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Maximal regularity of multistep fully discrete finite element methods for parabolic equations

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This article extends the semidiscrete maximal L^p-regularity results in Li (2019, Analyticity, maximal regularity and maximum-norm stability of semi-discrete finite element solutions of parabolic equations in nonconvex polyhedra. *Math. Comp.*, **88**, 1–44) to multistep fully discrete finite element methods for parabolic equations with more general diffusion coefficients in $W^{1,d+\beta}$, where d is the dimension of space and $\beta > 0$. The maximal angles of R-boundedness are characterized for the analytic semigroup e^{zA_h} and the resolvent operator $z(z-A_h)^{-1}$, respectively, associated to an elliptic finite element operator A_h. Maximal L^p-regularity, an optimal $\ell^p(L^q)$ error estimate and an $\ell^p(W^{1,q})$ estimate are established for fully discrete finite element methods with multistep backward differentiation formulae.

Keywords: parabolic equation; finite element method; backward differentiation formulae; maximal regularity; analytic semigroup; resolvent.

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$ be a polygonal or polyhedral domain. We consider the initial and boundary value problem for a linear parabolic partial differential equation (PDE)

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \displaystyle\sum_{i,j=1}^{d} \dfrac{\partial}{\partial x_i}\left(a_{ij}(x)\dfrac{\partial u(t,x)}{\partial x_j}\right) = f(t,x) & \text{for } (t,x) \in \mathbb{R}_+ \times \Omega, \\ u(t,x) = 0 & \text{for } (t,x) \in \mathbb{R}_+ \times \partial\Omega, \\ u(0,x) = u_0(x) & \text{for } x \in \Omega, \end{cases} \tag{1.1}$$

where $a_{ij} = a_{ji}$ are real-valued functions satisfying the following ellipticity and regularity conditions:

$$\lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \le \lambda|\xi|^2 \qquad \forall\, \xi = (\xi_1,\dots,\xi_d) \in \mathbb{R}^d,\ \ \forall\, x \in \Omega, \tag{1.2}$$

$$a_{ij} = a_{ji} \in W^{1,d+\beta} \qquad \text{for some constants } \lambda, \beta > 0. \tag{1.3}$$

It is known that the elliptic partial differential operator $A = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\big(a_{ij}(x)\frac{\partial}{\partial x_j}\big)$, under a Dirichlet boundary condition, generates a bounded analytic semigroup on the space $L^q := L^q(\Omega)$ for all $1 < q < \infty$; see Ouhabaz (1995, Theorem 2.4). When $u_0 = 0$, the solution of (1.1) possesses maximal

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L^p-regularity on L^q, namely

$$\|\partial_t u\|_{L^p(\mathbb{R}_+;L^q)} + \|Au\|_{L^p(\mathbb{R}_+;L^q)} \le C_{p,q}\|f\|_{L^p(\mathbb{R}_+;L^q)} \quad \forall\, 1 < p, q < \infty, \tag{1.4}$$

which is an important tool in studying well-posedness and regularity of solutions to nonlinear parabolic PDEs; see Amann (1995), Clément & Prüss (1992) and Li & Yang (2015). In the numerical solution of parabolic equations, it is also desirable to have a discrete analogue of this estimate, which has a number of applications in the analysis of stability and convergence of numerical methods for nonlinear parabolic problems, including semilinear parabolic equations with strong nonlinearities (Geissert, 2007), quasi-linear parabolic equations with nonsmooth coefficients (Li & Sun, 2015), optimal control and inverse problems (Leykekhman & Vexler, 2016; Leykekhman *et al.*, 2019) and so on.

Let S_h, $0 < h < h_0$ be a family of Lagrange finite element subspaces of $H_0^1(\Omega)$ consisting of all piecewise polynomials of degree $\le r$ subject to a quasi-uniform triangulation of the domain Ω, where $r \ge 1$ is any given integer. It is known that the semidiscrete finite element solutions of (1.1), defined by

$$\begin{cases} (\partial_t u_h, v_h) + \displaystyle\sum_{i,j=1}^{d} (a_{ij}\partial_j u_h, \partial_i v_h) = (f, v_h) \quad \forall\, v_h \in S_h,\ \forall\, t > 0, \\ u_h(0) = u_{h,0}, \end{cases} \tag{1.5}$$

where $\partial_i v_h = \frac{\partial v_h}{\partial x_i}$ for abbreviation, possesses maximal L^p-regularity similarly to the continuous problem. Namely, when $u_{h,0} = 0$ the solution of (1.5) satisfies

$$\|\partial_t u_h\|_{L^p(\mathbb{R}_+;L^q)} + \|A_h u_h\|_{L^p(\mathbb{R}_+;L^q)} \le C_{p,q}\|f\|_{L^p(\mathbb{R}_+;L^q)} \quad \forall\, 1 < p, q < \infty, \tag{1.6}$$

with a constant $C_{p,q}$ independent of the mesh size h, where $A_h : S_h \to S_h$ is the finite element approximation of the elliptic operator A, defined by

$$(A_h\phi_h, \varphi_h) = -\sum_{i,j=1}^{d} (a_{ij}\partial_j\phi_h, \partial_i\varphi_h) \quad \forall\, \phi_h, \varphi_h \in S_h. \tag{1.7}$$

Estimate (1.6) was proved in smooth domains with the Neumann boundary condition (Geissert, 2006; Li, 2015; Kashiwabara & Kemmochi, 2020) and in polyhedral domains with the Dirichlet boundary condition (Li & Sun, 2017a; Li, 2019). The proof of (1.6) is closely related to the proof of the following maximum-norm error estimate (cf. Schatz *et al.*, 1980, 1998; Nitsche & Wheeler, 1982; Thomée & Wahlbin, 2000; Leykekhman, 2004):

$$\|u - u_h\|_{L^\infty(0,T;L^\infty)} \le C\ell_h \inf_{\chi_h \in L^\infty(0,T;S_h)} \|u - \chi_h\|_{L^\infty(0,T;L^\infty)}, \tag{1.8}$$

with $\ell_h = \ln(2 + 1/h)$. The two results (1.6) and (1.8) are often proved simultaneously by similar techniques.

The extension of semidiscrete maximal L^p-regularity results to fully discrete finite element methods (FEMs) has been established for several different time discretization methods, including the backward Euler method (Ashyralyev *et al.*, 2002; Li & Sun, 2017b), discontinuous Galerkin method (Leykekhman

& Vexler, 2017), θ-schemes (Kemmochi & Saito, 2018) and A-stable multistep and Runge–Kutta methods (Kovács *et al.*, 2016). All these proofs use A-stability of the methods. It is known that A-stable multistep methods can be at most second-order accurate (cf. Hairer & Wanner, 1996, p. 247, Theorem 1.4). Hence, maximal L^p-regularity of fully discrete FEMs with higher-order multistep methods, including backward differentiation formulae (BDF), has not been proved so far.

The BDF method is one of the most popular high-order methods for solving parabolic equations owing to its ease of implementation. For $k = 1, \ldots, 6$, we denote by u^n and u_h^n, $n = k, \ldots, N$ the solution of the semidiscrete and fully discrete k-step BDF methods, given by

$$\frac{1}{\tau}\sum_{j=0}^{k} \delta_j u^{n-j} = Au^n + f^n, \quad n \ge k \tag{1.9}$$

and

$$\frac{1}{\tau}\sum_{j=0}^{k} \delta_j u_h^{n-j} = A_h u_h^n + f_h^n, \quad n \ge k, \tag{1.10}$$

respectively, where the starting values u^n and u_h^n, $n = 0, \ldots, k-1$ in the k-step BDF method are assumed to be given (obtained by other methods) and δ_j, $j = 0, \ldots, k$ are the coefficients in the polynomial

$$\delta(\zeta) := \sum_{j=1}^{k} \frac{1}{j}(1-\zeta)^j = \sum_{j=0}^{k} \delta_j \zeta^j. \tag{1.11}$$

It is known that the k-step BDF method is A(α_k)-stable with angles $\alpha_1 = \alpha_2 = 0.5\pi$, $\alpha_3 = 0.478\pi$, $\alpha_4 = 0.408\pi$, $\alpha_5 = 0.288\pi$ and $\alpha_6 = 0.099\pi$; see Hairer & Wanner (1996, Section V.2). Here, A(α)-stability is equivalent to

$$|\arg \delta(\zeta)| \le \pi - \alpha \quad \text{for } \zeta \in \mathbb{C} \text{ such that } |\zeta| \le 1,$$

and A-stability is equivalent to A(α)-stability with $\alpha = 0.5\pi$. In particular, only one-step and two-step BDF methods are A-stable. For $k > 6$, the BDF method is not A-stable, hence not useful.

Maximal L^p-regularity of the k-step BDF method, with $k = 3, \ldots, 6$, requires an additional angle condition, i.e.,

$$\left\{z(z-A)^{-1} \,:\, z \in \Sigma_{\theta+\frac{\pi}{2}}\right\} \text{ is } R\text{-bounded on } L^q \text{ for some angle } \theta > \tfrac{\pi}{2} - \alpha_k, \tag{1.12}$$

where Σ_θ is a sector on the complex plane defined by

$$\Sigma_\theta = \left\{z \in \mathbb{C}\backslash\{0\} : |\arg(z)| < \theta\right\},$$

and R-boundedness is a stronger concept than boundedness, required in the context of maximal L^p-regularity (see the rigorous definition in the next section). For the continuous operator A, condition (1.12) holds for all $\theta \in (0, \frac{\pi}{2})$, and therefore maximal L^p-regularity of semidiscretization in time holds for all BDF methods up to order 6. These results were proved in Kovács *et al.* (2016, Theorems 4.1–4.2 and Remark 4.2).

Similarly, the extension of maximal L^p-regularity to fully discrete FEMs, with the k-step BDF method in time, requires the following condition:

$$\left\{z(z-A_h)^{-1}P_h \,:\, z\in\Sigma_{\theta+\frac{\pi}{2}}\right\} \text{ is } R\text{-bounded on } L^q \text{ for some angle } \theta > \tfrac{\pi}{2}-\alpha_k, \tag{1.13}$$

with an R-bound independent of h, where P_h denotes the L^2-orthogonal projection onto the finite element space. However, (1.13) has only been proved for sufficiently small θ (see, for example, Li, 2019, Corollary 2.1 and Li & Sun, 2017a) for parabolic equations with constant coefficients. From the proof therein, it is not clear whether $\theta > \frac{\pi}{2}-\alpha_k$ for the k-step BDF method with $k=3,\ldots,6$ and how much regularity the diffusion coefficients a_{ij} should possess for (1.13) to be valid in general polygons and polyhedra.

In this article, we fill in the gap between semidiscrete FEMs and multistep fully discrete FEMs. In particular, by considering the complex-valued operator $e^{\mathrm{i}\theta}A_h$ in the proof of Li (2019), we show that (1.13) holds for all $\theta\in(0,\frac{\pi}{2})$ for diffusion coefficients $a_{ij}\in W^{1,d+\beta}(\Omega)$ with $\beta>0$. As a result, we obtain discrete maximal L^p-regularity of fully discrete FEMs with general k-step BDF methods. Namely, with zero starting values $u_h^n=0$, $n=0,\ldots,k-1$, the solution of (1.10) satisfies

$$\left\|(d_\tau u_h^n)_{n=k}^N\right\|_{\ell^p(L^q)} + \left\|(A_h u_h^n)_{n=k}^N\right\|_{\ell^p(L^q)} \le C_{p,q}\left\|(f_h^n)_{n=k}^N\right\|_{\ell^p(L^q)} \qquad \forall\, 1<p,q<\infty,$$

with $d_\tau u^n := (u^n-u^{n-1})/\tau$. In addition, we establish an error estimate between semidiscrete and fully discrete solutions in the temporally discrete $L^p(0,T;L^q)$ norm, for $1<p,q<\infty$:

$$\left\|(P_h u^n - u_h^n)_{n=k}^N\right\|_{\ell^p(L^q)} \le C_{p,q}\left\|(P_h u^n - R_h u^n)_{n=k}^N\right\|_{\ell^p(L^q)} + C_{p,q}\sum_{n=0}^{k-1}\left\|P_h u^n - u_h^n\right\|_{L^q},$$

where R_h denotes the Ritz projection onto the finite element space. In the case $p=q=\infty$, we obtain a fully discrete analogue of (1.8), i.e.,

$$\max_{1\le n\le N}\|P_h u^n - u_h^n\|_{L^\infty} \le C\ell_N \max_{1\le n\le N}\|P_h u^n - R_h u^n\|_{L^\infty} + C\max_{1\le n\le k-1}\left\|P_h u^n - u_h^n\right\|_{L^\infty},$$

with $\ell_N=\ln(1+N)$. The three results above hold in general polygons and polyhedra, with constants independent of h, τ and N. In convex polygons and polyhedra, we furthermore obtain the following temporally discrete $L^p(0,T;W^{1,q})$ estimate (for zero starting values $u_h^n=0$, $n=0,\ldots,k-1$):

$$\left\|(d_\tau u_h^n)_{n=k}^N\right\|_{\ell^p(W^{-1,q})} + \left\|(u_h^n)_{n=k}^N\right\|_{\ell^p(W^{1,q})} \le C_{p,q}\left\|(f_h^n)_{n=k}^N\right\|_{\ell^p(W^{-1,q})} \quad \forall\, 1<p,q<\infty.$$

The rest of this paper is organized as follows. In the next section we introduce basic notation and preliminary results to be used in this paper. The main theorems are presented in Section 3 with self-contained proofs for most of the results. The most technical proof (proof of Theorem 3.1) is postponed to Section 4, where we reformulate Theorem 3.1 into a form that can be proved in a similar way to Li (2019, Proof of Corollary 2.1). Then we sketch the proof by following the outline of Li (2019, Proof of Corollary 2.1) while highlighting the differences.

2. Preliminaries

We denote by $\Omega \subset \mathbb{R}^d$, with $d \in \{2, 3\}$, a general polygon or polyhedron (possibly nonconvex) unless otherwise stated. We denote by $W^{s,q}(\Omega)$ the conventional Sobolev space for $s \in \mathbb{R}$ and $1 \le q \le \infty$, with the abbreviations

$$W^{s,q} = W^{s,q}(\Omega), \quad L^q = W^{0,q}(\Omega) \quad \text{and} \quad H^s = W^{s,2}(\Omega).$$

The space of infinitely smooth functions with compact support in Ω is denoted by C_0^∞, and the completion of C_0^∞ in H^s is denoted by H_0^s. The space of Hölder continuous functions on $\overline{\Omega}$ is denoted by C^β.

For any $q \in [1, \infty]$, we denote by $q' \in [1, \infty]$ the number satisfying $1/q + 1/q' = 1$ and denote by

$$(u, v) = \int_\Omega u(x)v(x)\,\mathrm{d}x \quad \forall\, u \in L^q \text{ and } v \in L^{q'}, \tag{2.14}$$

the pairing between two real-valued functions on Ω.

For a sequence u^n, $n = 0, 1, \dots$, we denote

$$\dot{u}^n := \frac{1}{\tau}\sum_{j=0}^{k} \delta_j u^{n-j} \ \text{ for } n \ge k \quad \text{and} \quad d_\tau u^n := \frac{u^n - u^{n-1}}{\tau} \ \text{ for } n \ge 1.$$

For any Banach space X and any sequence $v^n \in X$, $n = 1, \dots, N$, we denote

$$\|(v^n)_{n=1}^N\|_{\ell^p(X)} := \begin{cases} \left(\tau \sum_{n=1}^{N} \|v^n\|_X^p\right)^{\frac{1}{p}} & \text{for } 1 \le p < \infty, \\ \sup_{1\le n\le N} \|v^n\|_X & \text{for } p = \infty. \end{cases}$$

We also denote

$$\|(v^n)_{n=1}^N\|_{\ell^{1,\infty}(X)} := \sup_{\lambda>0} \lambda\, \tau |\{n : \|v^n\|_X \ge \lambda\}|,$$

where $|\{n : \|v^n\|_X \ge \lambda\}|$ stands for the number of elements in the set $\{n : \|v^n\|_X \ge \lambda\}$. From the definitions above, it is straightforward to verify that if $\|v_n\|_X \le c_0 t_n^{-1}$ for some constant c_0 then

$$\|(v^n)_{n=1}^N\|_{\ell^{1,\infty}(X)} \le c_0.$$

In the case $X = \mathbb{R}$, we simply denote $\ell^p = \ell^p(\mathbb{R})$ and $\ell^{1,\infty} = \ell^{1,\infty}(\mathbb{R})$. The weak-type norm $\|\cdot\|_{\ell^{1,\infty}}$ is the same as the norm $\|\cdot\|_{L^{1,\infty}(X,\mu)}$ in Grafakos (2008, Section 1.1) when the measure space (X, μ) is chosen to be $X = \{1, \ldots, N\}$ and

$$\mu(\{1, \ldots, n\}) = n\tau \quad \text{for} \quad 1 \le n \le N.$$

For any given index set D, a collection of operators $\{M(z) : L^q \to L^q : z \in D\}$ is called *R-bounded* on L^q, with $1 < q < \infty$, if and only if the following inequality holds for all finite subcollections of operators $M(z_1), \ldots, M(z_m)$:

$$\left\| \left(\sum_{j=1}^m |M(z_j)v_j|^2 \right)^{\frac{1}{2}} \right\|_{L^q} \le C_R \left\| \left(\sum_{j=1}^m |v_j|^2 \right)^{\frac{1}{2}} \right\|_{L^q} \quad \forall\, v_1, \ldots, v_m \in L^q,$$

where the smallest constant C_R satisfying this inequality is called the R-bound of the collection. This characterization of R-boundedness on L^q is equivalent to the definition of R-boundedness of operators on a general Banach space; see Weis (2001a, Section 1.f).

It is known that maximal L^p-regularity of the semidiscrete BDF method (1.9) is related to the R-boundedness of the resolvent operators $z(z-A)^{-1}$, as shown in the following theorem.

THEOREM 2.1 (Kovács *et al.*, 2016, Theorems 4.1–4.2). If the collection of operators $\{z(z-A)^{-1} : z \in \Sigma_{\theta+\frac{\pi}{2}}\}$ is R-bounded on L^q for an angle $\theta > \frac{\pi}{2} - \alpha_k$ then the semidiscrete solution given by (1.9), with zero starting values $u^n = 0$ for $n = 0, \ldots, k-1$, satisfies the temporally discrete maximal L^p-regularity estimate

$$\|(\dot{u}^n)_{n=k}^N\|_{\ell^p(L^q)} + \|(Au^n)_{n=k}^N\|_{\ell^p(L^q)} \le C_{p,q} \|(f^n)_{n=k}^N\|_{\ell^p(L^q)} \quad \forall\, 1 < p, q < \infty,$$

where the constant $C_{p,q}$ is independent of τ and N.

Similarly, maximal L^p-regularity of fully discrete FEMs with BDF methods in time is summarized in the following theorem.

THEOREM 2.2 (Kovács *et al.*, 2016, Theorems 6.1). If the collection of operators $\{z(z-A_h)^{-1}P_h : z \in \Sigma_{\theta+\frac{\pi}{2}}\}$ is R-bounded on L^q for an angle $\theta > \frac{\pi}{2} - \alpha_k$ (with an R-bound independent of h) then the fully discrete finite element solution given by (1.10), with zero starting values $u_h^n = 0$ for $n = 0, \ldots, k-1$, satisfies the discrete maximal L^p-regularity estimate

$$\|(\dot{u}_h^n)_{n=k}^N\|_{\ell^p(L^q)} + \|(A_h u_h^n)_{n=k}^N\|_{\ell^p(L^q)} \le C_{p,q} \|(f_h^n)_{n=k}^N\|_{\ell^p(L^q)} \quad \forall\, 1 < p, q < \infty,$$

where the constant $C_{p,q}$ is independent of h, τ and N.

The condition of Theorem 2.1 was proved in Kovács *et al.* (2016, Remark 4.2), while the condition of Theorem 2.2 is proved in the current paper; see the third result of Theorem 3.1.

3. Main results

Throughout this article we assume that the coefficients a_{ij} satisfy the ellipticity and regularity conditions in (1.2)–(1.3) and that the triangulation is quasi-uniform so that the Lagrange finite element spaces have all the properties in Li (2019, Section 3.2), including shape regularity, quasi-uniformity, inverse inequality, the local approximation property and super-approximation property.

3.1 *Discrete semigroup, resolvent and maximal regularity*

For fully discrete FEMs with BDF methods in time, maximal L^p-regularity relies on the results in the following theorem. The proof of this theorem is presented in the next section. In Li (2019, Corollary 2.1), the results were shown for some small $\theta \in (0, \frac{\pi}{2})$ instead of all $\theta \in (0, \frac{\pi}{2})$, by proving and utilizing the following semigroup estimates:

$$\begin{aligned}
&\sup_{t>0} (\|E_h(t)v_h\|_{L^q} + t\|\partial_t E_h(t)v_h\|_{L^q}) \le C\|v_h\|_{L^q} && \forall\, v_h \in S_h, && \forall\, 1 \le q \le \infty,\\
&\left\| \sup_{t>0} |E_h(t)P_h|\,|v| \right\|_{L^q} \le C_q\|v\|_{L^q} && \forall\, v \in L^q, && \forall\, 1 < q \le \infty.
\end{aligned}$$

In this article, we shall consider the parabolic equation with a rotated elliptic operator $e^{i\theta}A$ for a general angle $\theta \in (0, \frac{\pi}{2})$ and estimate the semigroup $E_h^\theta(t)$ generated by the rotated operator $e^{i\theta}A_h$ directly. We shall prove the following semigroup estimates for all $\theta \in (0, \frac{\pi}{2})$ (see Lemma 4.1):

$$\begin{aligned}
&\sup_{t>0} (\|E_h^\theta(t)v_h\|_{L^q} + t\|\partial_t E_h^\theta(t)v_h\|_{L^q}) \le C_\theta\|v_h\|_{L^q} && \forall\, v_h \in S_h, && \forall\, 1 \le q \le \infty,\\
&\left\| \sup_{t>0} |E_h^\theta(t)P_h|\,|v| \right\|_{L^q} \le C_{\theta,q}\|v\|_{L^q} && \forall\, v \in L^q, && \forall\, 1 < q \le \infty.
\end{aligned}$$

These results would yield Theorem 3.1 for all $\theta \in (0, \frac{\pi}{2})$; see Section 4.4.

THEOREM 3.1 (Estimates for the analytic semigroup and resolvent). For all $\theta \in (0, \frac{\pi}{2})$, the following results hold.

1. The following collections of operators are all bounded on L^q for $1 \le q \le \infty$, and the bounds are independent of h (but may depend on θ):
   (i) the semigroup $\{e^{zA_h}P_h : z \in \Sigma_\theta\}$ and its derivative $\{zA_h e^{zA_h}P_h : z \in \Sigma_\theta\}$;
   (ii) the resolvent operators $\{z(z - A_h)^{-1}P_h : z \in \Sigma_{\theta+\frac{\pi}{2}}\}$.
2. The semigroup of operators $\{e^{zA_h}P_h : z \in \Sigma_\theta\}$ is R-bounded on L^q for $1 < q < \infty$, and the R-bound is independent of h (but may depend on θ and q).
3. The collection of resolvent operators $\{z(z - A_h)^{-1}P_h : z \in \Sigma_{\theta+\frac{\pi}{2}}\}$ is R-bounded on L^q for $1 < q < \infty$, and the R-bound is independent of h (but may depend on θ and q).

When the initial value $u_{h,0}$ is zero, by applying the Laplace transform in time to equation (1.5), one can express the solution of the semidiscrete FEM as

$$\partial_t u_h = \mathcal{L}_z^{-1}[z(z - A_h)^{-1}P_h(\mathcal{L}f)(z)] = \mathcal{F}_s^{-1}[is(is - A_h)^{-1}P_h(\mathcal{F}\tilde{f})(s)], \tag{3.1}$$

where $\mathcal{L}$ and $\mathcal{F}$ denote the Laplace and Fourier transforms in time, respectively, with $\tilde{f}$ denoting the zero extension of f in time to $t \in \mathbb{R}$. The notation $\mathcal{F}_s^{-1}$ standards for the inverse Fourier transform with respect to the variable s. The third result of Theorem 3.1 implies that the operator $M(s) = is(is - A_h)^{-1}P_h$ satisfies the R-boundedness on L^q of the two collections of operators

$$\{M(s) : S \in \mathbb{R}\backslash\{0\}\} \quad \text{and} \quad \{sM'(s) : S \in \mathbb{R}\backslash\{0\}\}.$$

This means that $M(s)$ is a Mihlin multiplier (with an R-bounded independent of h). The Mihlin multiplier theorem (cf. Weis, 2001a, Theorem 2.b) and the expression (3.1) immediately imply that

$$\|\partial_t u_h\|_{L^p(0,T;L^q)} \leq C_{p,q}\|f\|_{L^p(0,T;L^q)} \quad \forall\ 1 < p, q < \infty.$$

We summarize the result in the following corollary, which was shown in Li (2019, Theorem 2.1) using a different approach for the heat equation (with A_h replaced by Δ_h).

Corollary 3.2 If $u_{h,0} = 0$ and $f \in L^p(0,T;L^q)$ then the finite element solution given by (1.5) has maximal L^p-regularity:

$$\|\partial_t u_h\|_{L^p(0,T;L^q)} + \|A_h u_h\|_{L^p(0,T;L^q)} \leq C_{p,q}\|f\|_{L^p(0,T;L^q)} \quad \forall\ 1 < p, q < \infty, \tag{3.2}$$

where the constant $C_{p,q}$ is independent of h and T.

Corollary 3.2 immediately implies the following error estimate between semidiscrete FEM and the PDE problem, as shown in Li (2019, Corolloary 2.2).

Corollary 3.3 If $f \in L^p(0,T;L^q)$ then the solutions of the semidiscrete FEM (1.5) and the PDE problem (1.1) satisfy the following estimate:

$$\|u_h - P_h u\|_{L^p(0,T;L^q)} \leq C_{p,q}(\|u - R_h u\|_{L^p(0,T;L^q)} + \|u_h^0 - P_h u_0\|_{L^q}) \quad \forall\ 1 < p, q < \infty, \tag{3.3}$$

where the constant $C_{p,q}$ is independent of h and T.

We present a variant version of maximal L^p-regularity for semidiscrete BDF methods. This variant version is often more useful than the original result in Theorem 2.1 in analysis of nonlinear parabolic problems.

Theorem 3.4 If $u^n = 0$ for $n = 0, \ldots, k-1$ then the semidiscrete solution given by (1.9) satisfies the following estimates, with a constant $C_{p,q}$ independent of τ and N.

1. In a bounded Lipschitz domain, the maximal L^p-regularity estimate holds:

$$\left\|(d_\tau u^n)_{n=k}^N\right\|_{\ell^p(L^q)} + \left\|(Au^n)_{n=k}^N\right\|_{\ell^p(L^q)} \leq C_{p,q}\left\|(f^n)_{n=k}^N\right\|_{\ell^p(L^q)} \qquad \forall\, 1 < p, q < \infty.$$

2. In a convex domain, the following additional estimate holds:

$$\left\|(d_\tau u^n)_{n=k}^N\right\|_{\ell^p(W^{-1,q})} + \left\|(u^n)_{n=k}^N\right\|_{\ell^p(W^{1,q})} \leq C_{p,q}\left\|(f^n)_{n=k}^N\right\|_{\ell^p(W^{-1,q})} \quad \forall\, 1 < p, q < \infty.$$

Proof. The first result is a consequence of Theorem 2.1 and (1.12) (which holds for all $\theta \in (0, \frac{\pi}{2})$; cf. Kovács *et al.*, 2016, Remark 4.2), together with the following equivalence relation:

$$C^{-1}\|(\dot{u}^n)_{n=k}^N\|_{\ell^p(X)} \le \|(d_\tau u^n)_{n=k}^N\|_{\ell^p(X)} \le C\|(\dot{u}^n)_{n=k}^N\|_{\ell^p(X)}, \tag{3.4}$$

where X can be any UMD Banach space, including L^q and $W^{-1,q}$ for $1 < q < \infty$. This equivalence relation can be proved as follows.

We consider the generating functions of $d_\tau u^n$ and $\dot{u}^n$, respectively, i.e.,

$$\sum_{n=k}^{\infty} d_\tau u^n \zeta^n = \frac{1-\zeta}{\tau}\sum_{n=k}^{\infty} u^n \zeta^n,$$

$$\sum_{n=k}^{\infty} \dot{u}^n \zeta^n = \frac{1}{\tau}\delta(\zeta)\sum_{n=k}^{\infty} u^n \zeta^n,$$

where ζ is on the unit disk of the complex plane. The two equalities above imply

$$\sum_{n=k}^{\infty} d_\tau u^n \zeta^n = \frac{1-\zeta}{\delta(\zeta)}\sum_{n=k}^{\infty} \dot{u}^n \zeta^n.$$

Substituting $\zeta = e^{-i\theta}$ into the equality above, we obtain

$$\big[(d_\tau u^n)_{n=k}^\infty\big] = \mathcal{F}_\theta^{-1}\frac{1-e^{\mathrm{i}\theta}}{\delta(e^{\mathrm{i}\theta})}\mathcal{F}[(\dot{u}^n)_{n=k}^\infty](\theta),$$

$$\big[(\dot{u}^n)_{n=k}^\infty\big] = \mathcal{F}_\theta^{-1}\frac{\delta(e^{\mathrm{i}\theta})}{1-e^{\mathrm{i}\theta}}\mathcal{F}[(d_\tau u^n)_{n=k}^\infty](\theta),$$

where $\mathcal{F}$ represents the Fourier transform (which transforms a sequence to a Fourier series) and $\mathcal{F}_\theta^{-1}$ is its inverse transform (which transforms a function of θ to a sequence).

Since the polynomial $\delta(\zeta)$ associated to the k-step BDF method has only one zero at $\zeta = 1$ on the closed unit disk of the complex plane, it follows that the functions

$$M_1(\zeta) = \frac{1-\zeta}{\delta(\zeta)} \quad \text{and} \quad M_2(\zeta) = \frac{\delta(\zeta)}{1-\zeta}$$

are bounded from both below and above on the unit circle of the complex plane; therefore, both satisfy Blunck's multiplier conditions (cf. Jin *et al.*, 2018, Theorem 4 or Blunck, 2001, Theorem 1.3), i.e.,

$$|M_j(\zeta)| \le C \quad \text{and} \quad |(1+\zeta)(1-\zeta)M_j'(\zeta)| \le C, \quad j = 1, 2, \quad \forall\, \zeta \in \mathbb{C} \text{ such that } |\zeta| = 1,\ \zeta \ne 1.$$

Hence, the operators $\mathcal{F}_\theta^{-1}\frac{1-e^{i\theta}}{\delta(e^{i\theta})}\mathcal{F}$ and $\mathcal{F}_\theta^{-1}\frac{\delta(e^{i\theta})}{1-e^{i\theta}}\mathcal{F}$ are both bounded on $\ell^p(X)$. As a result, we have

$$\|(d_\tau u^n)_{n=k}^\infty\|_{\ell^p(X)} \le C\|(\dot{u}^n)_{n=k}^\infty\|_{\ell^p(X)} \quad \text{and} \quad \|(\dot{u}^n)_{n=k}^\infty\|_{\ell^p(X)} \le C\|(d_\tau u^n)_{n=k}^\infty\|_{\ell^p(X)}.$$

Modifying u^n for $n \geq N+1$, we can make $\dot{u}^n = 0$ for $n \geq N+1$ without changing the values of $d_\tau u^n$ for $k \leq n \leq N$. Then the first of the two inequalities above implies

$$\|(d_\tau u^n)_{n=k}^N\|_{\ell^p(X)} \leq C\|(\dot{u}^n)_{n=k}^N\|_{\ell^p(X)}.$$

Similarly, by modifying u^n for $n \geq N+1$, we can make $d_\tau u^n = 0$ for $n \geq N+1$ without changing the values of $\dot{u}^n$ for $k \leq n \leq N$. Then we obtain

$$\|(\dot{u}^n)_{n=k}^N\|_{\ell^p(X)} \leq C\|(d_\tau u^n)_{n=k}^N\|_{\ell^p(X)}.$$

This proves the norm equivalence (3.4) and completes the proof of the first result.

The second result was proved in Akrivis *et al.* (2017, Proposition 8.2) for $q_d' < q < q_d$, where q_d is the maximal number such that the solution of the Poisson equation

$$\begin{cases} Av = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

satisfies the estimate

$$\|v\|_{W^{1,q}} \leq C\|g\|_{W^{-1,q}} \quad \forall\, q_d' < q < q_d. \tag{3.6}$$

In a convex domain, we have $q_d = \infty$ (cf. Jia *et al.*, 2010, Theorem 1). This implies the second result. □

Theorem 2.2 and the third result of Theorem 3.1, together with the equivalence relation (3.4), imply the following result for the fully discrete FEM.

THEOREM 3.5 If $u_h^n = 0$ for $n = 0, \ldots, k-1$ then the fully discrete solution given by (1.10) satisfies the discrete maximal L^p-regularity estimate

$$\left\|(d_\tau u_h^n)_{n=k}^N\right\|_{\ell^p(L^q)} + \left\|(A_h u_h^n)_{n=k}^N\right\|_{\ell^p(L^q)} \leq C_{p,q}\left\|(f_h^n)_{n=k}^N\right\|_{\ell^p(L^q)} \quad \forall\, 1 < p, q < \infty, \tag{3.7}$$

where the constant $C_{p,q}$ is independent of h, τ and N.

3.2 *Error estimate between semidiscrete and fully discrete solutions*

Error estimates of semidiscrete BDF methods using maximal L^p-regularity were established for semilinear and quasi-linear parabolic equations in Akrivis & Li (2018) and Akrivis *et al.* (2017), Kunstmann *et al.* (2018), respectively. In this subsection we present a tool for establishing error estimates between fully discrete and semidiscrete solutions.

LEMMA 3.6 (Estimates of the solution operator for fully discrete BDF methods). The solution of the equations

$$\frac{1}{\tau}\sum_{j=0}^{k} \delta_j v_h^{n-j} - A_h v_h^n = 0, \quad n \geq k \tag{3.8}$$

can be represented by the starting values as

$$v_h^n = \sum_{j=0}^{k-1} E_h^{n,j} v^j \quad \text{for } n \geqslant k, \tag{3.9}$$

where $E_h^{n,j} : S_h \to S_h$ are some operators satisfying the following estimates (for some constant $\lambda_0 > 0$):

$$\begin{aligned}
&\|E_h^{n,j}\phi_h\|_{L^q} \le Ce^{-\lambda_0 t_{n+1}}\|\phi_h\|_{L^q} && \forall\, \phi_h \in S_h, \ \ \forall\, 1 \le q \le \infty,\\
&\|A_h E_h^{n,j}\phi_h\|_{L^q} \le Ce^{-\lambda_0 t_{n+1}}\|A_h\phi_h\|_{L^q} && \forall\, \phi_h \in S_h, \ \ \forall\, 1 \le q \le \infty,\\
&\|A_h E_h^{n,j}\phi_h\|_{L^q} \le Ce^{-\lambda_0 t_{n+1}} t_{n+1}^{-1}\|\phi_h\|_{L^q} && \forall\, \phi_h \in S_h, \ \ \forall\, 1 \le q \le \infty,
\end{aligned} \tag{3.10}$$

with a constant C independent of h, τ, $n \ge k$ and $1 \le q \le \infty$.

Proof. Setting $v_h^n = 0$ for $n = 0, \dots, k-1$ without changing the values of v_h^n for $n \ge k$, equation (3.8) can be rewritten as

$$\begin{cases}
\dfrac{1}{\tau}\displaystyle\sum_{j=0}^{k} \delta_j v_h^{n-j} - A_h v_h^n = g_h^n, & n \ge k,\\
v_h^n = 0 & n = 0, \dots, k-1,
\end{cases} \tag{3.11}$$

with

$$g_h^n = \begin{cases}
-\dfrac{1}{\tau}\displaystyle\sum_{j=n-k}^{k-1} \delta_{n-j} A_h^{-1}(P_h u^j - u_h^j) & \text{for } k \le n \le 2k-1,\\
0 & \text{for } n \ge 2k.
\end{cases}$$

Without loss of generality, we can also set $v_h^n = 0$ for $n \ge N+1$ without affecting equation (3.11) and g_h^n for $k \le n \le N$. In the following we derive an expression for v_h^n for $k \le n \le N$. But the expression turns out to be independent of N and therefore holds for all $n \ge k$.

We denote by

$$v(\zeta) = \sum_{n=k}^{\infty} v_h^n \zeta^n \quad \forall\, \zeta \in \mathbb{C}$$

the generating function of the sequence $(v_h^n)_{n=k}^{\infty}$. The series is well defined as an analytic function on the complex plane because there are only a finite number of v_h^n that are not zero (after modifying v_h^n to be zero for $n \geq N+1$). Multiplying (3.11) by ζ^n and summing the results for $n = k, k+1, \ldots$, we obtain

$$(\tau^{-1}\delta(\zeta) - A_h)v(\zeta) = \sum_{m=k}^{2k-1} g_h^m \zeta^m,$$

which implies

$$v(\zeta) = (\tau^{-1}\delta(\zeta) - A_h)^{-1} \sum_{m=k}^{2k-1} g_h^m \zeta^m. \tag{3.12}$$

It is known that, by choosing $\kappa \in (\frac{\pi}{2}, \pi)$ sufficiently close to $\frac{\pi}{2}$ (independent of τ; cf. Jin *et al.*, 2017, Lemma B.1), the polynomial $\delta(\zeta)$ associated to the k-step BDF method satisfies

$$\begin{aligned}
&\delta(e^{z\tau}) \in \Sigma_{\pi - \frac{1}{2}\alpha_k} && \forall z \in \Sigma_\kappa^\tau = \{z \in \Sigma_\kappa : |\mathrm{Im}(z)| \leq \pi/\tau\},\\
&C^{-1}|z| \leqslant |\tau^{-1}\delta(e^{z\tau})| \leqslant C|z| && \forall z \in -\lambda_0 + \Sigma_\kappa^\tau,\\
&|z - \tau^{-1}\delta(e^{z\tau})| \leqslant C|z|^2\tau && \forall z \in -\lambda_0 + \Sigma_\kappa^\tau,
\end{aligned} \tag{3.13}$$

for some sufficiently small constant λ_0; see Remark 3.7.

REMARK 3.7 In Jin *et al.* (2017, Lemma B.1), these estimates were all proved only for $z \in \Sigma_{\kappa_*}^\tau$ instead of $z \in -\lambda_0 + \Sigma_{\kappa_*}^\tau$, for all angles κ_* sufficiently close to $\frac{\pi}{2}$. Nevertheless, using Taylor's expansion one can see that these estimates are still correct when $|z|$ is sufficiently small (smaller than some constant independent of τ). Hence, these estimates also hold for $z \in -\lambda_0 + \Sigma_\kappa^\tau$ with a sufficiently small constant λ_0 and an angle κ slightly closer to $\frac{\pi}{2}$ than κ_*.

Theorem 3.1 and (3.13) imply that $(\tau^{-1}\delta(\zeta) - A_h)^{-1}$ is analytic and satisfies the following estimate for $z \in \Sigma_\kappa^\tau$:

$$\|(\tau^{-1}\delta(e^{z\tau}) - A_h)^{-1}\|_{L^q \to L^q} \leq C|z|^{-1}, \quad \|A_h(\tau^{-1}\delta(e^{z\tau}) - A_h)^{-1}\|_{L^q \to L^q} \leq C.$$

Since the largest eigenvalue of A_h is strictly negative and bounded away from zero (with an upper bound independent of h), it follows that in a small neighborhood of $z = 0$ the operator $(z - A_h)^{-1}$ is bounded analytic. As a result, the resolvent estimates above can be slightly improved as follows (for sufficiently small constant λ_0):

$$\begin{aligned}
&\|(\tau^{-1}\delta(e^{z\tau}) - A_h)^{-1}\|_{L^q \to L^q} \leq C(1+|z|)^{-1} \leq C|z+\lambda_0|^{-1} && \forall z \in -\lambda_0 + \Sigma_\kappa^\tau,\\
&\|A_h(\tau^{-1}\delta(e^{z\tau}) - A_h)^{-1}\|_{L^q \to L^q} \leq C && \forall z \in -\lambda_0 + \Sigma_\kappa^\tau.
\end{aligned} \tag{3.14}$$

By the Cauchy integral formula we have

$$
\begin{aligned}
v_h^n &= \frac{1}{2\pi \mathrm{i}} \int_{|\zeta|=1} (\tau^{-1}\delta(\zeta) - A_h)^{-1} \sum_{m=k}^{2k-1} g_h^m \zeta^m \frac{\mathrm{d}\zeta}{\zeta^{n+1}} \\
&= \frac{\tau}{2\pi \mathrm{i}} \int_{|\mathrm{Im}(z)|\le \frac{\pi}{\tau}} (\tau^{-1}\delta(e^{-\tau z}) - A_h)^{-1} \sum_{m=k}^{2k-1} g_h^m e^{-t_m z} e^{t_n z}\, \mathrm{d}z \qquad \text{(change of variable } \zeta = e^{-\tau z}) \\
&= \sum_{m=k}^{2k-1} \frac{\tau}{2\pi \mathrm{i}} \int_{\Gamma_{\kappa,n}^{\tau}} (\tau^{-1}\delta(e^{-\tau z}) - A_h)^{-1} g_h^m e^{t_{n-m} z}\, \mathrm{d}z \qquad \text{for } n \ge k, \qquad (3.15)
\end{aligned}
$$

where we have deformed the integration contour to

$$
\begin{aligned}
&\Gamma_{\kappa,n}^{\tau} = \Gamma_{\kappa,n}^{\tau,1} \cup \Gamma_{\kappa,n}^{\tau,2} \qquad (3.16) \\
&\text{with } \Gamma_{\kappa,n}^{\tau,1} = \{z \in \mathbb{C} : \arg(z+\lambda_0) = \pm\kappa,\ \ |z+\lambda_0| \ge t_{n+1}^{-1},\ \ |\mathrm{Im}(z)| \le \pi/\tau\} \\
&\text{and } \Gamma_{\kappa,n}^{\tau,2} = \{z \in \mathbb{C} : |\arg(z+\lambda_0)| \le \kappa,\ \ |z+\lambda_0| = t_{n+1}^{-1}\}.
\end{aligned}
$$

The deformation of the integration contour in (3.15) is legal due to the analyticity of the integrand for $z \in -\lambda_0 + \Sigma_\kappa^\tau$ and the periodicity in the imaginary part of z. We define the operator

$$
\begin{aligned}
M_{n-m} &= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\kappa,k+n-m}^{\tau}} (\delta(e^{-\tau z}) - A_h)^{-1} e^{t_{n-m} z}\, \mathrm{d}z \\
&= \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\kappa,n}^{\tau}} (\delta(e^{-\tau z}) - A_h)^{-1} e^{t_{n-m} z}\, \mathrm{d}z \qquad \text{(contour is deformed).} \qquad (3.17)
\end{aligned}
$$

Then from (3.15) we obtain

$$
\begin{aligned}
v_h^n &= \sum_{m=k}^{2k-1} \sum_{j=m-k}^{k-1} M_{n-m} \delta_{m-j} A_h^{-1} (P_h u^j - u_h^j) \\
&= \sum_{j=0}^{k-1} \sum_{m=k}^{j+k} M_{n-m} \delta_{m-j} A_h^{-1} (P_h u^j - u_h^j) \qquad \text{for } n \ge k.
\end{aligned}
$$

Therefore, the operator $E_h^{n,j}$ in (3.9) is given by

$$
E_h^{n,j} = \sum_{m=k}^{j+k} M_{n-m} \delta_{m-j}. \qquad (3.18)
$$

Using (3.14) and the property $\tau|z| \le C$ on $\Gamma^\tau_{\kappa,n}$ with $k \le m \le 2k-1$, from (3.17) we derive that

$$
\begin{aligned}
&e^{\lambda_0 t_{n+1}}\|M_{n-m}\phi_h\|_{L^q}\\
&\le C \sup_{z\in\Gamma^\tau_{\kappa,n}} e^{-(m+1)\tau \mathrm{Re}(z)} \int_{\Gamma^\tau_{\kappa,n}} \|(\delta(e^{-\tau z})-A_h)^{-1}\|_{L^q\to L^q} e^{t_{n+1}\mathrm{Re}(z+\lambda_0)}\|\phi_h\|_{L^q}|\mathrm{d}z|\\
&\le C\|\phi_h\|_{L^q}\bigg(\int_{\Gamma^{\tau,1}_{\kappa,n}} |z+\lambda_0|^{-1}e^{-Ct_{n+1}|z+\lambda_0|}|\mathrm{d}(z+\lambda_0)| + \int_{\Gamma^{\tau,2}_{\kappa,n}} |z+\lambda_0|^{-1}e^{t_{n+1}|z+\lambda_0|}|\mathrm{d}(z+\lambda_0)|\bigg)\\
&\le C\|\phi_h\|_{L^q}\bigg(\int_{t_{n+1}^{-1}}^{\frac{\pi}{\tau\sin(\theta)}} r^{-1}e^{-Ct_{n+1}r}\,\mathrm{d}r + \int_{-\kappa}^{\kappa} t_{n+1}e^{C}t_{n+1}^{-1}\,\mathrm{d}\varphi\bigg)\\
&\le C\|\phi_h\|_{L^q} \qquad \text{for } k\le m\le 2k-1 \text{ and } n\ge k.
\end{aligned}
\tag{3.19}
$$

Similarly, we have

$$
\begin{aligned}
&e^{\lambda_0 t_{n+1}}\|A_hM_{n-m}\phi_h\|_{L^q}\\
&\le C \sup_{z\in\Gamma^\tau_{\kappa,n}} e^{-(m+1)\tau \mathrm{Re}(z)} \int_{\Gamma^\tau_{\kappa,n}} \|(\delta(e^{-\tau z})-A_h)^{-1}\|_{L^q\to L^q} e^{t_{n+1}\mathrm{Re}(z+\lambda_0)}\|A_h\phi_h\|_{L^q}\,\mathrm{d}z\\
&\le C\|A_h\phi_h\|_{L^q}\bigg(\int_{\Gamma^{\tau,1}_{\kappa,n}} |z+\lambda_0|^{-1}e^{-Ct_{n+1}|z+\lambda_0|}|\mathrm{d}(z+\lambda_0)| + \int_{\Gamma^{\tau,2}_{\kappa,n}} |z+\lambda_0|^{-1}e^{t_{n+1}|z+\lambda_0|}|\mathrm{d}(z+\lambda_0)|\bigg)\\
&\le C\|A_h\phi_h\|_{L^q} \quad \text{for } k\le m\le 2k-1 \text{ and } n\ge k,
\end{aligned}
\tag{3.20}
$$

and

$$
\begin{aligned}
&e^{\lambda_0 t_{n+1}}\|A_hM_{n-m}\phi_h\|_{L^q}\\
&\le C \sup_{z\in\Gamma^\tau_{\kappa,n}} e^{-(m+1)\tau \mathrm{Re}(z)} \int_{\Gamma^\tau_{\kappa,n}} \|A_h(\delta(e^{-\tau z})-A_h)^{-1}\|_{L^q\to L^q} e^{t_{n+1}\mathrm{Re}(z+\lambda_0)}\|\phi_h\|_{L^q}\,\mathrm{d}z\\
&\le C\|\phi_h\|_{L^q}\bigg(\int_{\Gamma^{\tau,1}_{\kappa,n}} e^{-Ct_{n+1}|(z+\lambda_0)|}|\mathrm{d}(z+\lambda_0)| + \int_{\Gamma^{\tau,2}_{\kappa,n}} e^{t_{n+1}|(z+\lambda_0)|}|\mathrm{d}(z+\lambda_0)|\bigg)\\
&\le C\|\phi_h\|_{L^q}\bigg(\int_{t_{n+1}^{-1}}^{\infty} e^{-Ct_{n+1}r}\mathrm{d}r + \int_{-\kappa}^{\kappa} e^{C}t_{n+1}^{-1}\,\mathrm{d}\varphi\bigg)\\
&\le Ct_{n+1}^{-1}\|\phi_h\|_{L^q} \quad \text{for } k\le m\le 2k-1 \text{ and } n\ge k.
\end{aligned}
\tag{3.21}
$$

Substituting (3.19)–(3.21) into (3.18), we obtain the desired estimates in (3.10). □

THEOREM 3.8 If $f_h^n = P_h f^n$ then the fully discrete solution given by (1.10) and the semidiscrete solution given by (1.9) satisfy the following error estimate:

$$\big\|(P_h u^n - u_h^n)_{n=k}^N\big\|_{\ell^p(L^q)} \le C_{p,q}\big\|(P_h u^n - R_h u^n)_{n=k}^N\big\|_{\ell^p(L^q)} + C_{p,q}\sum_{n=0}^{k-1}\big\|P_h u^n - u_h^n\big\|_{L^q} \quad \forall\, 1 < p,q < \infty, \tag{3.22}$$

where $R_h : H_0^1 \to S_h$ is the Ritz projection defined by

$$\sum_{i,j=1}^{d}(a_{ij}\partial_j(w - R_h w), \partial_i v_h) = 0 \quad \forall\, v_h \in S_h,\ \ w \in H_0^1,$$

and the constant $C_{p,q}$ is independent of h, τ and N.

Proof. It is known that the Ritz projection defined above satisfies the following identity:

$$P_h A = A_h R_h \quad \text{on } H_0^1.$$

Applying the operator P_h to (1.9) and using the identity above, we obtain

$$\frac{1}{\tau}\sum_{j=0}^{k}\delta_j P_h u^{n-j} = A_h R_h u^n + P_h f^n, \quad n \ge k. \tag{3.23}$$

Then subtracting (1.10) from (3.23) we have

$$\frac{1}{\tau}\sum_{j=0}^{k}\delta_j(P_h u^{n-j} - u_h^{n-j}) - A_h(P_h u^n - u_h^n) = A_h(R_h u^n - P_h u^n), \quad n \ge k. \tag{3.24}$$

Applying the operator A_h^{-1} to the equation above we obtain

$$\frac{1}{\tau}\sum_{j=0}^{k}\delta_j[A_h^{-1}(P_h u^{n-j} - u_h^{n-j})] - A_h[A_h^{-1}(P_h u^n - u_h^n)] = R_h u^n - P_h u^n, \quad n \ge k. \tag{3.25}$$

We decompose the solution of (3.25) into two parts, i.e.,

$$A_h^{-1}(P_h u^n - u_h^n) = w_h^n + v_h^n, \tag{3.26}$$

with

$$\begin{cases} \dfrac{1}{\tau}\displaystyle\sum_{j=0}^{k}\delta_j w_h^{n-j} - A_h w_h^n = R_h u^n - P_h u^n, & n \ge k, \\ w_h^n = 0, & n = 0, \dots, k-1, \end{cases} \tag{3.27}$$

and

$$\begin{cases} \dfrac{1}{\tau}\displaystyle\sum_{j=0}^{k}\delta_j v_h^{n-j} - A_h v_h^n = 0, & n \ge k, \\ v_h^n = A_h^{-1}(P_h u^n - u_h^n), & n = 0, \dots, k-1. \end{cases} \tag{3.28}$$

Then Theorem 3.5 implies the following estimate for w_h^n:

$$\big\|(A_h w_h^n)_{n=k}^N\big\|_{\ell^p(L^q)} \le C_{p,q}\big\|(R_h u^n - P_h u^n)_{n=k}^N\big\|_{\ell^p(L^q)} \quad \forall\, 1 < p, q < \infty. \tag{3.29}$$

From Lemma 3.6 (or Thomée, 2006, Lemma 10.3), we know that the solution of (3.28) can be represented by the starting values as

$$v_h^n = \sum_{j=0}^{k-1} E_h^{n,j} A_h^{-1}(P_h u^j - u_h^j) \quad \text{for } n \geqslant k, \tag{3.30}$$

where $E_h^{n,j} : S_h \to S_h$ are some operators satisfying the estimates in (3.10). Substituting the second and third estimates of (3.10) into (3.30), we obtain

$$\|(A_h v_h)_{n=k}^N\|_{\ell^\infty(L^q)} \le C\sum_{j=0}^{k-1}\|P_h u^j - u_h^j\|_{L^q},$$

$$\|(A_h v_h)_{n=k}^N\|_{\ell^{1,\infty}(L^q)} \le C\sum_{j=0}^{k-1}\|A_h^{-1}(P_h u^j - u_h^j)\|_{L^q} \le C\sum_{j=0}^{k-1}\|P_h u^j - u_h^j\|_{L^q},$$

where $\ell^{1,\infty}$ is the weak-type ℓ^1 space defined in Section 2 (also see Grafakos, 2008, Section 1.1).

In Grafakos, (2008, Proposition 1.1.14), it is shown that if (X, μ) is a measure space and $f \in L^{1,\infty}(X, \mu) \cap L^\infty(X, \mu)$ then

$$\|f\|_{L^p(X,\mu)} \le C_p \|f\|_{L^{1,\infty}(X,\mu)}^{\frac{1}{p}} \|f\|_{L^\infty(X,\mu)}^{1-\frac{1}{p}} \quad \text{for all } 1 < p < \infty. \tag{3.31}$$

Here we choose the measure space (X, μ) to be $X = \{k, \dots, N\}$ with measure

$$\mu(\{k, \dots, n\}) = n\tau \quad \text{for } k \le n \le N.$$

Then any sequence $(f_n)_{n=k}^N$, with $f_n \in \mathbb{R}$, can be regarded as a function defined on X, and (3.31) can be written as

$$\|(f_n)_{n=k}^N\|_{\ell^p} \le C_p \|(f_n)_{n=k}^N\|_{\ell^{1,\infty}}^{\frac{1}{p}} \|(f_n)_{n=k}^N\|_{\ell^\infty}^{1-\frac{1}{p}} \quad \text{for all } 1 < p < \infty. \tag{3.32}$$

Choosing $f_n = \|A_h v_h\|_{L^q}$ in (3.32) immediately yields the following result:

$$\begin{aligned} \|(A_h v_h)_{n=k}^N\|_{\ell^p(L^q)} &\le C \|(A_h v_h)_{n=k}^N\|_{\ell^{1,\infty}(L^q)}^{\frac{1}{p}} \|(A_h v_h)_{n=k}^N\|_{\ell^\infty(L^q)}^{1-\frac{1}{p}} \\ &\le C \sum_{j=0}^{k-1} \|P_h u^j - u_h^j\|_{L^q}. \end{aligned} \tag{3.33}$$

Then, substituting (3.29) and (3.33) into (3.26), we obtain the desired estimate (3.22). □

In the proof of Lemma 3.6, the shift of the region from Σ_κ^τ to $-\lambda_0 + \Sigma_\kappa^\tau$ is to have the exponential factor $e^{-\lambda_0 t_{n+1}}$ in the estimates of (3.10). This exponential factor plays a role in establishing the $\ell^\infty(L^q)$ error estimate, which can be established using the following lemma.

LEMMA 3.9 The solution of (3.27) can be expressed as

$$w_h^n = \tau \sum_{j=k}^{n} E_h^{n-j}(R_h u^j - P_h u^j), \tag{3.34}$$

where the operators E_h^n satisfy the following estimates for $1 \le q \le \infty$ (for some positive constant λ_0):

$$\begin{aligned} &\|E_h^n \phi_h\|_{L^q} \le C e^{-\lambda_0 t_{n+1}} \|\phi_h\|_{L^q} && \forall\, \phi_h \in S_h, \\ &\|A_h E_h^n \phi_h\|_{L^q} \le C e^{-\lambda_0 t_{n+1}} t_{n+1}^{-1} \|\phi_h\|_{L^q} && \forall\, \phi_h \in S_h. \end{aligned} \tag{3.35}$$

Proof. Similarly, as in the proof of Lemma 3.6, we multiply (3.27) by ζ^n and sum the results for all $n \ge k$. This yields the following expression for the generating function $w(\zeta) = \sum_{n=k}^\infty w_h^n \zeta^n$:

$$w(\zeta) = (\tau^{-1}\delta(\zeta) - A_h)^{-1} \sum_{n=k}^{\infty} (R_h u^n - P_h u^n)\zeta^n. \tag{3.36}$$

Then, using Cauchy's integral formula, we can derive the following expression similarly to (3.15):

$$w_h^n = \sum_{m=k}^{n} \frac{\tau}{2\pi \mathrm{i}} \int_{\Gamma_{\kappa,n-m}^\tau} (\tau^{-1}\delta(e^{-\tau z}) - A_h)^{-1} (R_h u^m - P_h u^m) e^{t_{n-m} z}\, \mathrm{d}z \qquad \text{for } n \ge k, \tag{3.37}$$

where the integration contour $\Gamma_{\kappa,n}^\tau$ is defined in (3.16), and the summation is from $m = k$ to $m = n$ because for $m \ge n+1$ the integral above becomes zero (the solution w_h^n only depends on the right-hand side for $k \le m \le n$). Hence, the operator in (3.34) is given by

$$E_h^n = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_{\kappa,n}^\tau} (\tau^{-1}\delta(e^{-\tau z}) - A_h)^{-1} e^{t_n z}\, \mathrm{d}z \qquad \text{for } n \ge k. \tag{3.38}$$

Using the resolvent estimates in (3.14) and the similar estimation in (3.19)–(3.21) we have

$$\begin{aligned}
& e^{\lambda_0 t_{n+1}} \|E_h^n \phi_h\|_{L^q} \\
&\le C \sup_{z\in\Gamma_{\kappa,n}^\tau} e^{-\tau \mathrm{Re}(z)} \int_{\Gamma_{\kappa,n}^\tau} \|(\delta(e^{-\tau z}) - A_h)^{-1}\|_{L^q\to L^q} e^{t_{n+1}\mathrm{Re}(z+\lambda_0)} \|\phi_h\|_{L^q}\, \mathrm{d}z \\
&\le C\|\phi_h\|_{L^q} \bigg(\int_{\Gamma_{\kappa,n}^{\tau,1}} |z+\lambda_0|^{-1} e^{-Ct_{n+1}|z+\lambda_0|} |\mathrm{d}(z+\lambda_0)| + \int_{\Gamma_{\kappa,n}^{\tau,2}} |z+\lambda_0|^{-1} e^{t_{n+1}|z+\lambda_0|} |\mathrm{d}(z+\lambda_0)| \bigg) \\
&\le C\|\phi_h\|_{L^q} \bigg(\int_{t_{n+1}^{-1}}^\infty r^{-1} e^{-Ct_{n+1}r} \mathrm{d}r + \int_{-\kappa}^{\kappa} e^C\, \mathrm{d}\varphi \bigg) \\
&\le C\|\phi_h\|_{L^q} \qquad \text{for } n \ge k,
\end{aligned}$$

and

$$\begin{aligned}
e^{\lambda_0 t_{n+1}} \|A_h E_h^n \phi_h\|_{L^q} &\le C \sup_{z\in\Gamma_{\kappa,n}^\tau} e^{-\tau \mathrm{Re}(z)} \int_{\Gamma_{\kappa,n}^\tau} \|A_h(\delta(e^{-\tau z}) - A_h)^{-1}\|_{L^q\to L^q} e^{t_{n+1}\mathrm{Re}(z+\lambda_0)} \|\phi_h\|_{L^q}\, \mathrm{d}z \\
&\le C\|A_h\phi_h\|_{L^q} \bigg(\int_{\Gamma_{\kappa,n}^{\tau,1}} e^{-Ct_{n+1}|z+\lambda_0|} |\mathrm{d}(z+\lambda_0)| + \int_{\Gamma_{\kappa,n}^{\tau,2}} e^{t_{n+1}|z+\lambda_0|} |\mathrm{d}(z+\lambda_0)| \bigg) \\
&\le C\|\phi_h\|_{L^q} \bigg(\int_{t_{n+1}^{-1}}^\infty r^{-1} e^{-Ct_{n+1}r} \mathrm{d}r + \int_{-\kappa}^{\kappa} t_{n+1}^{-1} e^C\, \mathrm{d}\varphi \bigg) \\
&\le C t_{n+1}^{-1} \|\phi_h\|_{L^q} \quad \text{for } n \ge k.
\end{aligned}$$

This proves the desired estimates in (3.35). □

THEOREM 3.10 If $f_h^n = P_h f^n$ then the fully discrete solution given by (1.10) and the semidiscrete solution given by (1.9) satisfy the following error estimate for $1 \le q \le \infty$:

$$\max_{k\le n\le N} \|P_h u^n - u_h^n\|_{L^q} \le C\ell_N \max_{k\le n\le N} \|P_h u^n - R_h u^n\|_{L^q} + C \max_{0\le n\le k-1} \big\|P_h u^n - u_h^n\big\|_{L^q}, \tag{3.39}$$

where $\ell_N = \ln(1+N)$ and the constant C is independent of h, τ and N.

Proof. Again we use the error equation (3.25) and the decomposition (3.26). Then, substituting the second estimate of (3.10) into the expression (3.30), we obtain

$$\sup_{k\le n\le N} \|A_h v_h^n\|_{L^q} \le C \max_{0\le n\le k-1} \|P_h u^n - u_h^n\|_{L^q}. \tag{3.40}$$

Since the solution of (3.27) can be expressed as (3.34) with some operators E_h^n satisfying the estimates in (3.35), substituting the second estimate of (3.35) into (3.34) yields

$$\begin{aligned}\|A_h w_h^n\|_{L^q} &\le \tau \sum_{j=k}^{n} Ce^{-\lambda_0 t_{n+1-j}} t_{n+1-j}^{-1} \max_{k\le j\le n} \|R_h u^j - P_h u^j\|_{L^q} \\ &\le C\ln(1+n) \max_{k\le j\le n} \|R_h u^j - P_h u^j\|_{L^q}. \end{aligned} \tag{3.41}$$

This proves the desired result in Theorem 3.10. □

3.3 $\ell^p(W^{1,q})$ *estimate for fully discrete FEMs*

In this subsection we prove the following result using Theorems 3.4 and 3.8.

THEOREM 3.11 In a convex polygon or polyhedron, the solution of (1.10) with zero starting values $u_h^n = 0$, $n = 0, \ldots, k-1$, satisfies the estimate

$$\big\|(d_\tau u_h^n)_{n=k}^N\big\|_{\ell^p(W^{-1,q})} + \big\|(u_h^n)_{n=k}^N\big\|_{\ell^p(W^{1,q})} \le C_{p,q}\big\|(f_h^n)_{n=k}^N\big\|_{\ell^p(W^{-1,q})}, \tag{3.42}$$

where the constant $C_{p,q}$ is independent of h and τ.

Proof. Part 1. For $2 \le q < \infty$ the Ritz projection has the following approximation property in a convex polygon or polyhedron (see Remark 3.11):

$$\|P_h u^n - R_h u^n\|_{L^q} \le C_q h\|u^n\|_{W^{1,q}} \quad \forall\, 2 \le q < \infty. \tag{3.43}$$

Hence, choosing $f^n = f_h^n$ in (1.9), Theorem 3.8 implies that the solutions of (1.10) and (1.9) satisfy the error estimate

$$\begin{aligned}\big\|(P_h u^n - u_h^n)_{n=k}^N\big\|_{\ell^p(L^q)} &\le C_{p,q}\big\|(P_h u^n - R_h u^n)_{n=k}^N\big\|_{\ell^p(L^q)} \\ &\le C_{p,q} h\|(u^n)_{n=k}^N\|_{\ell^p(W^{1,q})} \\ &\le C_{p,q} h\|(f_h^n)_{n=k}^N\|_{\ell^p(W^{-1,q})},\end{aligned}$$

where the second-to-last inequality is due to (3.43) and the last inequality is due to the second result in Theorem 3.4. Using the inverse inequality we have

$$\big\|(P_h u^n - u_h^n)_{n=k}^N\big\|_{\ell^p(W^{1,q})} \le Ch^{-1}\big\|(P_h u^n - u_h^n)_{n=k}^N\big\|_{\ell^p(L^q)} \le C_{p,q}\|(f_h^n)_{n=k}^N\|_{\ell^p(W^{-1,q})}.$$

Then using the triangle inequality we obtain

$$\big\|(u_h^n)_{n=k}^N\big\|_{\ell^p(W^{1,q})} \le \big\|(P_h u^n)_{n=k}^N\big\|_{\ell^p(W^{1,q})} + \big\|(P_h u^n - u_h^n)_{n=k}^N\big\|_{\ell^p(W^{1,q})} \le C_{p,q}\|(f_h^n)_{n=k}^N\|_{\ell^p(W^{-1,q})}.$$

The last inequality uses the $W^{1,q}$ stability of P_h and the second result in Theorem 3.4.

Part 2. For $1 < q \le 2$ we express the solution of (1.10) as (see Lemma 3.9)

$$u_h^n = \tau \sum_{j=k}^{n} E_h^{n-j} f_h^j,$$

with E_h^n given by (3.38). By considering the gradient of the equality above, we have

$$\nabla u_h^n = (L_h \vec{g})^n := \tau \sum_{j=k}^{n} \nabla E_h^{n-j} P_h \nabla \cdot g^j, \quad n = k, \ldots, N, \tag{3.44}$$

where $g^j = a \nabla A^{-1} f_h^j$ and $\vec{g} = (g^j)_{j=k}^N$, with $a = (a_{ij})$ denoting the $(d \times d)$ matrix of the diffusion coefficients (thus $\nabla \cdot g^j = f_h^j$). It suffices to prove that the operator L_h is bounded on $\ell^p(L^q)$. To this end, we only need to prove the boundedness of its dual operator L_h' on $\ell^{p'}(L^{q'})$. We define a discrete space-time inner product

$$[\vec{g}, \vec{\eta}]_\tau = \tau \sum_{n=k}^{N} (g^n, \eta^n) \quad \text{for } \vec{g} = (g^j)_{j=k}^N \text{ and } \vec{\eta} = (\eta^j)_{j=k}^N.$$

Then, using the definition of L_h and integration by parts, we obtain $[L_h \vec{g}, \vec{\eta}]_\tau = [\vec{g}, L_h' \vec{\eta}]_\tau$ with

$$(L_h' \vec{\eta})^j = \tau \sum_{n=j}^{N} \nabla E_h^{n-j} P_h \nabla \cdot \vec{\eta}^n.$$

By a change of indices $j = N + k - j'$ and $n = N + k - n'$, we see that

$$(L_h' \vec{\eta})^{N+k-j'} = \tau \sum_{n'=k}^{j'} \nabla E_h^{j'-n'} P_h \nabla \cdot \eta^{N+k-n'}, \quad j' = k, \ldots, N. \tag{3.45}$$

Comparing (3.44) and (3.45) we see that $w_h^n := (L_h' \vec{\eta})^{N+k-n}$ is the 'gradient' of the solution to (1.10) with $f_h^n = P_h \nabla \cdot \vec{\eta}^{N+k-n}$. In Part 1 we showed that

$$\|(w_h^n)_{n=k}^N\|_{\ell^{p'}(L^{q'})} \le C \|(f_h^n)_{n=k}^N\|_{\ell^{p'}(W^{-1,q'})} = C \|(P_h \nabla \cdot \eta^n)_{n=k}^N\|_{\ell^{p'}(W^{-1,q'})},$$

where $2 \le q' < \infty$ for $1 < q \le 2$. Since P_h is symmetric and bounded on $W^{1,q}$, it follows that P_h is also bounded on $W^{-1,q'}$ (as the dual space of $W^{1,q}$). Hence, the inequality above reduces to

$$\|(w_h^n)_{n=k}^N\|_{\ell^{p'}(L^{q'})} \le C \|(\nabla \cdot \eta^n)_{n=k}^N\|_{\ell^{p'}(W^{-1,q'})} \le C \|(\eta^n)_{n=k}^N\|_{\ell^{p'}(L^{q'})}.$$

This proves the boundedness of L_h' on $\ell^{p'}(L^{q'})$. By the duality between $\ell^{p'}(L^{q'})$ and $\ell^p(L^q)$, the operator L_h must be bounded on $\ell^p(L^q)$. Using this boundedness of L_h, from (3.44) we derive that

$$\|(\nabla u_h^n)_{n=k}^N\|_{\ell^p(L^q)} \leq C\|(g^n)_{n=k}^N\|_{\ell^p(L^q)} = C\|(a\nabla A^{-1}f_h^n)_{n=k}^N\|_{\ell^p(L^q)}. \tag{3.46}$$

Using integration by parts and the symmetry of the operator A^{-1}, we have

$$\begin{aligned} |(a\nabla A^{-1}f_h^n, v)| = |(f_h^n, A^{-1}\nabla\cdot v)| &\leq \|f_h^n\|_{W^{-1,q}}\|A^{-1}\nabla\cdot(av)\|_{W^{1,q'}} \\ &\leq C\|f_h^n\|_{W^{-1,q}}\|\nabla\cdot(av)\|_{W^{-1,q'}} \\ &\leq C\|f_h^n\|_{W^{-1,q}}\|v\|_{L^{q'}}, \end{aligned}$$

where the second-to-last inequality is exactly (3.6). The inequality above implies (via duality)

$$\|a\nabla A^{-1}f_h^n\|_{L^q} \leq C\|f_h^n\|_{W^{-1,q}}.$$

Substituting this into (3.46) yields the desired result, i.e.,

$$\|(u_h^n)_{n=k}^N\|_{\ell^p(W^{1,q})} \leq \mathbb{C}\|(f_h^n)_{n=k}^N\|_{\ell^p(W^{-1,q})}. \tag{3.47}$$

Part 3. For $2 \leq q < \infty$ and $1 < q \leq 2$, we proved (3.47) in Parts 1 and 2, respectively. Now, testing (1.10) with v, we obtain

$$\begin{aligned} \left|\left(\frac{1}{\tau}\sum_{j=0}^k \delta_j u_h^{n-j}, v\right)\right| &= \left|\left(\frac{1}{\tau}\sum_{j=0}^k \delta_j u_h^{n-j}, P_h v\right)\right| \\ &= \left|-\sum_{i,j=1}^d (a_{ij}\partial_j u_h^n, \partial_i P_h v) + (f_h^n, P_h v)\right| \\ &\leq C\big(\|\nabla u_h^n\|_{\ell^p(L^q)} + \|f_h^n\|_{\ell^p(W^{-1,q})}\big)\|v\|_{\ell^{p'}(W^{1,q'})} \quad \forall\, v \in \ell^{p'}(W^{1,q'}). \end{aligned}$$

By the duality between $\ell^p(W^{-1,q})$ and $\ell^{p'}(W^{1,q'})$, the inequality above proves that

$$\left\|\left(\frac{1}{\tau}\sum_{j=0}^k \delta_j u_h^{n-j}\right)_{n=k}^N\right\|_{\ell^p(W^{-1,q})} \leq C\big(\|\nabla u_h^n\|_{\ell^p(L^q)} + \|f_h^n\|_{\ell^p(W^{-1,q})}\big).$$

Then using the equivalence relation (3.4) we obtain

$$\|(d_\tau u_h^n)_{n=k}^N\|_{\ell^p(W^{-1,q})} \leq C\big(\|\nabla u_h^n\|_{\ell^p(L^q)} + \|f_h^n\|_{\ell^p(W^{-1,q})}\big).$$

This completes the proof of Theorem 3.11. □

REMARK 3.12 In the proof of Theorem 3.11 we used (3.43). This can be proved by the following simple argument. For $2 \le q < \infty$ we define

$$\begin{cases} -Av = \operatorname{sign}\big(u - R_h u\big)\, \big|u - R_h u\big|^{q-1} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases}$$

and test the equation above with $u - R_h u$. Then via integration by parts, we obtain

$$\begin{aligned} \|u - R_h u\|_{L^q}^q = \sum_{i,j=1}^d (a_{ij}\partial_j(u - R_h u), \partial_i v) &= \sum_{i,j=1}^d (a_{ij}\partial_j(u - R_h u), \partial_i(v - P_h v)) \\ &\le \|u - R_h u\|_{W^{1,q}}\, Ch\|v\|_{W^{2,q'}}. \end{aligned}$$

Since $1 < q' \le 2$, using the $W^{2,q'}$ estimate of elliptic equations in a convex polygon or polyhedron (see Lemma 3.13), we have

$$\|v\|_{W^{2,q'}} \le C\|u - R_h u\|_{L^{(q-1)q'}}^{q-1} = C\|u - R_h u\|_{L^q}^{q-1}.$$

The last two estimates imply

$$\|u - R_h u\|_{L^q} \le Ch\|u - R_h u\|_{W^{1,q}} \quad \text{for } 2 \le q < \infty.$$

This implies (3.43). It remains to prove the following lemma on the $W^{2,q'}$ estimate.

LEMMA 3.13 In a convex polygon or polyhedron, there exists $q_0 > 2$ such that for any $1 < q < q_0$ and $g \in L^q$, the equation

$$\begin{cases} Av = g & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.48}$$

has a unique solution $v \in W^{2,q}$, which satisfies

$$\|v\|_{W^{2,q}} \le C\|g\|_{L^q}. \tag{3.49}$$

Proof. If the coefficients a_{ij} are constants or smooth, then Lemma 3.13 has been proved in Dauge, (1992, Corollaries 3.7 and 3.12) for some $q_0 > 2$. Using a perturbation argument, we prove that Lemma 3.13 still holds for the same q_0 when $a_{ij} \in W^{1,d+\beta}$.

When $a_{ij} \in W^{1,d+\beta} \hookrightarrow C^{1-\frac{d}{d+\beta}}$, the solution of (3.48) is given by

$$v(x) = \int_\Omega G_A(x, y) g(y)\, \mathrm{d}y,$$

where $G_A(x,y)$ is Green's function of the elliptic equation (3.48), satisfying the following estimate in a convex domain (cf. Grüter & Widman, 1982, Theorem 3.3):

$$|\nabla_x G_A(x,y)| \le C|x-y|^{1-d}.$$

Hence, we have

$$\begin{aligned}
\|\nabla v\|_{L^{p,\infty}} &= \left\| \int_\Omega \nabla_x G_A(x,y) g(y)\,\mathrm{d}y \right\| \\
&\le C \left\| \int_\Omega \frac{1}{|x-y|^{d-1}} g(y)\,\mathrm{d}y \right\| \\
&\le C \left\| \frac{1}{|x|^{d-1}} \right\|_{L^{\frac{d}{d-1},\infty}} \|g\|_{L^q} \\
&\le C\|g\|_{L^q} \qquad \text{with} \quad \frac{1}{p} = \frac{1}{q} - \frac{1}{d} < \frac{1}{q} - \frac{1}{d+\beta},
\end{aligned}$$

where the second-to-last inequality is Young's inequality of weak type and $L^{\frac{d}{d-1},\infty}$ is the weak-type $L^{\frac{d}{d-1}}$ space; see Grafakos, (2008, Theorem 1.2.13). The inequality above implies that

$$\nabla v \in L^{p,\infty} \hookrightarrow L^{q_*} \quad \text{for} \quad \frac{1}{q_*} := \frac{1}{q} - \frac{1}{d+\beta} \quad (\text{because } q_* < p). \tag{3.50}$$

For any point $x_0 \in \Omega$, we consider a smooth cut-off function ω_ε with the following properties:

$$\omega_\varepsilon(x) = \begin{cases} 1, & |x-x_0| < \varepsilon, \\ 0, & |x-x_0| \ge 2\varepsilon, \end{cases} \quad \text{and} \quad \|\nabla^m \omega_\varepsilon\|_{L^\infty} \le C\varepsilon^{-m} \text{ for } m \ge 0. \tag{3.51}$$

Since $a_{ij} \in W^{1,d+\beta} \hookrightarrow C^{1-\frac{d}{d+\beta}}$, it follows that

$$|a_{ij}(x) - a_{ij}(x_0)| \le C\varepsilon^{1-\frac{d}{d+\beta}} \quad \text{on the support of } \omega_\varepsilon. \tag{3.52}$$

Then multiplying (3.48) by ω_ε yields

$$\begin{cases} L(\omega_\varepsilon v) = \omega_\varepsilon g + \displaystyle\sum_{i,j=1}^d (a_{ij} v \partial_i\partial_j \omega_\varepsilon + a_{ij}\partial_j v \partial_i \omega_\varepsilon - \partial_i a_{ij} \omega_\varepsilon \partial_j v) & \text{in } \Omega, \\ \omega_\varepsilon v = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.53}$$

with

$$L = \sum_{i,j=1}^d a_{ij}(x_0)\partial_i\partial_j + \sum_{i,j=1}^d (a_{ij} - a_{ij}(x_0))\partial_i\partial_j =: L_0 + B.$$

In view of (3.52), we can choose a sufficiently small ε (smaller than some constant depending on $\|a_{ij}\|_{W^{1,d+\beta}}$) such that L becomes a small perturbation of L_0 as an operator from $W^{2,q}$ to L^q. Since the operator $L_0 : W^{2,q} \to L^q$ (with constant coefficients) is invertible when $1 < q < q_0$, it follows that $\omega_\varepsilon v \in W^{2,q}$ if the right-hand side of (3.53) is in L^q when $1 < q < q_0$. Indeed, the right-hand side is in L^q because

$$\|a_{ij} v \partial_{ij}\omega_\varepsilon + a_{ij}\partial_j v \partial_i \omega_\varepsilon - \partial_i a_{ij}\omega_\varepsilon \partial_j v\|_{L^q} \le C\varepsilon^{-2}\|a_{ij}\|_{W^{1,d+\beta}}\|v\|_{W^{1,q*}} \le C\varepsilon^{-2},$$

where the last inequality is due to (3.50) and the second-to-last inequality is due to the properties of the cut-off function ω_ε. This proves that for any $x_0 \in \Omega$ there is a neighborhood $B_\varepsilon(x_0) \cap \Omega$ (of constant radius ε) on which the solution v is in $W^{2,q}$. □

4. Proof of Theorem 3.1

In Section 3 we used the results of Theorem 3.1 to establish maximal L^p-regularity of multistep fully discrete FEMs. In this section we prove Theorem 3.1 by formulating it in a form that can be proved in a similar way to Li (2019, Proof of Corollary 2.1). Then we only sketch the proof by following the outline of Li (2019, Proof of Corollary 2.1) while highlighting the differences.

4.1 *Notation*

We use the same notation as Li (2019, Section 3), including the notation of function spaces, local approximation properties of the finite element spaces, delta function, regularized delta functions, Green's function, the regularized Green's function and dyadic decomposition of the domain. This notation will not be duplicated in the current paper. The only changes of notation in the current paper are

1. the dimension of space is denoted by d in this article (and N in Li, 2019);
2. the elliptic operator is $A = \sum_{i,j=1}^{d} \partial_i(a_{ij}\partial_j)$ in this article (and Laplacian Δ in Li, 2019).

We denote by

$$(u, v) = \int_\Omega u(x)v(x)\,\mathrm{d}x \quad \text{and} \quad \langle u, v\rangle = \int_\Omega u(x)\overline{v(x)}\,\mathrm{d}x \quad \forall\, u \in L^q,\ \ v \in L^{q'}, \tag{4.54}$$

the real and complex pairings between two complex-valued functions on Ω, respectively. This notation is consistent with (2.14) for real-valued functions.

4.2 *Complex Green's function and its regularized approximation*

We denote by $E(t) = e^{tA}$ the semigroup generated by A on L^q and denote by $E_h(t) = e^{tA_h}$ the semigroup generated by A_h on S_h. Since the semigroup $E(t)$ has a bounded analytic extension $E(z) : L^q \to L^q$ to $z \in \Sigma_\theta$ for all $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ (cf. Ouhabaz, 1995, Theorem 2.4), we can define

$$E^\theta(t) := E(te^{\mathrm{i}\theta}) \quad \text{for any angle } \theta \in (-\tfrac{\pi}{2}, \tfrac{\pi}{2}).$$

Then the function

$$u(t) = E^{\theta}(t)u_0 + \int_0^t E^{\theta}(t-s)f(s)\,\mathrm{d}s$$

is the solution of the complex-valued parabolic problem

$$\begin{cases} \partial_t u - e^{\mathrm{i}\theta} A u = f & \text{in } \mathbb{R}_+ \times \Omega, \\ u = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ u|_{t=0} = u_0 & \text{in } \Omega. \end{cases} \tag{4.55}$$

It is known that Green's function $G(t,x,y)$ defined by

$$\begin{cases} \partial_t G(t,x,y) - \displaystyle\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial G(t,x,y)}{\partial x_j}\right) = 0 & \text{in } \Omega \times (0,T], \\ G(t,x,y) = 0 & \text{on } \partial\Omega \times (0,T], \\ G(0,x,y) = \delta_y(x) & \text{in } \Omega \end{cases} \tag{4.56}$$

is symmetric with respect to x and y and has an analytic extension $G(z,x,y)$ for $z \in \Sigma_\varphi$, satisfying the following Gaussian estimate (cf. Davies, 1989, p. 103 or Ouhabaz, 1995, Proposition 2.3):

$$|G(z,x,y)| \le C_\varphi |z|^{-\frac{d}{2}} e^{-\frac{|x-y|^2}{C_\varphi |z|}} \quad \forall z \in \Sigma_\varphi,\ \forall x,y \in \Omega, \quad \forall \varphi \in (0,\pi/2), \tag{4.57}$$

where the constant C_φ depends only on φ. The Cauchy integral formula implies that

$$\partial_z^k G(z,x,y) = \frac{k!}{2\pi \mathrm{i}} \int_{|\zeta - z| = \frac{1}{2}|z|\sin(\varphi)} \frac{G(\zeta,x,y)}{(\zeta - z)^{k+1}}\,\mathrm{d}\zeta \qquad \forall z \in \Sigma_\varphi, \tag{4.58}$$

which further yields the following Gaussian pointwise estimate for the time derivatives of Green's function:

$$|\partial_z^k G(z,x,y)| \le \frac{C_{\varphi,k}}{z^{k+d/2}} e^{-\frac{|x-y|^2}{C_{\varphi,k}|z|}} \qquad \forall x,x_0 \in \Omega,\ \forall z \in \Sigma_\varphi,\ k = 0,1,2,\ldots. \tag{4.59}$$

Since $G(t,x,y) = G(t,y,x)$ (symmetry of Green's function) for $t > 0$, it follows that their analytic extensions are also equal, i.e.,

$$G(z,x,y) = G(z,y,x) \quad \forall x,y \in \Omega,\ \forall z \in \Sigma_\varphi. \tag{4.60}$$

It is straightforward to verify that $G^\theta(t,x,y) := G(te^{\mathrm{i}\theta},x,y)$ is the solution of the complex-valued parabolic equation

$$\begin{cases} \partial_t G^\theta(\cdot,\cdot,y) - e^{\mathrm{i}\theta} A G^\theta(\cdot,\cdot,y) = 0 & \text{in } \mathbb{R}_+ \times \Omega, \\ G^\theta(\cdot,\cdot,y) = 0 & \text{on } \mathbb{R}_+ \times \partial\Omega, \\ G^\theta(0,\cdot,y) = \delta_y & \text{in } \Omega. \end{cases} \tag{4.61}$$

Estimate (4.59) implies that

$$|\partial_t^k G^\theta(t,x,y)| \le \frac{C_{\theta,k}}{t^{k+d/2}} e^{-\frac{|x-y|^2}{C_{\theta,k} t}} \qquad \forall\, x,y \in \Omega, \ \ \forall\, t > 0, \ \ k = 0,1,2,\ldots, \tag{4.62}$$

where the constant $C_{\theta,k}$ is bounded when θ is bounded away from $\pm\frac{\pi}{2}$.

We define the regularized Green's function $\Gamma(t,x,y)$ by

$$\begin{cases} \partial_t \Gamma(t,x,y) - \displaystyle\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial \Gamma(t,x,y)}{\partial x_j}\right) = 0 & \text{in } \Omega \times (0,T], \\ \Gamma(t,x,y) = 0 & \text{on } \partial\Omega \times (0,T], \\ \Gamma(0,x,y) = \widetilde{\delta}_y(x) & \text{in } \Omega, \end{cases} \tag{4.63}$$

where $\widetilde{\delta}_y$ denotes the regularized delta function defined in Li (2019, Section 3.3). Similarly to the complex Green's function, we define $\Gamma^\theta(t,x,y) = \Gamma(te^{\mathrm{i}\theta},x,y)$, which is the solution of

$$\begin{cases} \partial_t \Gamma^\theta(t,x,y) - e^{\mathrm{i}\theta} \displaystyle\sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial \Gamma^\theta(t,x,y)}{\partial x_j}\right) = 0 & \text{in } \Omega \times (0,T], \\ \Gamma^\theta(t,x,y) = 0 & \text{on } \partial\Omega \times (0,T], \\ \Gamma^\theta(0,x,y) = \widetilde{\delta}_y(x) & \text{in } \Omega, \end{cases} \tag{4.64}$$

and can be represented by

$$\Gamma^\theta(t,x,y) = \int_\Omega G^\theta(t,x',x)\widetilde{\delta}_y(x')\,\mathrm{d}x' = \int_\Omega G^\theta(t,x,x')\widetilde{\delta}_y(x')\,\mathrm{d}x'. \tag{4.65}$$

From (4.65) and (4.59) we can derive the pointwise estimate

$$|\partial_t^k \Gamma^\theta(t,x,y)| \le \frac{C_{\theta,k}}{t^{k+d/2}} e^{-\frac{|x-y|^2}{C_{\theta,k} t}}, \quad k = 0,1,2,\ldots \tag{4.66}$$

for $x,y \in \Omega$ and $t > 0$ such that $\max(|x-y|,\sqrt{t}) \ge 2h$.

Similarly, for the discrete Green's function $\Gamma_h(t,\cdot,y) \in S_h$ defined by

$$\begin{cases} (\partial_t \Gamma_h(t,\cdot,y), v_h) + \displaystyle\sum_{i,j=1}^{d} (a_{ij}\partial_j \Gamma_h(t,\cdot,y), \partial_i v_h) = 0 \quad \forall\, v_h \in S_h,\ \forall\, t > 0, \\ \Gamma_h(0,\cdot,y) = \delta_{h,y}, \end{cases} \tag{4.67}$$

we define the complex-valued finite element function $\Gamma_h^\theta(t,x,y) = \Gamma_h(te^{\mathrm{i}\theta},x,y)$, which is the solution of

$$\begin{cases} (\partial_t \Gamma_h^\theta(t,\cdot,y), v_h) + e^{\mathrm{i}\theta} \displaystyle\sum_{i,j=1}^{d} (a_{ij}\partial_j \Gamma_h^\theta(t,\cdot,y), \partial_i v_h) = 0 \quad \forall\, v_h \in S_h,\ \forall\, t > 0, \\ \Gamma_h^\theta(0,\cdot,y) = \delta_{h,y}. \end{cases} \tag{4.68}$$

4.3 *A key lemma*

We fix an arbitrary angle $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and consider the finite element solution of the semidiscrete problem

$$\begin{cases} (\partial_t u_h(t), v_h) + e^{\mathrm{i}\theta} \displaystyle\sum_{i,j=1}^{d} (a_{ij}\partial_j u_h(t), \partial_i v_h) = (f_h(t), v_h) \quad \forall\, v_h \in S_h,\ \forall\, t > 0, \\ u_h(0) = u_{0,h}, \end{cases} \tag{4.69}$$

which is the finite element approximation of (4.55). The solution of (4.69) can be written as

$$u_h(t) = E_h^\theta(t)u_{0,h} + \int_0^t E_h^\theta(t-s)f_h(s)\,\mathrm{d}s,$$

where $E_h^\theta(t)$ is the semigroup generated by $e^{\mathrm{i}\theta}A_h$. Using the Green's functions defined in (4.61) and (4.68), the solutions of (4.55) and (4.69) can be represented by

$$u(t,y) = \int_\Omega G^\theta(t,x,y)u_0(x)\,\mathrm{d}x + \int_0^t \int_\Omega G^\theta(t-s,x,y)f(s,x)\,\mathrm{d}x\,\mathrm{d}s, \tag{4.70}$$

$$u_h(t,y) = \int_\Omega \Gamma_h^\theta(t,x,y)u_{0,h}(x)\,\mathrm{d}x + \int_0^t \int_\Omega \Gamma_h^\theta(t-s,x,y)f(s,x)\,\mathrm{d}x\,\mathrm{d}s \tag{4.71}$$

and

$$(E^\theta(t)v)(y) = \int_\Omega G^\theta(t,x,y)v(x)\,\mathrm{d}x, \qquad (E_h^\theta(t)v_h)(y) = \int_\Omega \Gamma_h^\theta(t,x,y)v_h(x)\,\mathrm{d}x. \tag{4.72}$$

Clearly, the operator $E_h^\theta(t)P_h : L^q \to S_h$ is an extension of $E_h^\theta(t) : S_h \to S_h$ to the domain L^q. Since $\Gamma_h^\theta(t,x,y)$ is a finite element function in x, it follows that

$$(E_h^\theta(t)P_h v)(y) = \int_\Omega \Gamma_h^\theta(t,x,y)P_h v(x)\,\mathrm{d}x = \int_\Omega \Gamma_h^\theta(t,x,y)v(x)\,\mathrm{d}x \quad \forall\, v \in L^q. \tag{4.73}$$

We shall prove the following result, which is the key to proving Theorem 3.1.

LEMMA 4.1 For any $\theta \in (0, \frac{\pi}{2})$, the following estimates hold:

$$\sup_{t>0} (\|E_h^\theta(t)v_h\|_{L^q} + t\|\partial_t E_h^\theta(t)v_h\|_{L^q}) \le C_\theta \|v_h\|_{L^q} \qquad \forall\, v_h \in S_h, \qquad \forall\, 1 \le q \le \infty, \tag{4.74}$$

$$\left\| \sup_{t>0} |E_h^\theta(t)P_h|\,|v| \right\|_{L^q} \le C_{\theta,q}\|v\|_{L^q} \qquad \forall\, v \in L^q, \qquad \forall\, 1 < q \le \infty, \tag{4.75}$$

where

$$(|E_h^\theta(t)|v)(y) := \int_\Omega |\Gamma_h^\theta(t,x,y)|v(x)\,\mathrm{d}x \quad \forall\, v \in L^q, \tag{4.76}$$

and the constants C and C_q are independent of h and T (bounded when θ is away from $\pm\frac{\pi}{2}$).

4.4 *Proof of Theorem* 3.1 *based on Lemma* 4.1

Proof of Theorem 3.1*(1)*. By the theory of analytic semigroups (Arendt *et al.*, 2011, Theorem 3.7.19), inequality (4.74) implies the existence of an angle $\varphi \in (0, \pi/2)$, such that the semigroup $\{E_h^\theta(t)\}_{t>0}$ extends to being a bounded analytic semigroup $\{E_h^\theta(z)\}_{z\in\Sigma_\varphi}$ in the sector Σ_φ, satisfying

$$\sup_{z\in\Sigma_\varphi} (\|E_h^\theta(z)v_h\|_{L^q} + |z|\|\partial_z E_h^\theta(z)v_h\|_{L^q}) \le C_\theta^* \|v_h\|_{L^q} \qquad \forall\, v_h \in S_h, \qquad \forall\, 1 \le q \le \infty. \tag{4.77}$$

From the proof of Arendt *et al.* (2011, Theorem 3.7.19), we see that both φ and C_θ^* in (4.77) depend only on the constant C_θ in (4.74) (thus they are independent of h and q). Then (4.77) and Arendt *et al.* (2011, Theorem 3.7.11) imply

$$\left\{ z(z - e^{\mathrm{i}\theta}A_h)^{-1} : z \in \Sigma_{\frac{\pi}{2}} \right\} \text{ is bounded on } L^q \text{ for } 1 \le q \le \infty, \tag{4.78}$$

with a bound depending only C_θ^* (thus independent of h). Rewriting the operator $z(z - e^{\mathrm{i}\theta}A_h)^{-1}$ as $e^{-\mathrm{i}\theta}z(e^{-\mathrm{i}\theta}z - A_h)^{-1}$ and replacing θ by $\pm\theta$ in (4.83), we obtain the following result (after adding a bounded operator P_h):

$$\left\{ z(z - A_h)^{-1}P_h : z \in \mathbb{C},\ -\theta - \tfrac{\pi}{2} \le \arg(z) \le -\theta + \tfrac{\pi}{2} \right\}$$
$$\text{and} \quad \left\{ z(z - A_h)^{-1}P_h : z \in \mathbb{C},\ \theta - \tfrac{\pi}{2} \le \arg(z) \le \theta + \tfrac{\pi}{2} \right\} \quad \text{are both bounded on } L^q,$$

for $1 \le q \le \infty$, with a bound independent of h. Since the union of the two bounded sets above is also bounded, we obtain the first result of Theorem 3.1.

Proof of Theorem 3.1(2). Estimate (4.75) immediately implies the maximal ergodic estimate

$$\left\| \sup_{t>0} \frac{1}{t} \int_0^t |E_h^\theta(s) P_h| |v| \mathrm{d}s \right\|_{L^q} \le C_{\theta,q} \|v\|_{L^q}, \quad \forall\, 1 < q \le \infty. \tag{4.79}$$

For $q \in (1, 2]$, according to Weis (2001a, Lemma 4.c), (4.79) implies the R-boundedness of the semigroup $\{E_h^\theta(z) P_h\}_{z \in \Sigma_\sigma}$ on L^q, with an angle $\sigma = \varphi q/4$ (thus independent of h); from Weis (2001a, Proofs of Lemma 4.c and Proposition 4.b), we see that the R-bound depends only on the constants C_θ^* and $C_{\theta,q}$ in (4.77)–(4.79) (thus it is independent of h).

For $q \in [2, \infty)$ we use the fact that the dual operator $E_h^\theta(z) P_h$ is itself, i.e.,

$$\big(E_h^\theta(z) P_h u_0, w_0\big) = \big(u_0, E_h^\theta(z) P_h w_0\big) \quad \forall\, u_0 \in L^q, \;\; w_0 \in L^{q'}. \tag{4.80}$$

This can be seen in the following way. The function $u_h(t) = E_h^\theta(z) P_h u_0$ is a solution of (4.69) with $u_{0,h} = P_h u_0$ and $f_h = 0$, and the function $w_h(T - t) = (E_h^\theta(z) P_h w_0)(T - t)$ is a solution of the backward equation

$$\begin{cases} (\partial_t w_h(t), v_h) - e^{\mathrm{i}\theta} \displaystyle\sum_{i,j=1}^{d} (a_{ij} \partial_j w_h(t), \partial_i v_h) = 0 \quad \forall\, v_h \in S_h, \;\forall\, t \in [0, T), \\ w_h(T) = P_h w_0. \end{cases} \tag{4.81}$$

Hence, substituting $f_h = 0$ and $v_h = w_h$ into (4.69) and using integration by parts in time, we obtain (4.80). Using the symmetry (4.80), we see that for $q \in [2, \infty)$ the dual operators $E_h^\theta(z)' = E_h^\theta(z)$, $z \in \Sigma_\sigma$ are R-bounded on $L^{q'}$ with angle $\sigma = \varphi q'/4$. This implies that the operators $E_h^\theta(z)$, $z \in \Sigma_\sigma$ are R-bounded on L^q (cf. Weis, 2001a, Remark 4.b. (ii) with $p = 2$).

Therefore, for $1 < q < \infty$, the following result holds:

$$E_h^\theta(t) \text{ has an } R\text{-bounded analytic extension } E_h^\theta(z) \text{ for } z \in \Sigma_\sigma, \tag{4.82}$$

where $\sigma = \varphi_\theta \min(q', q)/4$ and the R-bound is independent of h. This proves the second result of Theorem 3.1.

Proof of Theorem 3.1(3). Since

$$z(z - e^{\mathrm{i}\theta} A_h)^{-1} P_h = z \int_0^\infty e^{-zt} E_h^\theta(t) P_h \,\mathrm{d}t,$$

it follows that (cf. Weis, 2001b, Theorem 2.10)

$$\big\{ z(z - e^{\mathrm{i}\theta} A_h)^{-1} P_h \,:\, z \in \mathbb{C}, \; |\arg(z)| \le \tfrac{\pi}{2} \big\} \text{ is } R\text{-bounded on } L^q, \tag{4.83}$$

with an R-bound depending only on σ and the R-bound of $E_h^\theta(z)$ (thus independent of h). Rewriting $z(z - e^{\mathrm{i}\theta} A_h)^{-1} P_h$ as $e^{-\mathrm{i}\theta} z(e^{-\mathrm{i}\theta} z - A_h)^{-1} P_h$ and replacing θ by $\pm\theta$ in (4.83), we obtain the following result:

$$\left\{z(z-A_h)^{-1}P_h \,:\, z\in\mathbb{C},\ -\tfrac{\pi}{2}-\theta \le \arg(z) \le \tfrac{\pi}{2}-\theta\right\}$$

$$\text{and}\quad \left\{z(z-A_h)^{-1}P_h \,:\, z\in\mathbb{C},\ -\theta+\tfrac{\pi}{2} \le \arg(z) \le \theta+\tfrac{\pi}{2}\right\}\quad \text{are both } R\text{-bounded,}$$

with an R-bound independent of h. Since the union of the two R-bounded sets above is also R-bounded, we obtain the third result of Theorem 3.1.

It remains to prove Lemma 4.1.

4.5 *Proof of Lemma* 4.1

Lemma 4.1 can be proved in the same way as Li (2019, Theorem 2.1) using the following lemma. Hence, the proof of Lemma 4.1 is omitted.

LEMMA 4.2 The functions $\Gamma_h^\theta(t,x,x_0)$, $\Gamma^\theta(t,x,x_0)$ and $F^\theta(t,x,x_0) := \Gamma_h^\theta(t,x,x_0) - \Gamma^\theta(t,x,x_0)$ satisfy

$$\sup_{x_0\in\Omega}\sup_{t\in(0,\infty)} (\|\Gamma_h^\theta(t,\cdot,x_0)\|_{L^1(\Omega)} + t\|\partial_t\Gamma_h^\theta(t,\cdot,x_0)\|_{L^1(\Omega)}) \le C, \tag{4.84}$$

$$\sup_{x_0\in\Omega}\sup_{t\in(0,\infty)} (\|\Gamma^\theta(t,\cdot,x_0)\|_{L^1(\Omega)} + t\|\partial_t\Gamma^\theta(t,\cdot,x_0)\|_{L^1(\Omega)}) \le C, \tag{4.85}$$

$$\sup_{x_0\in\Omega}\left(\|\partial_t F^\theta(\cdot,\cdot,x_0)\|_{L^1((0,\infty)\times\Omega)} + \|t\partial_{tt}F^\theta(\cdot,\cdot,x_0)\|_{L^1((0,\infty)\times\Omega)}\right) \le C, \tag{4.86}$$

$$\sup_{x_0\in\Omega}\|\partial_t\Gamma_h^\theta(t,\cdot,x_0)\|_{L^1} \le Ce^{-\lambda_0 t} \qquad \forall\, t\ge 1, \tag{4.87}$$

where the constants C and λ_0 are independent of h.

Lemma 4.2 can be proved in the same way as Li (2019, Lemma 4.4) using the following two lemmas. Hence, the proof of Lemma 4.2 is omitted.

LEMMA 4.3 (Local energy error estimate for parabolic equations). If $\phi \in L^2(0,T;H_0^1)\cap H^1(0,T;L^2)$ and $\phi_h \in H^1(0,T;S_h)$ satisfy

$$(\partial_t(\phi-\phi_h),\chi_h) + e^{i\theta}\sum_{i,j=1}^d (a_{ij}\partial_j(\phi-\phi_h),\partial_i\chi_h) = 0 \quad \forall\, \chi_h\in S_h,\ \text{a.e. } t>0, \tag{4.88}$$

with $\phi(0)=0$ in Ω_j''. Then we have

$$\begin{aligned}
&|||\partial_t(\phi-\phi_h)|||_{Q_j} + d_j^{-1}|||\phi-\phi_h|||_{1,Q_j} \\
&\quad \le C\epsilon^{-3}\left(I_j(\phi_h(0)) + X_j(I_h\phi-\phi) + d_j^{-2}|||\phi-\phi_h|||_{Q_j'}\right) \\
&\qquad + (Ch^{1/2}d_j^{-1/2} + C\epsilon^{-1}hd_j^{-1} + \epsilon)\left(|||\partial_t(\phi-\phi_h)|||_{Q_j'} + d_j^{-1}|||\phi-\phi_h|||_{1,Q_j'}\right),
\end{aligned} \tag{4.89}$$

where

$$
\begin{aligned}
I_j(\phi_h(0)) &= \|\phi_h(0)\|_{1,\Omega_j'} + d_j^{-1}\|\phi_h(0)\|_{\Omega_j'}, \\
X_j(I_h\phi - \phi) &= d_j \big|\big|\big|\partial_t(I_h\phi - \phi)\big|\big|\big|_{1,Q_j'} + \big|\big|\big|\partial_t(I_h\phi - \phi)\big|\big|\big|_{Q_j'} \\
&\quad + d_j^{-1}\big|\big|\big|I_h\phi - \phi\big|\big|\big|_{1,Q_j'} + d_j^{-2}\big|\big|\big|I_h\phi - \phi\big|\big|\big|_{Q_j'},
\end{aligned}
$$

where $\epsilon \in (0,1)$ is an arbitrary positive constant and the positive constant C is independent of h, j and C_*; the norms $|||\cdot|||_{k,Q_j'}$ and $|||\cdot|||_{k,\Omega_j'}$ are defined in Li (2019, (3.23)).

Lemma 4.4 (Local estimates of the complex Green's function). There exists $\alpha \in (\frac{1}{2}, 1]$ and $C > 0$, independent of h and x_0, such that the complex Green's function G^θ defined in (4.61) and the complex regularized Green's function Γ^θ defined in (4.64) satisfy the following estimates:

$$
\begin{aligned}
& d_j^{\frac{d}{2}-4-\alpha}\|\Gamma^\theta(\cdot,\cdot,x_0)\|_{L^\infty(Q_j(x_0))} + d_j^{-4-\alpha}|||\nabla\Gamma^\theta(\cdot,\cdot,x_0)|||_{L^2(Q_j(x_0))} \\
&\quad + d_j^{-4}|||\Gamma^\theta(\cdot,\cdot,x_0)|||_{L^2H^{1+\alpha}(Q_j(x_0))} + d_j^{-2}|||\partial_t\Gamma^\theta(\cdot,\cdot,x_0)|||_{L^2H^{1+\alpha}(Q_j(x_0))} \\
&\quad + |||\partial_{tt}\Gamma^\theta(\cdot,\cdot,x_0)|||_{L^2H^{1+\alpha}(Q_j(x_0))} \le C d_j^{-\frac{d}{2}-4-\alpha},
\end{aligned} \tag{4.90}
$$

$$
\|G^\theta(\cdot,\cdot,x_0)\|_{L^\infty H^{1+\alpha}(\cup_{k\le j}Q_k(x_0))} + d_j^2\|\partial_t G^\theta(\cdot,\cdot,x_0)\|_{L^\infty H^{1+\alpha}(\cup_{k\le j}Q_k(x_0))} \le C d_j^{-\frac{d}{2}-1-\alpha}. \tag{4.91}
$$

The only difference between Lemma 4.3 and Li (2019, Lemma 5.1) is the presence of $e^{i\theta}$ in (4.88). This does not affect the proof of Lemma 4.3. Hence, the proof of Lemma 4.3 is also omitted. Lemma 4.4 can be proved in the same way as Li (2019, Lemma 4.1) based on the following regularity result.

Lemma 4.5 There exists a positive constant $\alpha \in (\frac{1}{2}, 1]$ such that the solution of the elliptic equation

$$
\begin{cases}
Av = g & \text{in } \Omega, \\
\varphi = 0 & \text{on } \partial\Omega,
\end{cases} \tag{4.92}
$$

satisfies

$$
\|v\|_{H^{1+\alpha}} \le C\|g\|_{H^{-1+\alpha}}.
$$

Proof. When $a_{ij} = a_{ji}$ = constants, Lemma 4.5 holds as explained in Li (2019, Lemma 4.1). When $a_{ij} = a_{ji} \in W^{1,d+\beta}$, it can be proved using a perturbation argument as shown below.

In fact, for any $x_0 \in \overline{\Omega}$, we can introduce a smooth cut-off function ω_ε satisfying (3.51)–(3.52) and reformulate equation (4.92) as

$$
\begin{cases}
\displaystyle\sum_{i,j=1}^{d} \partial_j\big[a_{ij}(x_0)\partial_i(\omega_\varepsilon v)\big] = g_\varepsilon & \text{in } \Omega, \\
\omega_\varepsilon v = 0 & \text{on } \partial\Omega,
\end{cases} \tag{4.93}
$$

with

$$g_\varepsilon = \omega_\varepsilon g + \sum_{i,j=1}^{d} \big[2\partial_j(a_{ij}v)\partial_i\omega_\varepsilon - \partial_j a_{ij} v \partial_i \omega_\varepsilon\big] + \sum_{i,j=1}^{d} \partial_j\big[(a_{ij}(x_0) - a_{ij})\omega_{2\varepsilon}\partial_i(\omega_\varepsilon v)\big].$$

Since the left-hand side of (4.93) has constant coefficients, we can apply the result of Lemma 4.5 to obtain

$$\begin{aligned}
&\|\omega_\varepsilon v\|_{H^{1+\alpha}} \\
&\quad \le C\|g_\varepsilon\|_{H^{-1+\alpha}} \\
&\quad \le C\|g\|_{H^{-1+\alpha}} + \sum_{i,j=1}^{d}\big(\|a_{ij}v\|_{H^\alpha} + \|\partial_j a_{ij} v\|_{L^2}\big) + C\sum_{i,j=1}^{d}\|(a_{ij}(x_0) - a_{ij})\omega_{2\varepsilon}\partial_i(\omega_\varepsilon v)\|_{H^\alpha} \\
&\quad \le C\|g\|_{H^{-1+\alpha}} + C\|v\|_{H^1} + C\sum_{i,j=1}^{d}\|(a_{ij}(x_0) - a_{ij})\omega_{2\varepsilon}\|_{W^{1,d+\beta/2}}\|\omega_\varepsilon v\|_{H^{1+\alpha}} \ \ \text{(see Lemma 4.6)}.
\end{aligned}$$

Using properties (3.51)–(3.52) and Hölder's inequality we have

$$\begin{aligned}
&\|(a_{ij}(x_0) - a_{ij})\omega_{2\varepsilon}\|_{W^{1,d+\beta/2}} \\
&\quad \le \|(a_{ij}(x_0) - a_{ij})\omega_{2\varepsilon}\|_{L^{d+\beta/2}} + \|\nabla a_{ij}\omega_{2\varepsilon}\|_{L^{d+\beta/2}} + \|(a_{ij}(x_0) - a_{ij})\nabla\omega_{2\varepsilon}\|_{L^{d+\beta/2}} \\
&\quad \le C\varepsilon^{1-\frac{d}{d+\beta}}\varepsilon^{\frac{d}{d+\beta/2}} + \|\nabla a_{ij}\|_{L^{d+\beta}}\|\omega_{2\varepsilon}\|_{L^s} + C\varepsilon^{1-\frac{d}{d+\beta}}\varepsilon^{-1}\varepsilon^{\frac{d}{d+\beta/2}} \quad \text{with } \tfrac{1}{s} = \tfrac{1}{d+\beta/2} - \tfrac{1}{d+\beta} \\
&\quad \le C\varepsilon^{\frac{1}{d+\beta/2} - \frac{1}{d+\beta}}.
\end{aligned}$$

Combining the last two estimates we obtain

$$\|\omega_\varepsilon v\|_{H^{1+\alpha}} \le C\|g\|_{H^{-1+\alpha}} + C\|v\|_{H^1} + C\varepsilon^{\frac{d}{d+\beta/2} - \frac{d}{d+\beta}}\|\omega_\varepsilon v\|_{H^{1+\alpha}}. \tag{4.94}$$

Choosing a sufficiently small ε, the last term above can be absorbed by the left-hand side. Hence, we obtain

$$\|\omega_\varepsilon v\|_{H^{1+\alpha}} \le C\|g\|_{H^{-1+\alpha}} + C\|v\|_{H^1}.$$

Since $\|v\|_{H^1} \le C\|g\|_{H^{-1}} \le \|g\|_{H^{-1+\alpha}}$ (standard H^1 estimate of elliptic equations), it follows that

$$\|\omega_\varepsilon v\|_{H^{1+\alpha}} \le C\|g\|_{H^{-1+\alpha}}.$$

Since this estimate holds in a neighborhood $B_\varepsilon(x_0) \cap \Omega$ of every point $x_0 \in \Omega$ with a constant radius ε (depending on $\|a_{ij}\|_{W^{1,d+\beta}}$), it follows that

$$\|v\|_{H^{1+\alpha}} \le C\|g\|_{H^{-1+\alpha}}.$$

This proves Lemma 4.5. In this proof we have used the following lemma. □

LEMMA 4.6 If $w \in W^{1,d+\beta/2}$ then the following inequality holds:

$$\|wv\|_{H^\alpha} \le C\|w\|_{W^{1,d+\beta/2}}\|v\|_{H^\alpha} \quad \forall\, v \in H^\alpha \text{ and } \alpha \in [0,1]. \tag{4.95}$$

Proof. Note that

$$\|wv\|_{L^2} \le C\|w\|_{L^\infty}\|v\|_{L^2} \le C\|w\|_{W^{1,d+\beta/2}}\|v\|_{L^2} \quad \text{because } W^{1,d+\beta/2} \hookrightarrow L^\infty,$$

and

$$\begin{aligned}\|\nabla(wv)\|_{L^2} &\le C(\|w\nabla v\|_{L^2} + \|\nabla w\, v\|_{L^2})\\ &\le C(\|w\|_{L^\infty}\|\nabla v\|_{L^2} + \|\nabla w\|_{L^{d+\beta/2}}\|v\|_{L^s}) \quad \text{with } s = \frac{1}{\frac{1}{2} - \frac{1}{d+\beta/2}} < \frac{2d}{d-2}\\ &\le C\|w\|_{W^{1,d+\beta/2}}\|v\|_{H^1},\end{aligned}$$

where the last inequality is due to the Sobolev embedding $H^1 \hookrightarrow L^s$ for $1 \le s < 2d/(d-2)$. Therefore, (4.95) holds for both $\alpha = 0$ and $\alpha = 1$. Since multiplying v by w is a linear operator on v, bounded on both L^2 and H^1, by the complex interpolation method this operator must also be bounded on H^α for $\alpha \in [0,1]$. This proves Lemma 4.6. □

The proof of Theorem 3.1 is complete.

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