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ERROR ANALYSIS OF A FULLY DISCRETE FINITE ELEMENT METHOD FOR VARIABLE DENSITY INCOMPRESSIBLE FLOWS IN TWO DIMENSIONS *

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Abstract. An error estimate is presented for a fully discrete, linearized and stabilized finite element method for solving the coupled system of nonlinear hyperbolic and parabolic equations describing incompressible flow with variable density in a two-dimensional convex polygon. In particular, the error of the numerical solution is split into the temporal and spatial components, separately. The temporal error is estimated by applying discrete maximal L^p-regularity of time-dependent Stokes equations, and the spatial error is estimated by using energy techniques based on the uniform regularity of the solutions given by semi-discretization in time.

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1. Introduction

The time-dependent incompressible flow with variable density is governed by the partial differential equations (PDEs)

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \tag{1.1}$$

$$\rho \partial_t \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \mu \Delta \mathbf{u} = 0, \tag{1.2}$$

$$\nabla \cdot \mathbf{u} = 0, \tag{1.3}$$

where ρ, $\mathbf{u}$ and p denote the density, velocity and pressure of the fluid, respectively, and $\mu > 0$ the viscosity constant. We consider (1.1)-(1.3) in a convex polygon $\Omega \subset \mathbb{R}^2$ up to a given time T, with the following boundary and initial conditions:

$$\begin{aligned} &\mathbf{u} = 0 && \text{on } \partial\Omega \times [0,T], \\ &\rho = \rho^0 \ \text{ and } \ \mathbf{u} = \mathbf{u}^0 && \text{at } t = 0, \end{aligned} \tag{1.4}$$

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where ρ^0 and $\mathbf{u}^0$ are given functions, and $\partial\Omega$ the boundary of the domain Ω. For given smooth initial data ρ^0 and $\mathbf{u}^0$ with positive density, i.e.,

$$\min_{x\in\Omega}\rho^0(x) > 0,$$

the existence and uniqueness of smooth solutions of (1.1)-(1.4) have been proved in [14,32,44]. In particular, this hyperbolic-parabolic system does not generate shock wave (at least in 2D).

Many efficient numerical methods have been developed for solving (1.1)-(1.4), including the projection finite element methods (FEMs) [5,7,24,41,46], the fractional-step methods with pressure Poisson equation [25,26], the BDF stepping methods [40], the finite volume method [11], and the discontinuous Galerkin (DG) method [42]. However, there are very few work on rigorous error analysis of the numerical methods.

As mentioned in [46], the variable density introduces considerable difficulties for the analysis of the accuracy of the numerical solutions, mainly due to the strong nonlinearities and the coupling of the equations. The main difficulty in the error analysis of such nonlinear problems is to prove certain boundedness of the numerical solutions uniformly with respect to the step size and mesh size, while the presence of the hyperbolic density equation increases the difficulty: it requires the numerical solution $\mathbf{u}_h^n$ of the velocity equation to be bounded in a very strong norm, i.e.,

$$\tau\sum_{n=1}^{N}\|\nabla\cdot\mathbf{u}_h^n\|_{L^\infty(\Omega)} \le c,$$

where c should be independent of the temporal step size τ and spatial mesh size h.

An error estimate for the velocity equations (1.2)-(1.3) was presented in [27] for the numerical methods developed in [25,26] by assuming that the numerical solutions ρ_h^n, $n = 1,\dots,N$, of the density equation have the following properties (cf. [27, Conjectures in Remark 4.2]):

(A1) Positivity and boundedness: $c_1 \le \min_{x\in\overline{\Omega}}\rho_h^n(x) \le \max_{x\in\overline{\Omega}}\rho_h^n(x) \le c_2$, $n = 1,2,\dots,N$, where c_1 and c_2 are positive constants independent of the step size and mesh size.

(A2) Error estimate of ρ_h^n in terms of $\mathbf{u}_h^n$ (numerical solution of the velocity equation):

$$\begin{aligned}
&\tau\sum_{n=1}^{N}\left(\|\rho(\cdot,t_n)-\rho_h^n\|_{L^2(\Omega)}^2 + \left\|\partial_t\rho(\cdot,t_n)-\frac{\rho_h^n-\rho_h^{n-1}}{\tau}\right\|_{H^{-1}(\Omega)}^2\right)\\
&\le c_\epsilon(\tau+h^m)^2 + \tau\sum_{n=1}^{N}\left(c_\epsilon\|\mathbf{u}(\cdot,t_n)-\mathbf{u}_h^n\|_{L^2(\Omega)}^2 + \epsilon\|\mathbf{u}(\cdot,t_n)-\mathbf{u}_h^n\|_{H^1(\Omega)}^2\right),
\end{aligned}$$

where $\epsilon\in(0,1)$ can be arbitrarily small, and the constant c_ϵ depends on ϵ.

However, a rigorous proof of (A1)-(A2) remains open. Thus a complete error estimate for the coupled system (1.1)-(1.4) is still missing.

The stability of the numerical solutions was investigated in [24, 41, 46] for different discretizations of the stabilized equations

$$\partial_t\rho + \mathbf{u}\cdot\nabla\rho + \frac{1}{2}\rho\nabla\cdot\mathbf{u} = 0, \tag{1.5}$$

$$\rho\partial_t\mathbf{u} + \rho\mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{2}\partial_t\rho\,\mathbf{u} + \frac{1}{2}\nabla\cdot(\rho\mathbf{u})\,\mathbf{u} - \Delta\mathbf{u} + \nabla p = 0, \tag{1.6}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{1.7}$$

which are theoretically equivalent to (1.1)-(1.3) but preserve the energy conservation upon discretization with the FEMs. The stability property of numerical schemes does not imply error estimates, but can be used to prove the convergence of DG via a compactness argument [42] (without explicit order of convergence).

In this article, we present a complete error estimate for a linearized and stabilized FEM for solving the coupled system (1.1)-(1.4). The techniques include discrete maximal L^p-regularity of parabolic equations recently established in [30] and an error splitting technique developed in [17, 35, 36].

The discrete maximal L^p-regularity, established in a series of articles [6, 18, 30, 33, 34, 38], is a mathematical tool for analysis of both time discretizations [3, 4, 31] and finite element spatial discretizations [37] of nonlinear parabolic equations. The technique used in this paper extends the result of [3, 4] to parabolic equations with a time-dependent coefficient in front of the time derivative (see Lemma 4.2). This extension is needed to treat the velocity equation in the presence of the variable density ρ in front of the time derivative.

The Helmholtz–Weyl decomposition, also known as Hodge decomposition, decomposes a vector field into a sum of its divergence-free and curl-free parts. It was used for numerical analysis of many different partial differential equations, including the Maxwell equations [9], the Ginzburg-Landau equations [39], and the Navier-Stokes equations with constant density [1, 12, 23, 45, 47–49]. In the presence of a variable density, the term $\rho\partial_t\mathbf{u}$ is no long divergence-free, and thus the projection of (1.2) onto the divergence-free subspace is more complicated.

The rest of this paper is organized as follows. In the next section, we introduce the notations, numerical scheme and main theorem. We present an overview for the proof of the main theorem in Section 3 and present the details in Sections 4 and 5. Numerical results are presented in Section 6 to support the theoretical analysis.

2. Notations and main results

2.1. Function spaces

For any integer $k \ge 0$ and real number $1 \le p \le \infty$, $W^{k,p}(\Omega)$ denotes the conventional Sobolev space of functions defined on the domain Ω, with the abbreviations $H^k(\Omega) = W^{k,2}(\Omega)$ and $L^p(\Omega) = W^{0,p}(\Omega)$, and

$$L^p_0(\Omega) = \{q \in L^p(\Omega) : \textstyle\int_\Omega q dx = 0\}.$$

The space of continuous functions on $\overline{\Omega}$ is denoted by $C(\overline{\Omega})$, and the space of infinitely differentiable functions with compact support in Ω is denoted by $C_0^\infty(\Omega)$. The closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$ is denoted by $W_0^{k,p}(\Omega)$, with the abbreviation $H_0^k(\Omega) = W_0^{k,2}(\Omega)$. The vector-valued extensions of these function spaces are denoted by

$$\mathbf{W}^{k,p}(\Omega) = W^{k,p}(\Omega) \times W^{k,p}(\Omega), \qquad \mathbf{W}_0^{k,p}(\Omega) = W_0^{k,p}(\Omega) \times W_0^{k,p}(\Omega),$$

with the abbreviations $\mathbf{H}^k(\Omega) = \mathbf{W}^{k,2}(\Omega)$ and $\mathbf{L}^p(\Omega) = \mathbf{W}^{0,p}(\Omega)$.

For the simplicity of notations, the inner products of both $L^2(\Omega)$ and $\mathbf{L}^2(\Omega)$ are denoted by $(\cdot,\cdot)$, namely,

$$\begin{aligned} (w, v) &= \int_\Omega u(x)v(x)\mathrm{d}x, && \forall\, w, v \in L^2(\Omega),\\ (\mathbf{w}, \mathbf{v}) &= \int_\Omega \mathbf{w}(x)\cdot\mathbf{v}(x)\mathrm{d}x, && \forall\, \mathbf{w}, \mathbf{v} \in \mathbf{L}^2(\Omega). \end{aligned}$$

Similarly, the norms of both $W^{k,p}(\Omega)$ and $\mathbf{W}^{k,p}(\Omega)$ are denoted by $\|\cdot\|_{W^{k,p}}$.

Following the notations of [4] (also see [3]), for any given sequence of functions $v^n \in W^{k,q}(\Omega)$, $n = 1, 2, \dots, m$, we define the norm

$$\|(v^n)_{n=1}^m\|_{L^p(W^{k,q}(\Omega))} = \begin{cases} \left(\tau \displaystyle\sum_{n=1}^m \|v^n\|^p_{W^{k,q}(\Omega)}\right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty,\\ \displaystyle\max_{1\le n\le m} \|v^n\|_{W^{k,q}(\Omega)} & \text{if } p = \infty. \end{cases} \tag{2.1}$$

Similarly, for a function v defined on $\Omega \times (0,T)$ we define the following Borchner norm:

$$\|v\|_{L^p(0,T;W^{k,q}(\Omega))} = \begin{cases} \left(\displaystyle\int_0^T \|v(\cdot,t)\|^p_{W^{k,q}(\Omega)} \mathrm{d}t\right)^{\frac{1}{p}} & \text{if } 1 \le p < \infty, \\ \operatorname*{ess\ sup}_{t\in(0,T)} \|v(\cdot,t)\|_{W^{k,q}(\Omega)} & \text{if } p = \infty. \end{cases} \tag{2.2}$$

2.2. Variational equations and stabilization

We define the bilinear form

$$B((\mathbf{u},p),(\mathbf{v},q)) := (\mu\nabla\mathbf{u},\nabla\mathbf{v}) - (p,\nabla\cdot\mathbf{v}) + (\nabla\cdot\mathbf{u},q), \qquad \forall\,(\mathbf{u},p),(\mathbf{v},q) \in \mathbf{H}^1_0(\Omega)\times L^2_0(\Omega). \tag{2.3}$$

Note that if $(\rho,\mathbf{u},p)$ is a solution of (1.1)-(1.3) then $(\rho,\mathbf{u},p+c)$ is also a solution of (1.1)-(1.3), where c can be an arbitrary constant. Besides, any sufficiently smooth solution of (1.1)-(1.3) satisfies (1.5)-(1.7) and thus the following variational equations:

$$\big(\partial_t\rho,\varphi\big) + (\mathbf{u}\cdot\nabla\rho,\varphi) + \frac{1}{2}(\nabla\cdot\mathbf{u}\ \rho,\varphi) = 0, \tag{2.4}$$

$$(\rho\partial_t\mathbf{u} + \rho\mathbf{u}\cdot\nabla\mathbf{u},\mathbf{v}) + \frac{1}{2}(\partial_t\rho\,\mathbf{u},\mathbf{v}) + \frac{1}{2}(\nabla\cdot(\rho\mathbf{u})\,\mathbf{u},\mathbf{v}) + B((\mathbf{u},p),(\mathbf{v},q)) = 0, \tag{2.5}$$

where $\varphi \in L^2(\Omega)$ and $(\mathbf{v},q) \in \mathbf{H}^1_0(\Omega)\times L^2_0(\Omega)$ are arbitrary test functions.

In the equations above, the extra terms $\frac{1}{2}(\nabla\cdot\mathbf{u}\ \rho,\varphi)$ and $\frac{1}{2}(\partial_t\rho\,\mathbf{u},\mathbf{v}) + \frac{1}{2}(\nabla\cdot(\rho\mathbf{u})\,\mathbf{u},\mathbf{v})$ vanish due to (1.3) and (1.1), respectively. These extra terms will stabilize the finite element solutions to be defined in the next subsection.

2.3. Spatial and temporal discretizations

Let $\mathscr{T}_h$ be a quasi-uniform triangulation of the convex polygon Ω into triangles $\mathcal{T}_j$, $j = 1,...,M$, with mesh size $h = \max_{1\le j\le M} \mathrm{diam}(\mathcal{T}_j)$. Let $P_r(\mathcal{T}_j)$ denote the space of polynomials of degree $\le r$ on the triangle $\mathcal{T}_j$, and define the following finite element spaces:

$$\mathbf{X}^r_h = \{u_h \in H^1_0(\Omega)^2 : u_h|_{\mathcal{T}_j} \in P_r(\mathcal{T}_j)^2,\ \forall\,\mathcal{T}_j \in \mathscr{T}_h\}, \tag{2.6}$$

$$M^r_h = \{q_h \in H^1(\Omega) : q_h|_{\mathcal{T}_j} \in P_r(\mathcal{T}_j),\ \forall\,\mathcal{T}_j \in \mathscr{T}_h\}, \tag{2.7}$$

$$\mathring{M}^r_h = \{q_h \in M^r_h : \textstyle\int_\Omega q_h(x)\mathrm{d}x = 0\}. \tag{2.8}$$

Consequently, $\mathbf{X}^2_h \times \mathring{M}^1_h$ is the lowest order Taylor–Hood finite element space, which satisfies the following inf-sup condition (cf. [8, Theorem 4.1])

$$\|q_h\|_{L^2(\Omega)} \le c \sup_{\mathbf{v}_h\in\mathbf{X}^2_h} \frac{|(\nabla q_h,\mathbf{v}_h)|}{\|\nabla\mathbf{v}_h\|_{L^2(\Omega)}}, \quad \forall\,q_h \in \mathring{M}^1_h. \tag{2.9}$$

We choose the finite element space $M^2_h \times \mathbf{X}^2_h \times \mathring{M}^1_h$ for the approximation of the solution $(\rho,\mathbf{u},p)$.

For any given function φ, we define the truncated function

$$\widehat{\varphi} = \max\left(\frac{1}{2}\min_{x\in\overline{\Omega}} \rho^0(x),\varphi\right). \tag{2.10}$$

Then $\widehat{\varphi} \geq \dfrac{1}{2}\min_{x\in\overline{\Omega}}\rho^0(x) > 0$. Let $t_n = n\tau$, $n = 0, 1, \ldots, N$, be a uniform partition of the time interval $[0, T]$ with step size $\tau = T/N$. Based on the variational equations (2.4)-(2.5), we approximate the solution $(\rho(\cdot, t_n), \mathbf{u}(\cdot, t_n), p(\cdot, t_n))$ by the finite element function $(\rho_h^n, \mathbf{u}_h^n, p_h^n) \in M_h^2 \times \mathbf{X}_h^2 \times \mathring{M}_h^1$, defined as the solution of the following equations:

$$(D_\tau\rho_h^n, \varphi_h) + (\mathbf{u}_h^{n-1} \cdot \nabla\rho_h^n, \varphi_h) + \frac{1}{2}(\nabla \cdot \mathbf{u}_h^{n-1}\, \rho_h^n, \varphi_h) = 0, \tag{2.11}$$

$$\begin{aligned}&(\widehat{\rho}_h^{n-1} D_\tau\mathbf{u}_h^n, \mathbf{v}_h) + \frac{1}{2}(D_\tau\widehat{\rho}_h^n\, \mathbf{u}_h^n, \mathbf{v}_h) + \frac{1}{2}(\nabla \cdot (\rho_h^n\mathbf{u}_h^{n-1})\, \mathbf{u}_h^n, \mathbf{v}_h)\\ &+ (\rho_h^n\mathbf{u}_h^{n-1} \cdot \nabla\mathbf{u}_h^n, \mathbf{v}_h) + B((\mathbf{u}_h^n, p_h^n), (\mathbf{v}_h, q_h)) = 0,\end{aligned} \tag{2.12}$$

where $(\varphi_h, \mathbf{v}_h, q_h) \in M_h^2 \times \mathbf{X}_h^2 \times \mathring{M}_h^1$ are test functions, and $D_\tau\rho_h^n := (\rho_h^n - \rho_h^{n-1})/\tau$ denotes the backward Euler difference. The initial data ρ_h^0 and $\mathbf{u}_h^0$ are defined as

$$\rho_h^0 = \Pi_h\rho^0 \ \text{ and } \ \mathbf{u}_h^0 = \mathbf{\Pi}_h\mathbf{u}^0, \tag{2.13}$$

where $\Pi_h : C(\overline{\Omega}) \to M_h^2$ and $\mathbf{\Pi}_h : \mathbf{C}(\overline{\Omega}) \to \mathbf{X}_h^2$ denote the scalar- and vector-valued Lagrange interpolation operators, respectively.

If $(\rho_h^k, \mathbf{u}_h^k)$ are given for $k = 0, \ldots, n-1$, then $(\rho_h^n, \mathbf{u}_h^n)$ can be solved in the following order:
(1) ρ_h^n can be solved from (2.11);
(2) $\widehat{\rho}_h^{n-1}$ and $\widehat{\rho}_h^n$ can be defined by (2.10);
(3) $(\mathbf{u}_h^n, p_h^n)$ can be solved from (2.12).

Remark 2.1. *The stabilization terms $\frac{1}{2}(\nabla \cdot \mathbf{u}_h^{n-1}\, \rho_h^n, \varphi_h)$ and $\frac{1}{2}(D_\tau\widehat{\rho}_h^n\, \mathbf{u}_h^n, \mathbf{v}_h) + \frac{1}{2}(\nabla \cdot (\rho_h^n\mathbf{u}_h^{n-1})\, \mathbf{u}_h^n, \mathbf{v}_h)$ in the numerical scheme (2.11)-(2.12) guarantees that the method is unconditionally stable, i.e., substituting $\varphi_h = \rho_h^n$ and $\mathbf{v}_h = \mathbf{u}_h^n$ into (2.11)-(2.12) yields the following energy equalities:*

$$\begin{aligned}&\|\rho_h^n\|^2 = \|\rho_h^{n-1}\|^2,\\ &\int_\Omega \frac{1}{2}\widehat{\rho}_h^n|\mathbf{u}_h^n|^2\mathrm{d}x + \tau\mu\|\nabla\mathbf{u}_h^n\|^2 = \int_\Omega \frac{1}{2}\widehat{\rho}_h^{n-1}|\mathbf{u}_h^{n-1}|^2\mathrm{d}x.\end{aligned}$$

Since $\widehat{\rho}_h^n$ and $\widehat{\rho}_h^{n-1}$ are bounded from both below and above due to the truncation in (2.10), the two equalities above prove that both ρ_h^n and $\mathbf{u}_h^n$ are bounded in the L^2 norm for all time levels without any restriction on the time stepsize τ and spatial mesh size h.

2.4. Convergence of the numerical solutions

In this work, we assume that the initial density $\rho^0(x)$ is positive (no vacuum) and that problem (1.1)-(1.4) has a sufficiently smooth solution, i.e.,

$$0 < \min_{x\in\overline{\Omega}}\rho^0(x) \leq \max_{x\in\overline{\Omega}}\rho^0(x) < \infty, \tag{2.14}$$

$$\begin{aligned}&\rho \in C([0, T]; H^3(\Omega) \cap W^{2,\infty}(\Omega)) \cap C^2([0, T]; H^2(\Omega)),\\ &\mathbf{u} \in C([0, T]; \mathbf{H}^3(\Omega)) \cap C^1([0, T]; \mathbf{W}^{2,q}(\Omega)) \cap C^2([0, T]; \mathbf{L}^\infty(\Omega)) \quad \text{for some } q > 2,\\ &p \in C([0, T]; H^2(\Omega)),\end{aligned} \tag{2.15}$$

and investigate the convergence of the numerical solutions defined in Section 2.3. In particular, we prove the following theorem.

Theorem 2.1. *Suppose that the solution of* (1.1)-(1.4) *is sufficiently smooth, satisfying* (2.14)-(2.15). *Then the discrete problem* (2.11)-(2.13) *has a unique finite element solution* $(\rho_h^n, \mathbf{u}_h^n, p_h^n) \in M_h^2 \times \mathbf{X}_h^2 \times \mathring{M}_h^1$, $n = 1, 2, \ldots, N$. *Moreover, there exist positive constants* τ_* *and* h_* *such that for* $\tau \le \tau_*$ *and* $h \le h_*$ *the following error estimates hold:*

$$\max_{1\le n\le N} (\|\rho(\cdot, t_n) - \rho_h^n\|_{L^2} + \|\mathbf{u}(\cdot, t_n) - \mathbf{u}_h^n\|_{L^2}) \le c(\tau + h^2), \tag{2.16}$$

$$\left(\tau \sum_{n=1}^{N} \|p(\cdot, t_n) - p_h^n\|_{L^2}^2\right)^{\frac{1}{2}} \le c\sqrt{\tau + h^2}, \tag{2.17}$$

where c *is some positive constant independent of the temporal step size* τ *and spatial mesh size* h *(possibly dependent on* T*).*

Remark 2.2. For the Navier–Stokes equations with constant density, the finite element solutions with the Taylor–Hood elements should have 3$^{\text{rd}}$-order convergence. However, for the model with variable density, finite element solutions of the scalar hyperbolic density equation generally have lower-order convergence (cf. [15, Remark 3.14], 2$^{\text{nd}}$-order or 2.5$^{\text{th}}$-order convergence, depending on the stabilization techniques), which further polluted the order of convergence of the velocity $\mathbf{u}_h^n$ through the coupling of the equations.

Remark 2.3. The truncation $\widehat{\rho}_h^n$ in (2.12) is used only to guarantee the existence and uniqueness of finite element solutions for large step size τ and mesh size h. For sufficiently small τ and h, we have $\widehat{\rho}_h^n = \rho_h^n$ (see the explanation in Section 5.3). Thus the truncation operation does not play a role in deriving the error estimates (2.16)-(2.17).

Remark 2.4. On the one hand, we have assumed that the solution is sufficiently smooth and investigate the convergence of numerical solutions. In particular, the H^3 regularity of solution is essential (due to the hyperbolic equation of ρ) in our analysis to obtain second-order convergence for the numerical solutions. On the other hand, we have assumed that the domain is convex polygonal in order to avoid approximating the boundary by piecewise linear lines (or quadratic curves), which leads to new errors that will make analysis more complicated. It is known that the solution can have H^2 regularity when the initial value is sufficiently smooth in a convex polygon such as rectangle. Whether the solution can have H^3 regularity in a convex polygon is not clear in general. In some special cases, the solution can have H^3 regularity if there are additional source terms in the equation (as shown in the numerical example of Section 6), which will not affect the error analysis in this paper.

Alternatively, if we assume that the domain is smooth and convex (instead of convex polygonal), then the solution can be sufficiently smooth as assumed in (2.14). In this case, the triangulated domain Ω_h is not equal to Ω. For a triangle $\mathcal{T}_j$ with two vertices on the boundary, we can define $\widetilde{\mathcal{T}}_j$ to be the extension of $\mathcal{T}_j$ to a curved triangle which fits the boundary $\partial\Omega$ exactly. For a triangle $\mathcal{T}_j$ with at most one vertex on the boundary, we simply denote $\widetilde{\mathcal{T}}_j = \mathcal{T}_j$. We assume that a diffeomorphic map from $\mathcal{G}_j : \mathcal{T}_j \to \widetilde{\mathcal{T}}_j$ is known, satisfying

$$\|\nabla^l \mathcal{G}_j\|_{L^\infty(\mathcal{T}_j)} \le C \quad \text{and} \quad \|\nabla^l \mathcal{G}_j^{-1}\|_{L^\infty(\widetilde{\mathcal{T}}_j)} \le C, \quad l = 1, \ldots, 3. \tag{2.18}$$

Such a map exists if the domain Ω is convex and the boundary $\partial\Omega$ is sufficiently smooth. We denote by $\mathcal{G} : \Omega_h \to \Omega$ the corresponding global map such that $\mathcal{G}|_{\mathcal{T}_j} = \mathcal{G}_j$. Then the finite element spaces

$$\begin{aligned}
\mathbf{X}_h^r &= \{u_h \in H_0^1(\Omega)^2 : (u_h \circ \mathcal{G})|_{\mathcal{T}_j} \in P_r(\mathcal{T}_j)^2, \ \forall \mathcal{T}_j \in \mathscr{T}_h\} \\
M_h^r &= \{q_h \in H^1(\Omega) : (q_h \circ \mathcal{G})|_{\mathcal{T}_j} \in P_r(\mathcal{T}_j), \ \forall \mathcal{T}_j \in \mathscr{T}_h\} \\
\mathring{M}_h^r &= \{q_h \in M_h^r : \textstyle\int_\Omega q_h(x)\mathrm{d}x = 0\}
\end{aligned}$$

have the same approximation properties (and satisfies the inf-sup condition) as the standard finite element spaces on a convex polygon. As a result, the error analysis presented in this paper also holds for such special finite elements in a smooth and convex domain.

To prove Theorem 2.1, we investigate the temporal and spatial discretizations in Section 4 and Section 5, separately. Throughout this paper, we denote by c a generic positive constant and ϵ is a generic small positive constant, both are independent of n, h and τ, possibly different at each occurrence.

3. An overview for the proof of Theorem 2.1

In this Section, we present an overview for the proof of Theorem 2.1 for the readers' convenience. The proof consists of analysis of temporal and spatial discretizations, respectively, which we briefly introduce below. The details are presented in Sections 4 and 5.

3.1. Analysis of temporal discretization

We introduce a semi-discrete problem

$$
\begin{aligned}
&D_\tau \rho_\tau^n + \mathbf{u}_\tau^{n-1} \cdot \nabla \rho_\tau^n = 0, && (3.1)\\
&\rho_\tau^{n-1} D_\tau \mathbf{u}_\tau^n + \rho_\tau^n \mathbf{u}_\tau^{n-1} \cdot \nabla \mathbf{u}_\tau^n + \nabla p_\tau^n - \mu \Delta \mathbf{u}_\tau^n = 0, && (3.2)\\
&\nabla \cdot \mathbf{u}_\tau^n = 0, && (3.3)
\end{aligned}
$$

with the boundary and initial conditions

$$
\begin{aligned}
&\mathbf{u}_\tau^n = 0 \quad \text{on } \partial\Omega, \quad n = 1, 2, \ldots, N,\\
&\rho_\tau^0 = \rho^0 \quad \text{and} \quad \mathbf{u}_\tau^0 = \mathbf{u}^0.
\end{aligned} \tag{3.4}
$$

Then the fully discrete solution defined by (2.11)-(2.12) can be viewed as the finite element solution of (3.1)-(3.4). The temporal discretization errors are denoted by

$$
e_\rho^n := \rho^n - \rho_\tau^n, \quad \mathbf{e}_\mathbf{u}^n := \mathbf{u}^n - \mathbf{u}_\tau^n \quad \text{and} \quad e_p^n := p^n - p_\tau^n, \tag{3.5}
$$

where

$$
\rho^n = \rho(\cdot, t_n), \quad \mathbf{u}^n = \mathbf{u}(\cdot, t_n), \quad p^n = p(\cdot, t_n).
$$

In Section 4, we prove the following estimates for the temporal discretization errors, together with some regularity estimates for $\rho_\tau^n, \mathbf{u}_\tau^n$ and p_τ^n uniformly with respect to the step size τ.

Proposition 3.1. *Under the assumptions of Theorem 2.1, there exists a positive constant τ_0 such that when $\tau \le \tau_0$, the time-discrete system (3.1)-(3.4) has a unique solution $(\rho_\tau, \mathbf{u}_\tau, p_\tau)$ satisfying the maximum principle*

$$
\min_{x\in\overline{\Omega}} \rho^0(x) \le \rho_\tau^n(x) \le \max_{x\in\overline{\Omega}} \rho^0(x), \quad \forall\, x \in \Omega, \quad n = 1, \ldots, N, \tag{3.6}
$$

the error estimates

$$
\begin{aligned}
&\max_{1\le n\le N} (\|e_\rho^n\|_{H^2} + \|\mathbf{e}_\mathbf{u}^n\|_{H^1} + \|\mathbf{e}_\mathbf{u}^n\|_{L^\infty}) \le c\tau, && (3.7)\\
&\sum_{n=1}^N \tau\big(\|D_\tau \mathbf{e}_\mathbf{u}^n\|_{L^2}^2 + \|\mathbf{e}_\mathbf{u}^n\|_{H^2}^2 + \|e_p^n\|_{H^1}^2\big) \le c\tau^2, && (3.8)
\end{aligned}
$$

and the following regularity estimates:

$$
\max_{0\le n\le N} \big(\|\mathbf{u}_\tau^n\|_{H^2} + \|\mathbf{u}_\tau^n\|_{W^{1,\infty}} + \|p_\tau^n\|_{H^1} + \|\rho_\tau^n\|_{H^2} + \|\rho_\tau^n\|_{W^{1,\infty}} + \|D_\tau \mathbf{u}_\tau^n\|_{L^\infty}\big) \le c, \tag{3.9}
$$

$$\sum_{n=1}^{N} \tau\big(\|D_\tau \mathbf{u}_\tau^n\|_{H^2}^2 + \|D_\tau p_\tau^n\|_{H^1}^2\big) \le c, \tag{3.10}$$

where the constant c is independent of the step size τ.

3.2. **Analysis of spatial discretization**

Proposition 3.1 gives the H^2 and $W^{1,\infty}$ regularity of the time-discrete solutions ρ_τ^n and $\mathbf{u}_\tau^n$ uniformly with respect to the step size τ. By using such uniform regularity of the time-discrete solutions, the following error estimates for the fully discrete finite element solutions will be proved in Section 5.

Proposition 3.2. *Under the assumption of Theorem 2.1, there exist positive constants $\tau_* \le \tau_0$ and h_* such that the fully discrete solutions given by (2.11)-(2.12) satisfy the following estimates when $\tau \le \tau_*$ and $h \le h_*$:*

$$\max_{1\le n\le N} \big(\|\mathbf{u}_\tau^n - \mathbf{u}_h^n\|_{L^2} + \|\rho_\tau^n - \rho_h^n\|_{L^2}\big) + h\bigg(\tau\sum_{n=1}^{N}\|p_\tau^n - p_h^n\|_{L^2}^2\bigg)^{\frac12} + \bigg(\tau\sum_{n=1}^{N}\|\mathbf{u}_\tau^n - \mathbf{u}_h^n\|_{H^1}^2\bigg)^{\frac12}$$
$$\le ch\sqrt{\tau + h^2}. \tag{3.11}$$

Propositions 3.1 and 3.2 together show that, in the case $\tau \le \tau_*$ and $h \le h_*$, the fully discrete finite element solutions obey the following error estimates:

$$\max_{1\le n\le N} \big(\|\mathbf{u}^n - \mathbf{u}_h^n\|_{L^2} + \|\rho^n - \rho_h^n\|_{L^2}\big) \le c(\tau + h^2), \tag{3.12}$$

$$\bigg(\tau\sum_{n=1}^{N}\|p^n - p_h^n\|_{L^2}^2\bigg)^{\frac12} \le c\sqrt{\tau + h^2}. \tag{3.13}$$

The proof of Theorem 2.1 is complete up to the proofs of Propositions 3.1 and 3.2, which are presented in Sections 4 and 5, respectively. □

Remark 3.1. *The factor h (independent of τ) in the energy error estimate (3.11) is the key to make the analysis go through. In fact, in the proof of Proposition 3.2, the uniform boundedness (uniform with respect to the step size τ and mesh size h) of*

$$\|\rho_\tau^n\|_{W^{1,\infty}} \quad \text{and} \quad \tau\sum_{n=1}^{N}\|\mathbf{u}_\tau^n\|_{W^{1,\infty}}^2 \qquad \text{(for the time-discrete solutions)} \tag{3.14}$$

$$\|\rho_h^n\|_{L^\infty}, \quad \|\mathbf{u}_h^n\|_{L^\infty} \quad \text{and} \quad \tau\sum_{n=1}^{N}\|\mathbf{u}_h^n\|_{W^{1,\infty}}^2 \qquad \text{(for the numerical solutions)} \tag{3.15}$$

are needed to control the strong nonlinearities in the coupling of the equations. The boundedness of (3.14) is proved in Proposition 3.1 by using discrete $L^p(W^{2,q})$ estimates. The boundedness of (3.15) can be proved by applying the inverse inequality to the error estimates (3.11), e.g.,

$$\begin{aligned}\|P_h\rho_\tau^n - \rho_h^n\|_{L^\infty} &\le ch^{-1}\|P_h\rho_\tau^n - \rho_h^n\|_{L^2}\\ &\le ch^{-1}(\|P_h\rho_\tau^n - \rho_\tau^n\|_{L^2} + \|\rho_\tau^n - \rho_h^n\|_{L^2})\\ &\le ch^{-1}(ch^2\|\rho_\tau^n\|_{H^2} + ch\sqrt{\tau+h^2})\\ &\le c\sqrt{\tau+h^2},\end{aligned} \tag{3.16}$$

where $P_h\rho_\tau^n$ denotes the L^2 projection of ρ_τ^n onto the finite element space M_h^2. The last inequality requires the factor h in the energy error estimate (3.11).

The proof of (3.11) requires (3.15) to hold at the $(n-1)$th step, and implies (3.15) at the nth step via the inverse inequality, e.g., (3.16). Thus both can be proved by using mathematical induction.

In order to have the error estimate (3.11), we need H^2 and $W^{1,\infty}$ regularity of the time-discrete solutions ρ_τ^n and $\mathbf{u}_\tau^n$ uniformly with respect to the step size τ. This is why we carry out the error estimates of ρ_τ^n and $\mathbf{u}_\tau^n$ in Proposition 3.1 in such strong norms, rather than the standard L^2 norm.

4. Proof of Proposition 3.1

The proof of Proposition 3.1 requires using the maximal L^p-regularity of time-discrete Stokes equations, which will be introduced in Section 4.2. The proof of Proposition 3.1 is presented in Section 4.3.

4.1. **Helmholtz–Weyl decomposition**

We define the divergence-free subspace and curl-free subspace of $\mathbf{L}^q(\Omega)$ as follows (cf. [16, Eq (III.1.4) and Theorem III.2.3]):

$$\mathbf{L}^q_\sigma(\Omega) = \{\mathbf{w} \in \mathbf{L}^q(\Omega) : \nabla\cdot\mathbf{w} = 0,\ \mathbf{w}\cdot\mathbf{n} = 0 \text{ on } \partial\Omega\}, \tag{4.1}$$

$$\mathbf{L}^q_\sigma(\Omega)^\perp = \{\nabla\phi : \phi \in W^{1,q}(\Omega)\}, \tag{4.2}$$

where $\mathbf{n}$ denotes the unit outward normal vector on the boundary $\partial\Omega$. We say that the Helmholtz decomposition exists on $\mathbf{L}^q(\Omega)$ if any function $\mathbf{v} \in \mathbf{L}^q(\Omega)$ has a unique decomposition

$$\mathbf{v} = \mathbf{P}_{\rm div}\mathbf{v} + \nabla\phi \tag{4.3}$$

with $\mathbf{P}_{\rm div}\mathbf{v} \in \mathbf{L}^q_\sigma(\Omega)$ and $\nabla\phi \in \mathbf{L}^q_\sigma(\Omega)^\perp$ such that

$$\|\mathbf{P}_{\rm div}\mathbf{v}\|_{L^q(\Omega)} + \|\nabla\phi\|_{L^q(\Omega)} \le C\|\mathbf{v}\|_{L^q(\Omega)}. \tag{4.4}$$

Here $\mathbf{P}_{\rm div} : \mathbf{L}^q(\Omega) \to \mathbf{L}^q_\sigma(\Omega)$ is called the Helmholtz projection (if the Helmholtz decomposition exists).

In fact, the Helmholtz decomposition on $\mathbf{L}^q(\Omega)$ exists if and only if the Neumann problem

$$\begin{cases} \Delta\phi = \nabla\cdot\mathbf{v} & \text{in } \Omega,\\ \nabla\phi\cdot\mathbf{n} = \mathbf{v}\cdot\mathbf{n} & \text{on } \partial\Omega, \end{cases} \tag{4.5}$$

has a unique weak solution $\phi \in W^{1,q}(\Omega)\backslash\mathbb{R}$ satisfying

$$\|\nabla\phi\|_{L^q(\Omega)} \le C\|\mathbf{v}\|_{L^q(\Omega)}, \quad \forall\, \mathbf{v} \in \mathbf{L}^q(\Omega). \tag{4.6}$$

Then $\mathbf{P}_{\rm div}\mathbf{v} = \mathbf{v} - \nabla\phi$ with ϕ given by (4.5).

In a smooth and convex domain, the $W^{1,q}$ solution of (4.5) exists (see [19, Theorem 1.2]) and the Helmholtz decomposition (4.4) exists on $\mathbf{L}^q(\Omega)$ for all $1 < q < \infty$. (see [19, Theorem 1.2] and [16, Lemma III.1.2]). Therefore, the operator $\mathbf{P}_{\rm div}$ is a bounded linear projection onto the divergence-free subspace $\mathbf{L}^q_\sigma(\Omega)$, satisfying

$$\|\mathbf{P}_{\rm div}\mathbf{v}\|_{L^q} \le c\|\mathbf{v}\|_{L^q}, \quad \forall\, \mathbf{v} \in \mathbf{L}^q(\Omega).$$

In particular, we have

$$\begin{aligned} &\mathbf{P}_{\rm div}\mathbf{v} = \mathbf{v}, && \forall\, \mathbf{v} \in \mathbf{L}^q_\sigma(\Omega),\\ &\mathbf{P}_{\rm div}(\nabla\phi) = 0, && \forall\, \phi \in W^{1,q}(\Omega). \end{aligned}$$

4.2. Discrete maximal L^p-regularity

In [14], it was shown that the operator $\mathbf{P}_{\rm div}\Delta$ has the maximal L^p-regularity on $\mathbf{L}^q_\sigma(\Omega)$, for $1<p,q<\infty$. Then [30, Theorem 3.1] (for the initial term involving v_0, see Remark 4.3) immediately implies the following result of discrete maximal L^p-regularity.

Lemma 4.1 (Discrete maximal L^p-regularity). *Let $1\le m\le N$, and let $a(x)$ be a function defined on Ω such that*

(1) $\kappa_0\le a(x)\le\kappa_1$ for some positive constants κ_0 and κ_1;
(2) $\|a\|_{W^{1,\sigma}(\Omega)}\le\kappa_2$ for some constants $\sigma>2$ and $\kappa_2>0$.

Then the solutions of the equations

$$a(x)D_\tau\mathbf{v}^n_\tau-\mathbf{P}_{\rm div}\Delta\mathbf{v}^n_\tau=\boldsymbol{f}^n,\quad n=1,2,\dots,m, \tag{4.7}$$

satisfy the following estimate:

$$\begin{aligned}&\|(D_\tau\mathbf{v}^n_\tau)_{n=1}^m\|_{L^p(L^q)}+\|(\mathbf{P}_{\rm div}\Delta\mathbf{v}^n_\tau)_{n=1}^m\|_{L^p(L^q)}\\&\le c\big(\|(\boldsymbol{f}^n)_{n=1}^m\|_{L^p(L^q)}+\tau^{\frac1p-1}\|\mathbf{v}^0_\tau\|_{L^q}+\tau^{\frac1p}\|\mathbf{P}_{\rm div}\Delta\mathbf{v}^0_\tau\|_{L^q}\big),\quad\forall\,1<p,q<\infty,\end{aligned} \tag{4.8}$$

where c is independent of τ, m and $a(x)$ (but may depend on κ_0, κ_1, κ_2, σ, p and q).

However, Lemma 4.1 cannot be directly applied to the physical equations considered in this paper, which requires analysis of the following type of equations:

$$\rho^{n-1}_\tau D_\tau\mathbf{u}^n_\tau-\mathbf{P}_{\rm div}\Delta\mathbf{u}^n_\tau=\boldsymbol{f}^n, \tag{4.9}$$

with the coefficient ρ^{n-1}_τ depending on n. In the following lemma, we extend Lemma 4.1 to this setting by using the ideas of [3,4].

Lemma 4.2 (Extension to time-dependent coefficients). *Let $1\le m\le N$, and let $a^{n-1}_\tau(x)$, $n=1,\dots,m$, be functions defined on Ω such that*

(1) $\kappa_0\le a^{n-1}_\tau(x)\le\kappa_1$ for some positive constants κ_0 and κ_1;
(2) $\max\limits_{1\le n\le m}\|a^{n-1}_\tau\|_{W^{1,\sigma}(\Omega)}\le\kappa_2$ for some constants $\sigma>2$ and $\kappa_2>0$;
(3) $\sum\limits_{n=1}^{m-1}\|a^n_\tau-a^{n-1}_\tau\|_{L^\infty}\le\kappa$.

Then the solution $\mathbf{v}^n_\tau$ of the equations

$$a^{n-1}_\tau D_\tau\mathbf{v}^n_\tau-\mathbf{P}_{\rm div}\Delta\mathbf{v}^n_\tau=\boldsymbol{f}^n,\quad n=1,\dots,m, \tag{4.10}$$

satisfies

$$\begin{aligned}&\|(D_\tau\mathbf{v}^n_\tau)_{n=1}^m\|_{L^p(L^q)}+\|(\mathbf{P}_{\rm div}\Delta\mathbf{v}^n_\tau)_{n=1}^m\|_{L^p(L^q)}\\&\le c(\|(\boldsymbol{f}^n)_{n=1}^m\|_{L^p(L^q)}+\tau^{\frac1p-1}\|\mathbf{v}^0_\tau\|_{L^q}+\tau^{\frac1p}\|\mathbf{P}_{\rm div}\Delta\mathbf{v}^0_\tau\|_{L^q}),\quad\forall\,1<p,q<\infty,\end{aligned} \tag{4.11}$$

where the constant c is independent of τ, m and a^{n-1}_τ, $n=1,\dots,m$, (but may depend on κ_0, κ_1, κ_2, σ, κ, p, q and T).

Proof. For $k=1,\dots,m$, the equation (4.10) can be rewritten as

$$a^{k-1}_\tau D_\tau\mathbf{v}^n_\tau-\mathbf{P}_{\rm div}\Delta\mathbf{v}^n_\tau=\boldsymbol{f}^n+(a^{k-1}_\tau-a^{n-1}_\tau)D_\tau\mathbf{v}^n_\tau. \tag{4.12}$$

Applying Lemma 4.1 to (4.12) yields

$$\begin{aligned} &\|(D_\tau \mathbf{v}_\tau^n)_{n=1}^k\|_{L^p(L^q)} + \|(\mathbf{P}_{\mathrm{div}} \Delta \mathbf{v}_\tau^n)_{n=1}^k\|_{L^p(L^q)} \\ &\le c(\|(\boldsymbol{f}^n)_{n=1}^k\|_{L^p(L^q)} + \tau^{\frac{1}{p}-1}\|\mathbf{v}_\tau^0\|_{L^q} + \tau^{\frac{1}{p}}\|\mathbf{P}_{\mathrm{div}}\Delta \mathbf{v}_\tau^0\|_{L^q}) \\ &\qquad\qquad + c\|((a_\tau^{k-1} - a_\tau^{n-1})D_\tau \mathbf{v}_\tau^n)_{n=1}^k\|_{L^p(L^q)}. \end{aligned} \tag{4.13}$$

Let $E_0 := 0$ and we define

$$\begin{aligned} E_k :&= \|(D_\tau \mathbf{v}_\tau^n)_{n=1}^k\|_{L^p(L^q)}^p + \|(\mathbf{P}_{\mathrm{div}} \Delta \mathbf{v}_\tau^n)_{n=1}^k\|_{L^p(L^q)}^p \\ &= \tau \sum_{n=1}^k \|D_\tau \mathbf{v}_\tau^n\|_{L^q}^p + \tau \sum_{n=1}^k \|\mathbf{P}_{\mathrm{div}}\Delta \mathbf{v}_\tau^n\|_{L^q}^p, \qquad \text{for } k = 1, \ldots, m. \end{aligned} \tag{4.14}$$

Then we have

$$\begin{aligned} \left\| \left((a_\tau^{k-1} - a_\tau^{n-1})D_\tau \mathbf{v}_\tau^n\right)_{n=1}^k \right\|_{L^p(L^q)}^p =& \tau \sum_{n=1}^k \left\|(a_\tau^{k-1} - a_\tau^{n-1})D_\tau \mathbf{v}_\tau^n\right\|_{L^q}^p \\ \le& \tau \sum_{n=1}^k \|a_\tau^{k-1} - a_\tau^{n-1}\|_{L^\infty}^p \|D_\tau \mathbf{v}_\tau^n\|_{L^q}^p \\ \le& \tau \sum_{n=1}^k \|a_\tau^{k-1} - a_\tau^{n-1}\|_{L^\infty}^p (\|D_\tau \mathbf{v}_\tau^n\|_{L^q}^p + \|\mathbf{P}_{\mathrm{div}}\Delta \mathbf{v}_\tau^n\|_{L^q}^p) \\ =& \sum_{n=1}^k \|a_\tau^{k-1} - a_\tau^{n-1}\|_{L^\infty}^p (E_n - E_{n-1}) \\ =& \sum_{n=1}^{k-1} (\|a_\tau^{k-1} - a_\tau^{n-1}\|_{L^\infty}^p - \|a_\tau^{k-1} - a_\tau^{n}\|_{L^\infty}^p) E_n \\ \le& c \sum_{n=1}^{k-1} \|a_\tau^{n-1} - a_\tau^n\|_{L^\infty} E_n. \end{aligned} \tag{4.15}$$

By denoting $F_k := \|(\boldsymbol{f}_n)_{n=1}^k\|_{L^p(L^q)}^p + \left(\tau^{\frac{1}{p}-1}\|\mathbf{v}_\tau^0\|_{L^q} + \tau^{\frac{1}{p}}\|\mathbf{P}_{\mathrm{div}}\Delta \mathbf{v}_\tau^0\|_{L^q}\right)^p$ and substituting the last inequality into the p^{th} power of (4.13), we obtain

$$E_k \le c \sum_{n=1}^{k-1} \|a_\tau^{n-1} - a_\tau^n\|_{L^\infty} E_n + cF_k, \quad k = 1, \ldots, m. \tag{4.16}$$

Then applying Gronwall's inequality (cf. [28, Lemma 5.1]) yields

$$E_m \le cF_m. \tag{4.17}$$

The proof of Lemma 4.2 is complete. □

4.3. **Proof of Proposition 3.1**

First, we assume that $\rho_\tau^n \in H^2(\Omega) \cap L^\infty(\Omega)$ and $\mathbf{u}_\tau^n \in \mathbf{W}^{1,\infty}(\Omega) \cap \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$ are given for $n = 0, \ldots, m-1$, such that $\nabla \cdot \mathbf{u}_\tau^n = 0$ and the following inequalities hold for $0 \le n \le m-1$ (induction assumption):

$$\|\mathbf{u}_\tau^n\|_{W^{1,\infty}} + \|\rho_\tau^n\|_{W^{1,\infty}} \le \|\mathbf{u}\|_{C([0,T];W^{1,\infty})} + \|\rho\|_{C([0,T];W^{1,\infty})} + 1, \tag{4.18}$$

$$\min_{x\in\overline{\Omega}} \rho_\tau^0(x) \le \rho_\tau^n(x) \le \max_{x\in\overline{\Omega}} \rho_\tau^0(x), \quad \forall\, x \in \Omega. \tag{4.19}$$

Under this assumption, for $n = m$ the hyperbolic equation (3.1) and the parabolic equations (3.2)-(3.3) have a unique solution (the proof is given in Appendix B)

$$\rho_\tau^m \in H^2(\Omega) \cap L^\infty(\Omega), \tag{4.20}$$
$$(\mathbf{u}_\tau^m, p_\tau^m) \in (\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)) \times (H^1(\Omega) \cap L_0^2(\Omega)), \tag{4.21}$$

respectively, obeying the following maximum principle:

$$\min_{x\in\overline{\Omega}} \rho_\tau^{m-1}(x) \le \rho_\tau^m(x) \le \max_{x\in\overline{\Omega}} \rho_\tau^{m-1}(x), \quad \forall\, x \in \Omega. \tag{4.22}$$

which shows that (4.19) also holds for $n = m$.

Next, we prove (4.18) for $n = m$ to complete the mathematical induction. In the mean time, we also obtain the estimates (3.6)-(3.10) in Proposition 3.1. To this end, we keep the generic constant c of this Section to be independent of m (but may depend on T). The proof consists of two parts: L^2-type estimates and $W^{1,\infty}$ estimates.

Part I: L^2, H^1 and H^2 estimates

We compare the time-discrete solution $(\rho_\tau^n, \mathbf{u}_\tau^n, p_\tau^n)$ with the PDE's solution $(\rho^n, \mathbf{u}^n, p^n)$. The latter satisfies the following equations:

$$D_\tau \rho^n + \mathbf{u}^{n-1} \cdot \nabla \rho^n = E^n, \tag{4.23}$$
$$\rho^{n-1} D_\tau \mathbf{u}^n + \rho^n \mathbf{u}^{n-1} \cdot \nabla \mathbf{u}^n + \nabla p^n - \mu \Delta \mathbf{u}^n = \mathbf{F}^n, \tag{4.24}$$
$$\nabla \cdot \mathbf{u}^n = 0, \tag{4.25}$$

where

$$E^n = D_\tau \rho^n - \partial_t \rho^n + (\mathbf{u}^{n-1} - \mathbf{u}^n) \cdot \nabla \rho^n \tag{4.26}$$

and

$$\mathbf{F}^n = \rho^{n-1} D_\tau \mathbf{u}^n - \rho^n \partial_t \mathbf{u}^n + \rho^n (\mathbf{u}^{n-1} - \mathbf{u}^n) \cdot \nabla \mathbf{u}^n \tag{4.27}$$

are truncation errors of the time discretization. Under the regularity assumption (2.15), we have

$$\|E^n\|_{L^\infty} + \|E^n\|_{H^2} + \|\mathbf{F}^n\|_{L^\infty} \le c\tau. \tag{4.28}$$

Using the notations of (3.5), subtracting (3.1)-(3.3) from (4.23)-(4.25) yields

$$D_\tau e_\rho^n + \mathbf{u}_\tau^{n-1} \cdot \nabla e_\rho^n + \mathbf{e}_\mathbf{u}^{n-1} \cdot \nabla \rho^n = E^n, \tag{4.29}$$

$$\begin{aligned}\rho_\tau^{n-1} D_\tau \mathbf{e}_\mathbf{u}^n + e_\rho^{n-1} D_\tau \mathbf{u}^n + e_\rho^n \mathbf{u}^{n-1} \cdot \nabla \mathbf{u}^n + \rho_\tau^n \mathbf{e}_\mathbf{u}^{n-1} \cdot \nabla \mathbf{u}^n + \rho_\tau^n \mathbf{u}_\tau^{n-1} \cdot \nabla \mathbf{e}_\mathbf{u}^n& \\ + \nabla e_p^n - \mu \Delta \mathbf{e}_\mathbf{u}^n = \mathbf{F}^n&,\end{aligned} \tag{4.30}$$

$$\nabla \cdot \mathbf{e}_\mathbf{u}^n = 0. \tag{4.31}$$

For a positive number $k \geq 1$, integrating (4.29) against $|e_\rho^n|^{k-1}e_\rho^n$ yields

$$\begin{aligned}
&\frac{\|e_\rho^n\|_{L^{k+1}}^{k+1} - (e_\rho^{n-1}, |e_\rho^n|^{k-1}e_\rho^n)}{\tau} + \int_\Omega \nabla\cdot\Big(\mathbf{u}_\tau^{n-1}\frac{1}{k+1}|e_\rho^n|^{k+1}\Big)\mathrm{d}x \\
&= (E^n - \mathbf{e}_\mathbf{u}^{n-1}\cdot\nabla\rho^n, |e_\rho^n|^{k-1}e_\rho^n) \\
&\leq \|E^n\|_{L^{k+1}}\|e_\rho^n\|_{L^{k+1}}^k + \|\mathbf{e}_\mathbf{u}^{n-1}\|_{L^{k+1}}\|\nabla\rho^n\|_{L^\infty}\|e_\rho^n\|_{L^{k+1}}^k,
\end{aligned} \tag{4.32}$$

where the last step is due to Hölder's inequality: $|(f,g)| \leq \|f\|_{L^{k+1}}\|g\|_{L^{\frac{k+1}{k}}}$. Since

$$\int_\Omega \nabla\cdot\big(\mathbf{u}_\tau^{n-1}\frac{1}{k+1}|e_\rho^n|^{k+1}\big)\mathrm{d}x = 0,$$

$$\frac{\|e_\rho^n\|_{L^{k+1}}^{k+1} - (e_\rho^{n-1}, |e_\rho^n|^{k-1}e_\rho^n)}{\tau} \geq \frac{\|e_\rho^n\|_{L^{k+1}}^{k+1} - \|e_\rho^{n-1}\|_{L^{k+1}}\|e_\rho^n\|_{L^{k+1}}^k}{\tau}, \quad \text{(Hölder's inequality again)}$$

it follows that (4.32) reduces to

$$\frac{\|e_\rho^n\|_{L^{k+1}} - \|e_\rho^{n-1}\|_{L^{k+1}}}{\tau} \leq c(\|E^n\|_{L^{k+1}} + \|\mathbf{e}_\mathbf{u}^{n-1}\|_{L^{k+1}}). \tag{4.33}$$

By summing up the last inequality for $n = 0, 1, 2, \ldots, m$, we obtain

$$\max_{0\leq n\leq m}\|e_\rho^n\|_{L^{k+1}} \leq c\sum_{n=1}^m \tau\big(\|E^n\|_{L^{k+1}} + \|\mathbf{e}_\mathbf{u}^{n-1}\|_{L^{k+1}}\big). \tag{4.34}$$

Taking $k \to \infty$ in the last inequality yields

$$\begin{aligned}
\max_{0\leq n\leq m}\|e_\rho^n\|_{L^\infty} &\leq c\sum_{n=1}^m \tau\big(\|E^n\|_{L^\infty} + \|\mathbf{e}_\mathbf{u}^{n-1}\|_{L^\infty}\big) \\
&\leq c\tau + c\sum_{n=1}^m \tau\|\mathbf{e}_\mathbf{u}^{n-1}\|_{L^\infty},
\end{aligned} \tag{4.35}$$

where we have used (4.28) in the last inequality.

The maximum principle (4.22) implies

$$\min_{x\in\overline{\Omega}}\rho^0(x) \leq \rho_\tau^n \leq \max_{x\in\overline{\Omega}}\rho^0(x). \tag{4.36}$$

Since $(\nabla e_p^n, D_\tau\mathbf{e}_\mathbf{u}^n) = -(e_p^n, D_\tau\nabla\cdot\mathbf{e}_\mathbf{u}^n) = 0$, multiplying (4.30) by $D_\tau\mathbf{e}_\mathbf{u}^n$ and integrating the result over Ω yield

$$\begin{aligned}
&D_\tau\Big(\frac{\mu}{2}\|\nabla\mathbf{e}_\mathbf{u}^n\|_{L^2}^2\Big) + \Big\|\sqrt{\rho_\tau^{n-1}}D_\tau\mathbf{e}_\mathbf{u}^n\Big\|_{L^2}^2 \\
&\leq |(e_\rho^{n-1}D_\tau\mathbf{u}^n, D_\tau\mathbf{e}_\mathbf{u}^n)| + |(e_\rho^n\mathbf{u}^{n-1}\cdot\nabla\mathbf{u}^n, D_\tau\mathbf{e}_\mathbf{u}^n)| + |(\rho_\tau^n\mathbf{e}_\mathbf{u}^{n-1}\cdot\nabla\mathbf{u}^n, D_\tau\mathbf{e}_\mathbf{u}^n)| \\
&\quad + |(\rho_\tau^n\mathbf{u}_\tau^{n-1}\cdot\nabla\mathbf{e}_\mathbf{u}^n, D_\tau\mathbf{e}_\mathbf{u}^n)| + |(\mathbf{F}^n, D_\tau\mathbf{e}_\mathbf{u}^n)| \\
&\leq \epsilon\|D_\tau\mathbf{e}_\mathbf{u}^n\|_{L^2}^2 + c\epsilon^{-1}\big(\|e_\rho^{n-1}\|_{L^2}^2\|D_\tau\mathbf{u}^n\|_{L^\infty}^2 + \|e_\rho^n\|_{L^2}^2\|\mathbf{u}^{n-1}\|_{L^\infty}^2\|\nabla\mathbf{u}^n\|_{L^\infty}^2\big) \\
&\quad + c\epsilon^{-1}\big(\|\rho_\tau^n\|_{L^\infty}^2\|\mathbf{e}_\mathbf{u}^{n-1}\|_{L^2}^2\|\nabla\mathbf{u}^n\|_{L^\infty}^2 + \|\rho_\tau^n\|_{L^\infty}^2\|\mathbf{u}_\tau^{n-1}\|_{L^\infty}^2\|\nabla\mathbf{e}_\mathbf{u}^n\|_{L^2}^2\big) + c\epsilon^{-1}\|\mathbf{F}^n\|_{L^2}^2.
\end{aligned} \tag{4.37}$$

By choosing a sufficiently small constant ϵ and using (4.18) and (4.36) to estimate $\|\rho_\tau^n\|_{L^\infty}$ and $\|\mathbf{u}_\tau^{n-1}\|_{L^\infty}$, we obtain

$$\begin{aligned}&\frac{1}{2}D_\tau\Big(\frac{\mu}{2}\|\nabla\mathbf{e}_{\mathbf{u}}^n\|_{L^2}^2\Big)+c^{-1}\big\|D_\tau\mathbf{e}_{\mathbf{u}}^n\big\|_{L^2}^2\\&\le c\big(\|e_\rho^{n-1}\|_{L^2}^2+\|e_\rho^n\|_{L^2}^2+\|\mathbf{e}_{\mathbf{u}}^{n-1}\|_{L^2}^2+\|\nabla\mathbf{e}_{\mathbf{u}}^n\|_{L^2}^2+\|\mathbf{F}^n\|_{L^2}^2\big),\quad n=1,\dots,m.\end{aligned}\tag{4.38}$$

Then, substituting (4.34) (with $k=1$) into the inequality above and summing up the resulting inequalities for $n=1,\dots,\ell$, (with $\ell\le m$) we have

$$\begin{aligned}&\max_{1\le n\le\ell}\|\nabla\mathbf{e}_{\mathbf{u}}^n\|_{L^2}^2+\sum_{n=1}^{\ell}\tau\big\|D_\tau\mathbf{e}_{\mathbf{u}}^n\big\|_{L^2}^2\\&\le c\sum_{n=1}^{\ell}\tau\big(\|\mathbf{e}_{\mathbf{u}}^{n-1}\|_{L^2}^2+\|\nabla\mathbf{e}_{\mathbf{u}}^n\|_{L^2}^2+\|E^n\|_{L^2}^2+\|\mathbf{F}^n\|_{L^2}^2\big)\\&\le c\sum_{n=1}^{\ell}\tau\big(\|\nabla\mathbf{e}_{\mathbf{u}}^n\|_{L^2}^2+\|E^n\|_{L^2}^2+\|\mathbf{F}^n\|_{L^2}^2\big),\qquad \ell=1,\dots,m,\end{aligned}\tag{4.39}$$

where we have used the inequality $\|\mathbf{e}_{\mathbf{u}}^{n-1}\|_{L^2}\le c\|\nabla\mathbf{e}_{\mathbf{u}}^{n-1}\|_{L^2}$ to derive the last inequality. When the step size τ is smaller than some constant, the last inequality reduces to (through applying Gronwall's inequality)

$$\max_{1\le n\le m}\|\nabla\mathbf{e}_{\mathbf{u}}^n\|_{L^2}^2+\sum_{n=1}^{m}\tau\|D_\tau\mathbf{e}_{\mathbf{u}}^n\|_{L^2}^2\le c\sum_{n=1}^{m}\tau(\|E^n\|_{L^2}^2+\|\mathbf{F}^n\|_{L^2}^2)\le c\tau^2,\tag{4.40}$$

where we have used (4.28) in the last inequality. Substituting the last inequality into (4.34) (with $k=1$) yields

$$\max_{0\le n\le m}\|e_\rho^n\|_{L^2}\le c\tau.\tag{4.41}$$

In order to derive an H^2 estimate of $\mathbf{e}_{\mathbf{u}}^n$, we consider (4.30)-(4.31), which can be rewritten as

$$\begin{aligned}-\mu\Delta\mathbf{e}_{\mathbf{u}}^n+\nabla e_p^n=&\big(\mathbf{F}^n-\rho_\tau^n\mathbf{e}_{\mathbf{u}}^{n-1}\cdot\nabla\mathbf{u}^n-\rho_\tau^n\mathbf{u}_\tau^{n-1}\cdot\nabla\mathbf{e}_{\mathbf{u}}^n\big)\\&+\big(-e_\rho^n\mathbf{u}^{n-1}\cdot\nabla\mathbf{u}^n-e_\rho^{n-1}D_\tau\mathbf{u}^n-\rho_\tau^{n-1}D_\tau\mathbf{e}_{\mathbf{u}}^n\big),\end{aligned}\tag{4.42}$$

$$\nabla\cdot\mathbf{e}_{\mathbf{u}}^n=0.\tag{4.43}$$

The standard H^2 estimate of Stokes equations (cf. [29]) implies

$$\begin{aligned}\sum_{n=1}^{m}\tau\|\Delta\mathbf{e}_{\mathbf{u}}^n\|_{L^2}^2\le&\sum_{n=1}^{m}\tau\Big\|\mathbf{F}^n-\rho_\tau^n\mathbf{e}_{\mathbf{u}}^{n-1}\cdot\nabla\mathbf{u}^n-\rho_\tau^n\mathbf{u}_\tau^{n-1}\cdot\nabla\mathbf{e}_{\mathbf{u}}^n\\&\qquad-e_\rho^n\mathbf{u}^{n-1}\cdot\nabla\mathbf{u}^n-e_\rho^{n-1}D_\tau\mathbf{u}^n-\rho_\tau^{n-1}D_\tau\mathbf{e}_{\mathbf{u}}^n\Big\|_{L^2}^2\\\le&\sum_{n=1}^{m}\tau\Big(\|\mathbf{F}^n\|_{L^2}+\|\rho_\tau^n\|_{L^\infty}\|\mathbf{e}_{\mathbf{u}}^{n-1}\|_{L^2}\|\nabla\mathbf{u}^n\|_{L^\infty}\\&\qquad+\|\rho_\tau^n\|_{L^\infty}\|\mathbf{u}_\tau^{n-1}\|_{L^\infty}\|\nabla\mathbf{e}_{\mathbf{u}}^n\|_{L^2}+\|e_\rho^n\|_{L^2}\|\mathbf{u}^{n-1}\|_{L^\infty}\|\nabla\mathbf{u}^n\|_{L^\infty}\\&\qquad+\|e_\rho^{n-1}\|_{L^2}\|D_\tau\mathbf{u}^n\|_{L^\infty}+\|\rho_\tau^{n-1}\|_{L^\infty}\|D_\tau\mathbf{e}_{\mathbf{u}}^n\|_{L^2}\Big)^2\end{aligned}$$

$$
\begin{aligned}
&\leq c\sum_{n=1}^{m}\tau\Big(\|\mathbf{F}^n\|_{L^2}^2+\|\mathbf{e}_{\mathbf{u}}^{n-1}\|_{L^2}^2+\|\nabla\mathbf{e}_{\mathbf{u}}^n\|_{L^2}^2+\|e_\rho^n\|_{L^2}^2+\|D_\tau\mathbf{e}_{\mathbf{u}}^n\|_{L^2}^2\Big)\\
&\leq c\tau^2, \qquad (4.44)
\end{aligned}
$$

where we have used the induction assumption (4.18) to control $\|\mathbf{u}_\tau^{n-1}\|_{L^\infty}$, (4.36) to control $\|\rho_\tau^n\|_{L^\infty}$, and (4.40)-(4.41) to obtain the last inequality. The inequality above implies the following H^2-norm estimate via the elliptic regularity (for the function $\mathbf{e}_{\mathbf{u}}^n$ satisfying the homogeneous Dirichlet boundary condition $\mathbf{e}_{\mathbf{u}}^n=0$ on $\partial\Omega$):

$$
\sum_{n=1}^{m}\tau\|\mathbf{e}_{\mathbf{u}}^n\|_{H^2}^2\leq c\sum_{n=1}^{m}\tau\|\Delta\mathbf{e}_{\mathbf{u}}^n\|_{L^2}^2\leq c\tau^2, \qquad (4.45)
$$

which further implies

$$
\sum_{n=1}^{m}\tau\|D_\tau\mathbf{e}_{\mathbf{u}}^n\|_{H^2}^2\leq\sum_{n=1}^{m}\tau\frac{2\|\mathbf{e}_{\mathbf{u}}^n\|_{H^2}^2+2\|\mathbf{e}_{\mathbf{u}}^n\|_{H^2}^2}{\tau^2}\leq c. \qquad (4.46)
$$

Note that (4.45) also implies $\|\mathbf{e}_{\mathbf{u}}^n\|_{H^2}\leq c\tau^{\frac12}$ and, as a consequence,

$$
\|\mathbf{u}_\tau^n\|_{H^2}^2\leq\|\mathbf{u}^n\|_{H^2}^2+\|\mathbf{e}_{\mathbf{u}}^n\|_{H^2}\leq\|\mathbf{u}^n\|_{H^2}^2+c\tau^{\frac12}\leq c, \qquad (4.47)
$$

which will be needed in Part II.

Part II: L^∞ and $W^{1,\infty}$ estimates

Since $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$, the H^2 estimate (4.45) implies

$$
\begin{aligned}
\sum_{n=1}^{m}\tau\|\mathbf{e}_{\mathbf{u}}^n\|_{L^\infty}&\leq\Big(\sum_{n=1}^{m}\tau\Big)^{\frac12}\Big(\sum_{n=1}^{m}\tau\|\mathbf{e}_{\mathbf{u}}^n\|_{L^\infty}^2\Big)^{\frac12} \qquad \text{(Hölder's inequality)}\\
&\leq cT^{\frac12}\Big(\sum_{n=1}^{m}\tau\|\mathbf{e}_{\mathbf{u}}^n\|_{H^2}^2\Big)^{\frac12}\leq c\tau. \qquad (4.48)
\end{aligned}
$$

Substituting the last inequality into (4.35) yields

$$
\max_{0\leq n\leq m}\|e_\rho^n\|_{L^\infty}\leq c\tau. \qquad (4.49)
$$

In view of Section 4.1, we have

$$
\mathbf{P}_{\rm div}(\rho_\tau^{n-1}D_\tau\mathbf{e}_{\mathbf{u}}^n)=\rho_\tau^{n-1}D_\tau\mathbf{e}_{\mathbf{u}}^n-\nabla\phi^n, \qquad (4.50)
$$

where $\phi\in W^{1,q}(\Omega)$, with $\int_\Omega\phi^n\,\mathrm{d}x=0$, is the solution of the equation

$$
\begin{cases}
\Delta\phi^n=\nabla\cdot(\rho_\tau^{n-1}D_\tau\mathbf{e}_{\mathbf{u}}^n) & \text{in } \Omega,\\
\nabla\phi^n\cdot\mathbf{n}=\rho_\tau^{n-1}D_\tau\mathbf{e}_{\mathbf{u}}^n\cdot\mathbf{n} & \text{on } \partial\Omega.
\end{cases} \qquad (4.51)
$$

Since $\nabla\cdot\mathbf{e}_{\mathbf{u}}^n=0$ in Ω and $\mathbf{e}_{\mathbf{u}}^n=0$ on $\partial\Omega$, the last equation is equivalent to

$$
\begin{cases}
\Delta\phi^n=\nabla\rho_\tau^{n-1}\cdot D_\tau\mathbf{e}_{\mathbf{u}}^n & \text{in } \Omega,\\
\nabla\phi^n\cdot\mathbf{n}=0 & \text{on } \partial\Omega,
\end{cases} \qquad (4.52)
$$

whose solution satisfies the following standard H^2 estimate (cf. [22, Theorem 3.1.3.1]):

$$\|\phi^n\|_{H^2} \leq c\|\nabla\rho_\tau^{n-1} \cdot D_\tau \mathbf{e}_\mathbf{u}^n\|_{L^2}. \tag{4.53}$$

By the two-dimensional Sobolev embedding $H^2(\Omega) \hookrightarrow W^{1,q}(\Omega)$, $\forall\, 2 < q < \infty$, we have

$$\|\nabla\phi^n\|_{L^q} \leq c\|\phi^n\|_{H^2} \leq c\|\rho_\tau^{n-1}\|_{W^{1,\infty}}\|D_\tau \mathbf{e}_\mathbf{u}^n\|_{L^2}. \tag{4.54}$$

The last inequality, together with (4.18) and (4.40), implies

$$\|(\nabla\phi^n)_{n=1}^m\|_{L^2(L^q)}^2 = \sum_{n=1}^m \tau\|\nabla\phi^n\|_{L^q}^2 \leq c\sum_{n=1}^m \tau\|D_\tau \mathbf{e}_\mathbf{u}^n\|_{L^2}^2 \leq c\tau^2, \quad \forall\, 2 < q < \infty. \tag{4.55}$$

By using (4.50), equation (4.42) can be rewritten in the following form:

$$\begin{aligned}\rho_\tau^{n-1} D_\tau \mathbf{e}_\mathbf{u}^n - \mu \mathbf{P}_{\rm div}\Delta \mathbf{e}_\mathbf{u}^n =& \mathbf{P}_{\rm div}\big(\mathbf{F}^n - \rho_\tau^n \mathbf{e}_\mathbf{u}^{n-1}\cdot\nabla\mathbf{u}^n - \rho_\tau^n \mathbf{u}_\tau^{n-1}\cdot\nabla\mathbf{e}_\mathbf{u}^n\big)\\ &+ \mathbf{P}_{\rm div}\big(- e_\rho^n \mathbf{u}^{n-1}\cdot\nabla\mathbf{u}^n - e_\rho^{n-1} D_\tau \mathbf{u}^n\big).\end{aligned} \tag{4.56}$$

Applying Lemma 4.2 (with $p = 2$) to the above equation yields

$$\begin{aligned}&\|(D_\tau \mathbf{e}_\mathbf{u}^n)_{n=1}^m\|_{L^2(L^q)} + \|(\mathbf{P}_{\rm div}\Delta \mathbf{e}_\mathbf{u}^n)_{n=1}^m\|_{L^2(L^q)} && (4.57)\\ &\leq c\|(\nabla\phi^n)_{n=1}^m\|_{L^2(L^q)} + c\big\|\big(\mathbf{F}^n - \rho_\tau^n \mathbf{e}_\mathbf{u}^{n-1}\cdot\nabla\mathbf{u}^n - \rho_\tau^n \mathbf{u}_\tau^{n-1}\cdot\nabla \mathbf{e}_\mathbf{u}^n\big)_{n=1}^m\big\|_{L^2(L^q)}\\ &\qquad + c\big\|\big(- e_\rho^n \mathbf{u}^{n-1}\cdot\nabla\mathbf{u}^n - e_\rho^{n-1} D_\tau \mathbf{u}^n\big)_{n=1}^m\big\|_{L^2(L^q)}\\ &\leq c\big(\|(\nabla\phi^n)_{n=1}^m\|_{L^2(L^q)} + \|(\mathbf{F}^n)_{n=1}^m\|_{L^2(L^q)}\big)\\ &\quad + c\|(\rho_\tau^n)_{n=1}^m\|_{L^\infty(L^\infty)}\|(\mathbf{e}_\mathbf{u}^{n-1})_{n=1}^m\|_{L^2(L^q)}\|(\nabla\mathbf{u}^n)_{n=1}^m\|_{L^\infty(L^\infty)}\\ &\quad + c\|(\rho_\tau^n)_{n=1}^m\|_{L^\infty(L^\infty)}\|(\mathbf{u}_\tau^{n-1})_{n=1}^m\|_{L^\infty(L^\infty)}\|(\nabla \mathbf{e}_\mathbf{u}^n)_{n=1}^m\|_{L^2(L^q)}\\ &\quad + c\|(e_\rho^n)_{n=1}^m\|_{L^2(L^q)}\|(\mathbf{u}^{n-1})_{n=1}^m\|_{L^\infty(L^\infty)}\|(\nabla\mathbf{u}^n)_{n=1}^m\|_{L^\infty(L^\infty)}\\ &\quad + c\|(e_\rho^{n-1})_{n=1}^m\|_{L^\infty(L^\infty)}\|(D_\tau \mathbf{u}^n)_{n=1}^m\|_{L^2(L^q)}\\ &=: J_1 + J_2 + J_3 + J_4 + J_5. && (4.58)\end{aligned}$$

By using the induction assumption (4.18) and (4.49), we have

$$\begin{aligned}&J_1 \leq c\tau, && \text{(use (4.55) and (4.28))}\\ &J_2 \leq c\|(\mathbf{e}_\mathbf{u}^{n-1})_{n=1}^m\|_{L^2(L^q)} \leq c\|(\mathbf{e}_\mathbf{u}^{n-1})_{n=1}^m\|_{L^2(H^2)} \leq c\tau, && \text{(use (4.36) and (4.45))}\\ &J_3 \leq c\|(\nabla \mathbf{e}_\mathbf{u}^{n-1})_{n=1}^m\|_{L^2(L^q)} \leq c\|(\mathbf{e}_\mathbf{u}^{n-1})_{n=1}^m\|_{L^2(H^2)} \leq c\tau, && \text{(use (4.18), (4.36) and (4.45))}\\ &J_4 \leq c\|(e_\rho^n)_{n=1}^m\|_{L^2(L^q)} \leq c\|(e_\rho^n)_{n=1}^m\|_{L^\infty(L^\infty)} \leq c\tau, && \text{(use (2.15) and (4.49))}\\ &J_5 \leq c\tau. && \text{(use (4.49))}\end{aligned}$$

By substituting these estimates into (4.57), we obtain

$$\|(D_\tau \mathbf{e}_\mathbf{u}^n)_{n=1}^m\|_{L^2(L^q)} + \|(\mathbf{P}_{\rm div}\Delta \mathbf{e}_\mathbf{u}^n)_{n=1}^m\|_{L^2(L^q)} \leq c\tau. \tag{4.59}$$

In any given smooth and convex domain Ω, there exists $q > 2$ such that ([22])

$$\|\mathbf{e}_\mathbf{u}^n\|_{W^{2,q}} \leq c\|\mathbf{P}_{\rm div}\Delta \mathbf{e}_\mathbf{u}^n\|_{L^q}.$$

Consequently, (4.59) further yields

$$\|(D_\tau \mathbf{e}_{\mathbf{u}}^n)_{n=1}^m\|_{L^2(L^q)} + \|(\mathbf{e}_{\mathbf{u}}^n)_{n=1}^m\|_{L^2(W^{2,q})} \le c\tau, \quad \text{for some } q>2 \text{ (depending on the domain } \Omega). \tag{4.60}$$

Then the Sobolev embedding $W^{2,q}(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ (for $q>2$) implies

$$\|(\mathbf{e}_{\mathbf{u}}^n)_{n=1}^m\|_{L^2(W^{1,\infty})} \le c\tau. \tag{4.61}$$

The last inequality implies $\sum_{n=1}^m \tau\|\mathbf{e}_{\mathbf{u}}^n\|_{W^{1,\infty}}^2 \le c\tau^2$. Since $\mathbf{u}_\tau^n = \mathbf{u}^n - \mathbf{e}_{\mathbf{u}}^n$ with $\mathbf{u}^n$ being the exact solution of the PDEs, it follows that (for sufficiently small step size τ)

$$\|\mathbf{u}_\tau^n\|_{W^{1,\infty}} \le \|\mathbf{u}^n\|_{W^{1,\infty}} + \|\mathbf{e}_{\mathbf{u}}^n\|_{W^{1,\infty}} \le \|\mathbf{u}^n\|_{W^{1,\infty}} + c\tau^{\frac{1}{2}} \le \|\mathbf{u}^n\|_{W^{1,\infty}} + 1, \quad n=1,\dots,m. \tag{4.62}$$

If we define $\widetilde{\mathbf{e}}_{\mathbf{u}}$, piecewise linear in time, such that

$$\widetilde{\mathbf{e}}_{\mathbf{u}}(x,t) = \frac{t_n - t}{\tau}\mathbf{e}_{\mathbf{u}}^{n-1}(x) + \frac{t-t_{n-1}}{\tau}\mathbf{e}_{\mathbf{u}}^n(x) \quad \text{for } t\in(t_{n-1},t_n],$$

then (4.60) implies that

$$\|\partial_t \widetilde{\mathbf{e}}_{\mathbf{u}}\|_{L^2(0,t_m;L^q)} + \|\widetilde{\mathbf{e}}_{\mathbf{u}}\|_{L^2(0,t_m;W^{2,q})} \le c\tau. \tag{4.63}$$

Since $\widetilde{\mathbf{e}}_{\mathbf{u}}(0)=0$, we can further define

$$\overline{\mathbf{e}}_{\mathbf{u}}(x,t) = \begin{cases} \overline{\mathbf{e}}_{\mathbf{u}}(x,t) & \text{for } t\in[0,t_m], \\ \overline{\mathbf{e}}_{\mathbf{u}}(x,2t_m - t) & \text{for } t\in[t_m,2t_m], \\ 0 & \text{for } t\in[2t_m,\infty), \end{cases}$$

which is a function defined for all $t\in\mathbb{R}_+$, satisfying

$$\|\partial_t \overline{\mathbf{e}}_{\mathbf{u}}\|_{L^2(\mathbb{R}_+;L^q)} + \|\overline{\mathbf{e}}_{\mathbf{u}}\|_{L^2(\mathbb{R}_+;W^{2,q})} \le c(\|\partial_t \widetilde{\mathbf{e}}_{\mathbf{u}}\|_{L^2(0,t_m;L^q)} + \|\widetilde{\mathbf{e}}_{\mathbf{u}}\|_{L^2(0,t_m;W^{2,q})}) \le c\tau. \tag{4.64}$$

Then the inhomogeneous Sobolev embedding ([43, Proposition 1.2.10]) implies

$$\|\overline{\mathbf{e}}_{\mathbf{u}}\|_{C([0,t_m];(L^q(\Omega),W^{2,q}(\Omega))_{1/2,1/2})} \le c(\|\partial_t \overline{\mathbf{e}}_{\mathbf{u}}\|_{L^2(\mathbb{R}_+;L^q)} + \|\overline{\mathbf{e}}_{\mathbf{u}}\|_{L^2(\mathbb{R}_+;W^{2,q})}) \le c\tau, \tag{4.65}$$

where the real interpolation space $(L^q(\Omega),W^{2,q}(\Omega))_{1/2,1/2}$ coincides with the Besov space $B^{1,q;2}(\Omega)$, which is embedded into $L^\infty(\Omega)$ for $q>2$ ([2, §7.32 and §7.34]). Consequently, (4.65) implies

$$\|\widetilde{\mathbf{e}}_{\mathbf{u}}\|_{C([0,t_m];L^\infty)} \le c\tau, \tag{4.66}$$

which is equivalent to

$$\max_{0\le n\le m} \|\mathbf{e}_{\mathbf{u}}^n\|_{L^\infty} \le c\tau. \tag{4.67}$$

In the following, we estimate $\|e_\rho^n\|_{W^{1,\infty}}$ and $\|e_\rho^n\|_{H^2}$.

Let $e_{\rho,x_j}^n := \partial_{x_j} e_\rho^n$ and $\mathbf{e}_{\mathbf{u},x_j}^{n-1} := \partial_{x_j}\mathbf{e}_{\mathbf{u}}^{n-1}$. By differentiating (4.29) with respect to x_j, we have

$$D_\tau e_{\rho,x_j}^n + \mathbf{u}_\tau^{n-1}\cdot\nabla e_{\rho,x_j}^n + \mathbf{u}_{\tau,x_j}^{n-1}\cdot\nabla e_\rho^n + \mathbf{e}_{\mathbf{u},x_j}^{n-1}\cdot\nabla\rho^n + \mathbf{e}_{\mathbf{u}}^{n-1}\cdot\nabla\rho_{x_j}^n = E_{x_j}^n. \tag{4.68}$$

Similar as (4.32)-(4.35), integrating (4.68) against $|e^n_{\rho,x_j}|^{k-1}e^n_{\rho,x_j}$ and taking $k\to\infty$, we have

$$
\begin{aligned}
&\max_{1\le n\le m}\|e^n_{\rho,x_j}\|_{L^\infty} &(4.69)\\
&\le c\tau\sum_{n=1}^{m}\left\|\mathbf{u}^{n-1}_{\tau,x_j}\cdot\nabla e^n_\rho+\mathbf{e}^{n-1}_{\mathbf{u},x_j}\cdot\nabla\rho^n+\mathbf{e}^{n-1}_{\mathbf{u}}\cdot\nabla\rho^n_{x_j}-E^n_{x_j}\right\|_{L^\infty}\\
&\le c\tau\sum_{n=1}^{m}\|\nabla\mathbf{u}^{n-1}_\tau\|_{L^\infty}\|\nabla e^n_\rho\|_{L^\infty}+c\tau\sum_{n=1}^{m}(\|\nabla\mathbf{e}^{n-1}_{\mathbf{u}}\|_{L^\infty}+\|\mathbf{e}^{n-1}_{\mathbf{u}}\|_{L^\infty})+c\tau\\
&\le c\tau\sum_{n=1}^{m}\|\nabla e^n_\rho\|_{L^\infty}+c\tau, &(4.70)
\end{aligned}
$$

where we have used (4.61) and (4.67) to estimate $c\tau\sum_{n=1}^{m}(\|\nabla\mathbf{e}^{n-1}_{\mathbf{u}}\|_{L^\infty}+\|\mathbf{e}^{n-1}_{\mathbf{u}}\|_{L^\infty})$, and (4.62) to estimate $\|\nabla\mathbf{u}^{n-1}_\tau\|_{L^\infty}$. Since the last inequality holds for $j=1,2$, it follows that

$$
\max_{1\le n\le m}\|\nabla e^n_\rho\|_{L^\infty}\le c\tau\sum_{n=1}^{m}\|\nabla e^n_\rho\|_{L^\infty}+c\tau. \tag{4.71}
$$

By using the discrete Gronwall's inequality, we obtain (for sufficiently small step size τ)

$$
\max_{1\le n\le m}\|\nabla e^n_\rho\|_{L^\infty}\le c\tau. \tag{4.72}
$$

In view of (4.62) and (4.72), for sufficiently small step size τ the mathematical induction (4.18) is closed. Consequently, the estimates (4.36), (4.40)-(4.41), (4.45)-(4.46), (4.67), (4.72) hold for $m=N$ with the same constants. These estimates can be summarized as follows:

$$
\begin{aligned}
&\min_{x\in\overline{\Omega}}\rho^0(x)\le\rho^n_\tau\le\max_{x\in\overline{\Omega}}\rho^0(x), \qquad n=1,\ldots,N, &(4.73)\\
&\max_{1\le n\le N}(\|e^n_\rho\|_{W^{1,\infty}}+\|\mathbf{e}^n_{\mathbf{u}}\|_{H^1}+\|\mathbf{e}^n_{\mathbf{u}}\|_{L^\infty})\le c\tau, &(4.74)\\
&\sum_{n=1}^{N}\tau(\|D_\tau\mathbf{e}^n_{\mathbf{u}}\|^2_{L^2}+\|\mathbf{e}^n_{\mathbf{u}}\|^2_{H^2})\le c\tau^2, &(4.75)\\
&\sum_{n=1}^{N}\tau\|D_\tau\mathbf{e}^n_{\mathbf{u}}\|^2_{H^2}\le c. &(4.76)
\end{aligned}
$$

Dropping off the exact solutions ρ^n and $\mathbf{u}^n$ from (4.74)-(4.75), and using (4.62), we have

$$
\max_{1\le n\le N}\left(\|\mathbf{u}^n_\tau\|_{H^2}+\|\mathbf{u}^n_\tau\|_{W^{1,\infty}}+\|\rho^n_\tau\|_{W^{1,\infty}}+\|D_\tau\mathbf{u}^n_\tau\|_{L^\infty}\right)\le c. \tag{4.77}
$$

It remains to estimate $\|e^n_\rho\|_{H^2}$ and $\|e^n_p\|_{H^1}$. To estimate the former, we consider the second-order partial derivatives of (4.29), i.e.,

$$
\begin{aligned}
D_\tau e^n_{\rho,x_ix_j}=&-\mathbf{u}^{n-1}_{\tau,x_ix_j}\cdot\nabla e^n_\rho-\mathbf{u}^{n-1}_{\tau,x_j}\cdot\nabla e^n_{\rho,x_i}-\mathbf{u}^{n-1}_{\tau,x_i}\cdot\nabla e^n_{\rho,x_j}-\mathbf{u}^{n-1}_\tau\cdot\nabla e^n_{\rho,x_ix_j}\\
&-\mathbf{e}^{n-1}_{\mathbf{u},x_ix_j}\cdot\nabla\rho^n-\mathbf{e}^{n-1}_{\mathbf{u},x_j}\cdot\nabla\rho^n_{x_i}\\
&-\mathbf{e}^{n-1}_{\mathbf{u},x_i}\cdot\nabla\rho^n_{x_j}-e^{n-1}_{\mathbf{u}}\cdot\nabla\rho^n_{x_ix_j}+E^n_{x_ix_j}.
\end{aligned}\tag{4.78}
$$

Integrating the equation above against e^n_{ρ,x_ix_j} and summing up the results for $n = 1, \dots, m$, we obtain

$$
\begin{aligned}
&\max_{1\le n\le m} \|\nabla^2 e^n_\rho\|^2_{L^2} \\
&\le \tau \sum_{i,j=1}^{2}\sum_{n=1}^{m} \Big\| \mathbf{u}^{n-1}_{\tau,x_ix_j}\cdot\nabla e^n_\rho + \mathbf{u}^{n-1}_{\tau,x_j}\cdot\nabla e^n_{\rho,x_i} + \mathbf{u}^{n-1}_{\tau,x_i}\cdot\nabla e^n_{\rho,x_j} \\
&\qquad\qquad + \mathbf{e}^{n-1}_{\mathbf{u},x_ix_j}\cdot\nabla\rho^n + \mathbf{e}^{n-1}_{\mathbf{u},x_j}\cdot\nabla\rho^n_{x_i} \\
&\qquad\qquad + \mathbf{e}^{n-1}_{\mathbf{u},x_i}\cdot\nabla\rho^n_{x_j} + \mathbf{e}^{n-1}_{\mathbf{u}}\cdot\nabla\rho^n_{x_ix_j} + E^n_{x_ix_j}\Big\|^2_{L^2} \\
&\le \tau\sum_{n=1}^{m}\big(\|\nabla^2\mathbf{u}^{n-1}_\tau\|^2_{L^2}\|\nabla e^n_\rho\|^2_{L^\infty} + \|\nabla \mathbf{u}^{n-1}_\tau\|^2_{L^\infty}\|\nabla^2 e^n_\rho\|^2_{L^2} \\
&\qquad + \|\nabla^2\mathbf{e}^{n-1}_{\mathbf{u}}\|^2_{L^2}\|\nabla\rho^n\|^2_{L^\infty} + \|\nabla\mathbf{e}^{n-1}_{\mathbf{u}}\|^2_{L^\infty}\|\nabla^2\rho^n\|^2_{L^2} \\
&\qquad + \|\nabla\mathbf{e}^{n-1}_{\mathbf{u}}\|^2_{L^\infty}\|\nabla^2\rho^n\|^2_{L^2} + \|\mathbf{e}^{n-1}_{\mathbf{u}}\|^2_{L^\infty}\|\nabla^3\rho^n\|^2_{L^2} + \|\nabla^2 E^n\|^2_{L^2}\big) \\
&=: I_1 + I_2 + I_3 + I_4 + I_5 + I_6 + I_7,
\end{aligned} \tag{4.79}
$$

where

$$
\begin{aligned}
&I_1 \le c\tau^2 && \text{(use (4.47) and (4.72))}\\
&I_2 \le c\tau\sum_{n=1}^{m}\|\nabla^2 e^n_\rho\|^2_{L^2} && \text{(use (4.62))}\\
&I_3 \le c\tau^2 && \text{(use (4.45))}\\
&I_4 \le c\tau^2 && \text{(use (4.61))}\\
&I_5 \le c\tau^2 && \text{(the same as } I_4)\\
&I_6 \le c\tau^2 && \text{(use (4.61))}\\
&I_7 \le c\tau^2 && \text{(use (4.28))}
\end{aligned}
$$

Substituting these estimates into (4.79) yields

$$
\max_{1\le n\le m}\|\nabla^2 e^n_\rho\|^2_{L^2} \le c\tau\sum_{n=1}^{m}\|\nabla^2 e^n_\rho\|^2_{L^2} + c\tau^2, \tag{4.80}
$$

which holds for all $m = 1, \dots, N$. By using Gronwall's inequality, we derive (for sufficiently small step size τ)

$$
\max_{1\le n\le N}\|\nabla^2 e^n_\rho\|_{L^2} \le c\tau. \tag{4.81}
$$

Finally, we estimate $\|\nabla e^n_p\|_{L^2}$. To this end, we rewrite (4.30) as

$$
\begin{aligned}
\nabla e^n_p =& -\rho^{n-1}_\tau D_\tau\mathbf{e}^n_{\mathbf{u}} - e^{n-1}_\rho D_\tau\mathbf{u}^n + \mu\varDelta\mathbf{e}^n_{\mathbf{u}} - e^n_\rho\mathbf{u}^{n-1}\cdot\nabla\mathbf{u}^n \\
&-\rho^n_\tau\mathbf{e}^{n-1}_{\mathbf{u}}\cdot\nabla\mathbf{u}^n - \rho^n_\tau\mathbf{u}^{n-1}_\tau\cdot\nabla\mathbf{e}^n_{\mathbf{u}} + \mathbf{F}^n.
\end{aligned} \tag{4.82}
$$

By using (4.18), (4.36) and (4.62), we have

$$
\begin{aligned}
\|\nabla e^n_p\|_{L^2} \le& c(\|\rho^{n-1}_\tau\|_{L^\infty}\|D_\tau\mathbf{e}^n_{\mathbf{u}}\|_{L^2} + \|D_\tau\mathbf{u}^n\|_{L^\infty}\|e^{n-1}_\rho\|_{L^2} + \|\mathbf{e}^n_{\mathbf{u}}\|_{H^2} \\
&+ \|\mathbf{u}^{n-1}\|_{L^\infty}\|\nabla\mathbf{u}^n\|_{L^\infty}\|e^n_\rho\|_{L^2} + \|\rho^n_\tau\|_{L^\infty}\|\nabla\mathbf{u}^n\|_{L^\infty}\|\mathbf{e}^{n-1}_{\mathbf{u}}\|_{L^2} \\
&+ \|\rho^n_\tau\|_{L^\infty}\|\mathbf{u}^{n-1}_\tau\|_{L^\infty}\|\nabla\mathbf{e}^n_{\mathbf{u}}\|_{L^2} + \|\mathbf{F}^n\|_{L^2})
\end{aligned}
$$

$$\leq c(\|D_\tau \mathbf{e}_{\mathbf{u}}^n\|_{L^2} + \|e_\rho^{n-1}\|_{L^2} + \|\mathbf{e}_{\mathbf{u}}^n\|_{H^2} + \|e_\rho^n\|_{L^2} + \|\mathbf{e}_{\mathbf{u}}^{n-1}\|_{L^2} + \|\nabla \mathbf{e}_{\mathbf{u}}^n\|_{L^2} + \|\mathbf{F}^n\|_{L^2}).$$

The last inequality, together with (4.40)-(4.41) and (4.45), implies

$$\tau \sum_{n=1}^{N} \|\nabla e_p^n\|_{L^2}^2 \leq c\tau^2. \tag{4.83}$$

Part III: Summary of the proof
(4.73) implies (3.6).
(4.74) and (4.81) imply (3.7).
(4.75) and (4.83) imply (3.8).
(4.77), (4.81) and (4.83) imply (3.9).
(4.76) and (4.83) imply (3.10).
The proof of Proposition 3.1 is complete. □

5. Proof of Proposition 3.2

In this section we estimate the spatial discretization errors by comparing the fully discrete finite element solution $(\rho_h^n, \mathbf{u}_h^n, p_h^n)$ with the semi-discrete solution $(\rho_\tau^n, \mathbf{u}_\tau^n, p_\tau^n)$. The solvability of the linear system (2.11)-(2.12) is proved in the next subsection.

5.1. Solvability of the linear system

If $(\rho_h^k, \mathbf{u}_h^k, p_h^k) \in M_h^2 \times \mathbf{X}_h^2 \times \mathring{M}_h^1$ is given for $k = 0, \dots, n-1$, then the discrete linear problem (2.11) has a unique solution $\rho_h^n \in M_h^2$ if and only if the corresponding homogeneous problem

$$(\tau^{-1}\Phi_h, \varphi_h) + (\mathbf{u}_h^{n-1} \cdot \nabla \Phi_h, \varphi_h) + \frac{1}{2}(\Phi_h \nabla \cdot \mathbf{u}_h^{n-1}, \varphi_h) = 0, \quad \forall \varphi_h \in M_h^2, \tag{5.1}$$

has only zero solution $\Phi_h = 0$. Indeed, the equation above can be rewritten as, through integration by parts,

$$(\tau^{-1}\Phi_h, \varphi_h) + \frac{1}{2}(\mathbf{u}_h^{n-1} \cdot \nabla \Phi_h, \varphi_h) - \frac{1}{2}(\mathbf{u}_h^{n-1} \cdot \nabla \varphi_h, \Phi_h) = 0. \tag{5.2}$$

Substituting $\varphi_h = \Phi_h$ into the equation above immediately yields $\Phi_h = 0$. This proves the unique solvability of equation (2.11).

After solving ρ_h^n from (2.11), the truncated functions $\widehat{\rho}_h^{n-1}$ and $\widehat{\rho}_h^n$ can be defined by (2.10), and the discrete linear problem (2.12) has a unique solution $(\mathbf{u}_h^n, p_h^n) \in \mathbf{X}_h^2 \times \mathring{M}_h^1$ if and only if the corresponding homogeneous problem

$$\begin{aligned}(\widehat{\rho}_h^{n-1}\tau^{-1}\mathbf{U}_h, \mathbf{v}_h) &+ \frac{1}{2}(\tau^{-1}(\widehat{\rho}_h^n - \widehat{\rho}_h^{n-1})\,\mathbf{U}_h, \mathbf{v}_h) + \frac{1}{2}(\nabla \cdot (\rho_h^n \mathbf{u}_h^{n-1})\,\mathbf{U}_h, \mathbf{v}_h) \\ &+ (\rho_h^n \mathbf{u}_h^{n-1} \cdot \nabla \mathbf{U}_h, \mathbf{v}_h) + B((\mathbf{U}_h, P_h), (\mathbf{v}_h, q_h)) = 0, \quad \forall (\mathbf{v}_h, q_h) \in \mathbf{X}_h^2 \times \mathring{M}_h^1,\end{aligned} \tag{5.3}$$

has only zero solution $(\mathbf{U}_h, P_h) = (\mathbf{0}, 0)$. Indeed, substituting $(\mathbf{v}_h, q_h) = (\mathbf{U}_h, P_h)$ into the equation above yields

$$\frac{1}{2\tau}((\widehat{\rho}_h^{n-1} + \widehat{\rho}_h^n)\mathbf{U}_h, \mathbf{v}_h) + (\mu \nabla \mathbf{U}_h, \nabla \mathbf{v}_h) = 0. \tag{5.4}$$

Since the truncation operation defined in (2.10) implies $\widehat{\rho}_h^{n-1} > 0$ and $\widehat{\rho}_h^n > 0$, the last equation implies $\mathbf{U}_h = \mathbf{0}$. Then (5.3) reduces to

$$(P_h, \nabla \cdot \mathbf{v}_h) = 0, \quad \forall \mathbf{v}_h \in \mathbf{X}_h^2, \tag{5.5}$$

which implies $\|P_h\|_{L^2} = 0$ in view of the inf-sup condition (2.9). This proves the unique solvability of equation (2.12).

5.2. Ritz projection, L^2 projections, and Lagrange interpolations

To obtain error estimates for the finite element spatial discretization, we need to use the Stokes–Ritz projection $(\mathbf{R}_h, Q_h) : \mathbf{H}_0^1(\Omega) \times L^2(\Omega) \to \mathbf{X}_h^2 \times \mathring{M}_h^1$, defined by

$$B((\mathbf{R}_h(\mathbf{u}, p), Q_h(\mathbf{u}, p)), (\mathbf{v}_h, q_h)) = B((\mathbf{u}, p), (\mathbf{v}_h, q_h)), \quad \forall\, (\mathbf{v}_h, q_h) \in \mathbf{X}_h^2 \times \mathring{M}_h^1.$$

The L^2 projections $P_h : L^2(\Omega) \to M_h^2$ and $\mathbf{P}_h : \mathbf{L}^2(\Omega) \to \mathbf{X}_h^2$ defined below will also be used:

$$\begin{aligned}
&(\mathbf{v} - \mathbf{P}_h\mathbf{v}, \mathbf{v}_h) = 0 && \forall\, \mathbf{v} \in \mathbf{L}^2(\Omega), && \forall\, \mathbf{v}_h \in \mathbf{X}_h^2,\\
&(v - P_h v, v_h) = 0 && \forall\, v \in L^2(\Omega), && \forall\, v_h \in M_h^2.
\end{aligned}$$

It is well known that the Ritz projection and the L^2 projections defined above satisfy the following standard estimates(cf. [10, Theorem 12.6.7], [21, Chapter II, Theorem 4.3], [20, Theorem 8.2], [13]):

$$\|\mathbf{R}_h(\mathbf{v}, q) - \mathbf{v}\|_{L^2} \le ch^{\ell+1}(\|\mathbf{v}\|_{H^{\ell+1}} + \|q\|_{H^\ell}), \qquad \ell = 1, 2, \tag{5.6}$$

$$\|\mathbf{R}_h(\mathbf{v}, q) - \mathbf{v}\|_{H^1} + \|Q_h(\mathbf{v}, q) - q\|_{L^2} \le ch^{\ell}(\|\mathbf{v}\|_{H^{\ell+1}} + \|q\|_{H^\ell}), \qquad \ell = 1, 2, \tag{5.7}$$

$$\|\mathbf{R}_h(\mathbf{v}, q)\|_{W^{1,\infty}} \le c(\|\mathbf{v}\|_{W^{1,\infty}} + \|q\|_{L^\infty}), \tag{5.8}$$

$$\|\mathbf{v} - \mathbf{P}_h\mathbf{v}\|_{L^2} \le ch^{\ell+1}\|\mathbf{v}\|_{H^{\ell+1}}, \qquad \ell = 0, 1, 2, \tag{5.9}$$

$$\|\varphi - P_h\varphi\|_{L^2} \le ch^{\ell+1}\|\varphi\|_{H^{\ell+1}}, \qquad \ell = 0, 1, 2, \tag{5.10}$$

$$\|P_h\varphi\|_{W^{k,q}} \le c\|\varphi\|_{W^{k,q}}, \qquad k = 0, 1,\ 1 \le q \le \infty, \tag{5.11}$$

$$\|\mathbf{P}_h\mathbf{w}\|_{L^2} \le c\|\mathbf{w}\|_{L^2}, \quad \|\mathbf{P}_h\mathbf{w}\|_{H^1} \le c\|\mathbf{w}\|_{H^1}, \tag{5.12}$$

where $(\mathbf{v}, q) \in (\mathbf{H}^{\ell+1}(\Omega) \cap \mathbf{H}_0^1(\Omega)) \times H^\ell(\Omega)$, $\varphi \in H^{\ell+1}(\Omega)$ and $\mathbf{w} \in \mathbf{H}_0^1(\Omega)$. Similarly, the Lagrangian interpolation operators satisfy

$$\|\Pi_h\varphi - \varphi\|_{L^2} + h\|\nabla(\Pi_h\varphi - \varphi)\|_{L^2} \le ch^{\ell+1}\|\varphi\|_{H^{\ell+1}}, \qquad \forall\, \varphi \in H^{\ell+1}(\Omega), \qquad \ell = 1, 2, \tag{5.13}$$

$$\|\mathbf{\Pi}_h\mathbf{v} - \mathbf{v}\|_{L^2} + h\|\nabla(\mathbf{\Pi}_h\mathbf{v} - \mathbf{v})\|_{L^2} \le ch^{\ell+1}\|\mathbf{v}\|_{H^{\ell+1}}, \qquad \forall\, \mathbf{v} \in \mathbf{H}^{\ell+1}(\Omega) \cap \mathbf{H}_0^1(\Omega), \qquad \ell = 1, 2, \tag{5.14}$$

$$\|\mathbf{\Pi}_h\mathbf{v} - \mathbf{v}\|_{L^\infty} \le ch\|\mathbf{v}\|_{W^{1,\infty}}, \qquad \forall\, \mathbf{v} \in \mathbf{W}^{1,\infty}(\Omega) \cap \mathbf{H}_0^1(\Omega). \tag{5.15}$$

The estimates (5.6)-(5.14) will be frequently used in this Section.

In view of (3.5), we also have the following estimates of the L^2 and Ritz projections:

$$\begin{aligned}
\|\rho_\tau^n - P_h\rho_\tau^n\|_{L^2} &\le \|e_\rho^n - P_h e_\rho^n\|_{L^2} + \|\rho^n - P_h\rho^n\|_{L^2}\\
&\le ch^2\|e_\rho^n\|_{H^2} + ch^3\|\rho^n\|_{H^3}\\
&\le ch^2\tau + ch^3, \qquad (\text{use (3.7) to estimate } \|e_\rho^n\|_{H^2}) \qquad (5.16)\\
\|\rho_\tau^n - P_h\rho_\tau^n\|_{H^1} &\le \|e_\rho^n - P_h e_\rho^n\|_{H^1} + \|\rho^n - P_h\rho^n\|_{H^1}\\
&\le ch\|e_\rho^n\|_{H^2} + ch^2\|\rho^n\|_{H^3}\\
&\le ch\tau + ch^2, \qquad (\text{use (3.7) to estimate } \|e_\rho^n\|_{H^2}) \qquad (5.17)\\
\|\mathbf{u}_\tau^n - \mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n)\|_{L^2} &\le \|\mathbf{e}_\mathbf{u}^n - \mathbf{R}_h(\mathbf{e}_\mathbf{u}^n, e_p^n)\|_{L^2} + \|\mathbf{u}^n - \mathbf{R}_h(\mathbf{u}^n, p^n)\|_{L^2}
\end{aligned}$$

$$
\begin{aligned}
&\le ch^2(\|\mathbf{e}_{\mathbf{u}}^n\|_{H^2} + \|e_p^n\|_{H^1}) + ch^3(\|\mathbf{u}^n\|_{H^3} + \|p^n\|_{H^2}) \\
&\le ch^2\tau^{\frac{1}{2}} + ch^3, \quad (\text{use (3.8) to estimate } \|\mathbf{e}_{\mathbf{u}}^n\|_{H^2} + \|e_p^n\|_{H^1}) \qquad (5.18) \\
\|\mathbf{u}_\tau^n - \mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n)\|_{H^1} &\le \|\mathbf{e}_{\mathbf{u}}^n - \mathbf{R}_h(\mathbf{e}_{\mathbf{u}}^n, e_p^n)\|_{H^1} + \|\mathbf{u}^n - \mathbf{R}_h(\mathbf{u}^n, p^n)\|_{H^1} \\
&\le ch(\|\mathbf{e}_{\mathbf{u}}^n\|_{H^2} + \|e_p^n\|_{H^1}) + ch^2(\|\mathbf{u}^n\|_{H^3} + \|p^n\|_{H^2}) \\
&\le ch\tau^{\frac{1}{2}} + ch^2. \quad (\text{use (3.8) to estimate } \|\mathbf{e}_{\mathbf{u}}^n\|_{H^2} + \|e_p^n\|_{H^1}) \qquad (5.19)
\end{aligned}
$$

5.3. Methodology

Instead of considering the original scheme (2.11)-(2.12) directly, we consider the following finite element equations:

$$
(D_\tau \rho_h^n, \varphi_h) + (\mathbf{u}_h^{n-1} \cdot \nabla \rho_h^n, \varphi_h) + \frac{1}{2}(\nabla \cdot \mathbf{u}_h^{n-1}\, \rho_h^n, \varphi_h) = 0, \qquad (5.20)
$$

$$
\begin{aligned}
&(\rho_h^{n-1} D_\tau \mathbf{u}_h^n, \mathbf{v}_h) + \frac{1}{2}(D_\tau \rho_h^n\, \mathbf{u}_h^n, \mathbf{v}_h) + \frac{1}{2}(\nabla \cdot (\rho_h^n \mathbf{u}_h^{n-1})\, \mathbf{u}_h^n, \mathbf{v}_h) \\
&+ (\rho_h^n \mathbf{u}_h^{n-1} \cdot \nabla \mathbf{u}_h^n, \mathbf{v}_h) + B((\mathbf{u}_h^n, p_h^n), (\mathbf{v}_h, q_h)) = 0, \qquad (5.21)
\end{aligned}
$$

which do not truncate ρ_h^{n-1} and ρ_h^n in (5.21). We shall prove that, for sufficiently small τ and h, the finite element solutions given by (5.20)-(5.21) satisfy

$$
\frac{1}{2}\min_{y\in\overline{\Omega}} \rho^0(y) \le \rho_h^n \le \frac{3}{2}\max_{y\in\overline{\Omega}} \rho^0(y), \quad n = 0, 1, \ldots, N, \qquad (5.22)
$$

which implies $\widehat{\rho}_h^n = \rho_h^n$ in view of the definition of the truncation (2.10). In other words, the solutions of (5.20)-(5.21) coincide with the solutions of (2.11)-(2.12). Thus it suffices to present error estimates for the solutions of (5.20)-(5.21).

For the solution $(\rho_h^n, \mathbf{u}_h^n, p_h^n)$ of (5.20)-(5.21), we denote

$$
e_{\rho,h}^n = P_h \rho_\tau^n - \rho_h^n, \qquad \mathbf{e}_{\mathbf{u},h}^n = \mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n) - u_h^n, \qquad e_{p,h}^n = Q_h(\mathbf{u}_\tau^n, p_\tau^n) - p_h^n.
$$

If we can prove (5.22) and

$$
\max_{1\le n\le N} \left(\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2} + \|e_{\rho,h}^n\|_{L^2}\right) + h\left(\tau \sum_{n=1}^N \|e_{p,h}^n\|_{L^2}^2\right)^{\frac{1}{2}} \le ch\sqrt{\tau + h^2}, \qquad (5.23)
$$

then $(\rho_h^n, \mathbf{u}_h^n, p_h^n)$, $n = 1, \ldots, N$, coincide with the solutions of (2.11)-(2.12) and

$$
\begin{aligned}
\|\rho^n - \rho_h^n\|_{L^2} &\le c(\|\rho^n - \rho_\tau^n\|_{L^2} + \|\rho_\tau^n - P_h\rho_\tau^n\|_{L^2} + \|e_{\rho,h}^n\|_{L^2} \\
&\le c\tau + ch^2\|\rho_\tau^n\|_{H^2} + ch\sqrt{\tau + h^2} \quad (\text{use (3.7), (3.9), (5.10) and (5.23)}) \\
&\le c(\tau + h^2), \\
\|\mathbf{u}^n - \mathbf{u}_h^n\|_{L^2} &\le c(\|\mathbf{u}^n - \mathbf{u}_\tau^n\|_{L^2} + \|\mathbf{u}_\tau^n - \mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n)\|_{L^2} + \|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2} \\
&\le c\tau + ch^2(\|\mathbf{u}_\tau^n\|_{H^2} + \|p_\tau^n\|_{H^1}) + ch\sqrt{\tau + h^2} \quad (\text{use (3.7), (3.9), (5.6) and (5.23)}) \\
&\le c(\tau + h^2), \\
\|p^n - p_h^n\|_{L^2} &\le c(\|p^n - p_\tau^n\|_{L^2} + \|p_\tau^n - Q_h(\mathbf{u}_\tau^n, p_\tau^n)\|_{L^2} + \|e_{p,h}^n\|_{L^2} \\
&\le c\|e_p^n\|_{H^1} + ch(\|\mathbf{u}_\tau^n\|_{H^2} + \|p_\tau^n\|_{H^1}) + c\|e_{p,h}^n\|_{L^2} \quad (\text{use (3.9) and (5.7)}) \\
&\le c\|e_p^n\|_{H^1} + c\|e_{p,h}^n\|_{L^2} + ch.
\end{aligned}
$$

The last inequality, (3.8) and (5.23) imply

$$\left(\tau\sum_{n=1}^{N}\|p^n-p_h^n\|_{L^2}^2\right)^{\frac12}\le c\sqrt{\tau+h^2}.$$

This proves the error estimate in Proposition 3.2.

It remains to prove (5.22) and (5.23). To this end, we integrate (3.1)-(3.3) against some test functions and reformulate the equations as

$$(D_\tau\rho_\tau^n,\varphi_h)+(\mathbf{u}_\tau^{n-1}\cdot\nabla\rho_\tau^n,\varphi_h)+\frac12(\nabla\cdot\mathbf{u}_\tau^{n-1}\,\rho_\tau^n,\varphi_h)=0, \tag{5.24}$$

$$\begin{aligned}&(\rho_\tau^{n-1}D_\tau\mathbf{u}_\tau^n,\mathbf{v}_h)+\frac12(D_\tau\rho_\tau^n\,\mathbf{u}_\tau^n,\mathbf{v}_h)+\frac12(\nabla\cdot(\rho_\tau^n\mathbf{u}_\tau^{n-1})\,\mathbf{u}_\tau^n,\mathbf{v}_h)\\&\qquad+(\rho_\tau^n\mathbf{u}_\tau^{n-1}\cdot\nabla\mathbf{u}_\tau^n,\mathbf{v}_h)+B((\mathbf{u}_\tau^n,p_\tau^n),(\mathbf{v}_h,q_h))=0,\end{aligned} \tag{5.25}$$

where $\varphi_h\in M_h^2$ and $(\mathbf{v}_h,q_h)\in\mathbf{X}_h^2\times\mathring{M}_h^1$ are arbitrary test functions. We have added the following stabilization terms to (5.24)-(5.25):

$$\frac12\nabla\cdot\mathbf{u}_\tau^{n-1}\,\rho_\tau^n=0,\qquad \frac12 D_\tau\rho_\tau^n\,\mathbf{u}_\tau^n+\frac12\nabla\cdot(\rho_\tau^n\mathbf{u}_\tau^{n-1})\,\mathbf{u}_\tau^n=0,$$

which are consequences of (3.3) and (3.1), respectively. Using integration by parts, (5.25) can be further written as

$$\begin{aligned}&(\rho_\tau^{n-1}D_\tau\mathbf{u}_\tau^n,\mathbf{v}_h)+\frac12(D_\tau\rho_\tau^n\,\mathbf{u}_\tau^n,\mathbf{v}_h)+\frac12(\rho_\tau^n\mathbf{u}_\tau^{n-1}\cdot\nabla\mathbf{u}_\tau^n,\mathbf{v}_h)\\&\qquad-\frac12(\rho_\tau^n\mathbf{u}_\tau^{n-1}\cdot\nabla\mathbf{v}_h,\mathbf{u}_\tau^n)+B((\mathbf{u}_\tau^n,p_\tau^n),(\mathbf{v}_h,q_h))=0.\end{aligned} \tag{5.26}$$

We shall prove (5.23) by considering the difference between (5.24)-(5.25) and (5.20)-(5.21).

The proof is by mathematical induction. Note that the initial data satisfy

$$\begin{aligned}&\|e_{\rho,h}^0\|_{L^\infty}=\|P_h\rho^0-\Pi_h\rho^0\|_{L^\infty}\le ch^{-1}\|P_h\rho^0-\Pi_h\rho^0\|_{L^2}\le c\|\rho^0\|_{H^2}h,\\&\|\mathbf{e}_{\mathbf{u},h}^0\|_{L^2}=\|\mathbf{R}_h(\mathbf{u}^0,p^0)-\mathbf{\Pi}_h\mathbf{u}^0\|_{L^2}\le c(\|\mathbf{u}^0\|_{\mathbf{H}^2}+\|p^0\|_{H^1})h,\\&\|\mathbf{e}_{\mathbf{u},h}^0\|_{L^\infty}=\|\mathbf{R}_h(\mathbf{u}^0,p^0)-\mathbf{\Pi}_h\mathbf{u}^0\|_{L^\infty}\le ch^{-1}\|\mathbf{R}_h(\mathbf{u}^0,p^0)-\mathbf{\Pi}_h\mathbf{u}^0\|_{L^2}\le c(\|\mathbf{u}^0\|_{\mathbf{H}^2}+\|p^0\|_{H^1})h,\\&\tau\|\mathbf{e}_{\mathbf{u},h}^0\|_{H^1}^2\le\tau\|\mathbf{R}_h(\mathbf{u}^0,p^0)-\mathbf{\Pi}_h\mathbf{u}^0\|_{H^1}^2\le c(\|\mathbf{u}^0\|_{H^2}^2+\|p^0\|_{H^1}^2)\tau h^2.\end{aligned} \tag{5.27}$$

For sufficiently small τ and h, the last four inequalities imply

$$\begin{aligned}&\|e_{\rho,h}^0\|_{L^\infty}\le\frac14\min_{x\in\overline{\Omega}}\rho^0(x),\\&\|\mathbf{e}_{\mathbf{u},h}^0\|_{L^2}\le\sqrt{\tau+h^2},\\&\|\mathbf{e}_{\mathbf{u},h}^0\|_{L^\infty}\le 1,\\&\tau\|\mathbf{e}_{\mathbf{u},h}^0\|_{H^1}^2\le h^2\sqrt{\tau+h^2}.\end{aligned} \tag{5.28}$$

Let $1 \le m \le N$, and assume that the data ρ_h^{n-1} and $\mathbf{u}_h^{n-1}$,, $n = 1, 2, ..., m$, are given and satisfying the following inequalities (induction assumption):

$$\begin{aligned}
&\max_{1\le n\le m} \|e_{\rho,h}^{n-1}\|_{L^\infty} \le \frac{1}{4}\min_{x\in\overline{\Omega}} \rho^0(x),\\
&\max_{1\le n\le m} \|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2} \le h(\tau + h^2)^{\frac{1}{4}},\\
&\max_{1\le n\le m} \|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^\infty} \le 1,\\
&\sum_{n=1}^{m} \tau \|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{H^1}^2 \le h^2\sqrt{\tau + h^2}.
\end{aligned} \tag{5.29}$$

Then we prove that the solution $(\rho_h^m, \mathbf{u}_h^m, p_h^m) \in M_h^2 \times \mathbf{X}_h^2 \times \mathring{M}_h^1$ given by (5.20)-(5.21) satisfies the following inequalities:

$$\begin{aligned}
&\max_{0\le n\le m} \|e_{\rho,h}^{n}\|_{L^\infty} \le \frac{1}{4}\min_{x\in\overline{\Omega}} \rho^0(x),\\
&\max_{0\le n\le m} \|\mathbf{e}_{\mathbf{u},h}^{n}\|_{L^2} \le h(\tau + h^2)^{\frac{1}{4}},\\
&\max_{0\le n\le m} \|\mathbf{e}_{\mathbf{u},h}^{n}\|_{L^\infty} \le 1,\\
&\sum_{n=0}^{m} \tau \|\mathbf{e}_{\mathbf{u},h}^{n}\|_{H^1}^2 \le h^2\sqrt{\tau + h^2}.
\end{aligned} \tag{5.30}$$

To use mathematical induction, we emphasize that all the generic constants below will be independent of m (but may depend on T).

Note that the induction assumption (5.29) implies that

$$\begin{aligned}
\|\rho_\tau^{n-1} - \rho_h^{n-1}\|_{L^\infty} &\le \|\rho_\tau^{n-1} - P_h\rho_\tau^{n-1}\|_{L^\infty} + \|e_{\rho,h}^{n-1}\|_{L^\infty}\\
&\le c\|\rho_\tau^{n-1}\|_{W^{1,\infty}} h + \frac{1}{4}\min_{x\in\overline{\Omega}} \rho^0(x)\\
&\le \frac{1}{2}\min_{x\in\overline{\Omega}} \rho^0(x) \qquad \text{when } h \text{ is sufficiently small,}
\end{aligned} \tag{5.31}$$

which further implies

$$\begin{aligned}
&\rho_h^{n-1}(x) \ge \min_{x\in\overline{\Omega}} \rho^0(x) - \|\rho_\tau^{n-1} - \rho_h^{n-1}\|_{L^\infty} \ge \frac{1}{2}\min_{x\in\overline{\Omega}} \rho^0(x), \qquad \forall\, x \in \Omega,\ \ n = 1, \dots, m,\\
&\rho_h^{n-1}(x) \le \max_{x\in\overline{\Omega}} \rho^0(x) + \|\rho_\tau^{n-1} - \rho_h^{n-1}\|_{L^\infty} \le \frac{3}{2}\max_{x\in\overline{\Omega}} \rho^0(x), \qquad \forall\, x \in \Omega,\ \ n = 1, \dots, m,
\end{aligned}$$

thus

$$\frac{1}{2}\min_{y\in\overline{\Omega}} \rho^0(y) \le \rho_h^{n-1} \le \frac{3}{2}\max_{y\in\overline{\Omega}} \rho^0(y), \qquad n = 1, \dots, m. \tag{5.32}$$

Besides, the induction assumption (5.29) implies that

$$\begin{aligned}
&\|\mathbf{u}_h^{n-1}\|_{L^\infty}\\
&\le \|\mathbf{R}_h(\mathbf{u}_\tau^{n-1}, p_\tau^{n-1})\|_{L^\infty} + \|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^\infty}
\end{aligned}$$

$$
\begin{aligned}
&\leq \|\mathbf{R}_h(\mathbf{u}_\tau^{n-1}, p_\tau^{n-1}) - \mathbf{\Pi}_h\mathbf{u}_\tau^{n-1}\|_{L^\infty} + \|\mathbf{\Pi}_h\mathbf{u}_\tau^{n-1} - \mathbf{u}_\tau^{n-1}\|_{L^\infty} + \|\mathbf{u}_\tau^{n-1}\|_{L^\infty} + \|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^\infty} \\
&\leq ch^{-1}\|\mathbf{R}_h(\mathbf{u}_\tau^{n-1}, p_\tau^{n-1}) - \mathbf{\Pi}_h\mathbf{u}_\tau^{n-1}\|_{L^2} + \|\mathbf{\Pi}_h\mathbf{u}_\tau^{n-1} - \mathbf{u}_\tau^{n-1}\|_{L^\infty} + \|\mathbf{u}_\tau^{n-1}\|_{L^\infty} + \|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^\infty} \\
&\leq c(\|\mathbf{u}_\tau^{n-1}\|_{H^2} + \|p_\tau^{n-1}\|_{H^1})h + c\|\mathbf{u}_\tau^{n-1}\|_{W^{1,\infty}}h + \|\mathbf{u}_\tau^{n-1}\|_{L^\infty} + 1 \quad \text{(use (5.6), (5.15) and (5.29))} \\
&\leq c \qquad \text{when } h \text{ is sufficiently small,} \quad n = 1, \dots, m.
\end{aligned} \tag{5.33}
$$

The estimates (5.32) and (5.33) will be frequently used in the following subsections.

5.4. Estimates of $e_{\rho,h}^n$.

Subtracting (5.20) from (5.24) yields

$$
\begin{aligned}
&(D_\tau(\rho_\tau^n - P_h\rho_\tau^n), \varphi_h) + (D_\tau e_{\rho,h}^n, \varphi_h) \\
&+ (\mathbf{u}_h^{n-1}\cdot\nabla(\rho_\tau^n - P_h\rho_\tau^n), \varphi_h) + \frac{1}{2}(\nabla\cdot\mathbf{u}_h^{n-1}(\rho_\tau^n - P_h\rho_\tau^n), \varphi_h) \\
&+ (\mathbf{u}_h^{n-1}\cdot\nabla e_{\rho,h}^n, \varphi_h) + \frac{1}{2}(\nabla\cdot\mathbf{u}_h^{n-1}e_{\rho,h}^n, \varphi_h) \\
&+ (\mathbf{e}_{\mathbf{u},h}^{n-1}\cdot\nabla\rho_\tau^n, \varphi_h) + \frac{1}{2}(\nabla\cdot\mathbf{e}_{\mathbf{u},h}^{n-1}\rho_\tau^n, \varphi_h) \\
&+ ((\mathbf{u}_\tau^{n-1} - \mathbf{R}_h(\mathbf{u}_\tau^{n-1}, p_\tau^{n-1}))\cdot\nabla\rho_\tau^n, \varphi_h) \\
&+ \frac{1}{2}(\rho_\tau^n\nabla\cdot(\mathbf{u}_\tau^{n-1} - \mathbf{R}_h(\mathbf{u}_\tau^{n-1}, p_\tau^{n-1})), \varphi_h) = 0, \qquad \forall\,\varphi_h \in M_h^2.
\end{aligned} \tag{5.34}
$$

Since $(D_\tau(\rho_\tau^n - P_h\rho_\tau^n), \varphi_h) = (D_\tau\rho_\tau^n - P_hD_\tau\rho_\tau^n, \varphi_h) = 0$, taking $\varphi_h = e_{\rho,h}^n$ in the equation above yields

$$
\begin{aligned}
\frac{1}{2}D_\tau(\|e_{\rho,h}^n\|_{L^2}^2) \leq &|(\mathbf{u}_h^{n-1}\cdot\nabla(\rho_\tau^n - P_h\rho_\tau^n), e_{\rho,h}^n)| \\
&+ |\frac{1}{2}(\nabla\cdot\mathbf{u}_h^{n-1}(\rho_\tau^n - P_h\rho_\tau^n), e_{\rho,h}^n)| \\
&+ |(\mathbf{u}_h^{n-1}\cdot\nabla e_{\rho,h}^n, e_{\rho,h}^n) + \frac{1}{2}(\nabla\cdot\mathbf{u}_h^{n-1}e_{\rho,h}^n, e_{\rho,h}^n)| \\
&+ |(\mathbf{e}_{\mathbf{u},h}^{n-1}\cdot\nabla\rho_\tau^n, e_{\rho,h}^n) + \frac{1}{2}(\nabla\cdot\mathbf{e}_{\mathbf{u},h}^{n-1}\rho_\tau^n, e_{\rho,h}^n)| \\
&+ |((\mathbf{u}_\tau^{n-1} - \mathbf{R}_h(\mathbf{u}_\tau^{n-1}, p_\tau^{n-1}))\cdot\nabla\rho_\tau^n, e_{\rho,h}^n)| \\
&+ |\frac{1}{2}(\rho_\tau^n\nabla\cdot(\mathbf{u}_\tau^{n-1} - \mathbf{R}_h(\mathbf{u}_\tau^{n-1}, p_\tau^{n-1})), e_{\rho,h}^n)| \\
=: &\sum_{k=1}^{6}|J_k|,
\end{aligned} \tag{5.35}
$$

where

$$
\begin{aligned}
J_1 \leq &\|\mathbf{u}_h^{n-1}\|_{L^\infty}\|\nabla(\rho_\tau^n - P_h\rho_\tau^n)\|_{L^2}\|e_{\rho,h}^n\|_{L^2} \\
\leq &\|\mathbf{u}_h^{n-1}\|_{L^\infty}(ch\tau + ch^2)\|e_{\rho,h}^n\|_{L^2} \qquad \text{(use (5.17))} \\
\leq &(ch\tau + ch^2)\|e_{\rho,h}^n\|_{L^2} \qquad \text{(use (5.33))} \\
\leq &c\epsilon^{-1}h^2(\tau^2 + h^2) + \epsilon\|e_{\rho,h}^n\|_{L^2}^2, \qquad \text{(use Hölder's inequality)}
\end{aligned} \tag{5.36}
$$

$$
\begin{aligned}
J_2 \leq &\|\nabla\mathbf{u}_h^{n-1}\|_{L^\infty}\|\rho_\tau^n - P_h\rho_\tau^n\|_{L^2}\|e_{\rho,h}^n\|_{L^2} \\
\leq &\|\nabla\mathbf{u}_h^{n-1}\|_{L^\infty}(ch^2\tau + ch^3)\|e_{\rho,h}^n\|_{L^2} \quad \text{(use (5.16))}
\end{aligned}
$$

$$
\begin{aligned}
&\le c\|\mathbf{u}_h^{n-1}\|_{L^\infty}(ch\tau + ch^2)\|e_{\rho,h}^n\|_{L^2}, && \text{(use inverse inequality)}\\
&\le (ch\tau + ch^2)\|e_{\rho,h}^n\|_{L^2}, && \text{(use (5.33))}\\
&\le c\epsilon^{-1}h^2(\tau^2 + h^2) + \epsilon\|e_{\rho,h}^n\|_{L^2}^2 && \text{(use Hölder's inequality)} \qquad (5.37)\\
J_3 &= 0, && (5.38)\\
J_4 &\le \|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2}\|\nabla\rho_\tau^n\|_{L^\infty}\|e_{\rho,h}^n\|_{L^2} + \|\nabla\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2}\|\rho_\tau^n\|_{L^\infty}\|e_{\rho,h}^n\|_{L^2}\\
&\le c(\|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2} + \|\nabla\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2})\|e_{\rho,h}^n\|_{L^2} && \text{(use (3.9))}\\
&\le c\epsilon^{-1}(\|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2}^2 + \|\nabla\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2}^2) + \epsilon\|e_{\rho,h}^n\|_{L^2}^2, && \text{(use Hölder's inequality)} \qquad (5.39)\\
J_5 &\le \|\mathbf{u}_\tau^{n-1} - \mathbf{R}_h(\mathbf{u}_\tau^{n-1}, p_\tau^{n-1})\|_{L^2}\|\nabla\rho_\tau^n\|_{L^\infty}\|e_{\rho,h}^n\|_{L^2}\\
&\le ch^2(\|\mathbf{u}_\tau^{n-1}\|_{H^2} + \|p_\tau^{n-1}\|_{H^1})\|\nabla\rho_\tau^n\|_{L^\infty}\|e_{\rho,h}^n\|_{L^2} && \text{(use (5.6))}\\
&\le ch^2\|e_{\rho,h}^n\|_{L^2} && \text{(use (3.9))}\\
&\le c\epsilon^{-1}h^4 + \epsilon\|e_{\rho,h}^n\|_{L^2}^2, && \text{(use Hölder's inequality)} \qquad (5.40)\\
J_6 &\le \|\rho_\tau^n\|_{L^\infty}\|\nabla\cdot(\mathbf{u}_\tau^{n-1} - \mathbf{R}_h(\mathbf{u}_\tau^{n-1}, p_\tau^{n-1}))\|_{L^2}\|e_{\rho,h}^n\|_{L^2}\\
&\le \|\rho_\tau^n\|_{L^\infty}(ch\tau^{\frac12} + ch^2)\|e_{\rho,h}^n\|_{L^2} && \text{(use (5.19))}\\
&\le (ch\tau^{\frac12} + ch^2)\|e_{\rho,h}^n\|_{L^2} && \text{(use (3.6))}\\
&\le c\epsilon^{-1}h^2(\tau + h^2) + \epsilon\|e_{\rho,h}^n\|_{L^2}^2. && \text{(use Hölder's inequality)} \qquad (5.41)
\end{aligned}
$$

Substituting $J_1, \ldots, J_6$ into (5.35) yields, for $n = 1, \ldots, m$,

$$
D_\tau\|e_{\rho,h}^n\|_{L^2}^2 \le c\epsilon^{-1}h^2(\tau + h^2) + c\epsilon^{-1}(\|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2}^2 + \|\nabla\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2}^2) + \epsilon\|e_{\rho,h}^n\|_{L^2}^2,
$$

which implies (by choosing a small ϵ and applying Grönwall's inequality)

$$
\begin{aligned}
\max_{1\le n\le k}\|e_{\rho,h}^n\|_{L^2}^2 &\le ch^2(\tau + h^2) + \sum_{n=1}^{k} c\tau\|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{H^1}^2 + c\|e_{\rho,h}^0\|_{L^2}^2, \qquad (5.42)\\
&\le ch^2(\tau + h^2) + \sum_{n=1}^{k} c\tau\|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{H^1}^2 + c\|P_h\rho^0 - \Pi_h\rho^0\|_{L^2}^2\\
&\le ch^2(\tau + h^2) + \sum_{n=1}^{k} c\tau\|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{H^1}^2 \qquad k = 1, \ldots, m.
\end{aligned}
$$

From the last inequality and the induction assumption (5.29), we derive

$$
\begin{aligned}
\max_{1\le n\le k}\|e_{\rho,h}^n\|_{L^2} &\le \left(ch^2(\tau + h^2) + \sum_{n=1}^{k} c\tau\|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{H^1}^2\right)^{\frac12} \le ch(\tau + h^2)^{\frac14}, \qquad (5.43)\\
\max_{1\le n\le m}\|e_{\rho,h}^n\|_{L^\infty} &\le ch^{-1}\max_{1\le n\le m}\|e_{\rho,h}^n\|_{L^2}\\
&\le c\sqrt{\tau + h^2} + ch^{-1}\left(\sum_{n=1}^{m} \tau\|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{H^1}^2\right)^{\frac12}\\
&\le c(\tau + h^2)^{\frac14}. \qquad (5.44)
\end{aligned}
$$

For sufficiently small τ and h the last inequality implies

$$\max_{1\le n\le m} \|e_{\rho,h}^n\|_{L^\infty} \le \frac{1}{4} \min_{x\in\overline{\Omega}} \rho^0(x). \tag{5.45}$$

Then the same argument as (5.31)-(5.32) shows that

$$\frac{1}{2}\min_{x\in\overline{\Omega}} \rho^0(x) \le \rho_h^n(x) \le \frac{3}{2}\max_{x\in\overline{\Omega}} \rho^0(x), \quad \forall\, x\in\Omega, \quad n=1,\dots,m. \tag{5.46}$$

5.5. Estimates of $D_\tau e_{\rho,h}^n$.

We estimate $\|D_\tau e_{\rho,h}^n\|_{L^2}$ by taking $\varphi_h = D_\tau e_{\rho,h}^n$ in (5.34). Since

$$(D_\tau(\rho_\tau^n - P_h\rho_\tau^n), D_\tau e_{\rho,h}^n) = (D_\tau\rho_\tau^n - P_h D_\tau\rho_\tau^n, D_\tau e_{\rho,h}^n) = 0,$$

we obtain

$$\begin{aligned}
\|D_\tau e_{\rho,h}^n\|_{L^2} \le& \|\mathbf{u}_h^{n-1}\cdot\nabla(\rho_\tau^n - P_h\rho_\tau^n)\|_{L^2} + \frac{1}{2}\|\nabla\cdot\mathbf{u}_h^{n-1}(\rho_\tau^n - P_h\rho_\tau^n)\|_{L^2}\\
&+ \|\mathbf{u}_h^{n-1}\cdot\nabla e_{\rho,h}^n\|_{L^2} + \frac{1}{2}\|\nabla\cdot\mathbf{u}_h^{n-1} e_{\rho,h}^n\|_{L^2}\\
&+ \|\mathbf{e}_{\mathbf{u},h}^{n-1}\cdot\nabla\rho_\tau^n\|_{L^2} + \frac{1}{2}\|\nabla\cdot\mathbf{e}_{\mathbf{u},h}^{n-1}\rho_\tau^n\|_{L^2}\\
&+ \|(\mathbf{u}_\tau^{n-1} - \mathbf{R}_h(\mathbf{u}_\tau^{n-1}, p_\tau^{n-1}))\cdot\nabla\rho_\tau^n\|_{L^2}\\
&+ \frac{1}{2}\|\rho_\tau^n\nabla\cdot(\mathbf{u}_\tau^{n-1} - \mathbf{R}_h(\mathbf{u}_\tau^{n-1}, p_\tau^{n-1}))\|_{L^2}\\
=:& \sum_{k=1}^{5} J_k^*.
\end{aligned} \tag{5.47}$$

where

$$\begin{aligned}
J_1^* \le& \|\mathbf{u}_h^{n-1}\|_{L^\infty} ch\|\rho_\tau^n\|_{H^2} + c\|\nabla\cdot\mathbf{u}_h^{n-1}\|_{L^\infty}\|\rho_\tau^n - P_h\rho_\tau^n\|_{L^2}\\
\le& \|\mathbf{u}_h^{n-1}\|_{L^\infty} ch\|\rho_\tau^n\|_{H^2} + ch^{-1}\|\mathbf{u}_h^{n-1}\|_{L^\infty} ch^2\|\rho_\tau^n\|_{H^2} \qquad \text{(inverse inequality)}\\
\le& ch, \qquad \text{(use (3.9) to estimate } \|\rho_\tau^n\|_{H^2}\text{, and (5.33) to estimate } \|\mathbf{u}_h^{n-1}\|_{L^\infty}) \qquad (5.48)\\
J_2^* \le& \|\mathbf{u}_h^{n-1}\|_{L^\infty}\|\nabla e_{\rho,h}^n\|_{L^2} + \|\nabla\cdot\mathbf{u}_h^{n-1}\|_{L^\infty}\|e_{\rho,h}^n\|_{L^2}\\
\le& \|\mathbf{u}_h^{n-1}\|_{L^\infty} ch^{-1}\|e_{\rho,h}^n\|_{L^2} + ch^{-1}\|\mathbf{u}_h^{n-1}\|_{L^\infty}\|e_{\rho,h}^n\|_{L^2} \qquad \text{(inverse inequality)}\\
\le& \|\mathbf{u}_h^{n-1}\|_{L^\infty} ch^{-1} ch(\tau+h^2)^{\frac{1}{4}} + ch^{-1}\|\mathbf{u}_h^{n-1}\|_{L^\infty} ch(\tau+h^2)^{\frac{1}{4}}\\
& \qquad \text{(use (5.43) to estimate } \|e_{\rho,h}^n\|_{L^2})\\
\le& c(\tau+h^2)^{\frac{1}{4}}, \qquad \text{(use (5.33) to estimate } \|\mathbf{u}_h^{n-1}\|_{L^\infty}) \qquad (5.49)\\
J_3^* \le& \|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2}\|\nabla\rho_\tau^n\|_{L^\infty} + \|\nabla\cdot\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2}\|\rho_\tau^n\|_{L^\infty}\\
\le& \|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2}\|\|\nabla\rho_\tau^n\|_{L^\infty} + ch^{-1}\|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2}\|\rho_\tau^n\|_{L^\infty} \qquad \text{(inverse inequality)}\\
\le& ch(\tau+h^2)^{\frac{1}{4}}\|\nabla\rho_\tau^n\|_{L^\infty} + ch^{-1}ch(\tau+h^2)^{\frac{1}{4}}\|\rho_\tau^n\|_{L^\infty}\\
& \qquad \text{(use (5.29) to estimate } \|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2})\\
\le& c(\tau+h^2)^{\frac{1}{4}}, \qquad \text{(use (3.9) to estimate } \|\nabla\rho_\tau^n\|_{L^\infty} \text{ and } \|\rho_\tau^n\|_{L^\infty}) \qquad (5.50)
\end{aligned}$$

$$
\begin{aligned}
J_4^* \leq & ch^2(\|\mathbf{u}_\tau^{n-1}\|_{H^2} + \|p_\tau^{n-1}\|_{H^1})\|\nabla\rho_\tau^n\|_{L^\infty} \leq ch^2, \qquad \text{(use (3.9))} && (5.51)\\
J_5^* \leq & c\|\rho_\tau^n\|_{L^\infty}\|\nabla\cdot(\mathbf{u}_\tau^{n-1} - \mathbf{R}_h(\mathbf{u}_\tau^{n-1}, p_\tau^{n-1}))\|_{L^2}\\
\leq & c\|\rho_\tau^n\|_{L^\infty} ch(\|\mathbf{u}_\tau^{n-1}\|_{H^2} + \|p_\tau^{n-1}\|_{H^1}) \qquad \text{(inverse inequality)}\\
\leq & ch. \qquad \text{(use (3.9) to estimate } \|\rho_\tau^n\|_{L^\infty},\ \|\mathbf{u}_\tau^{n-1}\|_{H^2} \text{ and } \|p_\tau^{n-1}\|_{H^1}) && (5.52)
\end{aligned}
$$

Substituting (5.48)-(5.52) into (5.47) yields

$$
\|D_\tau e_{\rho,h}^n\|_{L^2} \leq c(\tau + h^2)^{\frac{1}{4}}. \tag{5.53}
$$

5.6. Estimates of $\mathbf{e}_{\mathbf{u},h}^n$.

By using integration by parts, equation (5.21) can be rewritten as

$$
\begin{aligned}
&(\rho_h^{n-1}D_\tau\mathbf{u}_h^n, v_h) + \frac{1}{2}(D_\tau\rho_h^n\,\mathbf{u}_h^n, v_h) + \frac{1}{2}(\rho_h^n\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{u}_h^n, \mathbf{v}_h) - \frac{1}{2}(\rho_h^n\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{v}_h, \mathbf{u}_h^n)\\
&+ B((\mathbf{u}_h^n, p_h^n), (\mathbf{v}_h, q_h)) = 0.
\end{aligned} \tag{5.54}
$$

Subtracting (5.54) from (5.26), we obtain

$$
\begin{aligned}
&\Big[(\rho_h^{n-1}D_\tau\mathbf{e}_{\mathbf{u},h}^n, \mathbf{v}_h) + (\rho_h^{n-1}D_\tau(\mathbf{u}_\tau^n - \mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n)), \mathbf{v}_h)\\
&+ ((\rho_\tau^{n-1} - P_h\rho_\tau^{n-1})D_\tau\mathbf{u}_\tau^n, \mathbf{v}_h) + (e_{\rho,h}^{n-1}D_\tau\mathbf{u}_\tau^n, \mathbf{v}_h)\Big]\\
&+ \frac{1}{2}\Big[(D_\tau\rho_\tau^n\,\mathbf{e}_{\mathbf{u},h}^n, \mathbf{v}_h) + (D_\tau\rho_\tau^n\,(\mathbf{u}_\tau^n - \mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n)), \mathbf{v}_h)\\
&+ (D_\tau(\rho_\tau^n - P_h\rho_\tau^n)\,\mathbf{u}_h^n, \mathbf{v}_h) + (D_\tau e_{\rho,h}^n\,\mathbf{u}_h^n, \mathbf{v}_h)\Big]\\
&+ \frac{1}{2}\Big[(\rho_h^n\mathbf{u}_h^{n-1}\nabla\mathbf{e}_{\mathbf{u},h}^n, \mathbf{v}_h) + (\rho_h^n\mathbf{u}_h^{n-1}\nabla(\mathbf{u}_\tau^n - \mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n)), \mathbf{v}_h)\\
&+ (\rho_h^n(\mathbf{u}_\tau^{n-1} - \mathbf{R}_h(\mathbf{u}_\tau^{n-1}, p_\tau^{n-1}))\cdot\nabla\mathbf{u}_\tau^n, \mathbf{v}_h) + (\rho_h^n\mathbf{e}_{\mathbf{u},h}^{n-1}\cdot\nabla\mathbf{u}_\tau^n, \mathbf{v}_h)\\
&+ ((\rho_\tau^n - P_h\rho_\tau^n)\mathbf{u}_\tau^{n-1}\cdot\nabla\mathbf{u}_\tau^n, \mathbf{v}_h) + (e_{\rho,h}^n\mathbf{u}_\tau^{n-1}\cdot\nabla\mathbf{u}_\tau^n, \mathbf{v}_h)\Big]\\
&- \frac{1}{2}\Big[(\rho_h^n\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{v}_h, \mathbf{e}_{\mathbf{u},h}^n) + (\rho_h^n\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{v}_h, \mathbf{u}_\tau^n - \mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n))\\
&+ (\rho_h^n(\mathbf{u}_\tau^{n-1} - \mathbf{R}_h(\mathbf{u}_\tau^{n-1}, p_\tau^{n-1}))\cdot\nabla\mathbf{v}_h, \mathbf{u}_\tau^n) + (\rho_h^n\mathbf{e}_{\mathbf{u},h}^{n-1}\cdot\nabla\mathbf{v}_h, \mathbf{u}_\tau^n)\\
&+ ((\rho_\tau^n - P_h\rho_\tau^n)\mathbf{u}_\tau^{n-1}\cdot\nabla\mathbf{v}_h, \mathbf{u}_\tau^n) + (e_{\rho,h}^n\mathbf{u}_\tau^{n-1}\cdot\nabla\mathbf{v}_h, \mathbf{u}_\tau^n)\Big]\\
&+ B((\mathbf{e}_{\mathbf{u},h}^n, e_{p,h}^n), (\mathbf{v}_h, q_h))\\
=&0.
\end{aligned} \tag{5.55}
$$

Reformulating the last equation yields

$$
\begin{aligned}
&(\rho_h^{n-1}D_\tau\mathbf{e}_{\mathbf{u},h}^n, \mathbf{v}_h) + \frac{1}{2}(D_\tau\rho_h^n\,\mathbf{e}_{\mathbf{u},h}^n, \mathbf{v}_h) + B((\mathbf{e}_{\mathbf{u},h}^n, e_{p,h}^n), (\mathbf{v}_h, q_h))\\
=& - (\rho_h^{n-1}D_\tau(\mathbf{u}_\tau^n - \mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n)), \mathbf{v}_h)\\
&- \big[((\rho_\tau^{n-1} - P_h\rho_\tau^{n-1})D_\tau\mathbf{u}_\tau^n, \mathbf{v}_h) + (e_{\rho,h}^{n-1}D_\tau\mathbf{u}_\tau^n, \mathbf{v}_h)\big]\\
&- \frac{1}{2}(D_\tau e_{\rho,h}^n\,\mathbf{e}_{\mathbf{u},h}^n, \mathbf{v}_h) - \frac{1}{2}(D_\tau(\rho_\tau^n - P_h\rho_\tau^n)\,\mathbf{e}_{\mathbf{u},h}^n, \mathbf{v}_h) - \frac{1}{2}(D_\tau\rho_\tau^n\,(\mathbf{u}_\tau^n - \mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n)), \mathbf{v}_h)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\big[(D_\tau(\rho_\tau^n-P_h\rho_\tau^n)\,\mathbf{u}_h^n,\mathbf{v}_h)+(D_\tau e_{\rho,h}^n\,\mathbf{u}_h^n,\mathbf{v}_h)\big]\\
&-\frac{1}{2}\big[(\rho_h^n\mathbf{u}_h^{n-1}\nabla\mathbf{e}_{\mathbf{u},h}^n,\mathbf{v}_h)+(\rho_h^n\mathbf{u}_h^{n-1}\nabla(\mathbf{u}_\tau^n-\mathbf{R}_h(\mathbf{u}_\tau^n,p_\tau^n)),\mathbf{v}_h)\big]\\
&-\frac{1}{2}\big[(\rho_h^n(\mathbf{u}_\tau^{n-1}-\mathbf{R}_h(\mathbf{u}_\tau^{n-1},p_\tau^{n-1}))\cdot\nabla\mathbf{u}_\tau^n,\mathbf{v}_h)+(\rho_h^n\mathbf{e}_{\mathbf{u},h}^{n-1}\cdot\nabla\mathbf{u}_\tau^n,\mathbf{v}_h)\big]\\
&-\frac{1}{2}\big[((\rho_\tau^n-P_h\rho_\tau^n)\mathbf{u}_\tau^{n-1}\cdot\nabla\mathbf{u}_\tau^n,\mathbf{v}_h)+(e_{\rho,h}^n\mathbf{u}_\tau^{n-1}\cdot\nabla\mathbf{u}_\tau^n,\mathbf{v}_h)\big]\\
&+\frac{1}{2}\big[(\rho_h^n\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{v}_h,\mathbf{e}_{\mathbf{u},h}^n)+(\rho_h^n\mathbf{u}_h^{n-1}\cdot\nabla\mathbf{v}_h,\mathbf{u}_\tau^n-\mathbf{R}_h(\mathbf{u}_\tau^n,p_\tau^n))\big]\\
&+\frac{1}{2}\big[(\rho_h^n(\mathbf{u}_\tau^{n-1}-\mathbf{R}_h(\mathbf{u}_\tau^{n-1},p_\tau^{n-1}))\cdot\nabla\mathbf{v}_h,\mathbf{u}_\tau^n)+(\rho_h^n\mathbf{e}_{\mathbf{u},h}^{n-1}\cdot\nabla\mathbf{v}_h,\mathbf{u}_\tau^n)\big]\\
&+\frac{1}{2}\big[((\rho_\tau^n-P_h\rho_\tau^n)\mathbf{u}_\tau^{n-1}\cdot\nabla\mathbf{v}_h,\mathbf{u}_\tau^n)+(e_{\rho,h}^n\mathbf{u}_\tau^{n-1}\cdot\nabla\mathbf{v}_h,\mathbf{u}_\tau^n)\big]\\
=:&\sum_{j=1}^{10}I_j^*(\mathbf{v}_h). \qquad (5.56)
\end{aligned}
$$

Substituting $(\mathbf{v}_h,q_h)=(\mathbf{e}_{\mathbf{u},h}^n,e_{p,h}^n)$ and using the interpolation inequality

$$
\begin{aligned}
\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^4}^2 &\le c\|\mathbf{e}_{\mathbf{u},h}^n\|_{H^{\frac12}}^2 && \text{(Sobolev embedding, cf. [51, Eq. (32.7)])}\\
&\le c\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}\|\mathbf{e}_{\mathbf{u},h}^n\|_{H^1}, && \text{(interpolation inequality, cf. [51, Lemma 25.2])} \qquad (5.57)
\end{aligned}
$$

we have

$$
\begin{aligned}
|I_1^*(\mathbf{e}_{\mathbf{u},h}^n)| =&\big|(\rho_h^{n-1}D_\tau(\mathbf{u}_\tau^n-\mathbf{R}_h(\mathbf{u}_\tau^n,p_\tau^n)),\mathbf{e}_{\mathbf{u},h}^n)\big|\\
\le&\|\rho_h^{n-1}\|_{L^\infty}\|D_\tau\mathbf{u}_\tau^n-\mathbf{R}_h(D_\tau\mathbf{u}_\tau^n,D_\tau p_\tau^n)\|_{L^2}\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}\\
\le&c(\|D_\tau\mathbf{u}_\tau^n\|_{H^2}+\|D_\tau p_\tau^n\|_{H^1})h^2\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2} \qquad \text{(use (5.46))}\\
\le&c\epsilon^{-1}(\|D_\tau\mathbf{u}_\tau^n\|_{H^2}^2+\|D_\tau p_\tau^n\|_{H^1}^2)h^4+\epsilon\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2, \qquad (5.58)\\
|I_2^*(\mathbf{e}_{\mathbf{u},h}^n)| =&\big|((\rho_\tau^{n-1}-P_h\rho_\tau^{n-1})D_\tau\mathbf{u}_\tau^n,\mathbf{e}_{\mathbf{u},h}^n)+(e_{\rho,h}^{n-1}D_\tau\mathbf{u}_\tau^n,\mathbf{e}_{\mathbf{u},h}^n)\big|\\
\le&\|D_\tau\mathbf{u}_\tau^n\|_{L^\infty}\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}(\|\rho_\tau^{n-1}-P_h\rho_\tau^{n-1}\|_{L^2}+\|e_{\rho,h}^{n-1}\|_{L^2})\\
\le&\|D_\tau\mathbf{u}_\tau^n\|_{L^\infty}\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}(h^2\|\rho_\tau^{n-1}\|_{H^2}+\|e_{\rho,h}^{n-1}\|_{L^2})\\
\le&c\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}(h^2+\|e_{\rho,h}^{n-1}\|_{L^2}) \quad \text{(use (3.9) to control } \|\rho_\tau^{n-1}\|_{H^2} \text{ and } \|D_\tau\mathbf{u}_\tau^n\|_{L^\infty})\\
\le&c\epsilon^{-1}(h^4+\|e_{\rho,h}^{n-1}\|_{L^2}^2)+\epsilon\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2, \qquad (5.59)\\
|I_3^*(\mathbf{e}_{\mathbf{u},h}^n)| \le&c\|D_\tau e_{\rho,h}^n\|_{L^2}\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^4}^2+c\|D_\tau(\rho_\tau^n-P_h\rho_\tau^n)\|_{L^2}\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^4}^2\\
&+c\|D_\tau\rho_\tau^n\|_{L^\infty}\|\mathbf{u}_\tau^n-\mathbf{R}_h(\mathbf{u}_\tau^n,p_\tau^n)\|_{L^2}\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}\\
\le&c(\tau+h^2)^{\frac14}\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}\|\nabla\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}+ch^2\|D_\tau\rho_\tau^n\|_{H^2}\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}\|\nabla\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}\\
&\qquad \text{(use (5.53) and (5.10))}\\
&+c\|D_\tau\rho_\tau^n\|_{L^\infty}h^2(\|\mathbf{u}_\tau^n\|_{H^2}+\|p_\tau^n\|_{H^1})\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2} \qquad \text{(use (5.6))}\\
\le&c(\tau+h^2)^{\frac14}\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2+c(\tau+h^2)^{\frac14}\|\nabla\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2+c\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2+ch^4\\
&\qquad \text{(use (3.9) to control } \|\mathbf{u}_\tau^n\|_{H^2} \text{ and } \|D_\tau\rho_\tau^n\|_{L^\infty})\\
|I_5^*(\mathbf{e}_{\mathbf{u},h}^n)| \le&\big|(\rho_h^n\mathbf{u}_h^{n-1}\nabla\mathbf{e}_{\mathbf{u},h}^n,\mathbf{e}_{\mathbf{u},h}^n)+(\rho_h^n\mathbf{u}_h^{n-1}\nabla(\mathbf{u}_\tau^n-\mathbf{R}_h(\mathbf{u}_\tau^n,p_\tau^n)),\mathbf{e}_{\mathbf{u},h}^n)\big|\\
\le&c\|\rho_h^n\|_{L^\infty}\|\mathbf{u}_h^{n-1}\|_{L^\infty}\big(\|\nabla\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}+\|\nabla(\mathbf{u}_\tau^n-\mathbf{R}_h(\mathbf{u}_\tau^n,p_\tau^n))\|_{L^2}\big)\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&\le c\big(\|\nabla \mathbf{e}^n_{\mathbf{u},h}\|_{L^2} + \|\nabla(\mathbf{u}^n_\tau - \mathbf{R}_h(\mathbf{u}^n_\tau, p^n_\tau))\|_{L^2}\big)\|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2} \quad \text{(use (5.33) and (5.46))}\\
&\le \big(c\|\nabla \mathbf{e}^n_{\mathbf{u},h}\|_{L^2} + ch\tau^{\frac12} + ch^2\big)\|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2} \quad \text{(use (5.19))}\\
&\le \epsilon\|\nabla \mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} + c\epsilon^{-1}\|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} + c\epsilon^{-1}h^2(\tau+h^2), \qquad (5.60)\\
|I_6^*(\mathbf{e}^n_{\mathbf{u},h})| &\le \|\rho^n_h\|_{L^\infty}(\|\mathbf{u}^{n-1}_\tau - \mathbf{R}_h(\mathbf{u}^{n-1}_\tau, p^{n-1}_\tau))\|_{L^2} + \|\mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^2})\|\nabla \mathbf{u}^n_\tau\|_{L^\infty}\|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2}\\
&\le c(\|\mathbf{u}^{n-1}_\tau - \mathbf{R}_h(\mathbf{u}^{n-1}_\tau, p^{n-1}_\tau))\|_{L^2} + \|\mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^2})\|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2} \quad \text{(use (3.9) for } \|\nabla\mathbf{u}^n_\tau\|_{L^\infty})\\
&\qquad \text{(use (5.46) for } \|\rho^n_h\|_{L^\infty})\\
&\le c(h^2(\|\mathbf{u}^{n-1}_\tau\|_{H^2} + \|p^{n-1}_\tau\|_{H^1}) + \|\mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^2})\|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2} \quad \text{(use (5.6))}\\
&\le c(h^2 + \|\mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^2})\|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2} \quad \text{(use (3.9) to control } \|\mathbf{u}^{n-1}_\tau\|_{H^2} \text{ and } \|p^{n-1}_\tau\|_{H^1})\\
&\le ch^4 + c(\|\mathbf{e}^{n-1}_{\mathbf{u},h}\|^2_{L^2} + \|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2}), \qquad (5.61)\\
|I_7^*(\mathbf{e}^n_{\mathbf{u},h})| &\le (\|\rho^n_\tau - P_h\rho^n_\tau\|_{L^2} + \|e^n_{\rho,h}\|_{L^2})\|\mathbf{u}^{n-1}_\tau\|_{L^\infty}\|\nabla\mathbf{u}^n_\tau\|_{L^\infty}\|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2}\\
&\le (ch^2\|\rho^n_\tau\|_{H^2} + \|e^n_{\rho,h}\|_{L^2})\|\mathbf{u}^{n-1}_\tau\|_{L^\infty}\|\nabla\mathbf{u}^n_\tau\|_{L^\infty}\|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2} \quad \text{(use (5.10))}\\
&\le c(h^2 + \|e^n_{\rho,h}\|_{L^2})\|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2} \quad \text{(use (3.9) to control } \|\rho^n_\tau\|_{H^2} \text{ and } \|\nabla\mathbf{u}^n_\tau\|_{L^\infty})\\
&\le ch^4 + c(\|e^n_{\rho,h}\|^2_{L^2} + \|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2}), \qquad (5.62)\\
|I_8^*(\mathbf{e}^n_{\mathbf{u},h})| &\le \big|(\rho^n_h\mathbf{u}^{n-1}_h\cdot\nabla\mathbf{e}^n_{\mathbf{u},h}, \mathbf{e}^n_{\mathbf{u},h}) + (\rho^n_h\mathbf{u}^{n-1}_h\cdot\nabla\mathbf{e}^n_{\mathbf{u},h}, \mathbf{u}^n_\tau - \mathbf{R}_h(\mathbf{u}^n_\tau, p^n_\tau))\big|\\
&\le \|\rho^n_h\|_{L^\infty}\|\mathbf{u}^{n-1}_h\|_{L^\infty}\|\nabla\mathbf{e}^n_{\mathbf{u},h}\|_{L^2}(\|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2} + \|\mathbf{u}^n_\tau - \mathbf{R}_h(\mathbf{u}^n_\tau, p^n_\tau))\|_{L^2})\\
&\le \|\rho^n_h\|_{L^\infty}\|\mathbf{u}^{n-1}_h\|_{L^\infty}\|\nabla\mathbf{e}^n_{\mathbf{u},h}\|_{L^2}(\|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2} + ch^2(\|\mathbf{u}^{n-1}_\tau\|_{H^2} + \|p^{n-1}_\tau\|_{H^1}))\\
&\le c\|\nabla\mathbf{e}^n_{\mathbf{u},h}\|_{L^2}(\|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2} + h^2) \quad \text{(use (5.33) and (5.46) to control } \|\rho^n_h\|_{L^\infty}\|\mathbf{u}^{n-1}_h\|_{L^\infty})\\
&\qquad \text{(use (3.9) to control } \|\mathbf{u}^{n-1}_\tau\|_{H^2} \text{ and } \|p^{n-1}_\tau\|_{H^1})\\
&\le \epsilon\|\nabla\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} + c\epsilon^{-1}(\|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} + h^4), \qquad (5.63)\\
|I_9^*(\mathbf{e}^n_{\mathbf{u},h})| &\le \big|(\rho^n_h(\mathbf{u}^{n-1}_\tau - \mathbf{R}_h(\mathbf{u}^{n-1}_\tau, p^{n-1}_\tau))\cdot\nabla\mathbf{e}^n_{\mathbf{u},h}, \mathbf{u}^n_\tau) + (\rho^n_h\mathbf{e}^{n-1}_{\mathbf{u},h}\cdot\nabla\mathbf{e}^n_{\mathbf{u},h}, \mathbf{u}^n_\tau)\big|\\
&\le \|\rho^n_h\|_{L^\infty}(\|\mathbf{u}^{n-1}_\tau - \mathbf{R}_h(\mathbf{u}^{n-1}_\tau, p^{n-1}_\tau))\|_{L^2} + \|\mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^2})\|\nabla\mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^2}\|\mathbf{u}^n_\tau\|_{L^\infty}\\
&\le \|\rho^n_h\|_{L^\infty}(ch^2(\|\mathbf{u}^{n-1}_\tau\|_{H^2} + \|p^{n-1}_\tau\|_{H^1}) + \|\mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^2})\|\nabla\mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^2}\|\mathbf{u}^n_\tau\|_{L^\infty}\\
&\le (ch^2 + \|\mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^2})\|\nabla\mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^2}\\
&\le \epsilon\|\nabla\mathbf{e}^{n-1}_{\mathbf{u},h}\|^2_{L^2} + c\epsilon^{-1}(\|\mathbf{e}^{n-1}_{\mathbf{u},h}\|^2_{L^2} + h^4), \qquad (5.64)\\
|I_{10}^*(\mathbf{e}^n_{\mathbf{u},h})| &\le \big|((\rho^n_\tau - P_h\rho^n_\tau)\mathbf{u}^{n-1}_\tau\cdot\nabla\mathbf{e}^n_{\mathbf{u},h}, \mathbf{u}^n_\tau) + (e^n_{\rho,h}\mathbf{u}^{n-1}_\tau\cdot\nabla\mathbf{e}^n_{\mathbf{u},h}, \mathbf{u}^n_\tau)\big|\\
&\le (\|\rho^n_\tau - P_h\rho^n_\tau\|_{L^2} + \|e^n_{\rho,h}\|_{L^2})\|\mathbf{u}^{n-1}_\tau\|_{L^\infty}\|\nabla\mathbf{e}^n_{\mathbf{u},h}\|_{L^2}\|\mathbf{u}^n_\tau\|_{L^\infty}\\
&\le (ch^2\|\rho^n_\tau\|_{H^2} + \|e^n_{\rho,h}\|_{L^2})\|\mathbf{u}^{n-1}_\tau\|_{L^\infty}\|\nabla\mathbf{e}^n_{\mathbf{u},h}\|_{L^2}\|\mathbf{u}^n_\tau\|_{L^\infty}\\
&\le (ch^2 + \|e^n_{\rho,h}\|_{L^2})\|\nabla\mathbf{e}^n_{\mathbf{u},h}\|_{L^2}\\
&\le \epsilon\|\nabla\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} + c\epsilon^{-1}(h^4 + \|e^n_{\rho,h}\|^2_{L^2}). \qquad (5.65)
\end{aligned}
$$

It remains to estimate $|I_4^*(\mathbf{e}^n_{\mathbf{u},h})|$. To this end, we substitute $\varphi_h = P_h(\mathbf{u}^n_h\cdot\mathbf{e}^n_{\mathbf{u},h})$ into (5.34). Then we obtain

$$
\begin{aligned}
|(D_\tau e^n_{\rho,h}, P_h(\mathbf{u}^n_h\cdot\mathbf{e}^n_{\mathbf{u},h}))| \le& |(\mathbf{u}^{n-1}_h\cdot\nabla(\rho^n_\tau - P_h\rho^n_\tau), P_h(\mathbf{u}^n_h\cdot\mathbf{e}^n_{\mathbf{u},h}))|\\
&+ \frac12|(\nabla\cdot\mathbf{u}^{n-1}_h(\rho^n_\tau - P_h\rho^n_\tau), P_h(\mathbf{u}^n_h\cdot\mathbf{e}^n_{\mathbf{u},h}))|\\
&+ |(\mathbf{u}^{n-1}_h\cdot\nabla e^n_{\rho,h}, P_h(\mathbf{u}^n_h\cdot\mathbf{e}^n_{\mathbf{u},h}))|\\
&+ \frac12|(\nabla\cdot\mathbf{u}^{n-1}_h e^n_{\rho,h}, P_h(\mathbf{u}^n_h\cdot\mathbf{e}^n_{\mathbf{u},h}))|
\end{aligned}
$$

$$
\begin{aligned}
&+ |(\mathbf{e}_{\mathbf{u},h}^{n-1} \cdot \nabla \rho_\tau^n, P_h(\mathbf{u}_h^n \cdot \mathbf{e}_{\mathbf{u},h}^n))| \\
&+ \frac{1}{2}|(\nabla \cdot \mathbf{e}_{\mathbf{u},h}^{n-1} \rho_\tau^n, P_h(\mathbf{u}_h^n \cdot \mathbf{e}_{\mathbf{u},h}^n))| \\
&+ |((\mathbf{u}_\tau^{n-1} - \mathbf{R}_h(\mathbf{u}_\tau^{n-1}, p_\tau^{n-1})) \cdot \nabla \rho_\tau^n, P_h(\mathbf{u}_h^n \cdot \mathbf{e}_{\mathbf{u},h}^n))| \\
&+ \frac{1}{2}|(\rho_\tau^n \nabla \cdot (\mathbf{u}_\tau^{n-1} - \mathbf{R}_h(\mathbf{u}_\tau^{n-1}, p_\tau^{n-1})), P_h(\mathbf{u}_h^n \cdot \mathbf{e}_{\mathbf{u},h}^n))| \\
=:& \sum_{i=1}^{8} H_i. \qquad (5.66)
\end{aligned}
$$

By the decomposition $\mathbf{u}_h^n \cdot \mathbf{e}_{\mathbf{u},h}^n = (\mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n) - \mathbf{e}_{\mathbf{u},h}^n) \cdot \mathbf{e}_{\mathbf{u},h}^n = \mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n) \cdot \mathbf{e}_{\mathbf{u},h}^n - |\mathbf{e}_{\mathbf{u},h}^n|^2$, we have

$$
\begin{aligned}
H_1 \le& |(\mathbf{u}_h^{n-1} \cdot \nabla(\rho_\tau^n - P_h\rho_\tau^n), P_h(|\mathbf{e}_{\mathbf{u},h}^n|^2))| + |(\mathbf{u}_h^{n-1} \cdot \nabla(\rho_\tau^n - P_h\rho_\tau^n), P_h(\mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n) \cdot \mathbf{e}_{\mathbf{u},h}^n))| \\
\le& c\|\mathbf{u}_h^{n-1}\|_{L^\infty} \|\nabla(\rho_\tau^n - P_h\rho_\tau^n)\|_{L^2} (\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^4}^2 + \|\mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n)\|_{L^\infty} \|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}) \\
\le& c\|\mathbf{u}_h^{n-1}\|_{L^\infty} (ch\tau + ch^2)(\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^4}^2 + \|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}) \quad \text{(use (5.17) and (5.8))} \\
\le& (ch\tau + ch^2)(\|\nabla \mathbf{e}_{\mathbf{u},h}^n\|_{L^2} \|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2} + \|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}) \quad \text{(use (5.33) and (5.57))} \\
\le& ch(\tau + h)(\|\nabla \mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2 + \|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2) + ch^2(\tau^2 + h^2) + c\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2 \qquad (5.67) \\
H_2 \le& |(\nabla \cdot \mathbf{u}_h^{n-1}(\rho_\tau^n - P_h\rho_\tau^n), P_h(|\mathbf{e}_{\mathbf{u},h}^n|^2))| + |(\nabla \cdot \mathbf{u}_h^{n-1}(\rho_\tau^n - P_h\rho_\tau^n), P_h(\mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n) \cdot \mathbf{e}_{\mathbf{u},h}^n))| \\
\le& \|\nabla \mathbf{u}_h^{n-1}\|_{L^\infty} \|\rho_\tau^n - P_h\rho_\tau^n\|_{L^2} (\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^4}^2 + \|\mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n)\|_{L^\infty} \|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}) \\
\le& \|\nabla \mathbf{u}_h^{n-1}\|_{L^\infty} (ch^2\tau + ch^3)(\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^4}^2 + \|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}) \quad \text{(use (5.16))} \\
\le& \|\mathbf{u}_h^{n-1}\|_{L^\infty} (ch\tau + ch^2)(\|\nabla \mathbf{e}_{\mathbf{u},h}^n\|_{L^2} \|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2} + \|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}) \\
& \text{(use inverse inequality and (5.57))} \\
\le& (ch\tau + ch^2)(\|\nabla \mathbf{e}_{\mathbf{u},h}^n\|_{L^2} \|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2} + \|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}) \quad \text{(use (5.33))} \\
\le& ch(\tau + h)(\|\nabla \mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2 + \|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2) + ch^2(\tau^2 + h^2) + c\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2 \qquad (5.68) \\
H_3 \le& |(\mathbf{u}_h^{n-1} \cdot \nabla e_{\rho,h}^n, P_h(|\mathbf{e}_{\mathbf{u},h}^n|^2))| + |(\mathbf{u}_h^{n-1} \cdot \nabla e_{\rho,h}^n, P_h(\mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n) \cdot \mathbf{e}_{\mathbf{u},h}^n))| \\
=& |(e_{\rho,h}^n, \nabla \cdot (\mathbf{u}_h^{n-1} P_h(|\mathbf{e}_{\mathbf{u},h}^n|^2)))| + |(e_{\rho,h}^n, \nabla \cdot (\mathbf{u}_h^{n-1} P_h(\mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n) \cdot \mathbf{e}_{\mathbf{u},h}^n)))| \\
& \text{(integration by parts)} \\
=& |(e_{\rho,h}^n, \nabla \cdot \mathbf{u}_h^{n-1} P_h(|\mathbf{e}_{\mathbf{u},h}^n|^2)) + (e_{\rho,h}^n, \mathbf{u}_h^{n-1} \cdot \nabla P_h(|\mathbf{e}_{\mathbf{u},h}^n|^2))| \\
&+ |(e_{\rho,h}^n, \nabla \cdot \mathbf{u}_h^{n-1} P_h(\mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n) \cdot \mathbf{e}_{\mathbf{u},h}^n)) + (e_{\rho,h}^n, \mathbf{u}_h^{n-1} \cdot \nabla P_h(\mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n) \cdot \mathbf{e}_{\mathbf{u},h}^n))| \\
\le& c\|e_{\rho,h}^n\|_{L^\infty} \|\nabla \cdot \mathbf{u}_h^{n-1}\|_{L^\infty} \|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2 + c\|e_{\rho,h}^n\|_{L^\infty} \|\mathbf{u}_h^{n-1}\|_{L^\infty} \|\nabla |\mathbf{e}_{\mathbf{u},h}^n|^2\|_{L^1} \\
& \text{(use (5.11))} \\
&+ c\|e_{\rho,h}^n\|_{L^2} \|\nabla \cdot \mathbf{u}_h^{n-1}\|_{L^\infty} \|\mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n) \cdot \mathbf{e}_{\mathbf{u},h}^n\|_{L^2} \\
&+ c\|e_{\rho,h}^n\|_{L^2} \|\mathbf{u}_h^{n-1}\|_{L^\infty} \|\nabla(\mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n) \cdot \mathbf{e}_{\mathbf{u},h}^n)\|_{L^2} \\
\le& c\|\nabla \mathbf{u}_h^{n-1}\|_{L^\infty} \|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2 + c\|\nabla \mathbf{e}_{\mathbf{u},h}^n\|_{L^2} \|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2} \\
&+ c\|\nabla \mathbf{u}_h^{n-1}\|_{L^\infty} \|e_{\rho,h}^n\|_{L^2} \|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2} + c\|e_{\rho,h}^n\|_{L^2} \|\nabla \mathbf{e}_{\mathbf{u},h}^n\|_{L^2} \\
\le& c\|\nabla \mathbf{u}_h^{n-1}\|_{L^\infty} (\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2 + \|e_{\rho,h}^n\|_{L^2}^2) + c\epsilon^{-1}(\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2 + \|e_{\rho,h}^n\|_{L^2}^2) + \epsilon\|\nabla \mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2, \qquad (5.69)
\end{aligned}
$$

where the last inequality uses $\|\mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n)\|_{W^{1,\infty}}$, which is a consequence of (5.8) and (3.9). Since (5.8) implies

$$
\|\nabla \mathbf{u}_h^{n-1}\|_{L^\infty} \le \|\nabla \mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^\infty} + \|\nabla \mathbf{R}_h(\mathbf{u}_\tau^{n-1}, p_\tau^{n-1})\|_{L^\infty}
$$

$$\begin{aligned}&\leq \|\nabla \mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^\infty} + c\\ &\leq ch^{-1}\|\nabla \mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^2} + c,\end{aligned} \tag{5.70}$$

where the last step is due to the inverse inequality. Substituting the last inequality into the estimate of H_3 above, we obtain

$$H_3 \leq c(h^{-1}\|\nabla \mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^2} + 1)(\|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} + \|e^n_{\rho,h}\|^2_{L^2}) + c\epsilon^{-1}(\|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} + \|e^n_{\rho,h}\|^2_{L^2}) + \epsilon\|\nabla \mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2}. \tag{5.71}$$

Similarly, we have

$$\begin{aligned}
H_4 \leq& |(e^n_{\rho,h}\nabla\cdot \mathbf{u}^{n-1}_h, P_h(|\mathbf{e}^n_{\mathbf{u},h}|^2))| + |(e^n_{\rho,h}\nabla\cdot \mathbf{u}^{n-1}_h, P_h(\mathbf{R}_h(\mathbf{u}^n_\tau, p^n_\tau)\mathbf{e}^n_{\mathbf{u},h}))|\\
\leq& c\|\nabla \mathbf{u}^{n-1}_h\|_{L^\infty}(\|e^n_{\rho,h}\|_{L^\infty}\|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} + \|e^n_{\rho,h}\|_{L^2}\|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2})\\
\leq& c(h^{-1}\|\nabla \mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^2} + 1)(\|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} + \|e^n_{\rho,h}\|^2_{L^2}), \quad \text{(use (5.44) and (5.70))} && (5.72)\\
H_5 \leq& |(\mathbf{e}^{n-1}_{\mathbf{u},h}\cdot\nabla\rho^n_\tau, P_h(|\mathbf{e}^n_{\mathbf{u},h}|^2))| + |(\mathbf{e}^{n-1}_{\mathbf{u},h}\cdot\nabla\rho^n_\tau, P_h(\mathbf{R}_h(\mathbf{u}^n_\tau, p^n_\tau)\mathbf{e}^n_{\mathbf{u},h}))|\\
\leq& c\|\nabla\rho^n_\tau\|_{L^\infty}(\|\mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^\infty}\|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} + \|\mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^2}\|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2})\\
\leq& c(\|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} + \|\mathbf{e}^{n-1}_{\mathbf{u},h}\|^2_{L^2}), \quad \text{(use (3.9) and (5.30))} && (5.73)\\
H_6 \leq& |(\rho^n_\tau\nabla\cdot \mathbf{e}^{n-1}_{\mathbf{u},h}, P_h(|\mathbf{e}^n_{\mathbf{u},h}|^2))| + |(\rho^n_\tau\nabla\cdot \mathbf{e}^{n-1}_{\mathbf{u},h}, P_h(\mathbf{R}_h(\mathbf{u}^n_\tau, p^n_\tau)\mathbf{e}^n_{\mathbf{u},h}))|\\
\leq& c\|\rho^n_\tau\|_{L^\infty}(\|\nabla \mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^\infty}\|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} + \|\nabla \mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^2}\|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2})\\
\leq& c\|\rho^n_\tau\|_{L^\infty}(h^{-1}\|\nabla \mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^2}\|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} + \|\nabla \mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^2}\|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2}) \qquad \text{(inverse inequality)}\\
\leq& ch^{-1}\|\nabla \mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^2}\|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} + \epsilon\|\nabla \mathbf{e}^{n-1}_{\mathbf{u},h}\|^2_{L^2} + c\epsilon^{-1}\|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2}, \qquad \text{(use (5.46))}\\
H_7 \leq& |(\mathbf{u}^{n-1}_\tau - \mathbf{R}_h(\mathbf{u}^{n-1}_\tau, p^{n-1}_\tau))\cdot\nabla\rho^n_\tau, P_h(|\mathbf{e}^n_{\mathbf{u},h}|^2) - P_h(\mathbf{R}_h(\mathbf{u^n_\tau}, p^n_\tau)\mathbf{e}^n_{\mathbf{u},h}))|\\
\leq& \|\mathbf{u}^{n-1}_\tau - \mathbf{R}_h(\mathbf{u}^{n-1}_\tau, p^{n-1}_\tau)\|_{L^2}\|\nabla\rho^n_\tau\|_{L^\infty}(\|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^4} + \|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2})\\
\leq& ch^2(\|\mathbf{u}^{n-1}_\tau\|_{H^2} + \|p^{n-1}_\tau\|_{H^1})\|\nabla\rho^n_\tau\|_{L^\infty}(\|\nabla \mathbf{e}^n_{\mathbf{u},h}\|_{L^2}\|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2} + \|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2})\\
\leq& ch^2(\|\nabla \mathbf{e}^n_{\mathbf{u},h}\|_{L^2}\|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2} + \|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2})\\
\leq& ch^2(\|\nabla \mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} + \|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2}) + \epsilon\|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} + ch^4, && (5.74)\\
H_8 \leq& |(\rho^n_\tau\nabla\cdot(\mathbf{u}^{n-1} - \mathbf{R}_h(\mathbf{u}^{n-1}_\tau, p^{n-1}_\tau)), P_h(|\mathbf{e}^n_{\mathbf{u},h}|^2) - P_h(\mathbf{R}_h(\mathbf{u}^n_\tau, p^n_\tau)\mathbf{e}^n_{\mathbf{u},h}))|\\
\leq& \|\rho^n_\tau\|_{L^\infty}\|\nabla(\mathbf{u}^{n-1}_\tau - \mathbf{R}_h(\mathbf{u}^{n-1}_\tau, p^{n-1}_\tau))\|_{L^2}(\|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^4} + \|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2})\\
\leq& \|\rho^n_\tau\|_{L^\infty}(ch\tau^{\frac12} + ch^2)(\|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^4} + \|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2}) \qquad \text{(use (5.19))}\\
\leq& (ch\tau^{\frac12} + ch^2)(\|\nabla \mathbf{e}^n_{\mathbf{u},h}\|_{L^2}\|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2} + \|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2}) \qquad \text{(use (3.6) and (5.57))}\\
\leq& ch(\tau^{\frac12} + h)(\|\nabla \mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} + \|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2}) + ch^2(\tau + h^2) + c\|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2}. && (5.75)
\end{aligned}$$

Substituting (5.67)-(5.75) into (5.66) yields

$$\begin{aligned}|(D_\tau e^n_{\rho,h}, P_h(\mathbf{u}^n_h\cdot \mathbf{e}^n_{\mathbf{u},h}))| \leq& (\epsilon + ch\tau^{\frac12} + ch^2)(\|\nabla \mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} + \|\nabla \mathbf{e}^{n-1}_{\mathbf{u},h}\|^2_{L^2}) + c\epsilon^{-1}h^2(\tau + h^2)\\ &+ (ch^{-1}\|\nabla \mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^2} + c\epsilon^{-1})(\|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} + \|e^n_{\rho,h}\|^2_{L^2} + \|\mathbf{e}^{n-1}_{\mathbf{u},h}\|^2_{L^2} + \|e^{n-1}_{\rho,h}\|^2_{L^2}),\end{aligned}$$

and therefore

$$\begin{aligned}|I_4^*(\mathbf{e}^n_{\mathbf{u},h})| \leq& |(D_\tau(\rho^n_\tau - P_h\rho^n_\tau)\,\mathbf{u}^n_h, \mathbf{e}^n_{\mathbf{u},h})| + |(D_\tau e^n_{\rho,h}, P_h(\mathbf{u}^n_h\cdot \mathbf{e}^n_{\mathbf{u},h}))|\\ \leq& \|D_\tau\rho^n_\tau - P_hD_\tau\rho^n_\tau\|_{L^\infty}\|\mathbf{u}^n_h\|_{L^2}\|\mathbf{e}^n_{\mathbf{u},h}\|_{L^2} + |(D_\tau e^n_{\rho,h}, P_h(\mathbf{u}^n_h\cdot \mathbf{e}^n_{\mathbf{u},h}))|\\ \leq& (\epsilon + ch\tau^{\frac12} + ch^2)(\|\nabla \mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} + \|\nabla \mathbf{e}^{n-1}_{\mathbf{u},h}\|^2_{L^2}) + c\epsilon^{-1}h^2(\tau + h^2)\end{aligned}$$

$$+ (ch^{-1}\|\nabla \mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2} + c\epsilon^{-1})(\|\mathbf{e}_{\mathbf{u},h}^{n}\|_{L^2}^2 + \|e_{\rho,h}^{n}\|_{L^2}^2 + \|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2}^2 + \|e_{\rho,h}^{n-1}\|_{L^2}^2), \quad (5.76)$$

where $\|\mathbf{u}_h^n\|_{L^2}$ is bounded as explained in Remark 2.1.

Then substituting (5.58)-(5.65) and (5.76) into (5.56) yields

$$\begin{aligned}
&\frac{1}{2}D_\tau\Big\|\sqrt{\rho_h^n}\,\mathbf{e}_{\mathbf{u},h}^n\Big\|_{L^2}^2 + \|\nabla \mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2 \\
&\le |(\rho_h^{n-1}D_\tau \mathbf{e}_{\mathbf{u},h}^n, \mathbf{e}_{\mathbf{u},h}^n)| + |\frac{1}{2}(D_\tau\rho_h^n\,\mathbf{e}_{\mathbf{u},h}^n, \mathbf{e}_{\mathbf{u},h}^n)| + |B((\mathbf{e}_{\mathbf{u},h}^n, e_{p,h}^n), (\mathbf{e}_{\mathbf{u},h}^n, e_{p,h}^n))| \\
&\le (\epsilon + ch\tau^{\frac{1}{2}} + ch^2 + (\tau+h^2)^{\frac{1}{4}})(\|\nabla \mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2 + \|\nabla \mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2}^2) \\
&\quad + c\epsilon^{-1}h^2(\tau+h^2) + c\epsilon^{-1}(\|D_\tau \mathbf{u}_\tau^n\|_{H^2}^2 + \|D_\tau p_\tau^n\|_{H^1}^2)h^4 \\
&\quad + (ch^{-1}\|\nabla \mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2} + c\epsilon^{-1})(\|\mathbf{e}_{\mathbf{u},h}^{n}\|_{L^2}^2 + \|e_{\rho,h}^{n}\|_{L^2}^2 + \|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2}^2 + \|e_{\rho,h}^{n-1}\|_{L^2}^2). \quad (5.77)
\end{aligned}$$

By summing up the last inequality times τ for $n = 1, \dots, k$, and choosing sufficiently small ϵ, h and τ, we have (the first term on the right-hand side of (5.77) is absorbed by the left-hand side, except a starting term involving $\tau\|\nabla \mathbf{e}_{\mathbf{u},h}^0\|_{L^2}^2$)

$$\begin{aligned}
&\frac{1}{2}\Big\|\sqrt{\rho_h^k}\,\mathbf{e}_{\mathbf{u},h}^k\Big\|_{L^2}^2 + \sum_{n=0}^{k}\tau\|\nabla \mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2 \\
&\le \frac{1}{2}\Big\|\sqrt{\rho_h^0}\,\mathbf{e}_{\mathbf{u},h}^0\Big\|_{L^2}^2 + c\tau\|\nabla \mathbf{e}_{\mathbf{u},h}^0\|_{L^2}^2 + ch^2(\tau+h^2) \quad (5.78) \\
&\quad + \tau\sum_{n=1}^{k}(ch^{-1}\|\nabla \mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2} + c)(\|\mathbf{e}_{\mathbf{u},h}^{n}\|_{L^2}^2 + \|e_{\rho,h}^{n}\|_{L^2}^2 + \|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2}^2 + \|e_{\rho,h}^{n-1}\|_{L^2}^2)
\end{aligned}$$

for $k = 1, \dots, m$, where we have used (3.10) to estimate the third term on the right-hand side of (5.77), i.e.,

$$\tau\sum_{n=1}^{k} c(\|D_\tau \mathbf{u}_\tau^n\|_{H^2}^2 + \|D_\tau p_\tau^n\|_{H^1}^2)h^4 \le ch^4.$$

Since $\mathbf{e}_{\mathbf{u},h}^0 = \mathbf{R}_h(\mathbf{u}^0, p^0) - \mathbf{\Pi}_h\mathbf{u}^0$, the estimates (5.6)-(5.7) and (5.14) imply

$$\begin{aligned}
&\frac{1}{2}\Big\|\sqrt{\rho_h^0}\,\mathbf{e}_{\mathbf{u},h}^0\Big\|_{L^2}^2 \le ch^4(\|u^0\|_{H^2}^2 + \|p^0\|_{H^1}^2), \\
&\tau\|\nabla \mathbf{e}_{\mathbf{u},h}^0\|_{L^2}^2 \le c\tau h^2(\|u^0\|_{H^2}^2 + \|p^0\|_{H^1}^2).
\end{aligned}$$

Substituting the last two inequalities into (5.78) and considering ϵ(5.42)+(5.78), we have

$$\begin{aligned}
&\epsilon\|e_{\rho,h}^k\|_{L^2}^2 + \frac{1}{2}\Big\|\sqrt{\rho_h^k}\,\mathbf{e}_{\mathbf{u},h}^k\Big\|_{L^2}^2 + \frac{1}{2}\sum_{n=0}^{k}\tau\|\nabla \mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2 \\
&\le \epsilon\sum_{n=1}^{k}\tau\|\nabla \mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2}^2 + ch^2(\tau+h^2) \\
&\quad + \tau\sum_{n=1}^{k}(ch^{-1}\|\nabla \mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2} + c)(\|\mathbf{e}_{\mathbf{u},h}^{n}\|_{L^2}^2 + \|e_{\rho,h}^{n}\|_{L^2}^2 + \|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2}^2 + \|e_{\rho,h}^{n-1}\|_{L^2}^2), \quad k = 1, \dots, m.
\end{aligned}$$

Again, by choosing a sufficiently small parameter ϵ and using (5.46), the first term on the right-hand side above can be absorbed by the left-hand side. Then we obtain

$$
\begin{aligned}
&\|e^k_{\rho,h}\|^2_{L^2} + \|\mathbf{e}^k_{\mathbf{u},h}\|^2_{L^2} + \sum_{n=0}^{k} \tau \|\nabla \mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} \\
&\le ch^2(\tau + h^2) \\
&\quad + \tau \sum_{n=1}^{k} (ch^{-1}\|\nabla \mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^2} + c)(\|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} + \|e^n_{\rho,h}\|^2_{L^2} + \|\mathbf{e}^{n-1}_{\mathbf{u},h}\|^2_{L^2} + \|e^{n-1}_{\rho,h}\|^2_{L^2})
\end{aligned}
\tag{5.79}
$$

for $k = 1, \ldots, m$. The fourth inequality of (5.29) implies $\tau \sum_{n=1}^m (ch^{-1}\|\nabla \mathbf{e}^{n-1}_{\mathbf{u},h}\|_{L^2} + c) \le c$. Substituting this inequality into (5.79) and applying Gronwall's inequality, we obtain

$$
\max_{1\le n\le m} \left(\|e^n_{\rho,h}\|^2_{L^2} + \|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2}\right) + \sum_{n=0}^{m} \tau \|\nabla \mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} \le ch^2(\tau + h^2). \tag{5.80}
$$

For sufficiently small mesh size h and τ, the inequality above implies

$$
\begin{aligned}
&\|\mathbf{e}^m_{\mathbf{u},h}\|_{L^2} \le h(\tau + h^2)^{\frac{1}{4}}, \\
&\|\mathbf{e}^m_{\mathbf{u},h}\|_{L^\infty} \le ch^{-1}\|\mathbf{e}^m_{\mathbf{u},h}\|_{L^2} \le c\sqrt{\tau + h^2} \le 1, \\
&\sum_{n=0}^{m} \tau \|\nabla \mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} \le ch^2(\tau + h^2) \le h^2\sqrt{\tau + h^2}.
\end{aligned}
$$

This, together with the induction assumption (5.29), proves the second, third and fourth inequalities of (5.30). The first inequality of (5.30) has already been proved in (5.45). The mathematical induction is closed. Consequently, the estimates (5.30) and (5.80) hold for $m = N$ (with the same constants). These estimates imply

$$
\max_{0\le n\le N} (\|\rho^n_h\|_{L^\infty} + \|\mathbf{u}^n_h\|_{L^\infty}) \le c \tag{5.81}
$$

and

$$
\max_{1\le n\le N} \left(\|e^n_{\rho,h}\|^2_{L^2} + \|\mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2}\right) + \sum_{n=0}^{N} \tau \|\nabla \mathbf{e}^n_{\mathbf{u},h}\|^2_{L^2} \le ch^2(\tau + h^2). \tag{5.82}
$$

This proves (5.22) in view of the first inequality of (5.30) and the derivation of (5.32).

5.7. **Improving the estimate of $D_\tau e^n_{\rho,h}$.**

By using (5.82), we re-estimate $\|D_\tau e^n_{\rho,h}\|_{L^2}$ by taking $\varphi_h = D_\tau e^n_{\rho,h}$ in (5.34). Since

$$
(D_\tau(\rho^n_\tau - P_h\rho^n_\tau), D_\tau e^n_{\rho,h}) = (D_\tau\rho^n_\tau - P_h D_\tau \rho^n_\tau, D_\tau e^n_{\rho,h}) = 0,
$$

we obtain

$$
\begin{aligned}
\|D_\tau e^n_{\rho,h}\|_{L^2} \le& \|\mathbf{u}^{n-1}_h \cdot \nabla(\rho^n_\tau - P_h\rho^n_\tau)\|_{L^2} + \frac{1}{2}\|\nabla \cdot \mathbf{u}^{n-1}_h(\rho^n_\tau - P_h\rho^n_\tau)\|_{L^2} \\
&+ \|\mathbf{u}^{n-1}_h \cdot \nabla e^n_{\rho,h}\|_{L^2} + \frac{1}{2}\|\nabla \cdot \mathbf{u}^{n-1}_h e^n_{\rho,h}\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&+ \|\mathbf{e}_{\mathbf{u},h}^{n-1} \cdot \nabla \rho_\tau^n\|_{L^2} + \frac{1}{2}\|\nabla \cdot \mathbf{e}_{\mathbf{u},h}^{n-1} \rho_\tau^n\|_{L^2} \\
&+ \|(\mathbf{u}_\tau^{n-1} - \mathbf{R}_h(\mathbf{u}_\tau^{n-1}, p_\tau^{n-1})) \cdot \nabla \rho_\tau^n\|_{L^2} \\
&+ \frac{1}{2}\|\rho_\tau^n \nabla \cdot (\mathbf{u}_\tau^{n-1} - \mathbf{R}_h(\mathbf{u}_\tau^{n-1}, p_\tau^{n-1}))\|_{L^2} \\
=: &\sum_{k=1}^{5} J_k^*,
\end{aligned} \tag{5.83}
$$

where

$$
\begin{aligned}
J_1^* \le& \|\mathbf{u}_h^{n-1}\|_{L^\infty} ch \|\rho_\tau^n\|_{H^2} + c\|\nabla \cdot \mathbf{u}_h^{n-1}\|_{L^\infty} \|\rho_\tau^n - P_h \rho_\tau^n\|_{L^2} \\
\le& \|\mathbf{u}_h^{n-1}\|_{L^\infty} ch \|\rho_\tau^n\|_{H^2} + ch^{-1}\|\mathbf{u}_h^{n-1}\|_{L^\infty} ch^2 \|\rho_\tau^n\|_{H^2} \qquad \text{(inverse inequality)} \\
\le& ch, \qquad \text{(use (3.9) to estimate } \|\rho_\tau^n\|_{H^2}\text{, and (5.81) to estimate } \|\mathbf{u}_h^{n-1}\|_{L^\infty}) \qquad (5.84) \\
J_2^* \le& \|\mathbf{u}_h^{n-1}\|_{L^\infty} \|\nabla e_{\rho,h}^n\|_{L^2} + \|\nabla \cdot \mathbf{u}_h^{n-1}\|_{L^\infty} \|e_{\rho,h}^n\|_{L^2} \\
\le& \|\mathbf{u}_h^{n-1}\|_{L^\infty} ch^{-1} \|e_{\rho,h}^n\|_{L^2} + ch^{-1}\|\mathbf{u}_h^{n-1}\|_{L^\infty} \|e_{\rho,h}^n\|_{L^2} \qquad \text{(inverse inequality)} \\
\le& \|\mathbf{u}_h^{n-1}\|_{L^\infty} ch^{-1} ch(\tau^{\frac{1}{2}} + h) + ch^{-1}\|\mathbf{u}_h^{n-1}\|_{L^\infty} ch(\tau^{\frac{1}{2}} + h) \quad \text{(use (5.82) to estimate } \|e_\rho^n\|_{L^2}) \\
\le& c(\tau^{\frac{1}{2}} + h), \qquad \text{(use (5.81) to estimate } \|\mathbf{u}_h^{n-1}\|_{L^\infty}) \qquad (5.85) \\
J_3^* \le& \|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2} \|\nabla \rho_\tau^n\|_{L^\infty} + \|\nabla \cdot \mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2} \|\rho_\tau^n\|_{L^\infty} \\
\le& \|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2} \|\nabla \rho_\tau^n\|_{L^\infty} + ch^{-1}\|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2} \|\rho_\tau^n\|_{L^\infty} \qquad \text{(inverse inequality)} \\
\le& ch(\tau^{\frac{1}{2}} + h)\|\nabla \rho_\tau^n\|_{L^\infty} + ch^{-1} ch(\tau^{\frac{1}{2}} + h)\|\rho_\tau^n\|_{L^\infty} \qquad \text{(use (5.82) to estimate } \|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2}) \\
\le& c(\tau^{\frac{1}{2}} + h), \qquad \text{(use (3.9) to estimate } \|\nabla \rho_\tau^n\|_{L^\infty} \text{ and } \|\rho_\tau^n\|_{L^\infty}) \qquad (5.86) \\
J_4^* \le& ch^2(\|\mathbf{u}_\tau^{n-1}\|_{H^2} + \|p_\tau^{n-1}\|_{H^1}) \|\nabla \rho_\tau^n\|_{L^\infty} \le ch^2, \qquad \text{(use (3.9))} \qquad (5.87) \\
J_5^* \le& c\|\rho_\tau^n\|_{L^\infty} \|\nabla \cdot (\mathbf{u}_\tau^{n-1} - \mathbf{R}_h(\mathbf{u}_\tau^{n-1}, p_\tau^{n-1}))\|_{L^2} \\
\le& c\|\rho_\tau^n\|_{L^\infty} h(\|\mathbf{u}_\tau^{n-1}\|_{H^2} + \|p_\tau^{n-1}\|_{H^1}) \qquad \text{(use (5.7))} \\
\le& ch. \qquad \text{(use (3.9) to estimate } \|\rho_\tau^n\|_{L^\infty},\ \|\mathbf{u}_\tau^{n-1}\|_{H^2} \text{ and } \|p_\tau^{n-1}\|_{H^1}) \qquad (5.88)
\end{aligned}
$$

Substituting (5.84)-(5.88) into (5.83) yields

$$
\max_{1 \le n \le N} \|D_\tau e_{\rho,h}^n\|_{L^2} \le c(\tau^{\frac{1}{2}} + h). \tag{5.89}
$$

which improves the estimate (5.53) obtained in Section 5.5.

5.8. Estimates of $e_{p,h}^n$.

We estimate $\|D_\tau \mathbf{e}_{\mathbf{u},h}^n\|_{L^2}$ by taking $\mathbf{v}_h = D_\tau \mathbf{e}_{\mathbf{u},h}^n$ and $q_h = 0$ in (5.56). We obtain

$$
\begin{aligned}
&(\rho_h^{n-1} D_\tau \mathbf{e}_{\mathbf{u},h}^n, D_\tau \mathbf{e}_{\mathbf{u},h}^n) + B((\mathbf{e}_{\mathbf{u},h}^n, e_{p,h}^n), (D_\tau \mathbf{e}_{\mathbf{u},h}^n, 0)) \\
&= \Big(-\frac{1}{2} D_\tau \rho_h^n \, \mathbf{e}_{\mathbf{u},h}^n, D_\tau \mathbf{e}_{\mathbf{u},h}^n \Big) + \sum_{j=1}^{10} I_j^*(D_\tau \mathbf{e}_{\mathbf{u},h}^n) \\
&= \Big(-\frac{1}{2} D_\tau \rho_\tau^n \, \mathbf{e}_{\mathbf{u},h}^n, D_\tau \mathbf{e}_{\mathbf{u},h}^n \Big) + \sum_{j=1}^{10} (K_j, D_\tau \mathbf{e}_{\mathbf{u},h}^n).
\end{aligned} \tag{5.90}
$$

where $(K_j, \mathbf{v}_h) = I_j^*(\mathbf{v}_h)$ for $j \neq 3$ and

$$(K_3, \mathbf{v}_h) := \left(\frac{1}{2} D_\tau(\rho_\tau^n - \rho_h^n)\, \mathbf{e}_{\mathbf{u},h}^n, \mathbf{v}_h\right) + I_3^*(\mathbf{v}_h) = \left(- \frac{1}{2}(D_\tau \rho_\tau^n\, (\mathbf{u}_\tau^n - \mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n)), \mathbf{v}_h\right). \tag{5.91}$$

Thus

$$\begin{aligned}
\|\tfrac{1}{2} D_\tau \rho_\tau^n\, \mathbf{e}_{\mathbf{u},h}^n\|_{L^2} \leq& \tfrac{1}{2}\|D_\tau \rho_\tau^n\|_{L^\infty}\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}\\
\leq& \tfrac{1}{2}(\|D_\tau e_\rho^n\|_{L^\infty} + \|D_\tau \rho^n\|_{L^\infty})\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}\\
\leq& c(\|D_\tau e_\rho^n\|_{H^2} + \|D_\tau \rho^n\|_{H^2})\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}\\
\leq& ch(\tau^{\frac{1}{2}} + h), \quad \text{(use (3.7) and (5.82))} \qquad (5.92)\\
\|K_1\|_{L^2} =& \|\rho_h^{n-1} D_\tau(\mathbf{u}_\tau^n - \mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n))\|_{L^2}\\
\leq& \|\rho_h^{n-1}\|_{L^\infty}\|D_\tau(\mathbf{u}_\tau^n - \mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n))\|_{L^2}\\
=& \|\rho_h^{n-1}\|_{L^\infty}\|D_\tau \mathbf{u}_\tau^n - \mathbf{R}_h(D_\tau \mathbf{u}_\tau^n, D_\tau p_\tau^n)\|_{L^2}\\
\leq& \|\rho_h^{n-1}\|_{L^\infty} ch^2(\|D_\tau \mathbf{u}_\tau^n\|_{H^2} + \|D_\tau p_\tau^n\|_{H^1})\\
\leq& ch^2(\|D_\tau \mathbf{u}_\tau^n\|_{H^2} + \|D_\tau p_\tau^n\|_{H^1}), \qquad \text{(use (5.46) to control } \|\rho_h^{n-1}\|_{L^\infty}) \qquad (5.93)\\
\|K_2\|_{L^2} =& \|(\rho_\tau^{n-1} - P_h \rho_\tau^{n-1}) D_\tau \mathbf{u}_\tau^n + e_{\rho,h}^{n-1} D_\tau \mathbf{u}_\tau^n\|_{L^2}\\
\leq& (\|\rho_\tau^{n-1} - P_h \rho_\tau^{n-1}\|_{L^2} + \|e_{\rho,h}^{n-1}\|_{L^2})\|D_\tau \mathbf{u}_\tau^n\|_{L^\infty}\\
\leq& (ch^2\|\rho_\tau^{n-1}\|_{H^2} + \|e_{\rho,h}^{n-1}\|_{L^2})\|D_\tau \mathbf{u}_\tau^n\|_{L^\infty}\\
\leq& c(h^2 + \|e_{\rho,h}^{n-1}\|_{L^2}) \qquad \text{(use (3.9) to control } \|\rho_\tau^{n-1}\|_{H^2} \text{ and } \|D_\tau \mathbf{u}_\tau^n\|_{L^\infty})\\
\leq& ch(\tau^{\frac{1}{2}} + h), \qquad \text{(use (5.82) to estimate } \|e_{\rho,h}^{n-1}\|_{L^2}) \qquad (5.94)\\
\|K_3\|_{L^2} =& \|\tfrac{1}{2}(D_\tau \rho_\tau^n\, (\mathbf{u}_\tau^n - \mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n))\|_{L^2}\\
\leq& c\|D_\tau \rho_\tau^n\|_{L^\infty}\|\mathbf{u}_\tau^n - \mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n)\|_{L^2}\\
\leq& c(\|D_\tau e_\rho^n\|_{L^\infty} + \|D_\tau \rho^n\|_{L^\infty})\|\mathbf{u}_\tau^n - \mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n)\|_{L^2}\\
\leq& c(\|D_\tau e_\rho^n\|_{H^2} + \|D_\tau \rho^n\|_{H^2})h^2(\|\mathbf{u}_\tau^n\|_{H^2} + \|p_\tau^n\|_{H^1})\\
\leq& ch^2, \qquad \text{(use (3.7) to control } \|D_\tau e_\rho^n\|_{H^2})\\
& \qquad\qquad \text{(use (3.9) to control } \|\mathbf{u}_\tau^n\|_{H^2} + \|p_\tau^n\|_{H^1}) \qquad (5.95)\\
\|K_4\|_{L^2} =& \|D_\tau \rho_\tau^n - P_h D_\tau \rho_\tau^n)\, \mathbf{u}_h^n + D_\tau e_{\rho,h}^n\, \mathbf{u}_h^n\|_{L^2}\\
\leq& (\|D_\tau \rho_\tau^n - P_h D_\tau \rho_\tau^n\|_{L^2} + \|D_\tau e_{\rho,h}^n\|_{L^2})\|\mathbf{u}_h^n\|_{L^\infty}\\
\leq& (ch^2\|D_\tau \rho_\tau^n\|_{H^2} + \|D_\tau e_{\rho,h}^n\|_{L^2})\|\mathbf{u}_h^n\|_{L^\infty}\\
\leq& ch^2 + c\|D_\tau e_{\rho,h}^n\|_{L^2} \qquad \text{(use (3.7) to estimate } \|D_\tau \rho_\tau^n\|_{H^2})\\
& \qquad\qquad \text{(use (5.81) to estimate } \|\mathbf{u}_h^n\|_{L^\infty})\\
\leq& c(\tau^{\frac{1}{2}} + h), \qquad \text{(use (5.89))} \qquad (5.96)\\
\|K_5\|_{L^2} =& \|\rho_h^n \mathbf{u}_h^{n-1} \nabla \mathbf{e}_{\mathbf{u},h}^n + \rho_h^n \mathbf{u}_h^{n-1} \nabla(\mathbf{u}_\tau^n - \mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n))\|_{L^2}\\
\leq& \|\rho_h^n\|_{L^\infty}\|\mathbf{u}_h^{n-1}\|_{L^\infty}(\|\nabla \mathbf{e}_{\mathbf{u},h}^n\|_{L^2} + \|\nabla(\mathbf{u}_\tau^n - \mathbf{R}_h(\mathbf{u}_\tau^n, p_\tau^n))\|_{L^2})\\
\leq& \|\rho_h^n\|_{L^\infty}\|\mathbf{u}_h^{n-1}\|_{L^\infty}(\|\nabla \mathbf{e}_{\mathbf{u},h}^n\|_{L^2} + ch\tau^{\frac{1}{2}} + ch^2) \qquad \text{(use (5.19))}\\
\leq& \|\rho_h^n\|_{L^\infty}\|\mathbf{u}_h^{n-1}\|_{L^\infty}(ch^{-1}\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2} + ch\tau^{\frac{1}{2}} + ch^2) \quad \text{(use inverse inequality)}\\
\leq& c(\tau^{\frac{1}{2}} + h), \quad \text{(use (5.82) for } \|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}, \text{ and (5.81) for } \|\rho_h^n\|_{L^\infty} \text{ and } \|\mathbf{u}_h^{n-1}\|_{L^\infty}) \qquad (5.97)
\end{aligned}$$

$$
\begin{aligned}
\|K_6\|_{L^2} =&\|\rho_h^n(\mathbf{u}_\tau^{n-1}-\mathbf{R}_h(\mathbf{u}_\tau^{n-1},p_\tau^{n-1}))\cdot\nabla\mathbf{u}_\tau^n+\rho_h^n\mathbf{e}_{\mathbf{u},h}^{n-1}\cdot\nabla\mathbf{u}_\tau^n\|_{L^2}\\
\le&\|\rho_h^n\|_{L^\infty}(\|\mathbf{u}_\tau^{n-1}-\mathbf{R}_h(\mathbf{u}_\tau^{n-1},p_\tau^{n-1})\|_{L^2}+\|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2})\|\nabla\mathbf{u}_\tau^n\|_{L^\infty}\\
\le&\|\rho_h^n\|_{L^\infty}(ch^2(\|\mathbf{u}_\tau^{n-1}\|_{H^2}+\|p_\tau^{n-1}\|_{H^1})+\|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2})\|\nabla\mathbf{u}_\tau^n\|_{L^\infty}\\
\le&c(h^2+\|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2})\qquad\text{(use (5.81) to estimate }\|\rho_h^n\|_{L^\infty})\\
&\qquad\qquad\qquad\qquad\text{(use (3.9) to estimate }\|\nabla\mathbf{u}_\tau^n\|_{L^\infty})\\
\le&ch(\tau^{\frac12}+h),\qquad\qquad\text{(use (5.82))}
\end{aligned}\tag{5.98}
$$

$$
\begin{aligned}
\|K_7\|_{L^2} =&\|(\rho_\tau^n-P_h\rho_\tau^n)\mathbf{u}_\tau^{n-1}\cdot\nabla\mathbf{u}_\tau^n+e_{\rho,h}^n\mathbf{u}_\tau^{n-1}\cdot\nabla\mathbf{u}_\tau^n\|_{L^2}\\
\le&(\|\rho_\tau^n-P_h\rho_\tau^n\|_{L^2}+\|e_{\rho,h}^n\|_{L^2})\|\mathbf{u}_\tau^{n-1}\|_{L^\infty}\|\nabla\mathbf{u}_\tau^n\|_{L^\infty}\\
\le&(ch^2\|\rho_\tau^n\|_{H^2}+\|e_{\rho,h}^n\|_{L^2})\|\mathbf{u}_\tau^{n-1}\|_{L^\infty}\|\nabla\mathbf{u}_\tau^n\|_{L^\infty}\\
\le&c(h^2+\|e_{\rho,h}^n\|_{L^2})\quad\text{(use (3.9) to estimate }\|\rho_\tau^n\|_{H^2},\ \|\mathbf{u}_\tau^{n-1}\|_{L^\infty}\text{ and }\|\nabla\mathbf{u}_\tau^n\|_{L^\infty})\\
\le&ch(\tau^{\frac12}+h),\qquad\text{(use (5.82))}
\end{aligned}\tag{5.99}
$$

$$
\begin{aligned}
\|K_8\|_{L^2} =&\frac12\|\nabla\cdot(\rho_h^n\mathbf{u}_h^{n-1}\otimes\mathbf{e}_{\mathbf{u},h}^n)+\nabla\cdot(\rho_h^n\mathbf{u}_h^{n-1}\otimes(\mathbf{u}_\tau^n-\mathbf{R}_h(\mathbf{u}_\tau^n,p_\tau^n)))\|_{L^2}\\
\le&\|\rho_h^n\mathbf{u}_h^{n-1}\|_{L^\infty}\big(\|\nabla\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}+\|\nabla(\mathbf{u}_\tau^n-\mathbf{R}_h(\mathbf{u}_\tau^n,p_\tau^n))\|_{L^2}\big)\\
&+\|\nabla(\rho_h^n\mathbf{u}_h^{n-1})\|_{L^\infty}\big(\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}+\|\mathbf{u}_\tau^n-\mathbf{R}_h(\mathbf{u}_\tau^n,p_\tau^n)\|_{L^2}\big)\\
\le&\|\rho_h^n\mathbf{u}_h^{n-1}\|_{L^\infty}(ch^{-1}\|\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}+ch(\tau^{\frac12}+h))\qquad\text{(use inverse inequality and (5.19))}\\
&\|\nabla\cdot(\rho_h^n\mathbf{u}_h^{n-1})\|_{L^\infty}\big(ch(\tau^{\frac12}+h)+ch^2(\tau^{\frac12}+h)\big)\qquad\text{(use (5.80) and (5.18))}\\
\le&\|\rho_h^n\mathbf{u}_h^{n-1}\|_{L^\infty}(\tau^{\frac12}+h+ch(\tau^{\frac12}+h))\qquad\text{(use (5.80))}\\
&+\|\rho_h^n\mathbf{u}_h^{n-1}\|_{L^\infty}\big(c(\tau^{\frac12}+h)+ch(\tau^{\frac12}+h)\big)\qquad\text{(inverse inequality)}\\
\le&c(\tau^{\frac12}+h),\quad\text{(use (5.81) to estimate }\|\rho_h^n\|_{L^\infty}\text{ and }\|\mathbf{u}_h^{n-1}\|_{L^\infty})
\end{aligned}\tag{5.100}
$$

$$
\begin{aligned}
\|K_9\|_{L^2} =&\frac12\|\nabla\cdot(\rho_h^n(\mathbf{u}_\tau^{n-1}-\mathbf{R}_h(\mathbf{u}_\tau^{n-1},p_\tau^{n-1}))\otimes\mathbf{u}_\tau^n)+\nabla\cdot(\rho_h^n\mathbf{e}_{\mathbf{u},h}^{n-1}\otimes\mathbf{u}_\tau^n)\|_{L^2}\\
\le&\|\nabla(\rho_h^n(\mathbf{u}_\tau^{n-1}-\mathbf{R}_h(\mathbf{u}_\tau^{n-1},p_\tau^{n-1})))\|_{L^2}\|\mathbf{u}_\tau^n\|_{L^\infty}+\|\nabla(\rho_h^n\mathbf{e}_{\mathbf{u},h}^{n-1})\|_{L^2}\|\mathbf{u}_\tau^n\|_{L^\infty}\\
&+\|\rho_h^n(\mathbf{u}_\tau^{n-1}-\mathbf{R}_h(\mathbf{u}_\tau^{n-1},p_\tau^{n-1}))\|_{L^2}\|\nabla\mathbf{u}_\tau^n\|_{L^\infty}+\|\rho_h^n\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2}\|\nabla\mathbf{u}_\tau^n\|_{L^\infty}\\
\le&\|\rho_h^n\|_{L^\infty}\|\nabla(\mathbf{u}_\tau^{n-1}-\mathbf{R}_h(\mathbf{u}_\tau^{n-1},p_\tau^{n-1})))\|_{L^2}\|\mathbf{u}_\tau^n\|_{L^\infty}\\
&+\|\nabla\rho_h^n\|_{L^\infty}\|\mathbf{u}_\tau^{n-1}-\mathbf{R}_h(\mathbf{u}_\tau^{n-1},p_\tau^{n-1}))\|_{L^2}\|\mathbf{u}_\tau^n\|_{L^\infty}\\
&+\|\rho_h^n\|_{L^\infty}\|\nabla\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2}\|\mathbf{u}_\tau^n\|_{L^\infty}+\|\nabla\rho_h^n\|_{L^\infty}\|\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2}\|\mathbf{u}_\tau^n\|_{L^\infty}\\
&+\|\rho_h^n(\mathbf{u}_\tau^{n-1}-\mathbf{R}_h(\mathbf{u}_\tau^{n-1},p_\tau^{n-1}))\|_{L^2}\|\nabla\mathbf{u}_\tau^n\|_{L^\infty}+\|\rho_h^n\mathbf{e}_{\mathbf{u},h}^{n-1}\|_{L^2}\|\nabla\mathbf{u}_\tau^n\|_{L^\infty}\\
\le&c\|\rho_h^n\|_{L^\infty}(ch\tau^{\frac12}+ch^2)\|\mathbf{u}_\tau^n\|_{L^\infty}\qquad\text{(use (5.19))}\\
&+c\|\nabla\rho_h^n\|_{L^\infty}(ch^2\tau^{\frac12}+ch^3)\|\mathbf{u}_\tau^n\|_{L^\infty}\qquad\text{(use (5.18))}\\
&+c\|\rho_h^n\|_{L^\infty}(\tau^{\frac12}+h)\|\mathbf{u}_\tau^n\|_{L^\infty}+c\|\nabla\rho_h^n\|_{L^\infty}h(\tau^{\frac12}+h)\|\mathbf{u}_\tau^n\|_{L^\infty}\quad\text{(use (5.82))}\\
&+c\|\rho_h^n\|_{L^\infty}(ch^2\tau^{\frac12}+ch^3)\|\nabla\mathbf{u}_\tau^n\|_{L^\infty}+c\|\rho_h^n\|_{L^\infty}h(\tau^{\frac12}+h)\|\nabla\mathbf{u}_\tau^n\|_{L^\infty}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(use (5.18) and (5.82))}\\
\le&c(\tau^{\frac12}+h),\qquad\qquad\text{(use (3.9) to estimate }\|\mathbf{u}_\tau^n\|_{L^\infty}\text{ and }\|\nabla\mathbf{u}_\tau^n\|_{L^\infty})\\
&\qquad\qquad\qquad\qquad\text{(use inverse inequality to estimate }\|\nabla\rho_h^n\|_{L^\infty})\\
&\qquad\qquad\qquad\qquad\text{(use (5.81) to estimate }\|\rho_h^n\|_{L^\infty}),
\end{aligned}\tag{5.101}
$$

$$
\begin{aligned}
\|K_{10}\|_{L^2} =& \|\nabla\cdot((\rho_\tau^n - P_h\rho_\tau^n)\mathbf{u}_\tau^{n-1}\otimes\mathbf{u}_\tau^n) + \nabla\cdot(e_{\rho,h}^n\mathbf{u}_\tau^{n-1}\otimes\mathbf{u}_\tau^n)\|_{L^2}\\
\le& \|\nabla(\rho_\tau^n - P_h\rho_\tau^n)\|_{L^2}\|\mathbf{u}_\tau^{n-1}\|_{L^\infty}\|\mathbf{u}_\tau^n\|_{L^\infty} + \|\nabla e_{\rho,h}^n\|_{L^2}\|\mathbf{u}_\tau^{n-1}\|_{L^\infty}\|\mathbf{u}_\tau^n\|_{L^\infty}\\
&+ \|\rho_\tau^n - P_h\rho_\tau^n\|_{L^2}\|\nabla\mathbf{u}_\tau^{n-1}\|_{L^\infty}\|\mathbf{u}_\tau^n\|_{L^\infty} + \|e_{\rho,h}^n\|_{L^2}\|\nabla\mathbf{u}_\tau^{n-1}\|_{L^\infty}\|\mathbf{u}_\tau^n\|_{L^\infty}\\
&+ \|\rho_\tau^n - P_h\rho_\tau^n\|_{L^2}\|\mathbf{u}_\tau^{n-1}\|_{L^\infty}\|\nabla\mathbf{u}_\tau^n\|_{L^\infty} + \|e_{\rho,h}^n\|_{L^2}\|\mathbf{u}_\tau^{n-1}\|_{L^\infty}\|\nabla\mathbf{u}_\tau^n\|_{L^\infty}\\
\le& c(h\tau + h^2) + c(\tau^{\frac12} + h)\\
&\qquad \text{(use (5.16) to estimate } \|\rho_\tau^n - P_h\rho_\tau^n\|_{L^2} \text{ and (5.82) to estimate } \|e_{\rho,h}^n\|_{L^2})\\
&\qquad\qquad\qquad \text{(use (3.9) to estimate } \|\mathbf{u}_\tau^n\|_{L^\infty} \text{ and } \|\nabla\mathbf{u}_\tau^n\|_{L^\infty})\\
\le& c(\tau^{\frac12} + h).
\end{aligned}
\tag{5.102}
$$

Since

$$
\begin{aligned}
B((\mathbf{e}_{\mathbf{u},h}^n, e_{p,h}^n), (D_\tau\mathbf{e}_{\mathbf{u},h}^n, 0)) &= (\nabla\mathbf{e}_{\mathbf{u},h}^n, \nabla D_\tau\mathbf{e}_{\mathbf{u},h}^n) - (e_{p,h}^n, \nabla\cdot D_\tau\mathbf{e}_{\mathbf{u},h}^n)\\
&= (\nabla\mathbf{e}_{\mathbf{u},h}^n, \nabla D_\tau\mathbf{e}_{\mathbf{u},h}^n)\\
&\ge \frac12 D_\tau\|\nabla\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2,
\end{aligned}
\tag{5.103}
$$

substituting (5.92)-(5.102) into (5.90) yields

$$
\begin{aligned}
&(\rho_h^{n-1}D_\tau\mathbf{e}_{\mathbf{u},h}^n, D_\tau\mathbf{e}_{\mathbf{u},h}^n) + \frac12 D_\tau\|\nabla\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2\\
&\le \epsilon\|D_\tau\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2 + \sum_{j=0}^{10} c\epsilon^{-1}\|K_j\|_{L^2}^2\\
&\le \epsilon\|D_\tau\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2 + c\epsilon^{-1}(\tau + h^2) + c\epsilon^{-1}(\|D_\tau\mathbf{u}_\tau^n\|_{H^2}^2 + \|D_\tau p_\tau^n\|_{H^1}^2)h^4,
\end{aligned}
$$

which further implies (by choosing a sufficient small ϵ and using (5.32), and summing up the inequalities for $n = 1, \dots, N$)

$$
\begin{aligned}
&\tau\sum_{n=1}^N \|D_\tau\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2 + \max_{1\le n\le N}\|\nabla\mathbf{e}_{\mathbf{u},h}^n\|_{L^2}^2\\
&\le c\|\nabla\mathbf{e}_{\mathbf{u},h}^0\|_{L^2}^2 + c(\tau + h^2) + c\tau\sum_{n=1}^N(\|D_\tau\mathbf{u}_\tau^n\|_{H^2}^2 + \|D_\tau p_\tau^n\|_{H^1}^2)h^4\\
&\le c(\tau + h^2). \qquad \text{(use (5.27) and (3.10))}
\end{aligned}
\tag{5.104}
$$

Finally, we estimate $(e_{p,h}^n, \nabla\cdot\mathbf{v}_h)$ by taking $q_h = 0$ in (5.56). By using the K_j defined in (5.90), we have

$$
\begin{aligned}
&|(e_{p,h}^n, \nabla\cdot\mathbf{v}_h)|\\
&= \left|(\rho_h^{n-1}D_\tau\mathbf{e}_{\mathbf{u},h}^n, \mathbf{v}_h) + (\nabla\mathbf{e}_{\mathbf{u},h}^n, \nabla\mathbf{v}_h) + \left(\frac12 D_\tau\rho_\tau^n\,\mathbf{e}_{\mathbf{u},h}^n, \mathbf{v}_h\right) - \sum_{j=1}^{10}(K_j, \mathbf{v}_h)\right|\\
&\le \left(\|\rho_h^{n-1}D_\tau\mathbf{e}_{\mathbf{u},h}^n\|_{L^2} + \|\nabla\mathbf{e}_{\mathbf{u},h}^n\|_{L^2} + \left\|\frac12 D_\tau\rho_\tau^n\,\mathbf{e}_{\mathbf{u},h}^n\right\|_{L^2} + \sum_{j=1}^{10}\|K_j\|_{L^2}\right)\|\mathbf{v}_h\|_{H^1}\\
&\le \left(c\|D_\tau\mathbf{e}_{\mathbf{u},h}^n\|_{L^2} + \|\nabla\mathbf{e}_{\mathbf{u},h}^n\|_{L^2} + \left\|\frac12 D_\tau\rho_\tau^n\,\mathbf{e}_{\mathbf{u},h}^n\right\|_{L^2} + \sum_{j=1}^{10}\|K_j\|_{L^2}\right)\|\nabla\mathbf{v}_h\|_{L^2}
\end{aligned}
$$

$$\leq \Big(c\|D_\tau \mathbf{e}^n_{\mathbf{u},h}\|_{L^2} + \|\nabla \mathbf{e}^n_{\mathbf{u},h}\|_{L^2} + c(\tau^{\frac{1}{2}}+h) + ch^2(\|D_\tau \mathbf{u}^n_\tau\|_{H^2} + \|D_\tau p^n_\tau\|_{H^1})\Big)\|\nabla \mathbf{v}_h\|_{L^2},$$

where we have used (5.92)-(5.102) in the last inequality. The last inequality, together with the inf-sup condition (2.9), implies

$$\|e^n_{p,h}\|_{L^2} \leq c(\|D_\tau \mathbf{e}^n_{\mathbf{u},h}\|_{L^2} + \|\nabla \mathbf{e}^n_{\mathbf{u},h}\|_{L^2}) + c(\tau^{\frac{1}{2}}+h) + ch^2(\|D_\tau \mathbf{u}^n_\tau\|_{H^2} + \|D_\tau p^n_\tau\|_{H^1}).$$

By summing up the inequality above for $n = 1, \ldots, N$, and using (5.104) and (3.10), we have

$$\sum_{n=1}^{N} \tau \|e^n_{p,h}\|^2_{L^2} \leq c(\tau + h^2). \tag{5.105}$$

To summarize, (5.22) has been proved at the end of Section 5.6, and (5.82) and (5.105) imply (5.23). The proof of Proposition 3.2 is completed. □

6. Numerical results

In this section, we present numerical tests to support the theoretical result proved in Theorem 2.1. To this end, we solve the equations

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = f, \tag{6.1}$$
$$\rho \partial_t \mathbf{u} + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \mu \Delta \mathbf{u} = \mathbf{g}, \tag{6.2}$$
$$\nabla \cdot \mathbf{u} = 0, \tag{6.3}$$

with $\mu = 0.001$, in the unit square $\Omega = [0,1] \times [0,1]$ by the proposed method up to time $T = 0.5$, where f and $\mathbf{g}$ are determined by substituting the following exact solution into the equations (6.1)-(6.3):

$$\begin{aligned} \rho(x,y,t) &= 2 + x(x-1)\cos(\sin(t)) + y(y-1)\sin(\sin(t)), \\ u(x,y,t) &= \begin{pmatrix} t^3 y^2 (y-1) \\ t^2 x^2 (x-1) \end{pmatrix}, \\ p(x,y,t) &= tx + y - (t+1)/2. \end{aligned}$$

To illustrate the spatial order of convergence, we present the errors of the numerical solutions in Table 6.1 for different mesh size h with $\tau = h^2$, where we see that the numerical solutions has second-order convergence. To illustrate the temporal order of convergence, the computations are done for different step sizes τ, with a fixed sufficiently small mesh size h such that the error from the spatial discretization is negligible in observing the temporal order of convergence. The numerical results are presented in Table 6.2, where we see that the temporal order of convergence is also consistent with the error estimate presented in Theorem 2.1.

Table 6.1. Rates of convergence and error for different mesh size h with $\tau = h^2$.

h	$\frac{\|\rho^N - \rho_h^N\|_{L^2}}{\|\rho^N\|_{L^2}}$	order	$\frac{\|u^N - u_h^N\|_{L^2}}{\|u^N\|_{L^2}}$	order	$\frac{\|p^N - p_h^N\|_{L^2}}{\|p^N\|_{L^2}}$	order
1/8	4.870E-04	/	9.748E-03	/	3.72E-03	/
1/16	1.216E-04	2.00	2.505E-03	1.96	9.30E-04	2.00
1/32	3.039E-05	2.00	6.285E-04	1.99	2.33E-04	1.99
1/64	7.595E-06	2.00	1.598E-04	1.97	5.94E-05	1.97

Table 6.2. Rates of convergence and error in time.

τ	$\frac{\|\rho^N - \rho_h^N\|_{L^2}}{\|\rho^N\|_{L^2}}$	order	$\frac{\|u^N - u_h^N\|_{L^2}}{\|u^N\|_{L^2}}$	order	$\frac{\|p^N - p_h^N\|_{L^2}}{\|p^N\|_{L^2}}$	order
0.1	5.931E-03	/	2.940E-02	/	3.436E-02	/
$2^{-1} \times 0.1$	2.983E-03	0.99	1.477E-02	0.99	1.750E-02	0.97
$2^{-2} \times 0.1$	1.496E-03	0.99	7.403E-03	0.99	8.834E-03	0.98
$2^{-3} \times 0.1$	7.492E-04	0.99	3.707E-03	0.99	4.438E-03	0.99
$2^{-4} \times 0.1$	3.747E-04	0.99	1.855E-03	0.99	2.225E-03	0.99
$2^{-5} \times 0.1$	1.872E-04	1.00	9.273E-04	0.99	1.114E-03	0.99

Appendix A. Existence and uniqueness of solutions for (3.1)-(3.4)

We assume that $\rho_\tau^n \in H^2(\Omega) \cap L^\infty(\Omega)$ and $\mathbf{u}_\tau^n \in \mathbf{W}^{1,\infty}(\Omega) \cap \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$ are given for $n = 0, \dots, m-1$, such that $\nabla \cdot \mathbf{u}_\tau^n = 0$ and (4.18)-(4.19) hold. Under this assumption, we prove the existence and uniqueness of the solution $(\rho_\tau^m, \mathbf{u}_\tau^m, p_\tau^m)$.

Part I: Well-posedness of (3.1)

For the given $\rho_\tau^{m-1} \in H^2(\Omega) \cap L^\infty(\Omega)$ and $\mathbf{u}_\tau^{m-1} \in \mathbf{W}^{1,\infty}(\Omega) \cap \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$, with $\nabla \cdot \mathbf{u}_\tau^{m-1} = 0$, we can extend ρ_τ^{m-1} to $\widetilde{\rho}_\tau^{m-1} \in H^1(\mathbb{R}^2)$ such that $\widetilde{\rho}_\tau^{m-1} = \rho_\tau^{m-1}$ in Ω and the following estimates hold (cf. [50, Chapter VI, Theorem 5, pp. 181]):

$$\|\widetilde{\rho}_\tau^{m-1}\|_{L^2(\mathbb{R}^2)} \le c\|\rho_\tau^{m-1}\|_{L^2(\Omega)}, \tag{A.1}$$

$$\|\widetilde{\rho}_\tau^{m-1}\|_{H^1(\mathbb{R}^2)} \le c\|\rho_\tau^{m-1}\|_{H^1(\Omega)}. \tag{A.2}$$

The truncated function

$$\overline{\rho}_\tau^{m-1}(x) = \min\left(\max\left(\min_{y\in\overline{\Omega}} \rho_\tau^{m-1}(y), \widetilde{\rho}_\tau^{m-1}(x)\right), \max_{y\in\overline{\Omega}} \rho_\tau^{m-1}(y)\right), \quad x \in \mathbb{R}^2, \tag{A.3}$$

satisfies $\overline{\rho}_\tau^{m-1} \in H^1(\mathbb{R}^2)$ and

$$\min_{y\in\overline{\Omega}} \rho_\tau^{m-1}(y) \le \overline{\rho}_\tau^{m-1}(x) \le \max_{y\in\overline{\Omega}} \rho_\tau^{m-1}(y), \quad \forall\, x \in \mathbb{R}^2. \tag{A.4}$$

Similarly, the function

$$\widetilde{\mathbf{u}}_\tau^{m-1}(x) := \begin{cases} \mathbf{u}_\tau^{m-1}(x) & x \in \Omega, \\ 0 & x \in \mathbb{R}^2 \backslash \Omega, \end{cases} \tag{A.5}$$

extends $\mathbf{u}_\tau^{m-1} \in \mathbf{W}^{1,\infty}(\Omega) \cap \mathbf{H}_0^1(\Omega)$ to $\widetilde{\mathbf{u}}_\tau^{m-1} \in \mathbf{W}^{1,\infty}(\mathbb{R}^2)$.

With the extensions above, we define the bilinear form

$$a(\phi, \varphi) := \int_{\mathbb{R}^2} \big(\tau^{-1}\phi + \widetilde{\mathbf{u}}_\tau^{m-1} \cdot \nabla\phi\big)\varphi \,\mathrm{d}x + \int_{\mathbb{R}^2} \epsilon \nabla\phi \cdot \nabla\varphi \,\mathrm{d}x, \quad \forall\, \phi, \varphi \in H^1(\mathbb{R}^2),$$

which is coercive on $H^1(\mathbb{R}^2)$ for any fixed $\epsilon \in (0,1)$, i.e.,

$$a(\phi, \phi) = \tau^{-1}\|\phi\|_{L^2(\mathbb{R}^2)}^2 + \epsilon\|\nabla\phi\|_{L^2(\mathbb{R}^2)}^2 \ge c\epsilon\|\phi\|_{H^1(\mathbb{R}^2)}^2.$$

Consequently, the Lax–Milgram lemma implies that the equation

$$a(\rho^m_{\tau,\epsilon},\varphi)=\int_{\mathbb{R}^2}\tau^{-1}\overline{\rho}^{m-1}_\tau\varphi\mathrm{d}x,\quad \forall\,\varphi\in H^1(\mathbb{R}^2), \tag{A.6}$$

has a unique solution $\rho^m_{\tau,\epsilon}\in H^1(\mathbb{R}^2)$. In other words, the second-order elliptic equation

$$\tau^{-1}\rho^m_{\tau,\epsilon}+\widetilde{\mathbf{u}}^{m-1}_\tau\cdot\nabla\rho^m_{\tau,\epsilon}-\epsilon\varDelta\rho^m_{\tau,\epsilon}=\tau^{-1}\overline{\rho}^{m-1}_\tau\quad\text{in }\ \mathbb{R}^2 \tag{A.7}$$

has a unique weak solution $\rho^m_{\tau,\epsilon}\in H^1(\mathbb{R}^2)$. The equation above also implies

$$\epsilon\varDelta\rho^m_{\tau,\epsilon}=\tau^{-1}\rho^m_{\tau,\epsilon}+\widetilde{\mathbf{u}}^{m-1}_\tau\cdot\nabla\rho^m_{\tau,\epsilon}-\tau^{-1}\overline{\rho}^{m-1}_\tau\in L^2(\mathbb{R}^2). \tag{A.8}$$

Then the elliptic regularity (cf. [22, Theorem 3.2.1.2]) further implies $\rho^m_{\tau,\epsilon}\in H^2(\mathbb{R}^2)$.

Integrating (A.7) against the test function $\varphi=\max\big(0,\rho^m_{\tau,\epsilon}-\max_{y\in\overline{\varOmega}}\overline{\rho}^{m-1}_\tau(y)\big)$ yields

$$\tau^{-1}\|\varphi\|^2_{L^2(\mathbb{R}^2)}+\epsilon\|\nabla\varphi\|^2_{L^2(\mathbb{R}^2)}=0,$$

which implies $\varphi=0$. In other words, $\rho^m_{\tau,\epsilon}-\max_{y\in\overline{\varOmega}}\overline{\rho}^{m-1}_\tau(y)\le 0$. Similarly, Integrating (A.7) against the test function $\varphi=\min\big(0,\rho^m_{\tau,\epsilon}-\min_{y\in\overline{\varOmega}}\overline{\rho}^{m-1}_\tau(y)\big)$ yields $\rho^m_{\tau,\epsilon}-\min_{y\in\overline{\varOmega}}\overline{\rho}^{m-1}_\tau(y)\ge 0$. This proves that the weak solution of (A.7) obeys the maximum principle:

$$\min_{y\in\mathbb{R}^2}\overline{\rho}^{m-1}_\tau(y)\le\rho^m_{\tau,\epsilon}(x)\le\max_{y\in\mathbb{R}^2}\overline{\rho}^{m-1}_\tau(y),\quad\forall\,x\in\mathbb{R}^2. \tag{A.9}$$

Integrating (A.7) against $\rho^n_{\tau,\epsilon}$ yields

$$\|\rho^m_{\tau,\epsilon}\|_{L^2(\mathbb{R}^2)}\le\|\overline{\rho}^{m-1}_\tau\|_{L^2(\mathbb{R}^2)}\le c\|\overline{\rho}^{m-1}_\tau\|_{L^2(\varOmega)}, \tag{A.10}$$

where the last inequality is due to (A.1). By differentiating (A.7) with respect to x_j, we see that

$$\begin{aligned}&\tau^{-1}\partial_{x_j}\rho^m_{\tau,\epsilon}+\widetilde{\mathbf{u}}^{m-1}_\tau\cdot\nabla\partial_{x_j}\rho^m_{\tau,\epsilon}-\epsilon\varDelta\partial_{x_j}\rho^m_{\tau,\epsilon}\\&=-\partial_{x_j}\widetilde{\mathbf{u}}^{m-1}_\tau\cdot\nabla\rho^m_{\tau,\epsilon}+\tau^{-1}\partial_{x_j}\overline{\rho}^{m-1}_\tau.\end{aligned} \tag{A.11}$$

Integrating (A.11) against $\partial_{x_j}\rho^m_{\tau,\epsilon}$ yields

$$\begin{aligned}\sum_{j=1}^2\tau^{-1}\|\partial_{x_j}\rho^m_{\tau,\epsilon}\|^2_{L^2(\mathbb{R}^2)}&\le\sum_{j=1}^2\|\partial_{x_j}\widetilde{\mathbf{u}}^{m-1}_\tau\|^2_{L^\infty(\mathbb{R}^2)}\|\nabla\rho^m_{\tau,\epsilon}\|^2_{L^2(\mathbb{R}^2)}+\sum_{j=1}^2\tau^{-1}\|\partial_{x_j}\rho^{m-1}_\tau\|^2_{L^2(\mathbb{R}^2)}\\&\le\sum_{j=1}^2\|\partial_{x_j}\mathbf{u}^{m-1}_\tau\|^2_{L^\infty(\varOmega)}\|\nabla\rho^m_{\tau,\epsilon}\|^2_{L^2(\mathbb{R}^2)}+c\sum_{j=1}^2\tau^{-1}\|\partial_{x_j}\overline{\rho}^{m-1}_\tau\|^2_{L^2(\varOmega)},\end{aligned} \tag{A.12}$$

where the last inequality is due to (A.2) and (A.3). For sufficiently small τ, i.e.,

$$\tau\le\frac{1}{2\|\nabla\mathbf{u}^{m-1}_\tau\|^2_{L^\infty(\varOmega)}},$$

the first term on the right-hand side of (A.12) can be absorbed by the left-hand side. Consequently, we obtain

$$\|\nabla \rho_{\tau,\epsilon}^m\|_{L^2(\Omega)}^2 \le c\|\nabla \rho_\tau^{m-1}\|_{L^2(\Omega)}^2. \tag{A.13}$$

Since the estimates (A.10) and (A.13) are independent of ϵ, there exists a sequence $\epsilon_i \to 0$ such that ρ_{τ,ϵ_i}^m converges strongly in $L^2(B)$ and weakly in $H^1(B)$, for any bounded domain $B \subset \mathbb{R}^2$. The limit function $\rho_\tau^m = \lim_{\epsilon_i\to 0} \rho_{\tau,\epsilon_i}^m$ would satisfy

$$\int_{\mathbb{R}^2} (\tau^{-1}\rho_\tau^m + \widetilde{\mathbf{u}}_\tau^{m-1}\cdot\nabla\rho_\tau^m)\varphi\,\mathrm{d}x = \int_{\mathbb{R}^2} \tau^{-1}\rho_\tau^{m-1}\varphi\,\mathrm{d}x, \quad \forall\,\varphi \in C_0^\infty(\mathbb{R}^2). \tag{A.14}$$

Since both $\tau^{-1}\rho_\tau^m + \widetilde{\mathbf{u}}_\tau^{m-1}\cdot\nabla\rho_\tau^m$ and $\tau^{-1}\rho_\tau^{m-1}$ are in $L^2(\Omega)$ and $C_0^\infty(\Omega) \subset C_0^\infty(\mathbb{R}^2)$ is dense in $L^2(\Omega)$, it follows that

$$\int_{\Omega} (\tau^{-1}\rho_\tau^m + \mathbf{u}_\tau^{m-1}\cdot\nabla\rho_\tau^m)\varphi\,\mathrm{d}x = \int_{\Omega} \tau^{-1}\rho_\tau^{m-1}\varphi\,\mathrm{d}x, \quad \forall\,\varphi \in L^2(\Omega). \tag{A.15}$$

This proves the existence of a strong solution $\rho_\tau^m \in H^1(\Omega)\cap L^\infty(\Omega)$ of (3.1), obeying the maximum principle (in view of (A.4) and (A.9)):

$$\min_{y\in\overline{\Omega}} \rho_\tau^{m-1}(y) \le \rho_\tau^m(x) \le \max_{y\in\overline{\Omega}} \rho_\tau^{m-1}(y), \quad \forall\, x \in \Omega. \tag{A.16}$$

If there exist two such solutions $\rho_\tau^m, \widetilde{\rho}_\tau^m \in H^1(\Omega)$ for the equation (3.1), then we have

$$\tau^{-1}(\rho_\tau^m - \widetilde{\rho}_\tau^m) + \mathbf{u}_\tau^{m-1}\cdot\nabla(\rho_\tau^m - \widetilde{\rho}_\tau^m) = 0. \tag{A.17}$$

Integrating the above equation against $\rho_\tau^m - \widetilde{\rho}_\tau^m$ and using the divergence-free property of $\mathbf{u}_\tau^{m-1}$, we immediately obtain

$$\tau^{-1}\|\rho_\tau^m - \widetilde{\rho}_\tau^m\|_{L^2}^2 = 0, \tag{A.18}$$

which implies $\rho_\tau^m = \widetilde{\rho}_\tau^m$. The uniqueness is proved.

Part II: Well-posedness of (3.2)-(3.3)

For the given $\rho_\tau^{m-1}, \rho_\tau^m \in H^2(\Omega)\cap L^\infty(\Omega)$ and $\mathbf{u}_\tau^{m-1} \in \mathbf{W}^{1,\infty}(\Omega)\cap\mathbf{H}^2(\Omega)\cap\mathbf{H}_0^1(\Omega)$, with $\rho_\tau^{m-1} \ge 0$ and $\rho_\tau^m \ge 0$ (proved in Part I), we define the bilinear form

$$\begin{aligned} b(\mathbf{w},\mathbf{v}) :=& \frac{1}{2}\int_\Omega (\tau^{-1}(\rho_\tau^m - \rho_\tau^{m-1}) + \mathbf{u}_\tau^{m-1}\cdot\nabla\rho_\tau^m)\mathbf{w}\cdot\mathbf{v}\,\mathrm{d}x \\ &+ \int_\Omega \left(\tau^{-1}\rho_\tau^{m-1}\mathbf{w} + \rho_\tau^n\mathbf{u}_\tau^{m-1}\cdot\nabla\mathbf{w}\right)\cdot\mathbf{v}\,\mathrm{d}x \\ &+ \mu\int_\Omega \nabla\mathbf{w}\cdot\nabla\mathbf{v}\,\mathrm{d}x, \quad \forall\,\mathbf{w},\,\mathbf{v} \in \mathring{\mathbf{H}}_{\mathrm{div}}^1(\Omega), \end{aligned} \tag{A.19}$$

where $\mathring{\mathbf{H}}_{\mathrm{div}}^1(\Omega) = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega) : \nabla\cdot\mathbf{v} = 0\}$. It is easy to see that the bilinear form $b(\cdot,\cdot)$ is coercive on $\mathring{\mathbf{H}}_{\mathrm{div}}^1(\Omega)$, i.e.,

$$b(\mathbf{w},\mathbf{w}) = \frac{1}{2}\int_\Omega \tau^{-1}(\rho_\tau^m + \rho_\tau^{m-1})|\mathbf{w}|^2\,\mathrm{d}x + \mu\int_\Omega |\nabla\mathbf{w}|^2\,\mathrm{d}x \ge c\|\mathbf{w}\|_{H^1}^2. \tag{A.20}$$

Consequently, the Lax–Milgram lemma implies that the equation

$$b(\mathbf{u}_\tau^m, \mathbf{v}) := \int_\Omega \tau^{-1} \rho_\tau^{m-1} \mathbf{u}_\tau^{m-1} \cdot \mathbf{v} \mathrm{d}x, \quad \forall\, \mathbf{v} \in \mathring{\mathbf{H}}^1_{\mathrm{div}}(\Omega), \tag{A.21}$$

has a unique solution $\mathbf{u}_\tau^m \in \mathring{\mathbf{H}}^1_{\mathrm{div}}(\Omega)$, satisfying

$$\begin{aligned}
&\int_\Omega \left(\tau^{-1}\rho_\tau^{m-1}\mathbf{u}_\tau^m + \rho_\tau^m \mathbf{u}_\tau^{m-1} \cdot \nabla \mathbf{u}_\tau^m\right) \cdot \mathbf{v}\, \mathrm{d}x + \mu \int_\Omega \nabla \mathbf{u}_\tau^m \cdot \nabla \mathbf{v}\, \mathrm{d}x \\
&= \int_\Omega \tau^{-1} \rho_\tau^{m-1} \mathbf{u}_\tau^{m-1} \cdot \mathbf{v} \mathrm{d}x, \qquad\qquad \forall\, \mathbf{v} \in \mathring{\mathbf{H}}^1_{\mathrm{div}}(\Omega).
\end{aligned} \tag{A.22}$$

By using the notation of Section 4.1, the last equation implies, for $\mathbf{v} \in \mathbf{C}_0^\infty(\Omega)$,

$$\begin{aligned}
-(\mu \Delta \mathbf{u}_\tau^m, \mathbf{v}) &= -(\mu \mathbf{u}_\tau^m, \Delta \mathbf{v}) \\
&= -(\mu \mathbf{u}_\tau^m, \mathbf{P}_{\mathrm{div}} \Delta \mathbf{v}) \\
&= -(\mu \mathbf{u}_\tau^m, \Delta \mathbf{P}_{\mathrm{div}} \mathbf{v}) \\
&= (\mu \nabla \mathbf{u}_\tau^m, \nabla \mathbf{P}_{\mathrm{div}} \mathbf{v}) \\
&= \int_\Omega \tau^{-1} \rho_\tau^{m-1} \mathbf{u}_\tau^{m-1} \cdot \mathbf{P}_{\mathrm{div}} \mathbf{v} \mathrm{d}x - \int_\Omega \left(\tau^{-1}\rho_\tau^{m-1}\mathbf{u}_\tau^m + \rho_\tau^m \mathbf{u}_\tau^{m-1} \cdot \nabla \mathbf{u}_\tau^m\right) \cdot \mathbf{P}_{\mathrm{div}} \mathbf{v}\, \mathrm{d}x \\
&= \int_\Omega \mathbf{P}_{\mathrm{div}} \big(\tau^{-1}\rho_\tau^{m-1}\mathbf{u}_\tau^{m-1} - \tau^{-1}\rho_\tau^{m-1}\mathbf{u}_\tau^m - \rho_\tau^m \mathbf{u}_\tau^{m-1} \cdot \nabla \mathbf{u}_\tau^m\big) \mathbf{v}\, \mathrm{d}x,
\end{aligned} \tag{A.23}$$

where the last equality is due to the self-adjointness of the projection operator $\mathbf{P}_{\mathrm{div}}$. This last equality implies he following equations:

$$\begin{aligned}
&-\mu \Delta \mathbf{u}_\tau^m + \nabla \phi = \tau^{-1}\rho_\tau^{m-1}\mathbf{u}_\tau^{m-1} - \tau^{-1}\rho_\tau^{m-1}\mathbf{u}_\tau^m - \rho_\tau^m \mathbf{u}_\tau^{m-1} \cdot \nabla \mathbf{u}_\tau^m \in \mathbf{L}^2(\Omega), && \text{(A.24)} \\
&\nabla \cdot \mathbf{u}_\tau^m = 0. && \text{(A.25)}
\end{aligned}$$

The standard H^2 estimate for the Stokes equations (cf. [29]) implies $\mathbf{u}_\tau^m \in \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$, and (A.22) implies

$$\mathbf{P}_{\mathrm{div}}(\rho_\tau^{m-1} D_\tau \mathbf{u}_\tau^m + \rho_\tau^m \mathbf{u}_\tau^{m-1} \cdot \nabla \mathbf{u}_\tau^m - \mu \Delta \mathbf{u}_\tau^m) = 0.$$

By the Helmoholtz–Weyl decomposition (4.4), there exists a unique $p_\tau^m \in H^1(\Omega)$ such that $\int_\Omega p_\tau^m \mathrm{d}x = 0$ and

$$\rho_\tau^{m-1} D_\tau \mathbf{u}_\tau^m + \rho_\tau^m \mathbf{u}_\tau^{m-1} \cdot \nabla \mathbf{u}_\tau^m - \mu \Delta \mathbf{u}_\tau^m = -\nabla p_\tau^m. \tag{A.26}$$

This proves the existence and uniqueness of the solution

$$(\mathbf{u}_\tau^m, p_\tau^m) \in (\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)) \times (H^1(\Omega) \cap L_0^2(\Omega))$$

for (3.2)-(3.3) with $n = m$. □

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