MULTI-AGENT OPTIMIZATION AND EQUILIBRIUM

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- modeling of economic interactions under possible uncertainty
- more motivation for efforts on solving variational inequalities
- expansion of ideas related to games and MPEC formulations

Multi-Agent Optimization

game-like structures related to equilibrium modeling

Single agent as decision maker:

minimize f over $x \in C \subset \boldsymbol{R}^n$

Multiple agents k as decision makers with interactions:

minimize f_k over $x_k \in C_k \subset \boldsymbol{R}^{n_k}$ for $k = 1, \ldots, K$, where f_k and C_k may depend the decisions of the other agents

Organization of decisions: simultaneous? hierarchical? dynamical?

Connections with equilibrium concepts:

Nash equilibrium? economic equilibrium?
equilibrium constraints? variational inequalities?

Basic Framework

$x = (x_1, \ldots, x_k, \ldots, x_K) \in R^{n_1} \times \cdots R^{n_k} \times \cdots \times R^{n_K}$

$x = (x_k, x_{-k})$ where x_k is chosen by agent k, and

x_{-k} is chosen by the other agents

Agents' Costs and Constraints

Cost function for agent k: $f_k(x_k, x_{-k}) = f_k(x_1, \ldots, x_K)$

Feasible set for agent k: $C_k(x_{-k}) = \{x_k \,|\, (x_k, x_{-k}) \in C\}$

for a set C of vectors $(x_1, \ldots, x_K)$ in $R^{n_1} \times \cdots \times R^{n_K}$

Nash game case: $C = X_1 \times \cdots X_K$, so that $C_k(x_{-k}) \equiv X_k$

Definition of Equilibrium

$(\bar{x}_1, \ldots, \bar{x}_K)$ furnishes an equilibrium‘ $\iff$

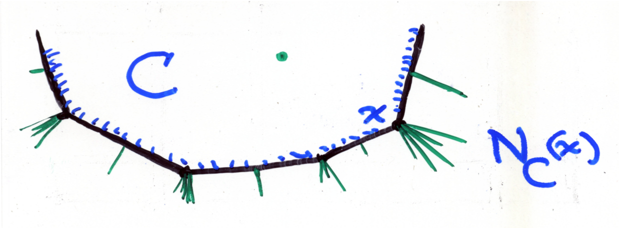
$\bar{x}_k$ “minimizes” $f_k(x_k, \bar{x}_{-k})$ over $x_k \in C_k(\bar{x}_{-k})$ for each k

Minimizes?? $\rightarrow$ global min? local min? “stationary point”?

even in ordinary computation such ambiguities come up

Variational Conditions for Use in Modeling

$-F(x) \in N_C(x)$, with $C \subset \mathbb{R}^N$ closed, $F : C \to \mathbb{R}^N$ continuous



Variational inequality case: C convex
Basic optimization case: $F = \nabla f$ (first-order condition)
Global minimum: both combined, with f convex

Quasi-variational extension: C becomes $C(x)$

Elaboration of Constraint Structure

Suppose the set C has the form

$$C = \{x \in X \,|\, g_i(x) \leq 0 \text{ for } i = 1, \ldots, m\}$$

Then $N_C(x)$ can be expressed by Lagrange multipliers in terms of the gradients $\nabla g_i(x)$ and the normal cone $N_X(x)$:

Lagrange Multiplier Rule Under a Basic Constraint Qualification

$$v \in N_C(x) \iff \begin{array}{l} \text{there exists } y = (y_1, \ldots, y_m) \text{ such that} \\ v - y_1 \nabla g_i(x) - \cdots - y_m \nabla g_m(x) \in N_X(x) \\ \text{with } \; g_i(x) \leq 0, \;\; y_i \geq 0, \;\; y_i g_i(x) = 0 \end{array}$$

$$-F(x) \in N_C(x) \iff -[F(x) + y_1 \nabla g_i(x) + \cdots + y_m \nabla g_m(x)] \in N_X(x)$$

Special case: $X = \text{whole space} \implies N_C(x) = \{0\}$

Variational Conditions for the Agents' Problems

Optimization context: agent k, given x_{-k} comprised of the decisions of the other agents, wishes to:

$$\text{minimize } f_k(x_k, x_{-k}) \text{ with respect to } x_k \in C_k(x_{-k})$$

Corresponding variational condition:

$$-F_k(x_k, x_{-k}) \in N_{C_k(x_{-k})}(x_k) \quad \text{for } F_k(x_k, x_{-k}) = \nabla_{x_k} f_k(x_k, x_{-k})$$

Equivalence with minimization: only holds in the **fully convex** case, where

$C_k(x_{-k})$ is a convex set, $f_k(x_k, x_{-k})$ is convex in x_k

But: maybe the variational condition is **more appropriate** than the minimization condition in formulating equilibrium!

Quasi-Variational Model of Equilibrium

The equilibrium condition on $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_K)$ that

$\bar{x}_k$ minimizes $f_k(x_k, \bar{x}_{-k})$ over $x_k \in C_k(\bar{x}_{-k})$ for each k

can be modeled in terms of $F_k(x_k, x_{-k}) = \nabla_{x_k} f_k(x_k, x_{-k})$ as

$$-F_k(\bar{x}_k, \bar{x}_{-k}) \in N_{C_k(\bar{x}_{-k})}(\bar{x}_k) \text{ for } k = 1, \ldots, K$$

This can be combined into a single quasi-variational condition:

$$\begin{aligned} -F(\bar{x}_1, \ldots, \bar{x}_K) &\in N_{C(\bar{x}_1,\ldots,\bar{x}_K)}(\bar{x}_1, \ldots, \bar{x}_K) \text{ with} \\ C(x_1, \ldots, x_K) &= C_1(x_{-1}) \times \cdots \times C_K(x_{-K}) \\ F(x_1, \ldots, x_K) &= (F_1(x_1, x_{-1}), \ldots, F_K(x_K, x_{-K})) \end{aligned}$$

Shortcoming, even in the fully convex case of the agents' problems:

- existence of a solution may be illusive
- quasi-variational conditions are hard to solve

Joint Constraint Structure Elaborated

Recall for agent k that the feasible set is:

$$C_k(x_{-k}) = \{x_k \mid (x_k, x_{-k}) \in C\}$$

for choice of a set $C \subset \boldsymbol{R}^n = \boldsymbol{R}^{n_1} \times \cdots \boldsymbol{R}^{n_K}$

Joint constraint assumption henceforth

$C =$ set of all $x = (x_1, \ldots, x_K)$ satisfying

$$g_i(x_1, \ldots, x_K) \leq 0 \ \text{ for } \ i = 1, \ldots, m,$$

$$\text{with } x_k \in X_k \text{ for } k = 1, \ldots, K,$$

where $X_k \subset \boldsymbol{R}^{n_k}$ is closed convex, g_i is continuously differentiable

Then $C_k(x_{-k}) = \{x_k \in X_k \mid g_i(x_k, x_{-k}) \leq 0 \text{ for } i = 1, \ldots, m\}$
and the normal cone to $C_k(x_{-k})$ at x_k has a formula involving the gradient vectors $\nabla_{x_k} g_i(x_k, x_{-k})$ and the normal cone $N_{X_k}(x_k)$

Lagrange Multipliers for Agent k

Under a **constraint qualification**, the first-order condition

$$-\nabla_{x_k} f_k(x_k, x_{-k}) \in N_{C_k(x_{-k})}(x_k)$$

holds if and only if there is a vector $y_k = (y_{k1}, \ldots, y_{km})$ such that

$$-\Big[\nabla_{x_k} f_k(x_k, x_{-k}) + \textstyle\sum_{i=1}^m y_{ki} \nabla_{x_k} g_i(x_k, x_{-k})\Big] \in N_{X_k}(x_k)$$

$$\text{with} \quad g_i(x_k, x_{-k}) \le 0, \quad y_{ki} \ge 0, \quad y_{ki} g_i(x_k, x_{-k}) = 0$$

Key fact to recall: in terms of $g(x) = (g_1(x), \ldots, g_m(x))$ the complementarity conditions can be written as $g(x) \in N_{R^m_+}(y_k)$

The pair of multiplier conditions $G(x) \in N_{R^m_+}(y_k)$ and

$$-\Big[\nabla_{x_k} f_k(x_k, x_{-k}) + \textstyle\sum_{i=1}^m y_{ki} \nabla_{x_k} g_i(x_k, x_{-k})\Big] \in N_{X_k}(x_k)$$

combine as **one variational inequality** for $(x_k, y_k) \in X_k \times R^m_+$

Why not model the role of agent k this way directly?

Equilibrium as a Variational Inequality, Initial Version

Let $G_k(x_k, x_{-k}, y_k) = \nabla_{x_k} f_k(x_k, x_{-k}) + \sum_{i=1}^m y_{ki} \nabla_{x_k} g_i(x_k, x_{-k})$.

The variational inequality for agent k requires

$$(-G_k(x_k, x_{-k}, y_k), g(x_1, \dots, x_K)) \in N_{X_k \times \mathbf{R}^m_+}(x_k, y_k)$$

Equilibrium model with separate multiplier vectors

Having $(G_k(\bar{x}_k, \bar{x}_{-k}, \bar{y}_k), g(\bar{x}_1, \dots, \bar{x}_K)) \in N_{X_k \times \mathbf{R}^m_+}(\bar{x}_k, \bar{y}_k)$ for all k corresponds to the single variational inequality

$$-G(\dots; \bar{x}_k, \bar{y}_k; \dots) \in N_D(\dots; \bar{x}_k, \bar{y}_k; \dots)$$

on the closed convex set $D = [X_1 \times \mathbf{R}^m_+] \times \cdots \times [X_K \times \mathbf{R}^m_+]$ with $G(\dots; x_k, y_k; \dots) = (\dots; G_k(x_k, x_{-k}, y_k), -g(x_1, \dots, x_K); \dots)$

Facchinei and Kanzow (2007) focus on this model, but restricted to full convexity, each constraint function g_i also being convex

The Market Significance of Lagrange Multipliers

Agent k wishes to: "minimize" $f_k(x_k, \bar{x}_{-k})$ subject to

$$x_k \in X_k \text{ and } g_i(x_k, \bar{x}_{-k}) \leq 0 \text{ for } i = 1, \ldots, m,$$

Interpretation of a Lagrange multiplier for g_i:

a shadow price for the resource represented by g_i, reflecting the effect on the objective f_k of shifts in availability

Equilibrium in an **economic** context:

Incorporate a "market" in which these prices act

But this means having the **same** multipliers/prices for each agent:

$$(\bar{y}_{k1}, \ldots, \bar{y}_{km}) = (\bar{y}_1, \ldots, \bar{y}_m) \text{ for } k = 1, \ldots, K$$

Challenge: can a variational inequality model be set up which

- enforces this multiplier/price equality, and
- guarantees the existence of such an equilibrium?

Equilibrium as a Variational Inequality, Better Version

To be determined: $\bar{x}_k$ for $k = 1, \ldots, K$ and $\bar{y} = (\bar{y}_1, \ldots, \bar{y}_m)$
Recall: $G_k(x_k, x_{-k}, y) = \nabla_{x_k} f_k(x_k, x_{-k}) + \sum_{i=1}^m y_i \nabla_{x_k} g_i(x_k, x_{-k})$

Equilibrium model of market type with shared multipliers

The previous model with the multipliers **shared** by all the agents corresponds to the variational inequality

$$-G^*(\bar{x}_1, , \ldots, \bar{x}_K, \bar{y}) \in N_{D^*}(\bar{x}_1, \ldots, \bar{x}_K, \bar{y})$$

for the closed convex set $D^* = X_1 \times \cdots \times X_K \times R^m_+$ and function $G^*(x_1, \ldots, x_K, y) = (\ldots; G_k(x_k, x_{-k}, y); \ldots; g(x_1, \ldots, x_K)\,)$

- A solution to this equilibrium model if one exists yields a solution to the previous equilibrium model, in particular.
- In the fully convex case this reverts to agents minimizing globally, but the model makes good sense even without that.

Existence of Equilibrium

Standard criterion for a solution to $-F(\bar{x}) \in N_C(\bar{x})$:

F continuous, C **convex** and **compact**.

For the market model $-G^*(\bar{x}_1, \ldots, \bar{x}_K, \bar{y}) \in N_{D^*}(\bar{x}_1, \ldots, \bar{x}_K, \bar{y})$ we have the continuity of G^*, and the closedness and convexity of the set $D^* = X_1 \times \cdots \times X_K \times R^m_+$ but D^* is **unbounded**

Remedy (one approach): restriction to the fully convex case:

$f_k(x_k, x_{-k})$ convex in x_k, $g_i(x)$ convex in $x = (x_1, \ldots, x_K)$

and utilize the Slater condition: there there is an

$\hat{x} \in X_1 \times \cdots \times X_K$ with $g_i(\hat{x}) < 0$ for $i = 1, \ldots, m$

Existence Theorem

In the **fully convex** case under the **Slater condition**, and with the sets $X_1, \ldots, X_K$ bounded, there is a solution $(\bar{x}_1, \ldots, \bar{x}_K, \bar{y})$ to the variational inequality for the market equilibrium model

Beyond Full Convexity: Modified Equilibrium Model

Suppose there are **exogenous prices** y_i^* at which the resources can be obtained from "the outside" if necessary. (The prices y_i introduced so far are **endogenous**, determined "locally")

Consequences:

- The constraint $g_i(x_1, \ldots, x_K) \leq 0$ can be exceeded by the agents if they are willing to pay the price y_i^* for any excess
- The endogenous prices y_i will have to satisfy $y_i \leq y_i^*$
- The multiplier space R_+^m becomes $[0, y_1^*] \times \cdots \times [0, y_m]$
- Except for this, the variational conditions remain the same

Existence Theorem Without (Much) Convexity

In the modified setting, an equilibrium exists merely under the assumption that the underlying sets X_k are convex and bounded. No convexity properties are required of the functions f_k or g_i.

Further elaboration: Take limit as $y_i^* \to \infty$. Get another result

Introduction of Uncertainty

Present: time 0, Future: time 1, states $s = 1, \ldots, S$, probs: $p(s)$

Agent k: choose x_k^0 for time 0, response $x_k^1(s)$ for time 1, state s

Expected cost for agent k to "minimize":

$$\sum_{s=1}^{S} p(s)\, f_k(x_k^0, x_k^1(s); x_{-k}^0, x_{-k}^1(s); s) \text{ (expected cost)}$$

Individual constraints for agent k : $(x_k^0, x_k^1(s)) \in X_k(s)$

Joint constraints for all agents at time 0:

$$g_i^0(x_1^0, \ldots, x_K^0) \leq 0 \text{ for } i = 1, \ldots, m_0$$

Joint constraints for all agents at time 1 in state s:

$$g_i^1(x_1^0, x_1^1(s); \ldots; x_K^0, x_K^1(s); s) \leq 0 \text{ for } i = 1, \ldots, m_1$$

These constraints refer to **shared resources** tradable in markets

Goal: an equilibrium with **prices** y_i^0, $y_i^1(s)$ for these resources

Optimality Conditions With Shared Multipliers

Lagrangian function for agent k: expected Lagrangian

$$\mathcal{L}_k(x_k^0, x_k^1; x_{-k}^0, x_{-k}^1; y^0, y^1) = \sum_{s=1}^S p(s)\, L_k(x_k^0, x_k^1(s); x_{-k}^0, x_{-k}^1(s); y^0, y^1(s))$$

where

$$L_k(x_k^0, x_k^1(s); x_{-k}^0, x_{-k}^1(s); y^0, y^1(s)) = f_k(x_k^0, x_k^1(s); x_{-k}^0, x_{-k}^1(s); s) + \sum_{i=1}^{m_0} y_i^0 g_i^0(x_k^0, x_{-k}^0) + \sum_{i=1}^{m_1} y_i^1(s) g_i^1(x_k^0, x_k^1(s); x_{-k}^0, x_{-k}^1(s); s)$$

for $(x_k^0, x_k^1) \in X_k$, meaning $(x_k^0, x_k^1(s)) \in X_k(s)$ for all s, and $(y^0, y^1) \in Y$, meaning $y^0 \in \mathbb{R}_+^{m_0}$ and $y^1(s) \in \mathbb{R}_+^{m_1}$ for all s

Corresponding first-order conditions: generalized KKT

$$-\nabla_{x^0,x^1}\mathcal{L}_k(x_k^0, x_k^1; x_{-k}^0, x_{-k}^1; y^0, y^1) \in N_{X_k}(x_k^0, x_k^1),$$
$$\nabla_{y^0,y^1}\mathcal{L}_k(x_k^0, x_k^1; x_{-k}^0, x_{-k}^1; y^0, y^1) \in N_Y(y_k^0, y_k^1)$$

Equilibrium: $\bar{y}^0$, $\bar{y}^1$, and $\bar{x}_k^0$, $\bar{x}_k^1(s)$ for $k = 1, \ldots, K$ such that the generalized KKT conditions hold **simultaneously** for all k

Variational Inequality for Equilibrium With Uncertainty

Assume each $X_k(s)$ is bounded and polyhedral convex, for reducing the X_k conditions to the given sets $X_k(s)$, and let

$$E_k(x_k^0, x_k^1; x_{-k}^0, x_{-k}^1; y^0, y^1) = \sum_{s=1}^S p(s) \nabla_{x_k^0} L_k(x_k^0, x_k^1(s); x_{-k}^0, x_{-k}^1(s); y^0, y^1(s))$$

$$e_k(x_k^0, x_k^1(s); x_{-k}^0, x_{-k}^1(s); y^0, y^1(s); s) = p(s) \nabla_{x_k^1(s)} L_k(x_k^0, x_k^1(s); x_{-k}^0, x_{-k}^1(s); y^0, y^1(s))$$

$$g^0 = (g_1^0, \ldots, g_{m_0}^0), \qquad g^1 = (g_1^1, \ldots, g_{m_1}^1)$$

Components of the desired variational inequality:

$$-\Big(E_k(\bar{x}_k^0, \bar{x}_k^1; \bar{x}_{-k}^0, \bar{x}_{-k}^1; \bar{y}^0, \bar{y}^1), e_k(\bar{x}_k^0, \bar{x}_k^1(s); \bar{x}_{-k}^0, \bar{x}_{-k}^1(s); \bar{y}^0, \bar{y}^1(s); s) \Big) \in N_{X_k(s)}(\bar{x}_k^0, \bar{x}_k^1(s)) \text{ for } k = 1, \ldots, K \text{ and } s = 1, \ldots, S,$$

$$g^0(\bar{x}_1^k, \ldots, \bar{x}_K^0) \in N_{\boldsymbol{R}_+^{m_0}}(\bar{y}^0),$$

$$g^1(\bar{x}_1^0, \bar{x}_1^1(s); \ldots; \bar{x}_K^0, \bar{x}_K^1(s); s) \in N_{\boldsymbol{R}_+^{m_1}}(\bar{y}^1(s)) \text{ for } s = 1, \ldots, S$$

Existence of equilibrium: Under full convexity and a Slater condition, or without that by introducing exogenous prices

Some References

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