CORRECTION TO: CONVERGENCE OF DZIUK'S SEMIDISCRETE FINITE ELEMENT METHOD FOR MEAN CURVATURE FLOW OF CLOSED SURFACES WITH HIGH-ORDER FINITE ELEMENTS

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Abstract. The proof of the main theorem in [B. Li, SIAM J. Numer. Anal., 59 (2021), pp. 1592–1617] is corrected. With the corrected proof, the main theorem in this published paper is still valid.

Key words. mean curvature flow, evolving surface, finite element method, convergence, error estimate

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1. Introduction. In [4, p. 1611], the author used the following formula:

$$\underline{D}_i\underline{D}_j u = \underline{D}_j\underline{D}_i u + Hn_i\underline{D}_j u - Hn_j\underline{D}_i u, \tag{1.1}$$

which is not correct and should be replaced by the following one (see [1, Lemma 2.6]):

$$\underline{D}_i\underline{D}_j u = \underline{D}_j\underline{D}_i u + n_i H_{jl}\underline{D}_l u - n_j H_{il}\underline{D}_l u. \tag{1.2}$$

However, with the formula in (1.2), the cancellations in Lemma 3,3, Proposition 4.1 and Proposition 4.2 do not hold, i.e.

$$\int_{\Gamma_h^\theta}[\mathrm{tr}(\nabla_{\Gamma_h^\theta}e_h^\theta)^2 - \mathrm{tr}(\nabla_{\Gamma_h^\theta}e_h^\theta\nabla_{\Gamma_h^\theta}e_h^\theta)] \neq 0, \tag{1.3}$$

$$\int_{\Gamma_h^*}[\mathrm{tr}(\nabla_{\Gamma_h^*}e_h^*)^2 - \mathrm{tr}(\nabla_{\Gamma_h^*}e_h^*\nabla_{\Gamma_h^*}e_h^*)] \neq 0. \tag{1.4}$$

According to these changes, the error equation now should be replaced by

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{e}\|_{\mathbf{M}(\mathbf{x})}^2 + 2\int_0^1\int_{\Gamma_h^\theta}|(\nabla_{\Gamma_h^\theta}e_h^\theta)\widehat{n}_h^\theta|^2\mathrm{d}\theta \\
&\lesssim ch^{2k-2} + c\epsilon^{-1}\|\mathbf{e}\|_{\mathbf{M}(\mathbf{x})}^2 + 2\epsilon\int_0^1\int_{\Gamma_h^\theta}|(\nabla_{\Gamma_h^\theta}e_h^\theta)\widehat{n}_h^\theta|^2\mathrm{d}\theta \\
&\quad + \Big|\int_0^1\int_{\Gamma_h^\theta}[\mathrm{tr}(\nabla_{\Gamma_h^\theta}e_h^\theta)^2 - \mathrm{tr}(\nabla_{\Gamma_h^\theta}e_h^\theta\nabla_{\Gamma_h^\theta}e_h^\theta)]\mathrm{d}\theta\Big|,
\end{aligned} \tag{1.5}$$

where the last term is missing in [4, Eq. (3.42)].

In the next section, we show that the last term in (1.5) differs from zero by a harmless lower-order correction term and an error term which can be absorbed into the left hand side of the error equation (1.5). This additional remainder will not influence the stability of the error equation. Therefore, the coercivity of $(\mathbf{A}(\mathbf{x})\mathbf{x} - \mathbf{A}(\mathbf{x}^*)\mathbf{x}^*)\cdot(\mathbf{x}-\mathbf{x}^*)$ can still be proved, leading to the correctness of the main theorem (i.e., [4, Theorem 2.1]).

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2. Correction. By using the commutator formula (1.2), integration by parts and the Einstein summation convention, we derive as follows on a closed smooth surface Γ for any smooth function $u \in C^\infty(\Gamma)$

$$\begin{aligned}\int_\Gamma \underline{D}_i u_j \underline{D}_j u_i &= -\int_\Gamma u_j \underline{D}_i \underline{D}_j u_i + \int_\Gamma u_j \underline{D}_j u_i H n_i \\ &= -\int_\Gamma u_j (\underline{D}_j \underline{D}_i u_i + n_i H_{jk} \underline{D}_k u_i - n_j H_{ik} \underline{D}_k u_i) + \int_\Gamma u_j \underline{D}_j u_i H n_i \\ &= \int_\Gamma \underline{D}_j u_j \underline{D}_i u_i - \int_\Gamma u_j \underline{D}_i u_i H n_j + \int_\Gamma u_j \underline{D}_j u_i H n_i \\ &\quad - \int_\Gamma u_j n_i H_{jk} \underline{D}_k u_i + \int_\Gamma u_j n_j H_{ik} \underline{D}_k u_i. \end{aligned} \tag{2.1}$$

By a density argument, (2.1) holds for $u \in H^1(\Gamma)$ as well. Then, by using the basic geometric perturbation estimates (cf. [4, Eq. (3.32)]), we have

$$\begin{aligned}&\Big|\int_{\Gamma_h^*} [\mathrm{tr}(\nabla_{\Gamma_h^*} e_h^*)^2 - \mathrm{tr}(\nabla_{\Gamma_h^*} e_h^* \nabla_{\Gamma_h^*} e_h^*)]\Big| \\ &= \Big| -\int_{\Gamma_h^*} \underline{D}_i e_{h,j}^* \underline{D}_j e_{h,i}^* + \int_{\Gamma_h^*} \underline{D}_j e_{h,j}^* \underline{D}_i e_{h,i}^* \Big| \\ &\le \Big| -\int_{\Gamma_h^*} \underline{D}_i e_{h,j}^* \underline{D}_j e_{h,i}^* + \int_\Gamma \underline{D}_i e_{h,j}^{*,l} \underline{D}_j e_{h,i}^{*,l} \Big| + \Big| \int_{\Gamma_h^*} \underline{D}_j e_{h,j}^* \underline{D}_i e_{h,i}^* - \int_\Gamma \underline{D}_j e_{h,j}^{*,l} \underline{D}_i e_{h,i}^{*,l} \Big| \\ &\quad + \Big| \int_\Gamma \underline{D}_i e_{h,j}^{*,l} \underline{D}_j e_{h,i}^{*,l} - \int_\Gamma \underline{D}_j e_{h,j}^{*,l} \underline{D}_i e_{h,i}^{*,l} \Big| \\ &\lesssim h^k \|e_h^*\|_{H^1(\Gamma_h^*)}^2 + \Big| -\int_\Gamma e_{h,j}^{*,l} \underline{D}_i e_{h,i}^{*,l} H n_j + \int_\Gamma e_{h,j}^{*,l} \underline{D}_j e_{h,i}^{*,l} H n_i \\ &\quad - \int_\Gamma e_{h,j}^{*,l} n_i H_{jk} \underline{D}_k e_{h,i}^{*,l} + \int_\Gamma e_{h,j}^{*,l} n_j H_{ik} \underline{D}_k e_{h,i}^{*,l} \Big| \quad \text{((2.1) is used here)} \end{aligned} \tag{2.2}$$

where the first term comes in a similar way as [4, Eq. (3.32)]. From integration by parts (cf. [4, Eq. (4.2)]), we continue the calculations with

$$\begin{aligned}&\Big| -\int_\Gamma e_{h,j}^{*,l} \underline{D}_i e_{h,i}^{*,l} H n_j + \int_\Gamma e_{h,j}^{*,l} \underline{D}_j e_{h,i}^{*,l} H n_i - \int_\Gamma e_{h,j}^{*,l} n_i H_{jk} \underline{D}_k e_{h,i}^{*,l} + \int_\Gamma e_{h,j}^{*,l} n_j H_{ik} \underline{D}_k e_{h,i}^{*,l} \Big| \\ &\lesssim \|e_h^*\|_{L^2(\Gamma_h^*)}^2 + \Big| \int_\Gamma \underline{D}_i e_{h,j}^{*,l} n_j H e_{h,i}^{*,l} + \int_\Gamma e_{h,j}^{*,l} H \underline{D}_j e_{h,i}^{*,l} n_i \\ &\quad - \int_\Gamma e_{h,j}^{*,l} H_{jk} \underline{D}_k e_{h,i}^{*,l} n_i - \int_\Gamma \underline{D}_k e_{h,j}^{*,l} n_j H_{ik} e_{h,i}^{*,l} \Big|, \end{aligned} \tag{2.3}$$

where $\|e_h^*\|_{L^2(\Gamma_h^*)}^2$ corresponds to the lower-order terms arising from the integration by parts formula [4, Eq. (4.2)]. Then, by changing n to $\hat{n}_h^{*,l}$ (normal vector of the interpolation surface Γ_h^*, lifted onto Γ) in the inequality above and using the geometric approximation relation $|\hat{n}_h^{*,l} - n| \lesssim h^k$ (as well as the inverse inequality), we further derive from (2.2)-(2.3) that

$$\begin{aligned}&\Big|\int_{\Gamma_h^*} [\mathrm{tr}(\nabla_{\Gamma_h^*} e_h^*)^2 - \mathrm{tr}(\nabla_{\Gamma_h^*} e_h^* \nabla_{\Gamma_h^*} e_h^*)]\Big| \\ &\lesssim (1+\epsilon^{-1}) \|e_h^*\|_{L^2(\Gamma_h^*)}^2 + \epsilon \|(\nabla_{\Gamma_h^*} e_h^*) \widehat{n}_h^*\|_{L^2(\Gamma_h^*)}^2. \end{aligned} \tag{2.4}$$

The induction hypothesis in [4, Eq. (3.2)] reads

$$\|e_h^*\|_{W^{1,\infty}(\Gamma_h^*)} \lesssim h^2 \tag{2.5}$$

and it holds that

$$\begin{aligned}
&\Big|\int_0^1 \|(\nabla_{\Gamma_h^\theta} e_h^\theta)\widehat{n}_h^\theta\|_{L^2(\Gamma_h^\theta)}^2 \mathrm{d}\theta - \|(\nabla_{\Gamma_h^*} e_h^*)\widehat{n}_h^*\|_{L^2(\Gamma_h^*)}^2\Big| \\
&= \Big|\int_0^1\int_0^\theta \frac{\mathrm{d}}{\mathrm{d}\alpha}\int_{\Gamma_h^\alpha} (\nabla_{\Gamma_h^\alpha} e_h^\alpha)\widehat{n}_h^\alpha \cdot (\nabla_{\Gamma_h^\alpha} e_h^\alpha)\widehat{n}_h^\alpha \mathrm{d}\alpha\mathrm{d}\theta\Big| \\
&= \Big|2\int_0^1\int_0^\theta \int_{\Gamma_h^\alpha} (\partial_\alpha^\bullet \nabla_{\Gamma_h^\alpha} e_h^\alpha)\widehat{n}_h^\alpha \cdot (\nabla_{\Gamma_h^\alpha} e_h^\alpha)\widehat{n}_h^\alpha \mathrm{d}\alpha\mathrm{d}\theta \\
&\quad - 2\int_0^1\int_0^\theta \int_{\Gamma_h^\alpha} \big((\nabla_{\Gamma_h^\alpha} e_h^\alpha)(\nabla_{\Gamma_h^\alpha} e_h^\alpha)\widehat{n}_h^\alpha\big) \cdot (\nabla_{\Gamma_h^\alpha} e_h^\alpha)\widehat{n}_h^\alpha \mathrm{d}\alpha\mathrm{d}\theta \\
&\quad + \int_0^1\int_0^\theta \int_{\Gamma_h^\alpha} \theta(\nabla_{\Gamma_h^\alpha}\cdot e_h^\alpha)(\nabla_{\Gamma_h^\alpha} e_h^\alpha)\widehat{n}_h^\alpha \cdot (\nabla_{\Gamma_h^\alpha} e_h^\alpha)\widehat{n}_h^\alpha \mathrm{d}\alpha\mathrm{d}\theta\Big| \\
&\lesssim \int_0^1\int_0^\theta \|\nabla_{\Gamma_h^\alpha} e_h^\alpha\|_{L^\infty(\Gamma_h^\alpha)} \|\nabla_{\Gamma_h^\alpha} e_h^\alpha\|_{L^2(\Gamma_h^\alpha)}^2 \mathrm{d}\alpha\mathrm{d}\theta \\
&\lesssim \|\nabla_{\Gamma_h^*} e_h^*\|_{L^\infty(\Gamma_h^*)} \|\nabla_{\Gamma_h^*} e_h^*\|_{L^2(\Gamma_h^*)}^2 \\
&\lesssim \|e_h^*\|_{L^2(\Gamma_h^*)}^2, \quad \text{((2.5) and inverse inequality are used)}
\end{aligned} \tag{2.6}$$

where in the second equality we have used distribution rule and the evolution equation of $\widehat{n}_h^\alpha$ on every curved triangle of the surface (cf. [3, p. 33]), i.e.

$$\partial_\alpha^\bullet \widehat{n}_{h,i}^\alpha = -\underline{D}_i e_{h,j}^\alpha \widehat{n}_{h,j}^\alpha, \tag{2.7}$$

and in the first inequality of (2.6) we have applied the following commutator formula (cf. [2, Lemma 2.6] and [4, Eq. (3.30)]) and Hölder's inequality:

$$\partial_\alpha^\bullet \underline{D}_i e_h^\alpha = -\theta(\underline{D}_i e_{h,j}^\alpha - \widehat{n}_{h,i}^\alpha \widehat{n}_{h,l}^\alpha \underline{D}_j e_{h,l}^\alpha)\underline{D}_j e_h^\alpha. \tag{2.8}$$

Thus, analogous to (2.4), we have the following estimate for the last term of (1.5), after integrating w.r.t. θ, with the help of fundamental theorem of calculus:

$$\begin{aligned}
&\Big|\int_0^1 \int_{\Gamma_h^\theta} [\mathrm{tr}(\nabla_{\Gamma_h^\theta} e_h^\theta)^2 - \mathrm{tr}(\nabla_{\Gamma_h^\theta} e_h^\theta \nabla_{\Gamma_h^\theta} e_h^\theta)]\mathrm{d}\theta\Big| \\
&\le \Big|\int_0^1 \int_{\Gamma_h^*} [\mathrm{tr}(\nabla_{\Gamma_h^*} e_h^*)^2 - \mathrm{tr}(\nabla_{\Gamma_h^*} e_h^* \nabla_{\Gamma_h^*} e_h^*)]\mathrm{d}\theta\Big| \\
&\quad + \Big|\int_0^1\int_0^\theta \frac{\mathrm{d}}{\mathrm{d}\alpha}\int_{\Gamma_h^\alpha} [\mathrm{tr}(\nabla_{\Gamma_h^\alpha} e_h^\alpha)^2 - \mathrm{tr}(\nabla_{\Gamma_h^\alpha} e_h^\alpha \nabla_{\Gamma_h^\alpha} e_h^\alpha)]\mathrm{d}\alpha\mathrm{d}\theta\Big| \\
&\lesssim (1+\epsilon^{-1})\|e_h^*\|_{L^2(\Gamma_h^*)}^2 + \epsilon\|(\nabla_{\Gamma_h^*} e_h^*)\widehat{n}_h^*\|_{L^2(\Gamma_h^*)}^2 + \|\nabla_{\Gamma_h^*} e_h^*\|_{L^\infty(\Gamma_h^*)} \|\nabla_{\Gamma_h^*} e_h^*\|_{L^2(\Gamma_h^*)}^2 \\
&\qquad\qquad \text{((2.4) and (2.8) are used here)} \\
&\lesssim (1+\epsilon^{-1})\|e_h^*\|_{L^2(\Gamma_h^*)}^2 + \epsilon\int_0^1 \|(\nabla_{\Gamma_h^\theta} e_h^\theta)\widehat{n}_h^\theta\|_{L^2(\Gamma_h^\theta)}^2 \mathrm{d}\theta,
\end{aligned} \tag{2.9}$$

where we have used induction hypothesis (2.5) and (2.6) in deriving the last inequality. The second term on the right hand side of (2.9) can be absorbed by the left hand side of

(1.5) by choosing a sufficiently small ϵ, and then the first term can be controlled by using Gronwall's inequality. Then the stability of the error equation [4, Eq. (3.42)-(3.43)] follows and therefore [4, Theorem 2.1] still holds.

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