mathcal{O}_{x_0}} \sum_{\Omega_j\in\mathcal{O}} \frac{\hbar(x_0)^2}{|x_0-z_0|^{1-s}d_i}\rho_j^{1-s}\mathtt{h}_j^{-1})\mathcal{M}\\
&\le C \max_{\widetilde{O}_i\in\mathcal{O}_{x_0}} \sum_{\Omega_j\in\mathcal{O}} \frac{\hbar(x_0)^2}{|x_0-z_0|^{1-s}d_i}\rho_j^{1-s}(h\rho_j^{1-\gamma})^{-1})\mathcal{M}\\
&\le C \max_{\widetilde{O}_i\in\mathcal{O}_{x_0}} \sum_{\Omega_j\in\mathcal{O}} \frac{\hbar(x_0)^2}{|x_0-z_0|^{1-s}d_i}h^{-1}\rho_j^{-(s-\gamma)}\mathcal{M}\\
&\le C \max_{\widetilde{O}_i\in\mathcal{O}_{x_0}} \frac{h|x_0-z_0|^{1-\gamma}\hbar(x_0)}{|x_0-z_0|^{1-s}d_i}h^{-1}|x_0-z_0|^{-(s-\gamma)}\mathcal{M}
\end{aligned}
$$

$$\leq C \max_{\widetilde{O}_i \in \mathcal{O}_{x_0}} \hbar(x_0) d_i^{-1} \mathcal{M}$$
$$\leq C\kappa^{-\gamma s}\mathcal{M}.$$

Substituting the estimates of F_1, F_2, F_3, F_4 and F_5 into (C.10), we obtain

$$\sum_{\Omega_j \in \mathcal{O}} \mathtt{h}_j^{-1} \|G - G_h\|_{L^2(\Omega_j)} \leq C\kappa^{\gamma(1-s)} + C\kappa^{-\gamma s}\mathcal{M}. \tag{C.11}$$

∎

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