On Numerical Solution of Hamilton-Jacobi-Bellman (HJB) Equations

Song Wang

The University of Western Australia

mailto:swang@maths.uwa.edu.au

Collaborators

L. Angermann, C.-S. Huang,

W. Li, K.L. Teo, X.Q. Yang

Outline

- The HJB equation arising in American option pricing
  - The mathematical formulation
  - Penalty Approach to the variational inequality problem
  - Fitted finite volume for the penalized equation
- The HJB equation arising from option pricing with proportional transaction costs
- The HJB equation governing a nonlinear obstacle problem

The American (put) option

Consider the following European put option

$$LV := -\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2(t)x^2\frac{\partial^2 V}{\partial x^2} - (r(t)x - D(x,t))\frac{\partial V}{\partial x} + rV = 0, \quad (1)$$

with the payoff and boundary conditions

$$V(x,t) = \max(K-x, 0),$$
$$V(0,t) = K\exp(-r(T-t)), \quad V(x,t) \to 0 \text{ as } x \to \infty.$$

If the option can be exercise anytime before T, the value determined by the above problem sometimes falls below its intrinsic value.

For example, when $x = 0$, the intrinsic value is K, but the above problem gives

$$V(x,t) = K\exp(-r(T-t)) < K,$$

which is not true. Therefore, we need to impose the following condition:

$$V(x,t) \geq \max(K-x,0)$$

Combining this with the above problem we have the following linear complementarity problem:

$$\begin{aligned} LV(x,t) &\geq 0 \qquad (2) \\ V(x,t) - V^*(x) &\geq 0 \qquad (3) \\ LV(x,t)\cdot(V(x,t)-V^*(x)) &= 0 \qquad (4) \end{aligned}$$

a.e. in $\Omega := I \times (0,T)$,

Note that this also contains the European option as special case if we assume V^* is sufficiently small (or $V^* = 0$).

Equivalent to the HJB equation

$$\min\{LV, V - V^*\} = 0.$$

Variational formulation

Before reformulating the complementarity problem (43)–(45) as a variational problem, we first transform it into an equivalent standard form satisfying homogeneous Dirichlet boundary conditions.

Let V_0 be the linear function satisfying the boundary conditions (**??**) and (**??**). It is easy to show that V_0 is given by

$$V_0(x) = \left(1 - \frac{x}{X}\right) g(t) = \left(1 - \frac{x}{X}\right) K \tag{5}$$

by (**??**). Introduce a new variable

$$U(x,t) = e^{\beta t}(V(x,t) - V_0(x))$$

where

$$\beta = \sup_{0<t<T} \sigma^2(t). \tag{6}$$

Taking LV_0 away from both sides of (43) and transforming V in (43)–(45) into the new variable U, it is easy to show that, the complementarity problem (43)–(45) then becomes

$$\begin{aligned} \mathcal{L}U(x,t) &\geq -f(x,t), && (7)\\ U(x,t) - U^*(x,t) &\geq 0, && (8)\\ \big(\mathcal{L}U(x,t) + f(x,t)\big) \cdot (U(x,t) - U^*(x,t)) &= 0, && (9) \end{aligned}$$

where

$$f(t) = e^{\beta t} LV_0(x), \qquad U^*(x,t) = e^{\beta t}(V^*(x) - V_0(x)) \tag{10}$$

and $\mathcal{L}U$ is the self-adjoint form of LU given by

$$\mathcal{L}U = -\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left[a(t)x^2\frac{\partial U}{\partial x} + b(t)xU\right] + c(t)U, \tag{11}$$

with

$$a = \frac{1}{2}\sigma^2, \tag{12}$$

$$b = r - \sigma^2, \tag{13}$$

$$c = r + b + \beta = 2r + \beta - \sigma^2. \tag{14}$$

From (**??**), (**??**), (5) and the definition of U, we see that the boundary conditions now become

$$U(0,t) = 0 = U(X,t), \quad t \in [0,T). \tag{15}$$

Finally, letting $u = -U$, the complementarity problem (7)–(9) can further be rewritten as the following standard form:

$$\begin{aligned} \mathcal{L}u(x,t) &\leq f(x,t), & (16) \\ u(x,t) - u^*(x,t) &\leq 0, & (17) \\ \big(\mathcal{L}u(x,t) - f(x,t)\big) \cdot (u(x,t) - u^*(x,t)) &= 0 & (18) \end{aligned}$$

in Q_T, where $Q_T = (0,X) \times (0,T)$,

$$f(x,t) := e^{\beta t} L V_0(x,t), \qquad u^*(x,t) := -e^{\beta t}(V^*(x) - V_0(x,t)) \tag{19}$$

and $\mathcal{L}u$ is the conservative form of Lu given by

$$\mathcal{L}u = -\frac{\partial u}{\partial t} - \frac{\partial}{\partial x}\left[a(t)x^2\frac{\partial u}{\partial x} + b(t)xu\right] + c(t)u, \tag{20}$$

with

$$a = \frac{1}{2}\sigma^2, \tag{21}$$

$$b = r - d - \sigma^2, \tag{22}$$

$$c = r + b - x\frac{\partial d}{\partial x} + \beta = 2r + \beta - \sigma^2 - \frac{\partial D}{\partial x}. \tag{23}$$

Here

$$\beta := \sup_{Q_T}\left[\sigma^2(t) + \frac{\partial D}{\partial x}\right]. \tag{24}$$

Let $\mathcal{K} := \{v \in H^1_{0,w}(\Omega) : v \leq u^*\}$. It is easy to verify that $\mathcal{K}$ is a convex and closed subset of $H^1_{0,w}(\Omega)$. Using $\mathcal{K}$, we define the following problem.

Find $u(t) \in \mathcal{K}$ such that, for all $v \in \mathcal{K}$,

$$\left(-\frac{\partial u(t)}{\partial t}, v - u(t)\right) + A(u(t), v - u(t); t) \geq (f(t), v - u(t)) \tag{25}$$

almost everywhere (a.e.) in $(0, T)$, where $A(\cdot, \cdot; t)$ be a bilinear form defined by

$$A(u, v; t) = (ax^2 u' + bxu, v') + (cu, v), \quad u, v \in H^1_{0,w}(\Omega). \tag{26}$$

Here $H^1_{0,w}(\Omega)$ is a weighted Sobolev space.

Lemma

There exist positive constants C and M, independent of v, such that for any $v, w \in H^1_{0,w}(I)$,

$$A(v, v; t) \geq C||v||_A^2 \quad (27)$$

$$A(v, w; t) \leq M||v||_A||w||_A \quad (28)$$

for $t \in (0, T)$, where $|| \cdot ||_A$ is a weighted energy norm.

The Power Penalty Approach

Consider now the following semilinear equation

$$\mathcal{L}u_\lambda(x,t) + \lambda[u_\lambda(x,t) - u^*(x,t)]_+^{1/k} = f(x,t), \quad (x,t) \in Q_T \quad (29)$$

with the given boundary and final conditions

$$u_\lambda(0,t) = 0 = u_\lambda(X,t) \quad \text{and} \quad u_\lambda(x,T) = u^*(x,T),$$

where $\lambda > 0$ and $k > 0$ are parameters, and $[\cdot]_+$ denotes the positive part of a function.

When $k = 1$, it becomes a linear penalty method. In this case, large values of λ are needed which causes computational problems.

The corresponding variational form is

Find $u_\lambda(t) \in H^1_{0,w}(\Omega)$ such that, for all $v \in H^1_{0,w}(\Omega)$,

$$\left(-\frac{\partial u_\lambda(t)}{\partial t}, v\right) + A(u_\lambda(t), v; t) + \lambda\left([u_\lambda(t) - u^*(t)]_+^{1/k}, v\right) = (f(t), v) \tag{30}$$

a.e. in $(0, T)$ and $u_\lambda(x, T) = g_3(x)$.

The above problem has a unique solution.

The proof (by Wang, Yang & Teo) is essentially to show

$$\int_0^T \Big[\big(\mathcal{L}(v_1 - v_2), v_1 - v_2\big) + \lambda \Big([v_1 - u^*]_+^{1/k} - [v_2 - u^*]_+^{1/k}, v_1 - v_2 \Big) \Big] d\tau \geq C ||v_1 - v_2||^2_{L^2(0,T;H_0^1(I))}.$$

Convergence of the penalty approach

Let u and u_λ be the solutions to (25) and (30), respectively. If $\frac{\partial u}{\partial t} \in L^{k+1}(Q_T)$, then there exists a non-negative constant $C > 0$, independent of u, u_λ and λ, such that

$$\|u - u_\lambda\|_{L^\infty(0,T;L^2(\Omega))} + \|u - u_\lambda\|_{L^2(0,T;H^1_{0,w}(\Omega))} \leq \frac{C}{\lambda^{k/2}}, \quad (31)$$

where k is the parameter used in (30) and $Q_T = (0, X) \times (0, T)$.

The significance of this result is that the convergence rate can be of any order, depending on the choice of k*.*

Therefore, we can choose rather small λ, but large k.

Discretisation of the nonlinear PDE

Consider

$$-\frac{\partial u}{\partial t}(x,t) - \frac{\partial}{\partial x}\left[a(t)x^2\frac{\partial u}{\partial x}(x,t) + b(x,t)xu(x,t)\right]$$
$$+c(x,t)u(x,t) + \Psi(u(x,t),t) = f(x,t) \qquad (32)$$

Mesh:

$$I_i := (x_i, x_{i+1}), \quad i = 0, \ldots, N-1.$$

Integrating the equation (32) over $\Omega_i := (x_{i-1/2}, x_{i+1/2})$, $i = 1, \ldots, N-1$,

$$-\int_{\Omega_i} \dot{u}\,dx - [x\rho(u)]_{x_{i-1/2}}^{x_{i+1/2}} + \int_{\Omega_i} cu\,dx + \int_{\Omega_i} \Psi(u,\cdot)dx = \int_{\Omega_i} f\,dx\,, \qquad (33)$$

where $\rho = axv' + bv$ denotes the flux.

The flux can be approximated locally by solving

$$(axv' + b_{i+1/2}v)' = 0, \quad x \in I_i \tag{34}$$

$$v(x_i) = u_i, \quad v(x_{i+1}) = u_{i+1}, \tag{35}$$

where $b_{i+1/2} = b(x_{i+1/2}, t)$. This gives

$$\rho_i(u) = b_{i+1/2}\frac{x_{i+1}^{\alpha_i}u_{i+1} - x_i^{\alpha_i}u_i}{x_{i+1}^{\alpha_i} - x_i^{\alpha_i}}, \tag{36}$$

where $\alpha_i = b_{i+1/2}/a$.

Substituting into (33),

$$-\frac{\partial u_i(t)}{\partial t}l_i + E_i(t)\boldsymbol{u}(t) + d_i(u_i(t)) = f_i(t)l_i, \tag{37}$$

for $i = 1, 2, ..., N - 1$, where

$$E_i = (0, .., 0, e_{i,i-1}(t), e_{i,i}(t), e_{i,i+1}(t), 0, ..., 0).$$

Time discretisation

2-level discretisation:

$$\frac{u_i^{m+1} - u_i^m}{-\Delta t_m} l_i + \theta \left[E_i^{m+1} \boldsymbol{u}^{m+1} + d_i(u_i^{m+1}) \right]$$
$$+ (1-\theta) \left[E_i^m \boldsymbol{u}^m + d_i(u_i^m) \right] = (\theta f_i^{m+1} + (1-\theta) f_i^m) l_i$$

for $m = 0, 1, ..., M-1$, where $\theta \in [1/2, 1]$.

$\theta = 1$ – Backward Euler; $\theta = 1/2$ – Crank-Nicolson.

Properties of the system matrix

The system matrix is an M-matrix.

The discretization is monotone, which *guarantees* the solution is non-negative.

Error estimates for the spatial discretization

Let $\Theta \in [1/2, 1]$, $U^0 = I_h u_0$ *and* $k = 1$. *Then, if* $u, \dot{u} \in L^2(0, T; H^1(\Omega))$, $\ddot{u} \in L^2(0, T; L^2(\Omega))$ *and* $\rho(u) \in C(0, T; W^1_\infty(\Omega))$, *the following estimate holds:*

$$\|I_h u(0) - U^M\|_{0,h} \leq C(h + \Delta t).$$

A proof is given in Angermann & Wang, Numer. Math. (2007).

Error estimates for the spatial discretization

Let $\Theta \in [1/2, 1]$, $U^0 = I_h u_0$ *and* $k = 1$. *Then, if* $u, \dot{u} \in L^2(0,T; H^1(\Omega))$, $\ddot{u} \in L^2(0,T; L^2(\Omega))$ *and* $\rho(u) \in C(0,T; W^1_\infty(\Omega))$, *the following estimate holds:*

$$\|I_h u(0) - U^M\|_{0,h} \leq C(h + \Delta t).$$

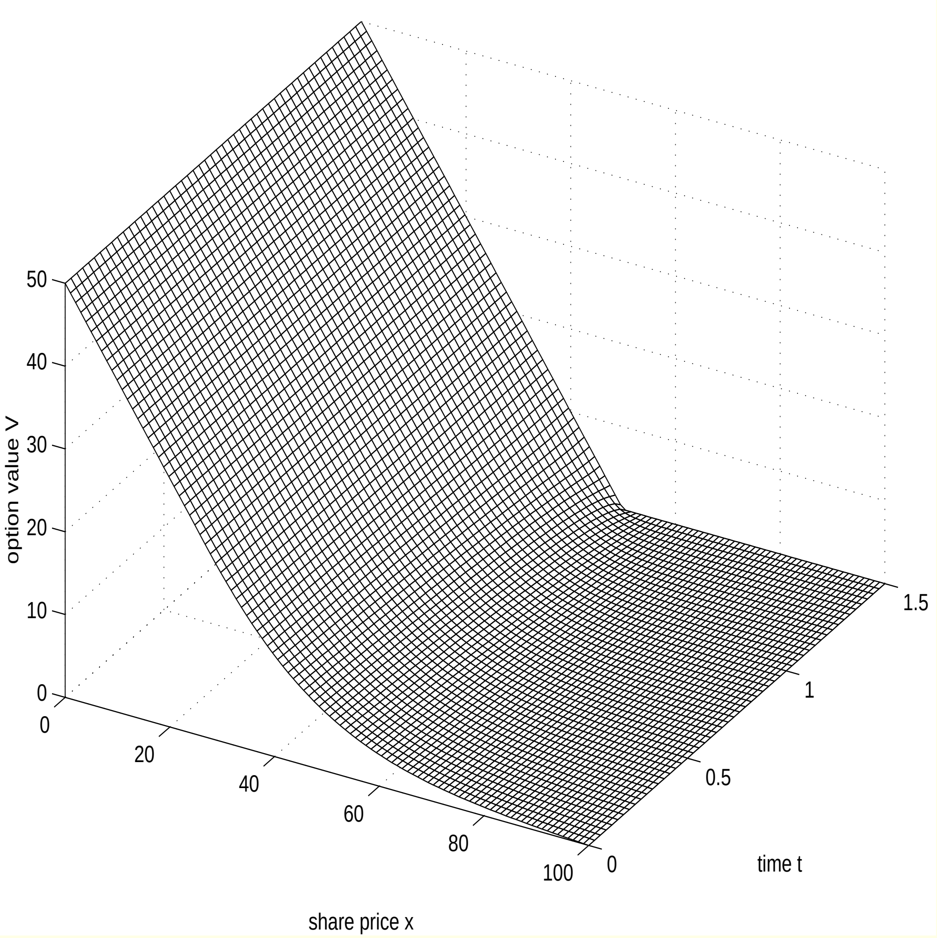
A proof is given in Angermann & Wang, Numer. Math. (2007).

Remark: Under stronger smoothness assumptions on u than those in the above theorem and by more detailed considerations, it can also be shown that the Crank–Nicolson method ($\Theta = 1/2$) is of 2nd order accuracy in Δt.

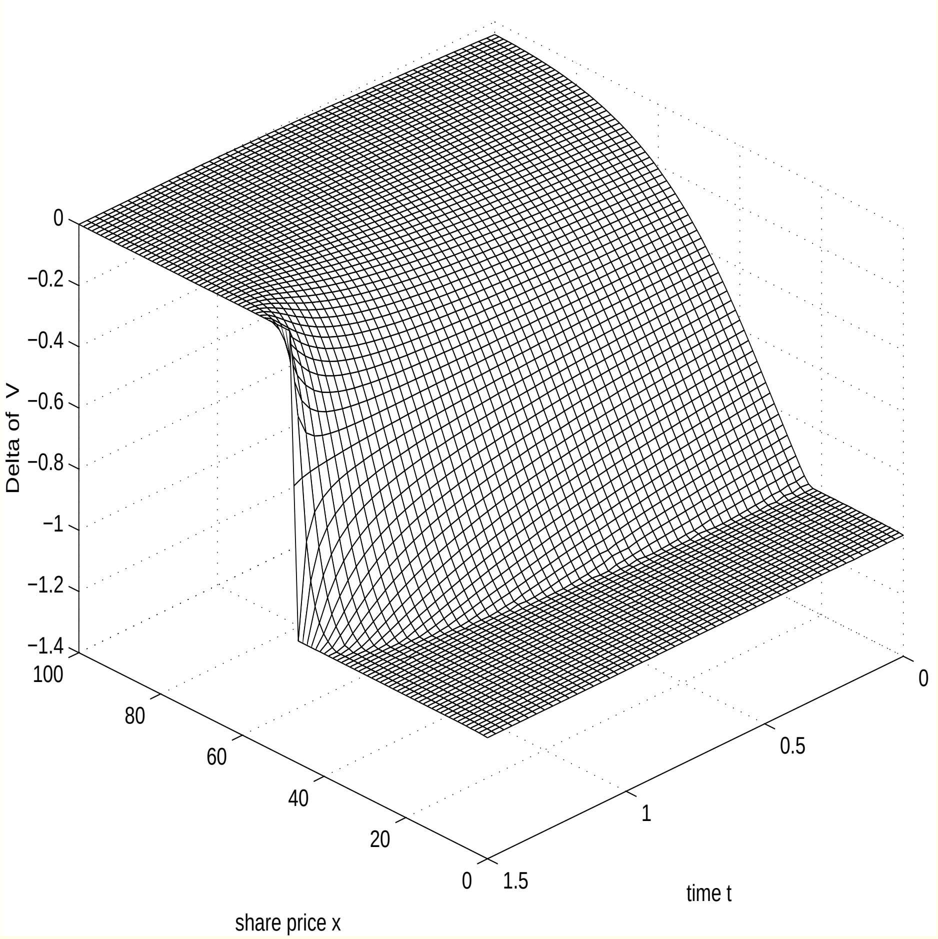
Numerical experiments

$\theta = 1/2$ – Crank-Nicholson scheme for the time stepping.

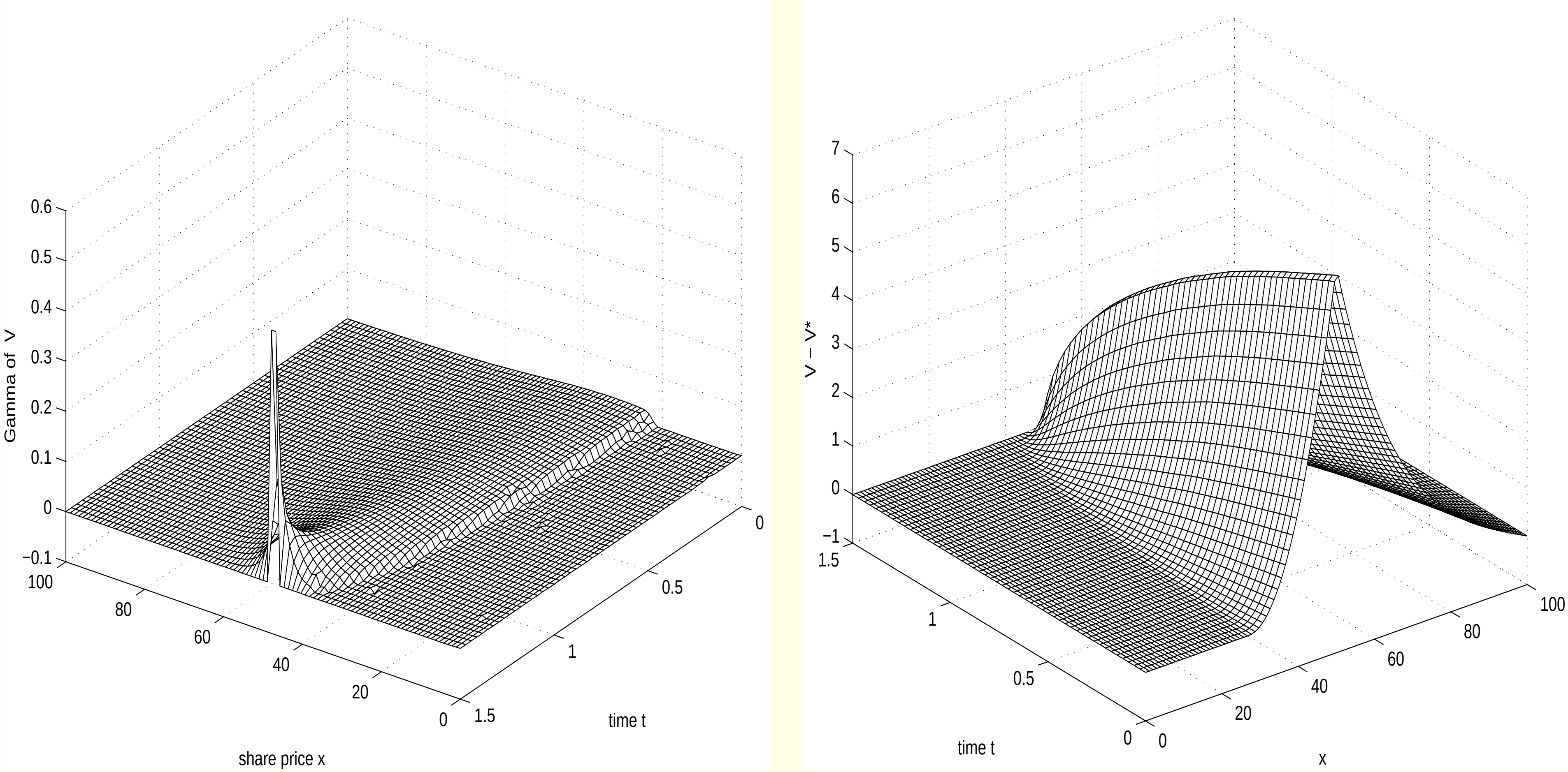
Test Problem: American Put option. Parameters: $X = 100$, $T = 1.5$, $r = 0.03, \sigma = 0.4$ and $K = 50$.



V

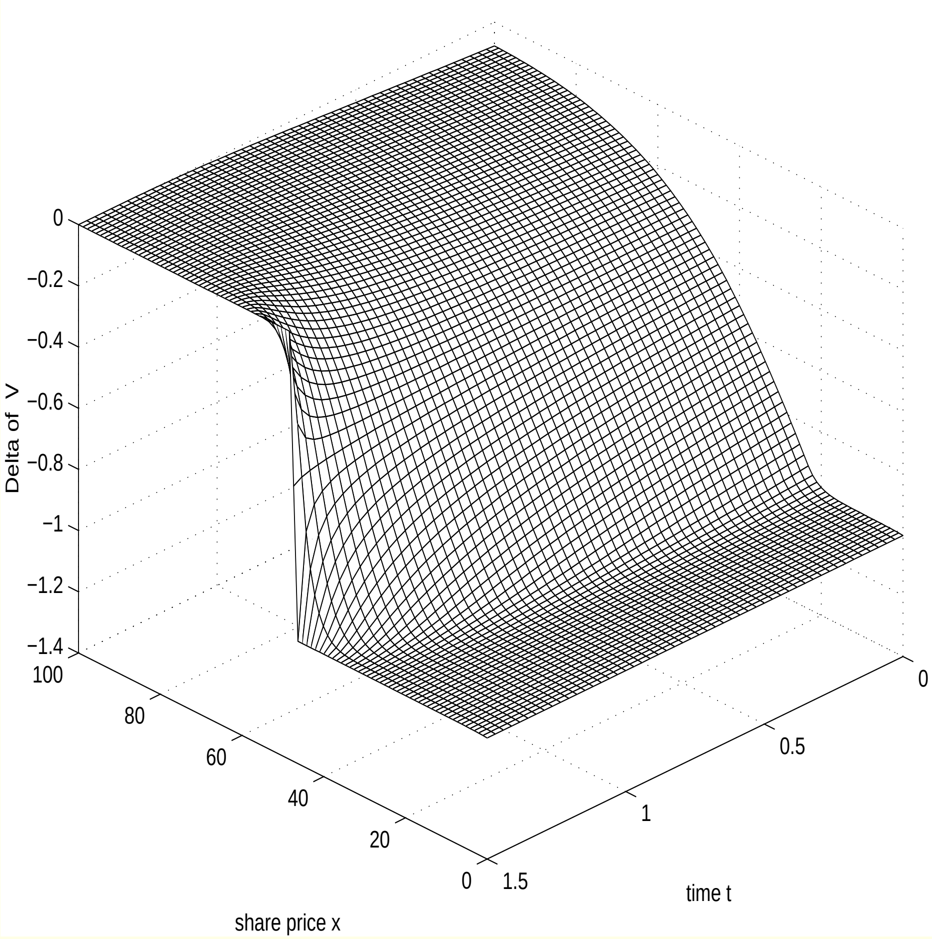


Δ

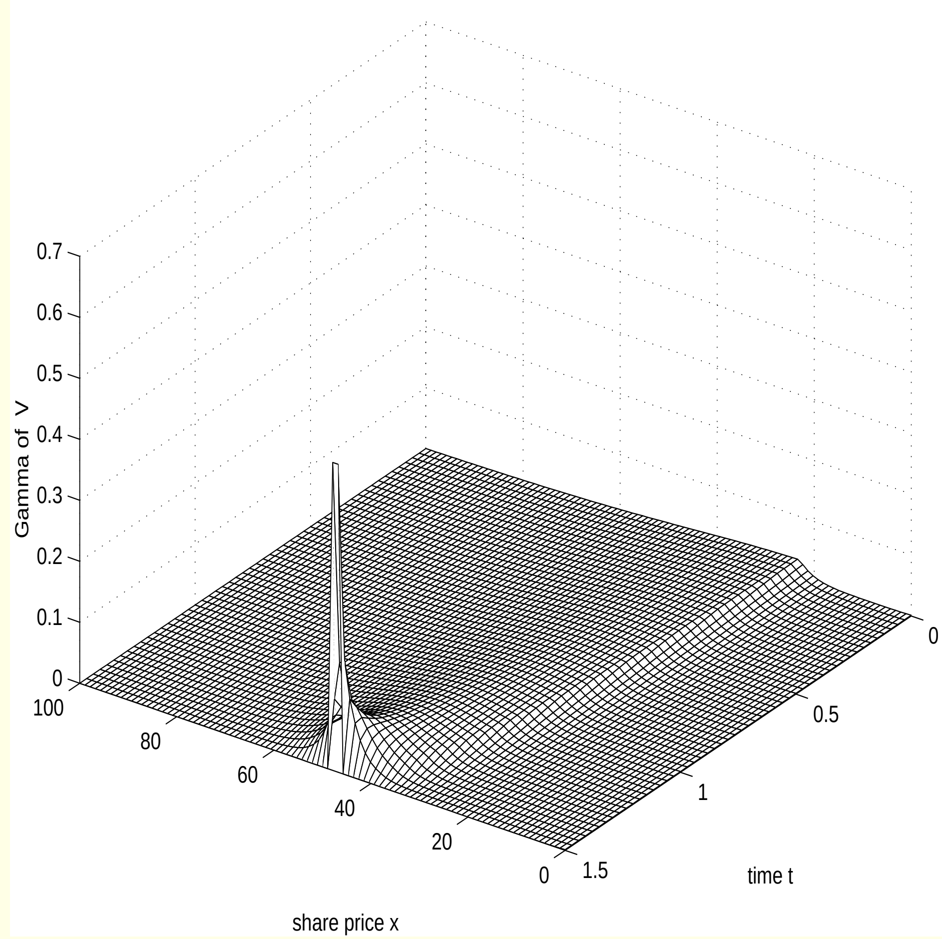


Γ $\qquad\qquad\qquad$ $V - V^*$

Figure 1: Computed value V, Δ and Γ of the option, and the constraint $V - V^*$ for $k = 4$ and $\lambda = 10$.

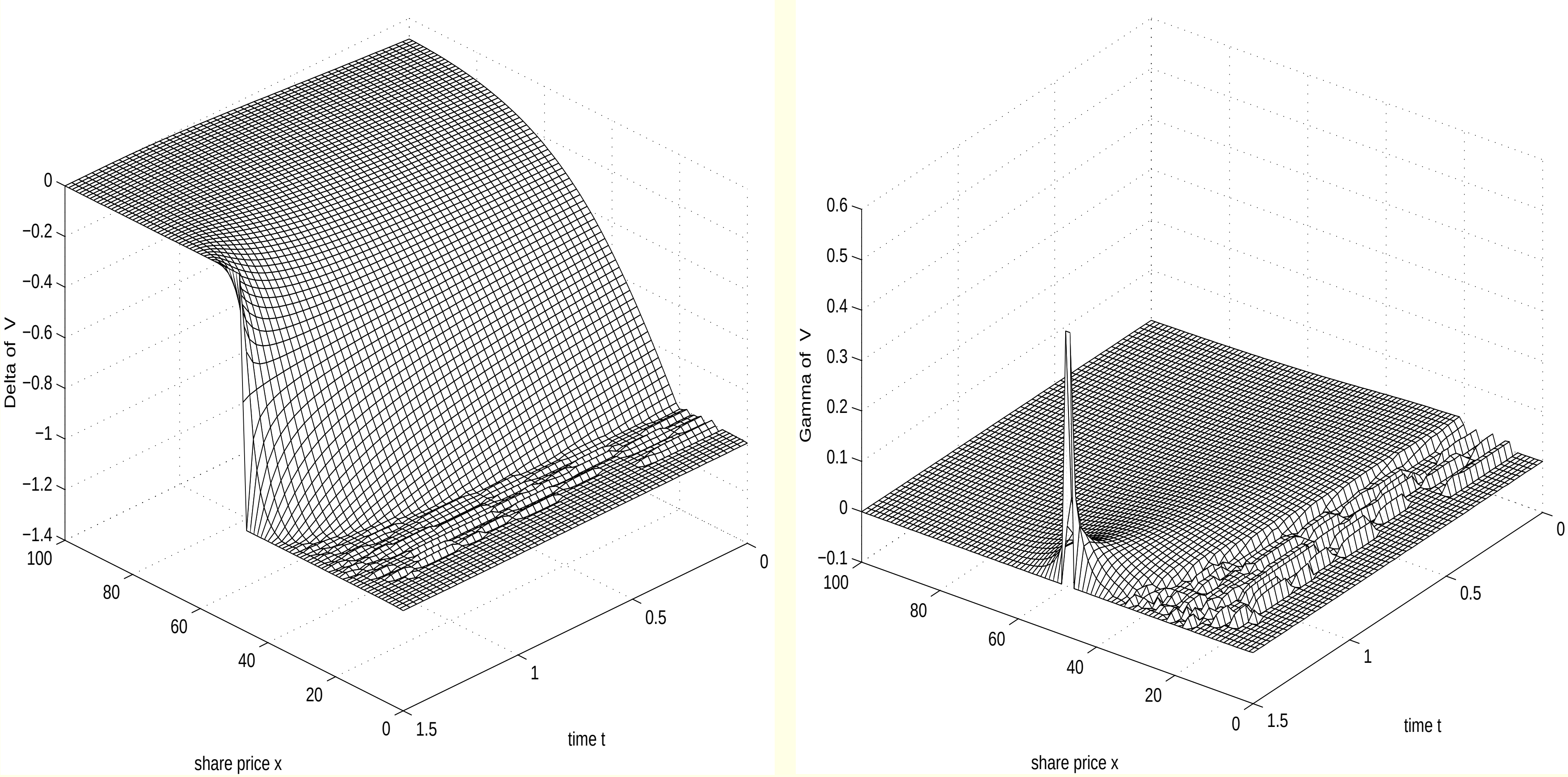


Δ



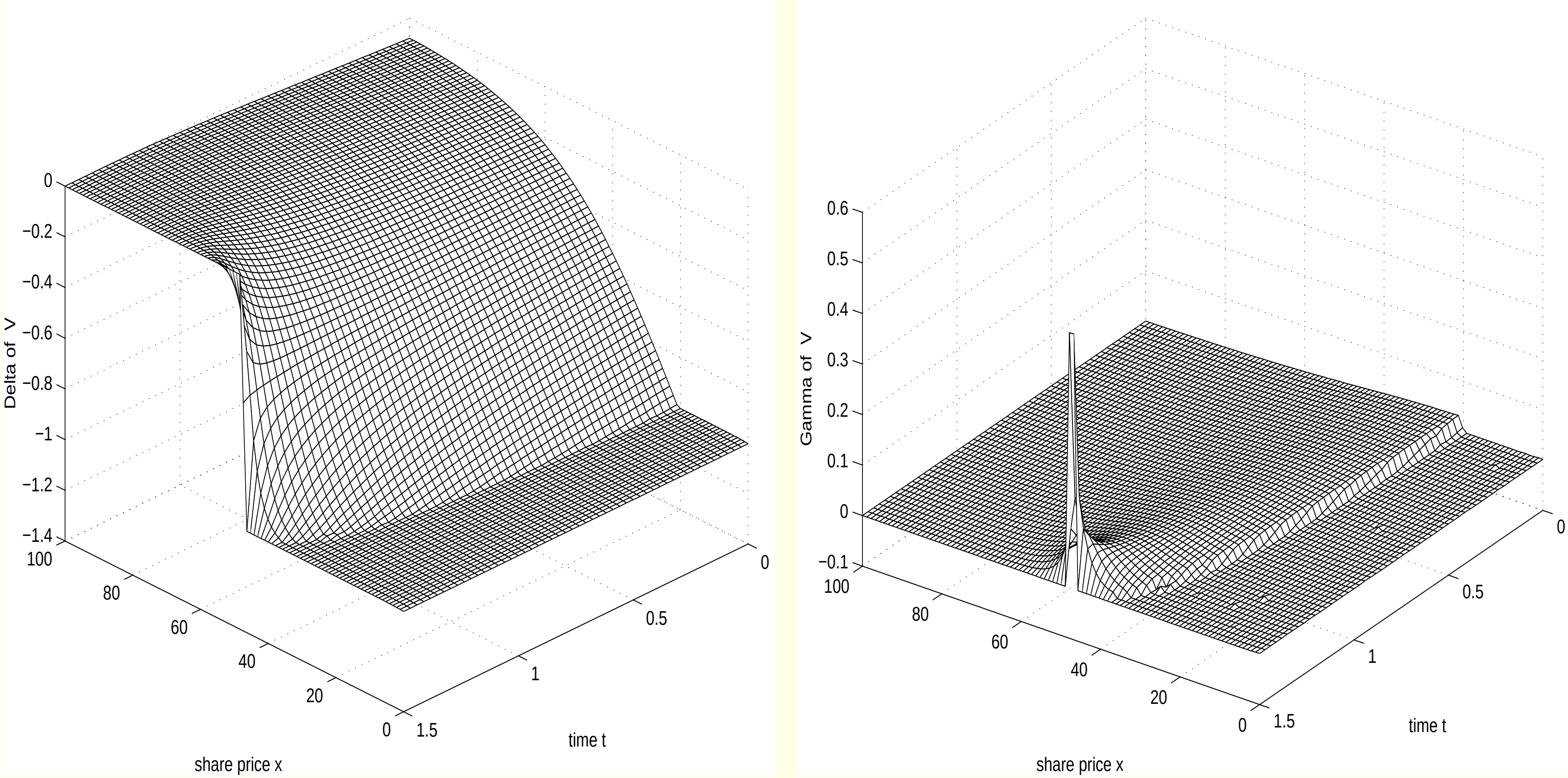
Γ

Figure 2: Computed Δ and Γ of the option for $k = 1$ and $\lambda = 10$.



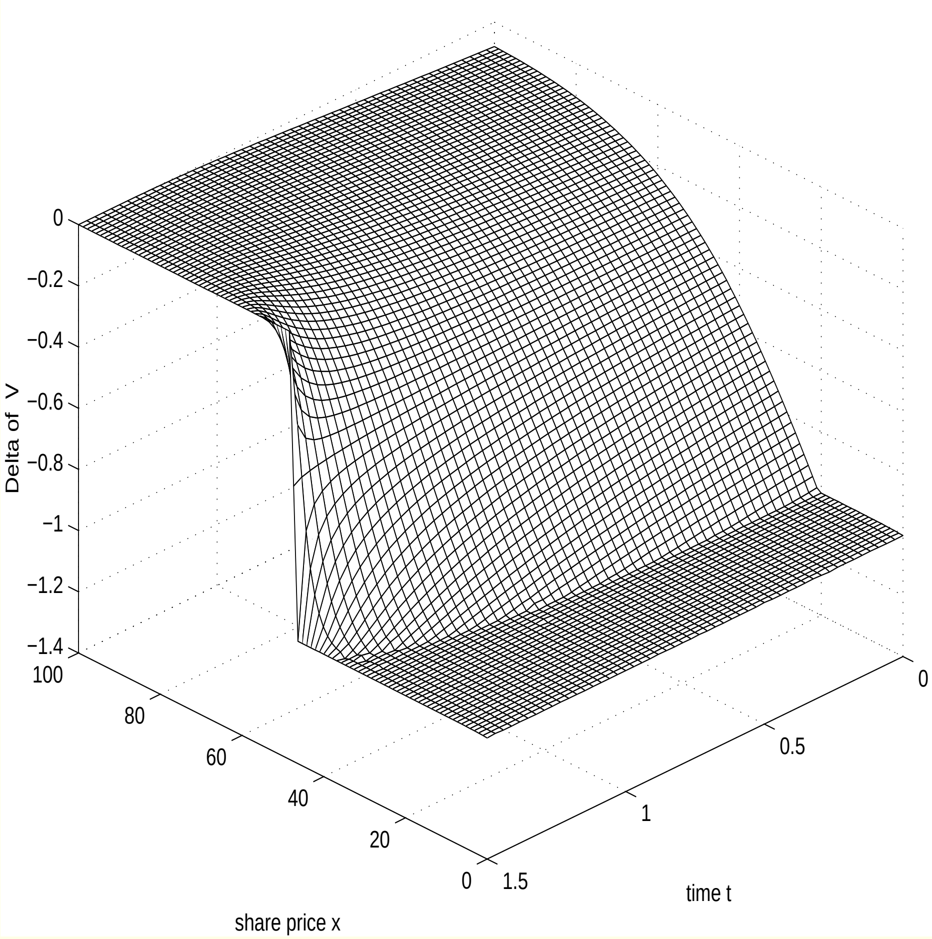
Δ Γ

Figure 3: Computed Δ and Γ of the option for $k = 2$ and $\lambda = 10$.

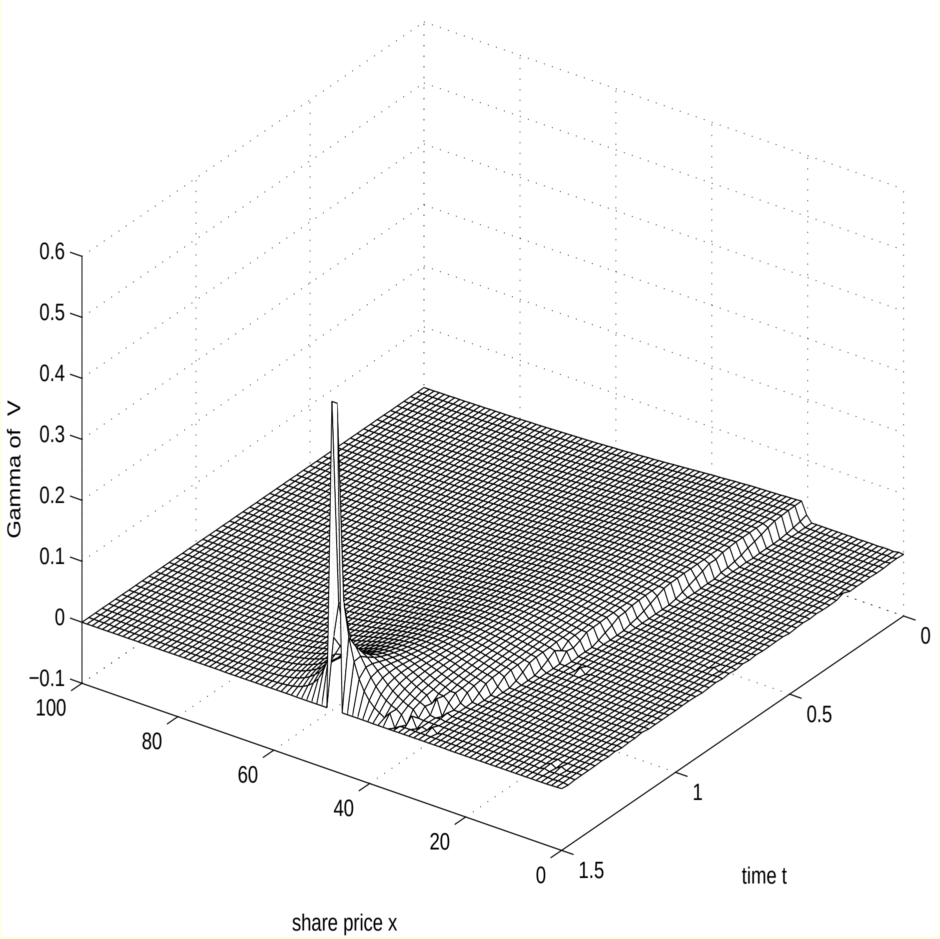


Δ Γ

Figure 4: Computed Δ and Γ of the option for $k = 6$ and $\lambda = 10$.



Δ



Γ

Figure 5: Computed Δ and Γ of the option for $k = 8$ and $\lambda = 4$.

HJB equation:

$$\min\{LV, V - V^*\} = 0,$$

where

$$\begin{aligned} LV &= -\frac{\partial V}{\partial t} - \frac{1}{2}\left[\sigma_1^2 x^2 \frac{\partial^2 V}{\partial x^2} + 2\rho\sigma_1\sigma_2 xy \frac{\partial^2 V}{\partial x \partial y} + \sigma_2^2 y^2 \frac{\partial^2 V}{\partial y^2}\right] \\ &\quad -r\left[x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y}\right] + rV, \end{aligned}$$

Options with proportional transaction costs

Question: how can we define the issuing price of a call or put option?

Consider the following problems.

Problem 1. *Utility maximization for an investor without an option*: Suppose that the investor trades only in the underlying stock and the bond. At time $t \in [0, T]$, the investor holds β dollars in the bond and α shares of the stock whose price is S. The objective of the investor is to maximize the expected utility of wealth at terminal time T.

Problem 2. *Utility maximization for an investor buying an option* Assume that the investor trades in the market for the stock and the bond, and in addition, purchases a cash-settling European call option written on the stock with strike price K and expiry date T. Then, the investor's time t expected utility of terminal wealth is to be maximized over the set of feasible strategies.

Problem 3 *Utility maximization for an investor writing an option*: If the investor trades in the market for the stock and the bond, and, in addition, sells a cash-settling European call option written on the stock with strike price K and expiry date T. Then, the investor wishes to maximize expected utility of terminal wealth over the set of feasible strategies.

Using these and the no arbitrage principle we can define the values of the put and call options.

The HJB equations:

$$\min\{\mathcal{L}_1 V, \mathcal{L}_2 V, \mathcal{L}_3 V\} = 0, \tag{38}$$

where

$$\mathcal{L}_1 = -\left(\frac{\partial}{\partial t} + r\beta\frac{\partial}{\partial \beta} + \mu S\frac{\partial}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2}\right), \tag{39}$$

$$\mathcal{L}_2 = -\frac{\partial}{\partial \alpha} + (1+\theta)S\frac{\partial}{\partial \beta}, \tag{40}$$

$$\mathcal{L}_3 = \frac{\partial}{\partial \alpha} - (1-\theta)S\frac{\partial}{\partial \beta}. \tag{41}$$

with different terminal and boundary conditions.

α – value of shares.

β – value of bonds.

Penalty approach:

$$\mathcal{L}_1 V_{\lambda_1,\lambda_2} + \lambda_1 [\mathcal{L}_2 V_{\lambda_1,\lambda_2}]^- + \lambda_2 [\mathcal{L}_3 V_{\lambda_1,\lambda_2}]^- = 0. \tag{42}$$

We have shown that there exists a unique viscosity solution to (42) and $V_{\lambda_1,\lambda_2} \to V$ as $\lambda_1, \lambda_2 \to \infty$.

We have also constructed a numerical method for solving (42).

A nonlinear obstacle problem

$$\begin{aligned}
\frac{\partial u}{\partial t} + T(u(x,t)) - f(x,t) &\leq 0 && (43)\\
u(x,t) - u^*(x,t) &\leq 0 && (44)\\
\left(\frac{\partial u}{\partial t} + T(u(x,t)) - f(x,t)\right) \cdot (u(x,t) - u^*(x,t)) &= 0 && (45)
\end{aligned}$$

for $(x,t) \in \Omega \times (0,T] =: Q$ with given initial and boundary conditions, where

$$T(u(x,t)) = -\nabla \cdot (A(x)\nabla u(x,t)) + G(u(x,t)), \qquad (46)$$

$A(x) = (a_{ij}(x))$ is an $n \times n$ matrix, and

$G(\cdot) : \mathbb{R} \mapsto \mathbb{R}$ is monotone.

Penalty approach

$$\frac{\partial u_\lambda}{\partial t} + T(u_\lambda(x,t)) + \lambda\left(\phi(u_\lambda(x,t)) + \varepsilon\right)^{1/k} = f(x,t) + \lambda\varepsilon^{1/k}, \quad (47)$$

for $(x,t) \in \Omega$ where

$$\phi(v(x,t)) = [v(x,t) - u^*(x,t)]_+ = \max\{v(x,t) - u^*(x,t), 0\}.$$

Convergence Result

Let u and u_λ be the solutions to the original and the penalized problems, respectively. If $\frac{\partial u}{\partial t} \in L^{k+1}(Q)$, then there exists a constant $C > 0$, independent of u, u_λ and λ, such that

$$||u - u_\lambda||_{L^\infty(0,T;L^2(\Omega))} + ||u - u_\lambda||_{L^2(0,T;H_0^1(\Omega))}$$

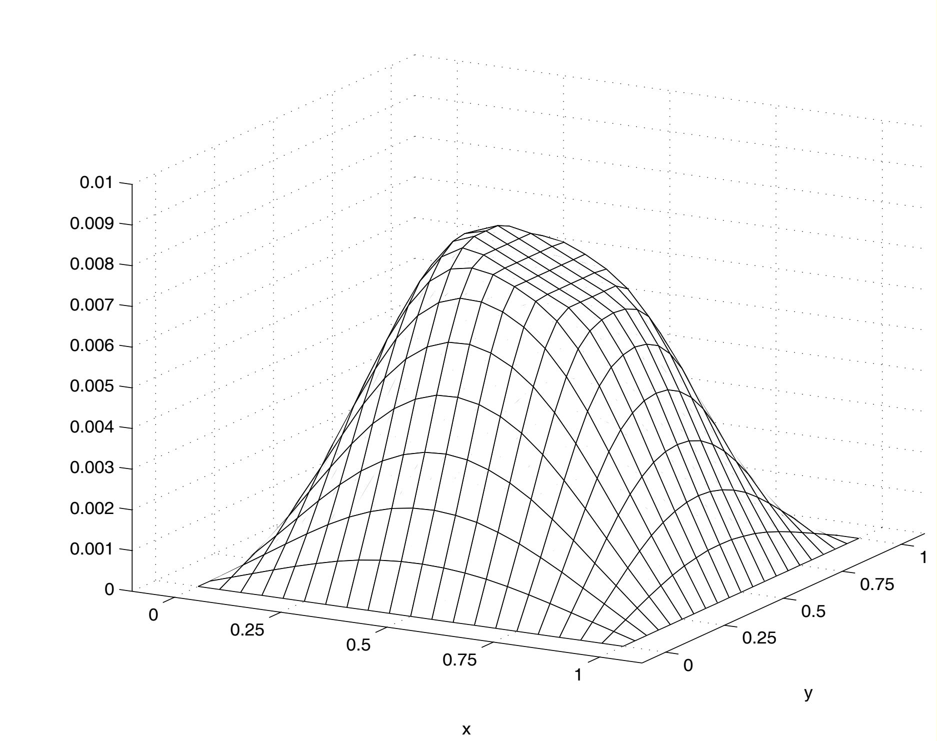
$$\leq C \left[\frac{1}{\lambda^k} + \varepsilon(\lambda\varepsilon^{1/k} + 1) \right]^{1/2},$$

If $\lambda\varepsilon^{1/k} \leq \mathcal{O}(1)$, then

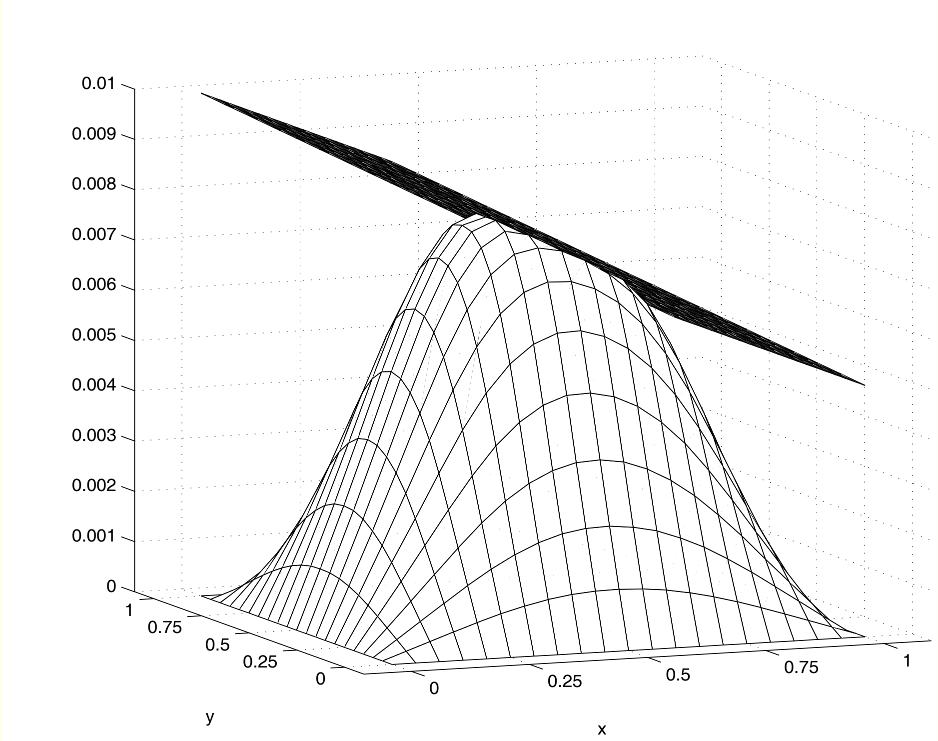
$$||u - u_\lambda||_{L^\infty(0,T;L^2(\Omega))} + ||u - u_\lambda||_{L^2(0,T;H_0^1(\Omega))} \leq \frac{C}{\lambda^{k/2}}.$$

Numerical Results

Example 1. $a_{11} = a_{22} = 1$, $a_{12} = a_{21} = 0$ and $G(u) = u^2$, $u^* = -0.05(x-1) + 0.05$. $\varepsilon = 10^{-8}$.



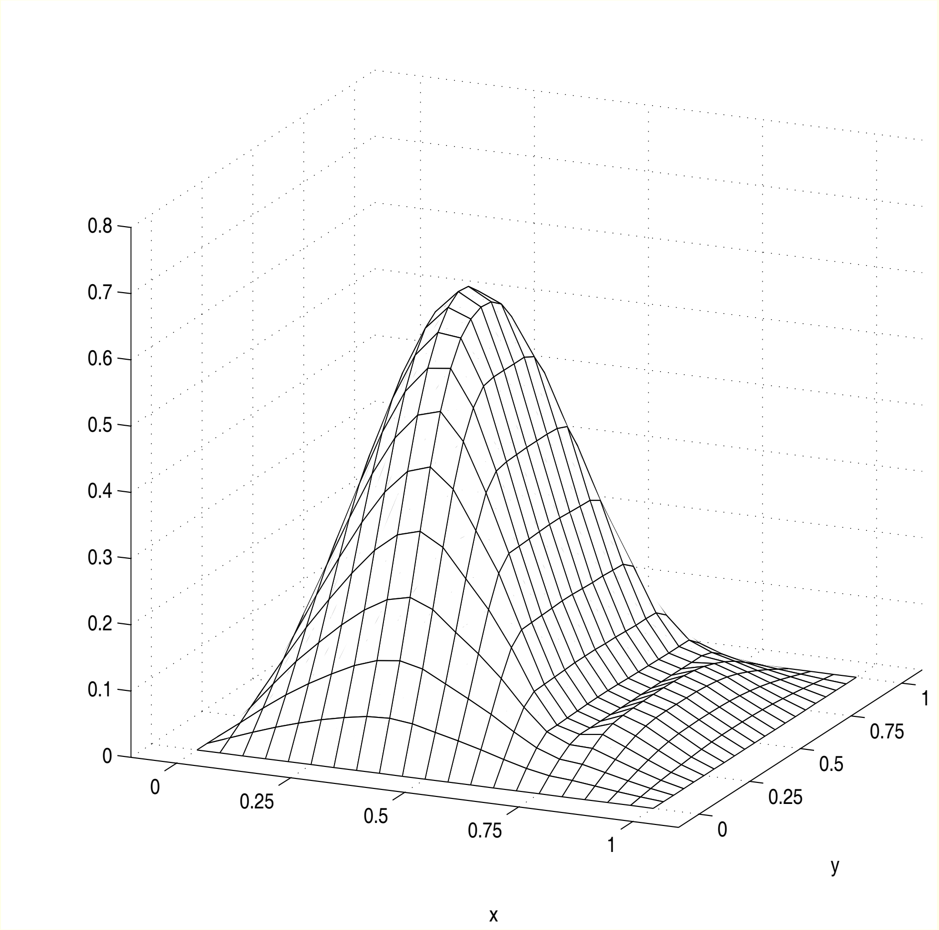
u_λ at $t = 1$.



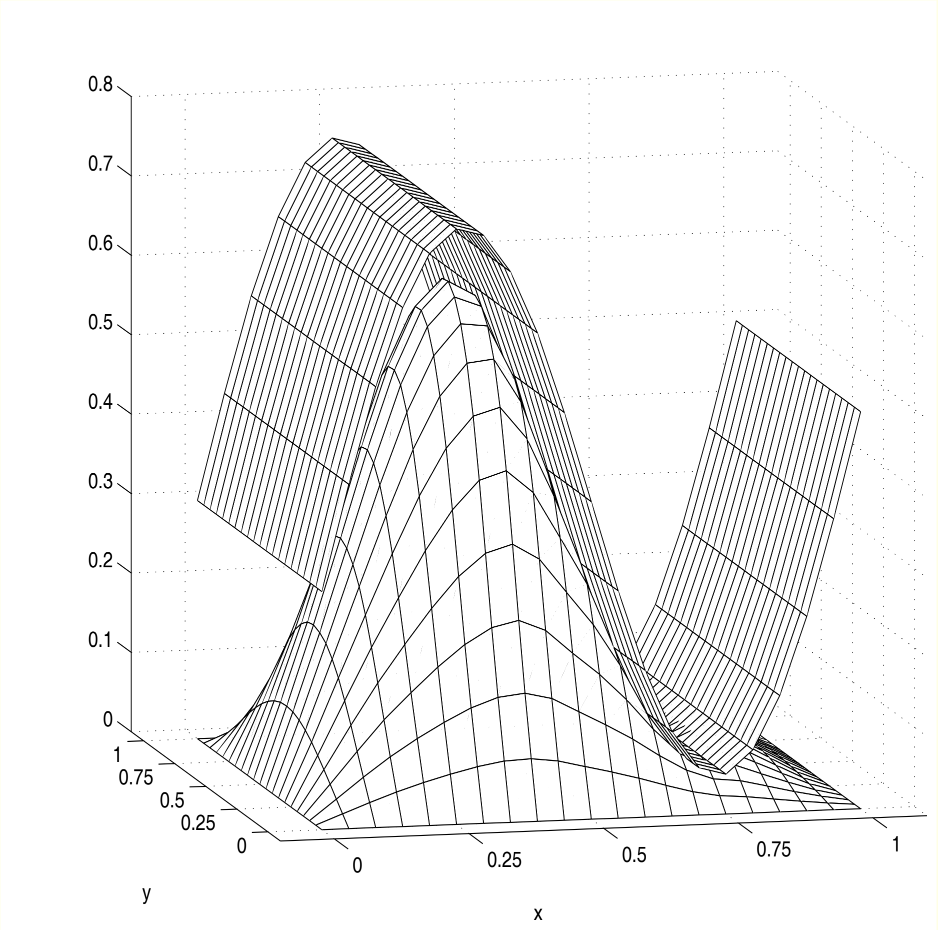
u_λ and u^* at $t = 1$.

Figure 6: Computed solution for Example 1.

Example 2. $a_{11} = 3 + \sin(2\pi y)$, $a_{22} = 3 + \sin(2\pi x)$, $a_{12} = a_{21} = 0$, $G(u) = u^2$, $u^* = 0.2(x + 2\sin(2\pi x)) + 0.3$.



u_λ at $t = 1$.



u_λ and u^* at $t = 1$.

Figure 7: Computed solution for Example 2.