

Error analysis of linearized semi-implicit Galerkin finite element methods for nonlinear parabolic equations

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Abstract

This paper is concerned with the time-step condition of commonly-used linearized semi-implicit schemes for nonlinear parabolic PDEs with Galerkin finite element approximations. In particular, we study the time-dependent nonlinear Joule heating equations. We present optimal error estimates of the semi-implicit Euler scheme in both the L^2 norm and the H^1 norm without any time-step restriction. Theoretical analysis is based on a new splitting of the error and precise analysis of a corresponding time-discrete system. The method used in this paper can be applied to more general nonlinear parabolic systems and many other linearized (semi)-implicit time discretizations for which previous works often require certain restriction on the time-step size τ .

Keywords: Nonlinear parabolic system, unconditionally optimal error estimate, linearized semi-implicit scheme, Galerkin method.

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1 Introduction

In the last several decades, numerous effort has been devoted to the development of efficient numerical schemes for nonlinear parabolic PDEs arising from a variety of physical applications. A key issue to those schemes is the time-step condition. Usually, fully implicit schemes are unconditionally stable. However, at each time step, one has to solve a system of nonlinear equations. An explicit scheme is much easy in computation. But it suffers the severely restricted time-step size for convergence. A popular and widely-used approach is a linearized (semi)-implicit scheme, such as linearized semi-implicit Euler scheme. At each time step, the scheme only requires the solution of a linear system. To study the error estimate of linearized (semi)-implicit schemes, the boundedness of numerical solution (or error function) in L^∞ norm or a stronger norm is often required. If a priori estimate for numerical solution in such a norm cannot be obtained, one may employ the induction method with inverse inequality to bound the numerical solution, such as

$$\|R_h u^n - U_h^n\|_{L^\infty} \leq Ch^{-d/2} \|R_h u^n - U_h^n\|_{L^2} \leq Ch^{-d/2} (\tau^p + h^{r+1}), \quad (1.1)$$

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where u^n and U_h^n are the exact solution and numerical solution, respectively, R_h is some projection operator and d is the dimension. The above inequality, however, results in a time-step restriction, particularly for problems in three dimensional space. Such a technique has been widely used in error analysis for many different nonlinear parabolic PDEs, *e.g.*, see [1, 16, 18, 20, 21] for Navier-Stokes equations, [2, 11, 36] for nonlinear Joule heating problems, [15, 25, 27] for porous media flows, [7, 12, 13, 28] for viscoelastic fluid flow, [22, 35] for KdV equations and [10, 29] for some other equations. In all these works, error estimates were established under certain time-step restrictions. We believe that these time-step restrictions may not be necessary in most cases. In this paper, we only focus our attention on a time-dependent and nonlinear Joule heating system by a linearized semi-implicit scheme. However, our approach is applicable for more general nonlinear parabolic PDEs and many other time discretizations to obtain optimal error estimates unconditionally.

The time-dependent nonlinear Joule heating system is defined by

$$\frac{\partial u}{\partial t} - \Delta u = \sigma(u)|\nabla\phi|^2, \quad (1.2)$$

$$-\nabla \cdot (\sigma(u)\nabla\phi) = 0, \quad (1.3)$$

for $x \in \Omega$ and $t \in [0, T]$, where Ω is a bounded smooth domain in \mathbb{R}^d , $d = 2, 3$. The initial and boundary conditions are given by

$$\begin{aligned} u(x, t) = 0, \quad \phi(x, t) = g(x, t) & \quad \text{for } x \in \partial\Omega, \quad t \in [0, T], \\ u(x, 0) = u_0(x) & \quad \text{for } x \in \Omega. \end{aligned} \quad (1.4)$$

The nonlinear system above describes the model of electric heating of a conducting body, where u is the temperature, ϕ is the electric potential, and σ is the temperature-dependent electric conductivity. Following the previous works [11, 36], we assume that $\sigma \in W^{1,\infty}(\mathbb{R})$ and

$$\kappa \leq \sigma(s) \leq K, \quad (1.5)$$

for some positive constants κ and K .

Theoretical analysis for the Joule heating system was done by several authors [3, 5, 8, 34, 31, 32, 33]. Among these works, Yuan [33] proved existence and uniqueness of a C^α solution in three-dimensional space. Based on this result, further regularity can be derived with suitable assumption on the initial and boundary conditions. Numerical methods and analysis for the Joule heating system can be found in [2, 4, 11, 30, 36, 37, 38]. For the system in two-dimensional space, optimal L^2 error estimate of a mixed finite element method with the linearized semi-implicit Euler scheme was obtained in [36] under a weak time-step condition. Error analysis for the three-dimensional model was given in [11], in which the linearized semi-implicit Euler scheme with a linear Galerkin FEM was used. An optimal L^2 -error estimate was presented under the time step restriction $\tau = O(h^{d/6})$. A more general time discretization with higher-order finite element approximations was studied in [2]. An optimal L^2 -norm error estimate was given under the conditions $\tau = O(h^{d/2p})$ and $r \geq 2$ where p is the order of the discrete scheme in time direction and r is the degree of piecewise polynomial approximations used. No optimal error estimates in H^1 -norm have been obtained.

The main idea in this paper is a splitting of the numerical error into the temporal direction and the spatial direction by introducing a corresponding time-discrete parabolic

system (or elliptic system). Error bounds of the Galerkin finite element methods for the time-discrete parabolic equations in certain norm is dependent only upon the spatial mesh size h and independent of the time-step size τ . If a suitable regularity of the solution of the time-discrete equations can be proved, numerical solution in the L^∞ norm (or stronger norm) is bounded unconditionally by the induction assumption together with the inverse inequality

$$\|R_h U^n - U_h^n\|_{L^\infty} \leq Ch^{-d/2} \|R_h U^n - U_h^n\|_{L^2} \leq Ch^{r+1-d/2}, \quad (1.6)$$

where U^n is the solution of the time-discrete equations. With the boundedness, optimal error estimates can be established for the fully discrete scheme without any time-step restriction. In this paper, we analyze the linearized (semi-implicit) backward Euler scheme with the standard Galerkin approximation in spatial directions for the nonlinear Joule heating system (1.2)-(1.4). With the splitting, we present unconditionally optimal error estimates in both the L^2 norm and the H^1 norm.

The rest of the paper is organized as follows. In Section 2, we present the linearized semi-implicit Euler scheme with a linear Galerkin finite element approximation in spatial direction and our main results. After introducing the corresponding time-discrete parabolic system, we provide in Section 3 a priori estimates and optimal error estimates for the time-discrete solution, which imply the suitable regularity of the time-discrete solution. With the regularity obtained, we present optimal error estimates of the Galerkin finite element solution in L^2 -norm without any time-step restriction, and the optimal error estimate in H^1 norm follows immediately due to the nature of our approach. The concluding remarks are presented in Section 4. Extension to r -order Galerkin finite element approximation is straightforward with the corresponding assumptions of regularity.

2 Galerkin methods and main results

Let Ω be a bounded convex and smooth domain in \mathbb{R}^d ($d = 2, 3$). For any integer $k \geq 0$ and $1 \leq p < \infty$. Let $W_p^k(\Omega)$ be the Sobolev space with the norm

$$\|f\|_{W_p^k} = \left(\sum_{|\beta| \leq k} \int_{\Omega} |D^\beta f|^p dx \right)^{\frac{1}{p}},$$

where

$$D^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_d^{\beta_d}}$$

for the multi-index $\beta = (\beta_1, \dots, \beta_d)$, $\beta_1 \geq 0, \dots, \beta_d \geq 0$, and $|\beta| = \beta_1 + \dots + \beta_d$. For any integer $k \geq 0$ and $0 < \alpha < 1$, let $C^{k+\alpha}(\bar{\Omega})$ denote the usual Hölder space with the norm

$$\|f\|_{C^{k+\alpha}} = \sum_{|\beta| \leq k} \|D^\beta f\|_{C(\bar{\Omega})} + \sum_{|\beta|=k} \sup_{x, y \in \Omega} \frac{|D^\beta f(x) - D^\beta f(y)|}{|x - y|^\alpha}$$

and let $C_0(\bar{\Omega})$ be the space of continuous functions on $\bar{\Omega}$ vanishing on the boundary $\partial\Omega$. For any Banach space X and function $f : [0, T] \rightarrow X$, we define the norm

$$\|f\|_{L^p((0, T); X)} = \begin{cases} \left(\int_0^T \|f(t)\|_X^p dt \right)^{\frac{1}{p}}, & 1 \leq p < \infty, \\ \text{ess sup}_{t \in (0, T)} \|f(t)\|_X, & p = \infty. \end{cases}$$

With the boundary conditions (1.4), the weak formulation of the system (1.2)-(1.3) is defined by

$$(u_t, \xi_u) + (\nabla u, \nabla \xi_u) = (\sigma(u)|\nabla \phi|^2, \xi_u), \quad (2.1)$$

$$(\sigma(u)\nabla \phi, \nabla \xi_\phi) = 0 \quad (2.2)$$

for any $\xi_u, \xi_\phi \in H_0^1(\Omega)$ and a.e. $t \in (0, T)$.

Let π_h be a regular division of Ω into triangles in \mathbb{R}^2 or tetrahedras in \mathbb{R}^3 , i.e. $\Omega = \cup_j \Omega_j$, and denote by $h = \max_j \{\text{diam } \Omega_j\}$ the mesh size. For a triangle Ω_j at the boundary, we define $\tilde{\Omega}_j$ as the triangle with one curved side (or a tetrahedra with one curved face in \mathbb{R}^3) with the same vertices as Ω_j , and set $D_j = \tilde{\Omega}_j \setminus \Omega_j$. For an interior triangle, we set $\tilde{\Omega}_j = \Omega_j$ and $D_j = \emptyset$. For a given division π_h , we define the finite element spaces [26]:

$$V_h = \{v_h \in C(\bar{\Omega}) : v_h|_{\Omega_j} \text{ is linear for each element and } v_h = 0 \text{ on } D_j\},$$

$$S_h = \{v_h \in C(\bar{\Omega}) : v_h|_{\tilde{\Omega}_j} \text{ is linear for each element}\}.$$

It follows that V_h is a subspace of $H_0^1(\Omega)$ and S_h is a subspace of $H^1(\Omega)$. For any function $v \in S_h$, we define $\Lambda_h v$ as the function which satisfies $\Lambda_h v = 0$ on D_j and $\Lambda_h v = v$ on T_j . We define $\tilde{\Pi}_h : C(\bar{\Omega}) \rightarrow S_h$ to be the Lagrangian interpolation operator, i.e. $\tilde{\Pi}_h v$ coincides with v at each vertex of the triangular division of Ω , and set $\Pi_h = \Lambda_h \tilde{\Pi}_h$. Clearly, Π_h is a projection operator from $C_0(\bar{\Omega})$ onto V_h .

Let $\{t_n\}_{n=0}^N$ be a partition in the time direction with $t_n = n\tau$, $T = N\tau$ and

$$u^n = u(x, t_n), \quad \phi^n = \phi(x, t_n).$$

For any sequence of functions $\{f^n\}_{n=0}^N$, we define

$$D_t f^{n+1} = \frac{f^{n+1} - f^n}{\tau}.$$

For simplicity, we assume that $g \in H^1(\Omega)$. The fully discrete finite element scheme is to find $U_h^n, \Phi_h^n - g^n \in V_h$ for $n = 0, 1, \dots, N$ such that for all $\xi_u, \xi_\phi \in V_h$

$$(D_t U_h^{n+1}, \xi_u) + (\nabla U_h^{n+1}, \nabla \xi_u) = (\sigma(U_h^n)|\nabla \Phi_h^n|^2, \xi_u), \quad (2.3)$$

$$(\sigma(U_h^n)\nabla \Phi_h^n, \nabla \xi_\phi) = 0, \quad (2.4)$$

with the initial conditions $U_h^0 = I_h u^0$, where I_h is the Lagrangian interpolation operator.

In the rest part of this paper, we always assume that the solution to the initial/boundary value problem (1.2)-(1.4) exists and satisfies

$$\begin{aligned} & \|u\|_{L^\infty((0,T);H^2)} + \|u_t\|_{L^\infty((0,T);L^2)} + \|u_t\|_{L^2((0,T);H^2)} + \|u_{tt}\|_{L^2((0,T);L^2)} + \|u_0\|_{H^2} \\ & + \|\phi\|_{L^\infty((0,T);W^{2,12/5})} + \|\phi_t\|_{L^2((0,T);H^1)} + \|\nabla \phi\|_{L^\infty((0,T);C^\alpha)} \\ & + \|g\|_{L^\infty((0,T);W^{2,12/5})} + \|g_t\|_{L^2((0,T);H^1)} + \|\nabla g\|_{L^\infty((0,T);C^\alpha)} \leq C. \end{aligned} \quad (2.5)$$

We denote by C a generic positive constant, which is independent of n , h and τ and ϵ a generic small positive constant. We present our main results in the following theorem.

Theorem 2.1 *Suppose that the system (1.2)-(1.3) with the initial and boundary conditions (1.4) has a unique solution (u, ϕ) satisfying (2.5). Then there exist positive constants τ_0*

and h_0 such that when $\tau < \tau_0$ and $h < h_0$, the finite element system (2.3)-(2.4) admits a unique solution (U_h^n, Φ_h^n) , $n = 1, \dots, N$, such that

$$\max_{1 \leq n \leq N} \|U_h^n - u^n\|_{L^2} + \max_{1 \leq n \leq N} \|\Phi_h^n - \phi^n\|_{L^2} \leq C(\tau + h^2), \quad (2.6)$$

$$\max_{1 \leq n \leq N} \|U_h^n - u^n\|_{H^1} + \max_{1 \leq n \leq N} \|\Phi_h^n - \phi^n\|_{H^1} \leq C(\tau + h). \quad (2.7)$$

For $U^0 = u_0$ and Φ^0 , we define U^n and Φ^n to be the solution of the following discrete parabolic system (or elliptic system)

$$D_t U^{n+1} - \Delta U^{n+1} = \sigma(U^n) |\nabla \Phi^n|^2, \quad 0 \leq n \leq N-1, \quad (2.8)$$

$$-\nabla \cdot (\sigma(U^n) \nabla \Phi^n) = 0, \quad 0 \leq n \leq N, \quad (2.9)$$

with the boundary conditions

$$U^{n+1}(x) = 0, \quad \Phi^n(x) = g(x, t_n) \quad \text{for } x \in \partial\Omega. \quad (2.10)$$

We will present the proof of Theorem 2.1 in the next two sections. The key to our proof is the following error splitting

$$\begin{aligned} \|U_h^n - u^n\| &\leq \|e^n\| + \|e_h^n\| + \|U^n - R_h U^n\|, \\ \|\Phi_h^n - \phi^n\| &\leq \|\eta^n\| + \|\eta_h^n\| + \|\Phi^n - P_h^n \Phi^n\| \end{aligned}$$

for any norm $\|\cdot\|$, where

$$\begin{aligned} e^n &= U^n - u^n, & e_h^n &= U_h^n - R_h U^n, \\ \eta^n &= \Phi^n - \phi^n, & \eta_h^n &= \Phi_h^n - P_h^n \Phi^n, \end{aligned}$$

with $R_h : H_0^1(\Omega) \rightarrow V_h$ being the Riesz projection operator, i.e.

$$(\nabla(v - R_h v), \nabla w) = 0, \quad \text{for all } v \in H_0^1(\Omega) \text{ and } w \in V_h.$$

and $P_h^n \Phi^n = g(\cdot, t_n) + \Pi_h(\Phi^n - g(\cdot, t_n))$ for $n = 0, 1, 2, \dots, N$.

With the definition of the operator Π_h , P_h^n and R_h , the following estimates hold [23]: for any $2 \leq p < \infty$, there exists a positive constant C (independent of the function v) such that

$$\|v - \Pi_h v\|_{L^p} + h \|v - \Pi_h v\|_{W^{1,p}} \leq Ch^2 \|v\|_{W^{2,p}}, \quad (2.11)$$

$$\|\Phi^n - P_h^n \Phi^n\|_{L^p} + h \|\Phi^n - P_h^n \Phi^n\|_{W^{1,p}} \leq Ch^2 \|\Phi^n - g^n\|_{W^{2,p}}, \quad (2.12)$$

$$\|v - R_h v\|_{L^p} + h \|v - R_h v\|_{W^{1,p}} \leq Ch^2 \|v\|_{W^{2,p}}, \quad (2.13)$$

for $v \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$.

3 Error estimates

We analyze the error function (e^n, η^n) from the linearized semi-implicit Euler scheme (time-discrete system) and the errors function (e_h^n, η_h^n) of the Galerkin finite element method for the time-discrete system in the following two subsections, respectively.

3.1 The time-discrete solution

In this subsection, we prove the existence and uniqueness of the time-discrete system (2.8)-(2.10) and establish the error bounds for (e^n, η^n) .

Theorem 3.1 *Suppose that the system (1.2)-(1.4) has a unique solution (u, ϕ) satisfying (2.5). Then there exists a positive constant τ_0 such that when $\tau < \tau_0$, the time-discrete system (2.8)-(2.10) admits a unique solution (U^n, Φ^n) such that*

$$\max_{1 \leq n \leq N} \|U^n\|_{H^2} + \max_{1 \leq n \leq N} \|D_t U^n\|_{L^2} + \left(\sum_{n=1}^N \tau \|D_t U^n\|_{H^2}^2 \right)^{1/2} \leq C, \quad (3.1)$$

$$\max_{1 \leq n \leq N} \|\Phi^n\|_{W^{2,12/5}} + \max_{1 \leq n \leq N} \|\nabla \Phi^n\|_{L^\infty} \leq C \quad (3.2)$$

and

$$\max_{1 \leq n \leq N} \|e^n\|_{H^1} + \max_{1 \leq n \leq N} \|\eta^n\|_{H^1} \leq C\tau. \quad (3.3)$$

Proof We rewrite the system (1.2)-(1.3) by

$$D_t u^{n+1} - \Delta u^{n+1} = \sigma(u^n) |\nabla \phi^n|^2 + R_1^{n+1}, \quad (3.4)$$

$$-\nabla \cdot (\sigma(u^n) \nabla \phi^n) = 0, \quad (3.5)$$

where R_1^{n+1} is the truncation errors due to the time discretization, i.e.

$$R_1^{n+1} = D_t u^{n+1} - \frac{\partial u}{\partial t} \Big|_{t=t_{n+1}} + (\sigma(u^{n+1}) - \sigma(u^n)) |\nabla \phi^{n+1}|^2 + \sigma(u^n) \nabla(\phi^{n+1} + \phi^n) \cdot \nabla(\phi^{n+1} - \phi^n).$$

With the regularity given in (2.5), we have

$$\|R_1^{n+1}\|_{L^2} \leq C, \quad \sum_{n=0}^{N-1} \|R_1^{n+1}\|_{L^2}^2 \tau \leq C\tau^2. \quad (3.6)$$

Subtracting the equations (3.4)-(3.5) from the equations (2.8)-(2.9), respectively, we obtain

$$D_t e^{n+1} - \Delta e^{n+1} = (\sigma(U^n) - \sigma(u^n)) |\nabla \phi^n|^2 + \sigma(U^n) (\nabla \phi^n + \nabla \Phi^n) \cdot \nabla \eta^n + R_1^{n+1}, \quad (3.7)$$

$$-\nabla \cdot (\sigma(U^n) \nabla \eta^n) = \nabla \cdot [(\sigma(u^n) - \sigma(U^n)) \nabla \phi^n]. \quad (3.8)$$

An alternative to the last equation is

$$-\nabla \cdot (\sigma(u^n) \nabla \eta^n) = \nabla \cdot [(\sigma(u^n) - \sigma(U^n)) (\nabla \phi^n + \nabla \eta^n)]. \quad (3.9)$$

Multiplying the equation (3.8) by η^{n+1} and integrating the result over Ω , we have

$$\|\nabla \eta^n\|_{L^2}^2 \leq C \|e^n\|_{L^2} \|\nabla \eta^n\|_{L^2}$$

which leads to

$$\|\nabla \eta^n\|_{L^2} \leq C \|e^n\|_{L^2}. \quad (3.10)$$

Similarly, multiplying (3.7) by e^{n+1} and integrating it over Ω gives

$$\begin{aligned} & D_t \left(\frac{1}{2} \|e^{n+1}\|_{L^2}^2 \right) + \|\nabla e^{n+1}\|_{L^2}^2 \\ & \leq C \|e^n\|_{L^2} \|e^{n+1}\|_{L^2} \|\nabla \phi^n\|_{L^\infty} + (\sigma(U^n)(\nabla \phi^n + \nabla \Phi^n)e^{n+1}, \nabla \eta^n) \\ & \quad + \|R_1^{n+1}\|_{L^2} \|e^{n+1}\|_{L^2}. \end{aligned}$$

By (2.9) and using integrating by part,

$$\begin{aligned} & |(\sigma(U^n)(\nabla \phi^n + \nabla \Phi^n)e^{n+1}, \nabla \eta^n)| \\ & \leq |(\sigma(U^n)e^{n+1}\nabla \phi^n, \nabla \eta^n)| \\ & \quad + |(e^{n+1}\nabla \cdot (\sigma(U^n)\nabla \Phi^n) + \sigma(U^n)\nabla \Phi^n \cdot \nabla e^{n+1}, \eta^n)| \\ & \leq C(\|e^{n+1}\|_{L^2} \|\nabla \eta^n\|_{L^2} + \|\nabla \phi^n\|_{L^\infty} \|\nabla e^{n+1}\|_{L^2} \|\eta^n\|_{L^2} + \|\nabla \eta^n\|_{L^2} \|\nabla e^{n+1}\|_{L^2} \|\eta^n\|_{L^\infty}). \end{aligned}$$

Applying the maximum principle to the elliptic equation (2.9) shows that $\|\Phi^n\|_{L^\infty} \leq C$ and therefore,

$$\|\eta^n\|_{L^\infty} \leq C,$$

for $n = 0, 1, 2, \dots$. It follows that

$$\begin{aligned} & D_t \left(\frac{1}{2} \|e^{n+1}\|_{L^2}^2 \right) + \frac{1}{2} \|\nabla e^{n+1}\|_{L^2}^2 \\ & \leq C \|e^n\|_{L^2}^2 + C \|e^{n+1}\|_{L^2}^2 + C \|\eta^n\|_{H^1}^2 + C \|R_1^{n+1}\|_{L^2}^2 \\ & \leq C \|e^n\|_{L^2}^2 + C \|e^{n+1}\|_{L^2}^2 + C \|R_1^{n+1}\|_{L^2}^2, \end{aligned}$$

where we have used (3.10) in the last step. By applying Gronwall's inequality, combined with (3.6), we derive that there exists a small positive constant τ_0 such that when $\tau < \tau_0$,

$$\max_{1 \leq n \leq N} \|e^n\|_{L^2}^2 + \max_{1 \leq n \leq N} \|\eta^n\|_{H^1}^2 + \sum_{n=1}^N \|e^n\|_{H^1}^2 \tau \leq C\tau^2. \quad (3.11)$$

In particular, the above estimate implies that

$$\|U^n\|_{H^1}^2 \leq C \quad (3.12)$$

and

$$\|D_t U^{n+1}\|_{L^2} \leq \|D_t u^{n+1}\|_{L^2} + \|D_t e^{n+1}\|_{L^2} \leq C.$$

With the above inequalities, we derive from (2.8) that

$$\|U^{n+1}\|_{H^2} \leq C + C \|\nabla \Phi^n\|_{L^4}^2. \quad (3.13)$$

Since $H^2(\Omega) \hookrightarrow C^\alpha(\bar{\Omega})$ in \mathbb{R}^d with $d = 2, 3$, $\|e^n\|_{C^\alpha} \leq C$. By applying the $W^{1,4}$ estimate [6, 24] to (3.9), we get

$$\begin{aligned} \|\nabla \eta^n\|_{L^4} & \leq \|(\sigma(u^n) - \sigma(U^n))\nabla \Phi^n\|_{L^4} \\ & \leq C_0 \|e^n\|_{L^\infty} (\|\nabla \phi^n\|_{L^4} + \|\nabla \eta^n\|_{L^4}) \end{aligned}$$

where C_0 is some positive constant. By assuming that $C_0 \|e^n\|_{L^\infty} < 1/2$, we derive that

$$\|\nabla \eta^n\|_{L^4} \leq C$$

and (3.13) implies that

$$\|e^{n+1}\|_{H^2} \leq \|u^{n+1}\|_{H^2} + \|U^{n+1}\|_{H^2} \leq C \quad (3.14)$$

and

$$\|e^{n+1}\|_{L^\infty} \leq \|e^{n+1}\|_{H^1}^{1/2} \|e^{n+1}\|_{H^2}^{1/2} \leq C\tau^{1/4}.$$

From the above derivation, one can see that there exists $\tau_0 > 0$ such that if $\tau < \tau_0$, then $C_0\|e^n\|_{L^\infty} < 1/2$ implies $C_0\|e^{n+1}\|_{L^\infty} < 1/2$ as well as (3.14). In addition, we see that $\|\nabla\Phi^n\|_{L^4} \leq C$ and therefore,

$$\max_{1 \leq n \leq N} \|U^n\|_{C^\alpha} \leq C. \quad (3.15)$$

By applying Schauder's estimates ([9], page 74) to (2.9), we derive that

$$\max_{1 \leq n \leq N} \|\nabla\Phi^n\|_{C^\alpha} \leq C, \quad (3.16)$$

which together with (3.12) and (2.9) implies that

$$\max_{1 \leq n \leq N} \|\Phi^n\|_{W^{2,12/5}} \leq C. \quad (3.17)$$

Multiplying (3.7) by $-\Delta e^{n+1}$ and summing up the equations for $n = 0, 1, \dots, N-1$, we obtain

$$\begin{aligned} & \max_{1 \leq n \leq N} \|e^n\|_{H^1}^2 + \sum_{n=1}^N \tau \|\Delta e^n\|_{L^2}^2 \\ & \leq \sum_{n=0}^{N-1} \tau \left(\|(\sigma(U^n) - \sigma(u^n))|\nabla\phi^n|^2\|_{L^2}^2 + \|\sigma(U^n)(\nabla\phi^n + \nabla\Phi^n) \cdot \nabla\eta^n\|_{L^2}^2 + \|R_1^{n+1}\|_{L^2}^2 \right) \leq C\tau^2. \end{aligned}$$

It follows that

$$\max_{1 \leq n \leq N} \|e^n\|_{H^1} \leq C\tau,$$

and

$$\sum_{n=1}^N \tau \|\Delta D_t e^n\|_{L^2}^2 \leq C\tau^{-2} \sum_{n=1}^N \tau \|\Delta e^n\|_{L^2}^2 \leq C.$$

By the theory of elliptic equations [9, 14], $\|D_t e^n\|_{H^2} \leq C\|\Delta D_t e^n\|_{L^2}$ for $n = 1, \dots, N$, and so

$$\sum_{n=1}^N \tau \|D_t e^n\|_{H^2}^2 \leq C. \quad (3.18)$$

The proof of Theorem 3.1 is complete. ■

3.2 The fully-discrete finite element solution

Here we study the error (e_h^n, η_h^n) of the Galerkin finite element method for the time-discrete system (2.8)-(2.10).

Theorem 3.2 Suppose that the system (1.2)-(1.4) has a solution (u, ϕ) satisfying (2.5). Then there exist positive constants h_0 and τ_0 such that when $h < h_0$ and $\tau < \tau_0$, the fully-discrete system (2.3)-(2.4) admits a unique solution (U_h^n, Φ_h^n) such that

$$\|e_h^n\|_{L^2} + \|\eta_h^n\|_{L^2} \leq Ch^2, \quad (3.19)$$

$$\|e_h^n\|_{H^1} + \|\eta_h^n\|_{H^1} \leq Ch. \quad (3.20)$$

Note that the condition of $\tau < \tau_0$ is to ensure that Theorem 3.1 holds. For the given U_h^n , the error estimate for the equation (2.4) is given in the following Lemma.

Lemma 3.1 Suppose that the system (1.2)-(1.4) has a unique solution (u, ϕ) satisfying (2.5). Then

$$\begin{aligned} \|\nabla(\Phi_h^n - \Phi^n)\|_{L^2} &\leq C(h + \|e_h^n\|_{L^2}), \\ \|\Phi_h^n - \Phi^n\|_{L^2} &\leq C(h^2 + \|e_h^n\|_{L^2} + h^{-d/6}\|e_h^n\|_{L^2}^2), \end{aligned}$$

where (U_h^n, Φ_h^n) and (U^n, Φ^n) are the solution of the finite element system (2.3)-(2.4) and the time-discrete system (2.8)-(2.10), respectively.

Remark 3.1 The proof of the above lemma is similar as that of Lemma 3.2 in [11], in which the factor $h^{-d/6}$ appears when $\|e_h^n\|_{L^3}$ reduces to $\|e_h^n\|_{L^2}$ via the inverse inequality. More important is that in [11], e_h^n is the difference between the exact solution of the system (1.2)-(1.3) and the fully discrete finite element solution. The restriction for the time-step size, $\tau \leq k_0 h^{d/6}$, was required when the preliminary error bound $\|e_h^n\|_{L^2} \leq C(\tau + h^2)$ was used by induction in the second inequality of Lemma 3.1. However, in our approach, e_h^n is the difference between the solution of the time-discrete system (2.8)-(2.10) and the fully discrete finite element solution. Thus, the induction assumption shows that $\|e_h^n\|_{L^2} \leq Ch^2$ and then, we can prove the optimal error bound of the scheme unconditionally.

Proof of Theorem 3.2 At each time step of the scheme, one only needs to solve two uncoupled linear discrete systems. Due to the assumption (1.5), it is easy to see that coefficient matrices in both systems are symmetric and positive definite. Existence and uniqueness of the Galerkin finite element solution follows immediately. It is seen that the inequality (3.20) follows from (3.19) via the inverse inequality. Therefore, it suffices to prove (3.19).

The weak formulation of the time-discrete system (2.8)-(2.10) is

$$(D_t U^{n+1}, \xi_u) + (\nabla U^{n+1}, \nabla \xi_u) = (\sigma(U^n) |\nabla \Phi^n|^2, \xi_u), \quad (3.21)$$

$$(\sigma(U^n) \nabla \Phi^n, \nabla \xi_\phi) = 0, \quad (3.22)$$

for any $\xi_u, \xi_\phi \in V_h$. From the above equations and the finite element system (2.3)-(2.4), we find that the error function (e_h^n, η_h^n) satisfies

$$\begin{aligned} &(D_t e_h^{n+1}, \xi_u) + (\nabla e_h^{n+1}, \nabla \xi_u) \\ &= (D_t(U^{n+1} - R_h U^{n+1}), \xi_u) + ((\sigma(U_h^n) - \sigma(U^n)) |\nabla \Phi^n|^2, \xi_u) \end{aligned}$$

$$\begin{aligned}
& + 2((\sigma(U_h^n) - \sigma(U^n))\nabla\Phi^n \cdot \nabla(\Phi_h^n - \Phi^n), \xi_u) \\
& + (\sigma(U_h^n)|\nabla(\Phi_h^n - \Phi^n)|^2, \xi_u) \\
& + 2(\sigma(U^n)\nabla\Phi^n \cdot \nabla(\Phi_h^n - \Phi^n), \xi_u) \\
& := (\bar{R}_1^{n+1}, \xi_u) + (\bar{R}_2^{n+1}, \xi_u) + (\bar{R}_3^{n+1}, \xi_u) + (\bar{R}_4^{n+1}, \xi_u) + (\bar{R}_5^{n+1}, \xi_u),
\end{aligned} \tag{3.23}$$

and

$$\begin{aligned}
(\sigma(U_h^n)\nabla\eta_h^n, \nabla\xi_\phi) & = ((\sigma(U^n) - \sigma(U_h^n))\nabla\Phi^n, \nabla\xi_\phi) \\
& + (\sigma(U_h^n)\nabla(\Phi^n - P_h^n\Phi^n), \nabla\xi_\phi)
\end{aligned} \tag{3.24}$$

for all $\xi_u, \xi_\phi \in V_h$.

Since $\eta_h^n = 0$ on $\partial\Omega$, we can take $\xi_\phi = \eta_h^n$ in (3.24) to get

$$\|\nabla\eta_h^n\|_{L^2} \leq C\|e_h^n\|_{L^2} + Ch, \tag{3.25}$$

where we have noted the fact that $\|\nabla(\phi^n - P_h^n\phi^n)\|_{L^2} \leq Ch$. With the above inequality, from Lemma 3.1 we derive that

$$\|\eta_h^n\|_{L^2} \leq Ch^2 + C\|e_h^n\|_{L^2} + h^{-d/6}\|e_h^n\|_{L^2}^2. \tag{3.26}$$

Taking $\xi_u = e_h^{n+1}$ in (3.23), the right-hand side is estimated by

$$\begin{aligned}
(\bar{R}_1^{n+1}, e_h^{n+1}) & \leq \epsilon\|e_h^{n+1}\|_{H^1}^2 + C\epsilon^{-1}\|D_t U^{n+1} - R_h D_t U^{n+1}\|_{L^2}^2 \\
& \leq \epsilon\|e_h^{n+1}\|_{H^1}^2 + C\epsilon^{-1}\|D_t U^{n+1}\|_{H^2}^2 h^4,
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
(\bar{R}_2^{n+1}, e_h^{n+1}) & \leq C\|e_h^{n+1}\|_{L^2}(\|e_h^n\|_{L^2} + \|U^n - R_h U^n\|_{L^2}) \\
& \leq \epsilon\|e_h^{n+1}\|_{L^2}^2 + C\epsilon^{-1}(\|e_h^n\|_{L^2}^2 + h^4),
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
(\bar{R}_3^{n+1}, e_h^{n+1}) & \leq C\|e_h^{n+1}\|_{L^6}(\|e_h^n\|_{L^2} + \|U^n - R_h U^n\|_{L^2})(\|\nabla\eta_h^{n+1}\|_{L^3} + \|\Phi^n - P_h^n\Phi^n\|_{L^3}) \\
& \leq C\|e_h^{n+1}\|_{H^1}(\|e_h^n\|_{L^2} + Ch^2)(\|\nabla\eta_h^n\|_{L^3} + Ch) \\
& \leq \epsilon\|e_h^{n+1}\|_{H^1}^2 + C\epsilon^{-1}(\|e_h^n\|_{L^2} + Ch^2)^2(Ch^{-d/6}\|\nabla\eta_h^n\|_{L^2} + Ch)^2 \\
& \leq \epsilon\|e_h^{n+1}\|_{H^1}^2 + C\epsilon^{-1}(\|e_h^n\|_{L^2}^2 + h^4)(Ch^{-d/6}\|e_h^n\|_{L^2} + Ch^{1-d/6})^2,
\end{aligned} \tag{3.29}$$

$$(\bar{R}_5^{n+1}, e_h^{n+1}) = -2(\sigma(U^n)\nabla\Phi^n \cdot (\Phi_h^n - \Phi^n), \nabla e_h^{n+1}) \tag{3.30}$$

$$\begin{aligned}
& \leq C\|\Phi_h^n - \Phi^n\|_{L^2}\|\nabla e_h^{n+1}\|_{L^2} \\
& \leq \epsilon\|\nabla e_h^{n+1}\|_{L^2}^2 + C\epsilon^{-1}(\|e_h^n\|_{L^2}^2 + h^{-d/3}\|e_h^n\|_{L^2}^4 + h^4)
\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}
(\bar{R}_4^{n+1}, e_h^{n+1}) & \leq C\|e_h^{n+1}\|_{L^\infty}(\|\nabla\eta_h^n\|_{L^2}^2 + \|\nabla(\Phi^n - P_h^n\Phi^n)\|_{L^2}^2) \\
& \leq Ch^{-1/2}\|e_h^{n+1}\|_{H^1}(\|e_h^n\|_{L^2}^2 + h^2) \\
& \leq \epsilon\|e_h^{n+1}\|_{H^1}^2 + Ch^{-1}\|e_h^n\|_{L^2}^4 + Ch^3.
\end{aligned} \tag{3.32}$$

With the above estimates, (3.23) reduces to

$$\begin{aligned}
D_t(\|e_h^{n+1}\|_{L^2}^2) + \|\nabla e_h^{n+1}\|_{L^2}^2 & \leq C(\|e_h^n\|_{L^2}^2 + h^{-1}\|e_h^n\|_{L^2}^4) \\
& + Ch^3 + C\epsilon^{-1}\|D_t U^{n+1}\|_{H^2}^2 h^4,
\end{aligned} \tag{3.33}$$

which holds for $0 \leq n \leq N-1$.

Now we prove that

$$\|e_h^n\|_{L^2} \leq h^{1/2} \quad \text{for } 0 \leq n \leq N \quad (3.34)$$

by using mathematical induction. Clearly, this inequality holds for $n = 0$. If we assume that this inequality holds for $0 \leq n \leq k$, then the inequality (3.33) reduces to

$$D_t (\|e_h^{n+1}\|_{L^2}^2) + \|\nabla e_h^{n+1}\|_{L^2}^2 \leq C\|e_h^n\|_{L^2}^2 + Ch^3 + C\epsilon^{-1}\|D_t U^{n+1}\|_{H^2}^2 h^4 \quad (3.35)$$

for $0 \leq n \leq k$. By applying Gronwall's inequality, we derive that

$$\|e_h^{k+1}\|_{L^2}^2 \leq C_1 h^3 \leq h^{1/2} \quad \text{if } h < 1/C_1^{2/5}. \quad (3.36)$$

This completes the induction.

With (3.34), we can apply Gronwall's inequality to (3.33) and get

$$\max_{1 \leq n \leq N} \|e_h^n\|_{L^2} \leq Ch^{3/2}. \quad (3.37)$$

Since $\eta_h^{n+1} \in H_0^1(\Omega)$, from the estimates (3.25)-(3.26) we see that

$$\max_{1 \leq n \leq N} \|\eta_h^n\|_{H^1} \leq Ch. \quad (3.38)$$

which implies that $\|\nabla \eta_h^n\|_{L^6} \leq Ch^{-d/3} \|\nabla \eta_h^n\|_{L^2} \leq C$ and so $\|\nabla \Phi_h^n\|_{L^6} \leq C$.

Finally, we rewrite (3.24) as

$$\begin{aligned} & (\sigma(U^n) \nabla \eta_h^n, \nabla \xi_\phi) \\ &= ((\sigma(U^n) - \sigma(U_h^n)) \nabla \Phi_h^n, \nabla \xi_\phi) + (\sigma(U^n) \nabla (\Phi^n - P_h^n \Phi^n), \nabla \xi_\phi), \quad \forall \xi_\phi \in V_h, \end{aligned} \quad (3.39)$$

and apply the $W^{1,p}$ estimate [23] to the above equation. Then we get

$$\begin{aligned} \|\nabla \eta_h^n\|_{L^{12/5}} &\leq C \|(\sigma(U^n) - \sigma(U_h^n)) \nabla \Phi_h^n\|_{L^{12/5}} + C \|\nabla (\Phi^n - P_h^n \Phi^n)\|_{L^{12/5}} \\ &\leq C \|U_h^n - U^n\|_{L^4} \|\nabla \Phi_h^n\|_{L^6} + C \|\nabla (\Phi^n - P_h^n \Phi^n)\|_{L^{12/5}} \\ &\leq Ch^{-d/4} (\|e_h^n\|_{L^2} + h^2 \|U^n\|_{H^2}) + Ch \|\Phi^n\|_{W^{2,12/5}} \\ &\leq C \|e_h^n\|_{L^2}^{1/2} + Ch. \end{aligned} \quad (3.40)$$

Therefore, we obtain a refined estimate:

$$\begin{aligned} (\bar{R}_4^{n+1}, e_h^{n+1}) &\leq C \|e_h^{n+1}\|_{L^6} (\|\nabla \eta_h^n\|_{L^{12/5}}^2 + \|\nabla (\Phi^n - P_h^n \Phi^n)\|_{L^{12/5}}^2) \\ &\leq \epsilon \|e_h^{n+1}\|_{L^6}^2 + C\epsilon^{-1} \|\nabla \eta_h^n\|_{L^{12/5}}^4 + C\epsilon^{-1} \|\nabla (\Phi^n - P_h^n \Phi^n)\|_{L^{12/5}}^4 \\ &\leq \epsilon \|e_h^{n+1}\|_{H^1}^2 + C\epsilon^{-1} \|e_h^n\|_{L^2}^2 + C\epsilon^{-1} h^4. \end{aligned} \quad (3.41)$$

With the estimates (3.27)-(3.31) and (3.41), the equation (3.23) reduces to

$$D_t (\|e_h^{n+1}\|_{L^2}^2) + \|\nabla e_h^{n+1}\|_{L^2}^2 \leq C\|e_h^n\|_{L^2}^2 + Ch^4 + C\|D_t U^{n+1}\|_{H^2}^2 h^4.$$

By applying Gronwall's inequality, we get

$$\max_{1 \leq n \leq N} \|e_h^n\|_{L^2}^2 \leq Ch^4. \quad (3.42)$$

The L^2 error estimate of η_h^n follows from (3.26) and (3.42). The proof of Theorem 3.2 is complete. ■

Theorem 2.1 follows immediately from Theorem 3.2 and Theorem 3.1. ■

4 Conclusions

We have presented an approach to obtain optimal error estimates and unconditional stability of linearized (semi) implicit schemes with a Galerkin finite element method for the three-dimensional nonlinear Joule heating equations. The analysis is based on a new splitting of the error into the time direction and the spatial direction, by which the numerical solution (or its error) in a strong norm can be bounded by induction assumption and the inverse inequalities without any restrictions on the time-step size. In most existing approaches, a time-step condition has to be enforced to bound the numerical solution in a stronger norm. Clearly, our analysis can be extended to many other nonlinear parabolic systems, while we only focus on the electric heating model in the present paper.

In this paper, we only considered a linear Galerkin finite element approximation. The extension to high-order Galerkin finite element methods can be done similarly. For simplicity, we have assumed that the function g is defined in the domain Ω instead of on the boundary $\partial\Omega$. If the function g is defined only on the boundary $\partial\Omega$, a similar analysis can be given by taking the boundary terms into consideration, see [11] for reference. Optimal error estimates still can be proved without any condition on the time-step size.

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