ON A STRUCTURED CLASS OF BILEVEL PROGRAMS

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a basic pricing framework

$$
\begin{aligned}
\max_{t \in T, x, x^q} \quad & \sum_{a \in \mathcal{A}} t_a x_a & \\
\text{s.t.} \quad & t_a = 0 & a \in \mathcal{A}_1 \\
& \min_{x, x^q} \quad \sum_{a \in A} (c_a + t_a) x_a & \\
& x_a = \sum_q x_a^q & a \in \mathcal{A} \\
& x^k \in X^k & q \in \mathcal{Q}
\end{aligned}
$$

X^k : set of demand-feasible flows associated with OD pair q

properties

- combinatorial problem (MIP reformulation)
- strongly NP-hard
- network structure

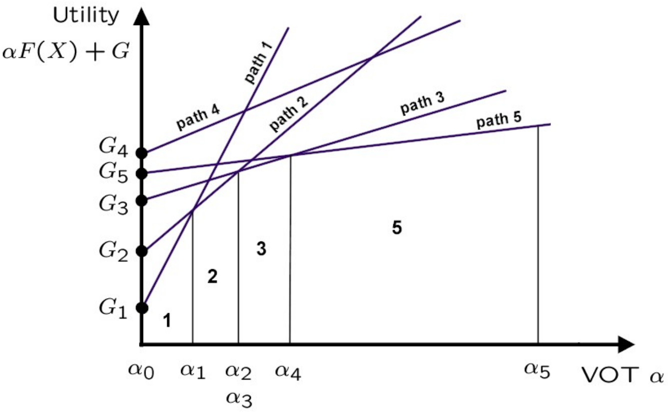
resolution

- branch-and-bound
- branch-and-cut
- penalties (exact or not) and local optimization
- metaheuristics (tabu search)
- inverse optimization
- trust region methods

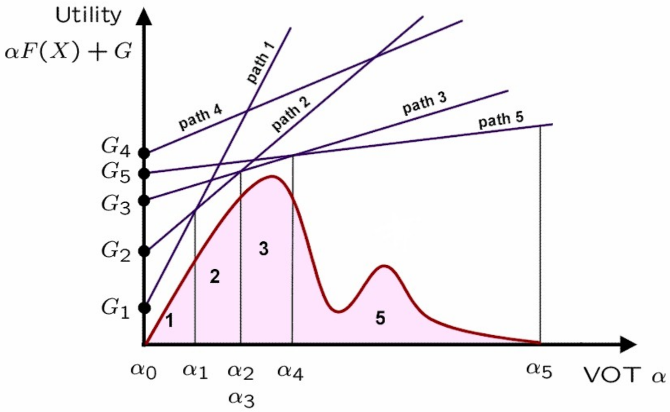
variants

- congestion (MPEC)
- pricing and design
- highway (product line design)
- pricing and capacity setting (revenue management)
- multiclass (demand segmentation)

continuous multiclass



continuous multiclass



a random utility (logit) variant

route choice model based on utility maximization

$$
\begin{aligned}
\max_{x,t} \quad & \sum_{q\in\mathcal{Q}} D_q \sum_{r\in\mathcal{R}^q} x_r^q t_r^q \\
\text{s.t.} \quad & x_r^q = \frac{\exp[-\theta(c_r^q + t_r^q)]\; t_r^q}{\sum\limits_{\ell\in\mathcal{R}^q} \exp[-\theta(c_\ell^q + t_\ell^q)]} && q \in \mathcal{Q}, \quad r \in \mathcal{R}^q \\
& t_r^q = \sum_{a\in r} t_a && q \in \mathcal{Q} \quad r \in \mathcal{R}^q \\
& t \in \mathbb{R}^{|\mathcal{A}^m|}
\end{aligned}
$$

$$
\begin{aligned}
\max_{x,t} \quad & F(t,x) = \sum_{q\in\mathcal{Q}} x^q \cdot \Omega^q\, t^q \\
\text{s.t.} \quad & t \in T \\
& \min_{x} \qquad f(t,x) = \sum_{q\in\mathcal{Q}} \left[(c^q + \Omega^q\, t^q) \cdot x^q + \frac{1}{\theta} x^q \cdot \log x^q \right] \\
& \qquad\qquad \sum_{r\in\mathcal{R}^q} x_r^q = D_q \qquad\qquad\qquad\qquad\qquad\qquad q \in \mathcal{Q} \\
& \qquad\qquad x^q \geq 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad q \in \mathcal{Q}
\end{aligned}
$$

second-order approximation of the lower level

$$
\begin{aligned}
\max_{x,t} \quad & F(t,x) = \sum_{q\in\mathcal{Q}} x^q \cdot \ \Omega^q \, t^q \\
\text{s.t.} \quad & t \in T \cap \hat{T} \\
& \min_{x} \quad \sum_{q\in\mathcal{Q}} (c^q + \Omega^q \, t^q)\cdot x^q + \frac{1}{\theta}\left[(x^q \cdot \mathrm{diag}(1/\hat{x}^q)\cdot x^q + (e + \log(\hat{x}^q) - \hat{x}^q)\cdot x^q\right] \\
& \qquad \sum_{r\in\mathcal{R}^q} x^q_r = D_q \qquad q \in \mathcal{Q} \\
& \qquad x^q \geq 0 \qquad q \in \mathcal{Q}
\end{aligned}
$$

lower level KKT conditions

$$
\begin{aligned}
\max_{x,t} \quad & F(t,x) = \sum_{q \in \mathcal{Q}} x^q \cdot \ \Omega^q \ t^q & \\
\text{s.t.} \quad & t \in T \cap \hat{T} & \\
& \frac{1}{\theta}\mathrm{diag}(1/\hat{x}^q)\, x^q + c^q + \Omega^q t^q + \frac{1}{\theta}\log(\hat{x}^q) - \pi_q \, e \geq 0 & q \in \mathcal{Q} \\
& \frac{1}{\theta}\mathrm{diag}(1/\hat{x}^q)\, x^q + c^q + \Omega^q t^q + \frac{1}{\theta}\log(\hat{x}^q) - \pi_q \, e \perp x^q & q \in \mathcal{Q} \\
& \sum_{r \in \mathcal{R}^q} x_r^q = D_q & q \in \mathcal{Q} \\
& x^q \geq 0 & q \in \mathcal{Q}
\end{aligned}
$$

concave approximation

$$
\begin{aligned}
\max_{x,t} \quad & \sum_{q \in \mathcal{Q}} -c^q \cdot\, x^q - \frac{1}{\theta} x^q \cdot \mathrm{diag}(1/\hat{x}^q)\, x^q - \frac{1}{\theta} x^q \cdot \log(\hat{x}^q) + D_q\, \pi_q \\
\text{s.t.} \quad & t \in T \cap \hat{T} \\
& \frac{1}{\theta}\, \mathrm{diag}(1/\hat{x}^q)\, x^q + c^q + \Omega^q t^q + \frac{1}{\theta} \log(\hat{x}^q) + \pi_q\, e \geq 0 && q \in \mathcal{Q} \\
& \frac{1}{\theta} \mathrm{diag}(1/\hat{x}^q)\, x^q + c^q + \Omega^q t^q + \frac{1}{\theta} \log(\hat{x}^q) - \pi_q\, e \perp x^q && q \in \mathcal{Q} \\
& \sum_{r \in \mathcal{R}^q} x_r^q = D_q && q \in \mathcal{Q} \\
& x^q \geq 0 && q \in \mathcal{Q}
\end{aligned}
$$

flows always positive

$$
\begin{aligned}
\max_{x,t} \quad & \sum_{q\in\mathcal{Q}} -c^q \cdot\; x^q - \frac{1}{\theta} x^q \cdot \mathrm{diag}(1/\hat{x}^q)\, x^q - \frac{1}{\theta} x^q \cdot \log(\hat{x}^q) + D_q\, \pi_q \\
\text{s.t.} \quad & t \in T \cap \hat{T} \\
& \frac{1}{\theta}\, \mathrm{diag}(1/\hat{x}^q)\, x^q + c^q + \Omega^q t^q + \frac{1}{\theta} \log(\hat{x}^q) + \pi_q\, e = 0 & q \in \mathcal{Q} \\
& \frac{1}{\theta} \mathrm{diag}(1/\hat{x}^q)\, x^q + c^q + \Omega^q t^q + \frac{1}{\theta} \log(\hat{x}^q) - \pi_q\, e \perp x^q & q \in \mathcal{Q} \\
& \sum_{r\in\mathcal{R}^q} x_r^q = D_q & q \in \mathcal{Q} \\
& x^q \geq 0 & q \in \mathcal{Q}
\end{aligned}
$$

numerical results : objective

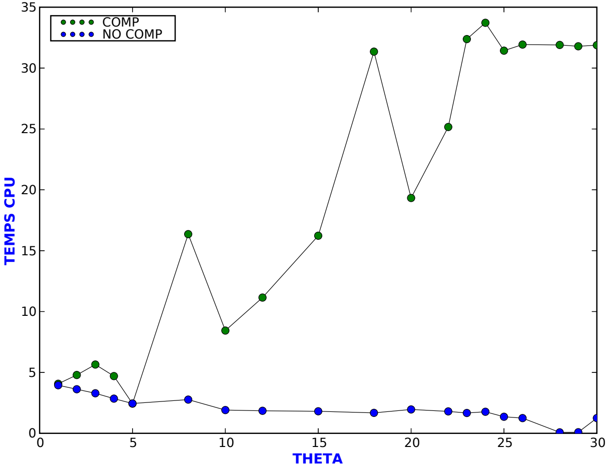


FIGURE: Instance 19, $|\mathcal{Q}| = 5$, $\mathcal{R} = 206$, $|\mathcal{A}^m| = 25$

numerical results : CPU time

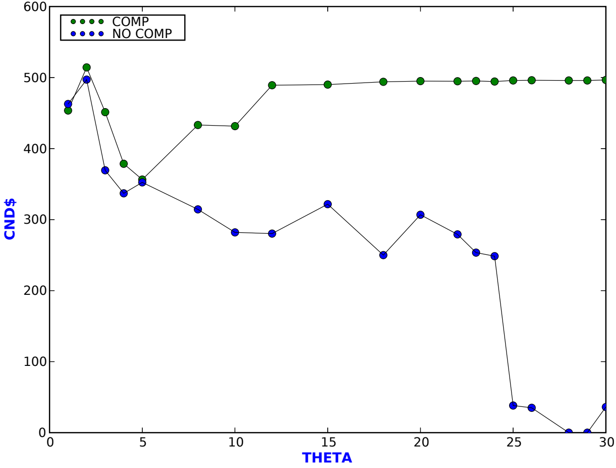


FIGURE: Instance 19, $|\mathcal{Q}| = 5$, $\mathcal{R} = 206$, $|\mathcal{A}^m| = 25$

lower level piecewise linear approximation

$$
\begin{aligned}
\max_{x,t} \quad & \sum_{q\in\mathcal{Q}} x^q \cdot \ \Omega^q\ t^q \\
\text{s.t.} \quad & \min_{x} \sum_{q\in\mathcal{Q}} \left[(c^q + \Omega^q\ t^q)x + \frac{1}{\theta} e \cdot w^q \right] \\
& \text{s.t. } w^q \geq (\log\alpha + 1)x^q + -\alpha\ e \qquad & q \in \mathcal{Q} \\
& \qquad \sum_{r\in\mathcal{R}^q} x_r^q = D_q \qquad & q \in \mathcal{Q} \\
& \qquad x^q \geq 0 \qquad & q \in \mathcal{Q} \\
& \qquad t \in T
\end{aligned}
$$

lower level KKT

$$
\begin{array}{lll}
\max\limits_{x,t,\varphi,\pi} & \sum\limits_{q\in\mathcal{Q}} \left[-c^q \cdot x^q - \frac{1}{\theta} e \cdot w^q + \frac{1}{\theta} \sum\limits_{n\in N} \sum\limits_{r\in\mathcal{R}^q} \varphi_{rqn}\, b_n \right] + d \cdot \pi & \\
\text{s.t.} & \sum\limits_{n\in N} \varphi_{rqn} = 1 & q \in \mathcal{Q}, \quad r \in \mathcal{R}^q \\
& w^q \geq a_n x^q + b_n\, e & q \in \mathcal{Q}, \quad n \in N \\
& w^q - a_n x^q - b_n\, e \perp \varphi_{\cdot qn} & q \in \mathcal{Q}, \quad n \in N \\
& -\frac{1}{\theta} \sum\limits_{n} a_n\, \varphi_{\cdot qn} + \pi_q e - c^q - \Omega^q\, t^q \geq 0 & q \in \mathcal{Q}, \\
& -\frac{1}{\theta} \sum\limits_{n} a_n\, \varphi_{\cdot qn} + \pi_q e - c^q - \Omega^q\, t^q \perp x^q & q \in \mathcal{Q}, \\
& \sum\limits_{r\in\mathcal{R}^q} x_r^q = D_q & q \in \mathcal{Q} \\
& x^q \geq 0 & q \in \mathcal{Q} \\
& \varphi \geq 0 & \\
& t \in T &
\end{array}
$$

flows always positive

$$
\begin{aligned}
\max_{x,t,\varphi,\pi} \quad & \sum_{q\in\mathcal{Q}} \left[-c^q \cdot x^q - \frac{1}{\theta} e \cdot w^q + \frac{1}{\theta} \sum_{n\in N} \sum_{r\in\mathcal{R}^q} \varphi_{rqn}\, b_n \right] + d \cdot \pi & \\
\text{s.t.} \quad & \sum_{n\in N} \varphi_{rqn} = 1 & q \in \mathcal{Q}, \quad r \in \mathcal{R}^q \\
& w^q \geq a_n x^q + b_n\, e & q \in \mathcal{Q}, \quad n \in N \\
& w^q - a_n x^q - b_n\, e \perp \varphi_{\cdot qn} & q \in \mathcal{Q}, \quad n \in N \\
& -\frac{1}{\theta} \sum_{n} a_n\, \varphi_{\cdot qn} + \pi_q e - c^q - \Omega^q\, t^q = 0 & q \in \mathcal{Q}, \\
& -\frac{1}{\theta} \sum_{n} a_n\, \varphi_{\cdot qn} + \pi_q e - c^q - \Omega^q\, t^q \perp x^q & q \in \mathcal{Q}, \\
& \sum_{r\in\mathcal{R}^q} x^q_r = D_q & q \in \mathcal{Q} \\
& x^q \geq 0 & q \in \mathcal{Q} \\
& \varphi \geq 0 & \\
& t \in T &
\end{aligned}
$$

0 tangent plane

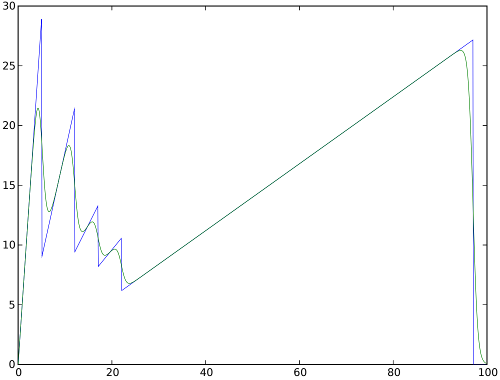


FIGURE: Instance 10, $|\mathcal{Q}| = 5$, $\mathcal{R} = 10$, $|\mathcal{A}^m| = 1$

5 tangent planes (50 constraints)

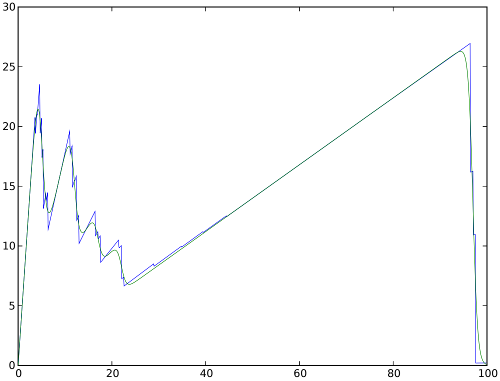


FIGURE: Instance 10, $|\mathcal{Q}| = 5$, $\mathcal{R} = 10$, $|\mathcal{A}^m| = 1$

10 tangent planes (100 constraints)

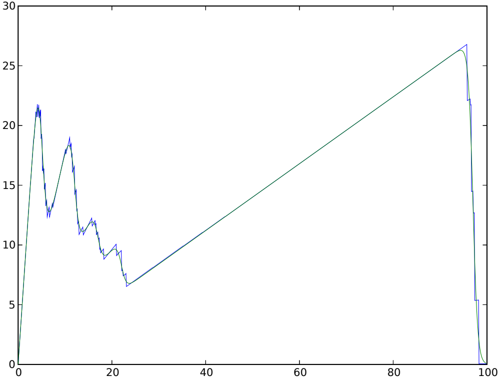


FIGURE: Instance 10, $|\mathcal{Q}| = 5$, $\mathcal{R} = 10$, $|\mathcal{A}^m| = 1$

20 tangent planes (200 constraints)

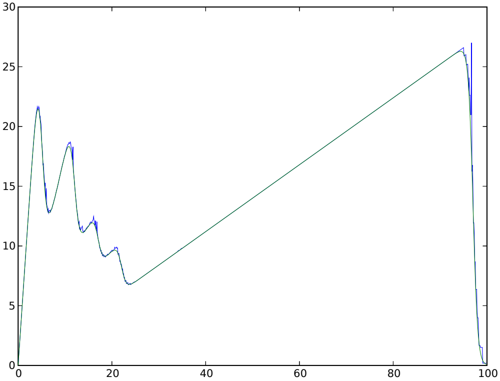


FIGURE: Instance 10, $|\mathcal{Q}| = 5$, $\mathcal{R} = 10$, $|\mathcal{A}^m| = 1$

numerical results : objective

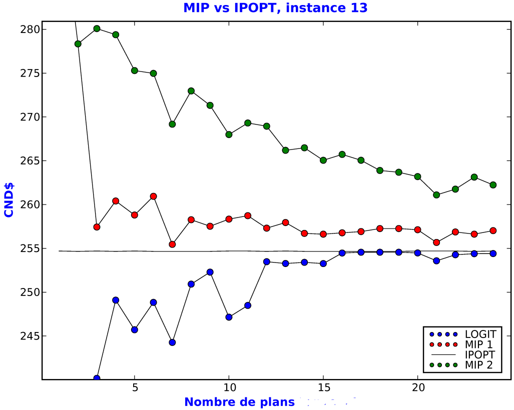


FIGURE: Ipopt launched from 100 randomized points achieved 209.30 ($\theta = 5$)

varying the scale parameter

Algorithm	CPU	F^*	F^*_{logit}	+LOC	+TR	+TR+LOC
LOC	13.65		430.18			
TR	19.70	478.56	487.81	498.00		
DET	0.00	615.00	210.23	466.33	236.94	466.33
PL3	29.37	595.98	303.13	497.33	495.88	497.33
PL5	990.97	560.71	486.88	497.12	496.61	497.12

TABLE: Instance 19, $|\mathcal{Q}| = 5$, $|\mathcal{A}^m| = 25$,$|\mathcal{R}| = 91$, $\theta = 0.5$

Algorithm	CPU	F^*	F^*_{logit}	+LOC	+TR	+TR+LOC
LOC	0.77		395.61			
TR	5.71	503.48	503.50	511.94		
DET	0.00	615.00	207.51	0.00	478.29	0.00
PL3	0.11	626.98	600.40	633.69	633.69	633.69
PL5	0.31	641.99	607.25	633.69	633.69	633.89

TABLE: Instance 19, $|\mathcal{Q}| = 5$, $|\mathcal{A}^m| = 25$,$|\mathcal{R}| = 91$, $\theta = 10.0$

varying the set of candidate routes

Algorithm	CPU	F^*	F^*_{logit}	+LOC	+TR	+TR+LOC
LOC	0.20		167.18			
TR	3.26	200.51	202.13	202.18		
DET	0.00	352.00	91.40	181.62	142.53	143.46
PL3	0.15	287.19	98.28	202.19	155.25	168.51
PL5	2.43	242.60	187.77	202.18	202.14	202.18

TABLE: Instance 13, $|\mathcal{Q}| = 4$, $|\mathcal{A}^m| = 7, |\mathcal{R}| = 29$, $\theta = 1.0$

Algorithm	CPU	F^*	F^*_{logit}	+LOC	+TR	+TR+LOC
LOC	0.19		167.183			
TR	4.88	187.63	189.76	199.41		
DET	0.00	384.00	68.80	152.97	152.83	152.97
PL3	45.79	276.19	102.53	199.40	151.03	152.97
PL5	48.24	237.36	184.71	199.40	199.26	199.40

TABLE: Instance 13, $|\mathcal{Q}| = 4$, $|\mathcal{A}^m| = 7, |\mathcal{R}| = 263$, $\theta = 1.0$

finite capacities

$$
\begin{aligned}
\max_{x,t} \quad & \sum_{q\in\mathcal{Q}} D_q \sum_{r\in\mathcal{R}^q} x_r^q \; t_r^q \\
\text{s.à} \quad & x_r^q = \frac{\exp[-\theta(c_r^q + t_r^q)]\; t_r^q}{\sum_{\ell\in\mathcal{R}^q} \exp[-\theta(c_\ell^q + t_\ell^q)]} && q\in\mathcal{Q}, \quad r\in\mathcal{R}^q \\
& t_r^q = \sum_{a\in r} t_a && q\in\mathcal{Q} \quad r\in\mathcal{R}^q \\
& \sum_{q\in\mathcal{Q}} \sum_{r\in\mathcal{R}^q} \delta_{ar}^q x_r^q \le U_a && a\in\mathcal{A}^m \\
& t\in\mathbb{R}^{|\mathcal{A}|}
\end{aligned}
$$

piecewise linear approximation

$$
\begin{aligned}
\max_{x,t} \quad & F(t,x) = \sum_{q\in\mathcal{Q}} x^q \cdot \; \Omega^q \, t^q \\
\text{s.t.} \quad & t \in T \\
& \min_{x} \quad f(t,x) = \sum_{q\in\mathcal{Q}} \left[(c^q + \Omega^q \, t^q) \cdot x^q + \frac{1}{\theta} x^q \cdot \log x^q \right] \\
& \qquad \sum_{q\in\mathcal{Q}} \Lambda^q x^q \le U \\
& \qquad \sum_{r\in R^q} x_r^q = D_q \qquad\qquad q \in \mathcal{Q} \\
& \qquad x^q \ge 0 \qquad\qquad q \in \mathcal{Q}
\end{aligned}
$$

where Λ^q is the incidence matrix for market q.

substituting KKT

$$
\begin{array}{lll}
\max\limits_{x,t,\gamma,\varphi,\pi} & \sum\limits_{q\in\mathcal{Q}}\left[-c^q\cdot x^q - \frac{1}{\theta}e\cdot w^q + \frac{1}{\theta}\sum\limits_{n\in N}\sum\limits_{r\in\mathcal{R}^q}\varphi_{rqn}\, b_n\right] + D\cdot\pi + U\cdot\gamma & \\
\text{s.à} & \sum\limits_{n\in N}\varphi_{rqn} = 1 & q\in\mathcal{Q},\, r\in\mathcal{R}^q \\
 & w^q \geq a_n x^q + b_n\, e & q\in\mathcal{Q},\, n\in N \\
 & w^q - a_n x^q - b_n\, e \perp \varphi_{\cdot qn} & q\in\mathcal{Q},\, n\in N \\
 & -\frac{1}{\theta}\sum\limits_{n} a_n\, \varphi_{\cdot qn} + \pi_q e - c^q - \Omega^q\, t^q - {\Lambda^q}^T\gamma = 0 & q\in\mathcal{Q}, \\
 & \sum\limits_{q\in\mathcal{Q}}\Lambda^q x^q \leq U & \\
 & \sum\limits_{q\in\mathcal{Q}}\Lambda^q x^q - U \perp \gamma & \\
 & \sum\limits_{r\in\mathcal{R}^q} x_r^q = D_q & q\in\mathcal{Q} \\
 & t\in T, x^q\geq 0, \gamma\geq 0, \varphi\geq 0 & q\in\mathcal{Q}
\end{array}
$$

numerical results (1)

Algorithm	CPU	F^*	F^*_{grav}	F^*_{iter}
TR	73.51	155.10	153.04	153.05
DET	0.01	179.00	102.28	102.27
DET+TR	76.96	143.40	138.20	138.19
PL3	0.90	189.13	106.30	106.27
PL3+TR	76.75	146.58	136.18	136.15
PL5	2.83	170.19	136.59	136.56
PL5+TR	46.28	148.89	145.48	145.45
PL10	334.65	164.27	148.72	148.71
PL10+TR	365.19	156.38	153.07	153.07

TABLE: Instance 13, $|\mathcal{Q}| = 4$, $|\mathcal{A}^m| = 7$,$|\mathcal{R}| = 29$, $\theta = 1.0$

numerical results (2)

Algorithm	CPU	F^*	F^*_{grav}	F^*_{iter}
TR	131.58	169.67	168.41	168.40
DET	0.01	179.00	98.39	98.36
DET+TR	28.60	165.70	165.45	165.44
PL3	0.13	176.21	150.37	150.34
PL3+RC	38.36	173.54	170.61	170.59
PL5	0.22	175.70	150.40	150.35
PL5+TR	57.79	172.21	172.21	172.20
PL10	0.53	171.66	166.68	166.65
PL10+TR	20.15	172.76	171.22	171.21

TABLE: Instance 13, $|\mathcal{Q}| = 4$, $|\mathcal{A}^m| = 7$,$|\mathcal{R}| = 29$, $\theta = 5.0$

numerical results (3)

Algorithm	CPU	F^*	F^*_{grav}	F^*_{iter}
TR	123.11	290.36	286.09	295.21
DET	0.02	337.00	239.64	238.92
DET+TR	70.44	327.13	231.24	230.72
PL3	0.74	346.32	237.65	231.00
PL3+TR	91.28	327.93	306.38	304.32
PL5	11.51	336.34	306.71	304.43
PL5+TR	24.59	329.95	299.85	297.70
PL10	27.7	333.50	321.92	324.85
PL10+TR	39.63	328.01	324.28	320.66

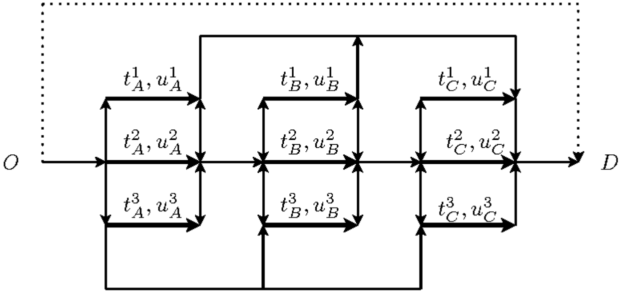
TABLE: Instance 19, $|\mathcal{Q}| = 5$, $|\mathcal{A}^m| = 25$,$|\mathcal{R}| = 93$, $\theta = 2.0$

discretization

$$
\begin{array}{lll}
\max\limits_{x,t,\gamma,\varphi,\pi,z,y} & \sum\limits_{q\in\mathcal{Q}} \left(-c^q \cdot x^q - \frac{1}{\theta} e \cdot w^q + \frac{1}{\theta} e \cdot \varphi^q \, b \right) + d \cdot \pi + e \cdot y \; \alpha & \\
\text{s.t.} & y_{am} = \gamma_a \, z_{am} & a \in \mathcal{A}^m, \quad m \in M \\
 & \sum\limits_{a\in\mathcal{A}^m} \sum\limits_{m\in M} z_{am}\alpha_m \leq U & \\
 & \sum\limits_{n\in N} \varphi^q_{rn} = 1 & q \in \mathcal{Q}, \quad r \in \mathcal{R}^q \\
 & \sum\limits_{m\in M} z_{am} = 1 & a \in \mathcal{A}^m \\
 & w^q \geq a_n x^q + b_n \, e & q \in \mathcal{Q}, \quad n \in N \\
 & w^q - a_n x^q - b_n \, e \perp \varphi^q_{\cdot n} & q \in \mathcal{Q}, \quad n \in N \\
 & -\frac{1}{\theta} \sum\limits_{n} a_n \, \varphi^q_{\cdot n} + \pi_q \, e - c^q - \Omega^q \, t^q - {\Lambda^q}^T \gamma \geq 0 & q \in \mathcal{Q} \\
 & -\frac{1}{\theta} \sum\limits_{n} a_n \, \varphi^q_{n} + \pi_q \, e - c^q - \Omega^q \, t^q - {\Lambda^q}^T \gamma \perp x^q & q \in \mathcal{Q} \\
 & \sum\limits_{q\in\mathcal{Q}} \Lambda^q x^q \leq \sum\limits_{m\in M} \alpha_m \, z_{\cdot m} & \\
 & \sum\limits_{q\in\mathcal{Q}} \Lambda^q x^q - \sum\limits_{m\in M} \alpha_m \, z_{\cdot m} \perp \gamma & \\
 & \sum\limits_{r\in\mathcal{R}^q} x^q_r = D_q & q \in \mathcal{Q} \\
 & t \in T, \gamma \geq 0, \varphi \geq 0, z \in \{0,1\}^{|\mathcal{A}| \times |M|} & q \in \mathcal{Q}
\end{array}
$$

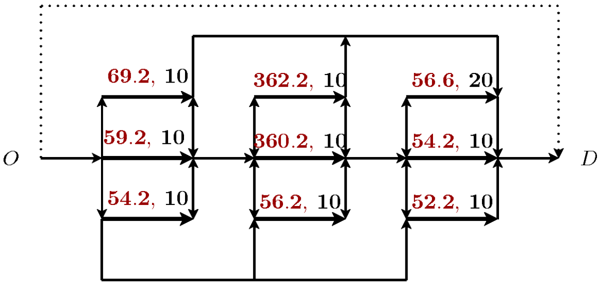
a product line design example

- 100 products
- 3 types (luxury levels)
- more than 150 possible choices



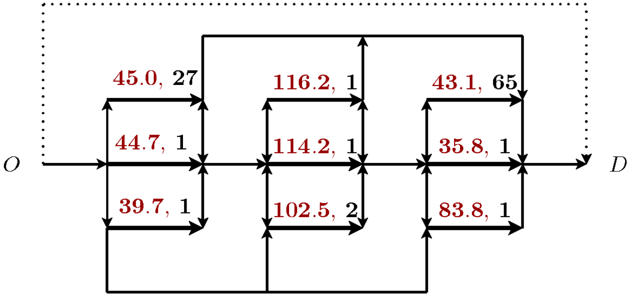
numerical results (1)

θ	N	CPU	F^*	F^*_{grav}	F^*_{iter}
0.1	5	5998	524	273	273

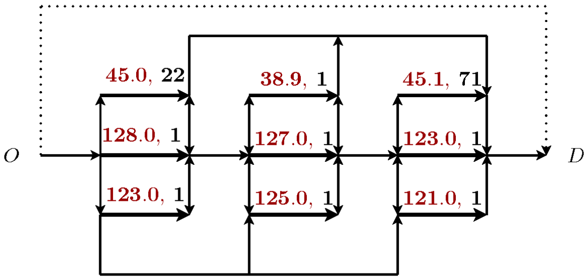


numerical results (2)

θ	N	CPU	F^*	F^*_{grav}	F^*_{iter}
0.25	5	96	436	345	343



θ	N	CPU	F^*	F^*_{grav}	F^*_{iter}
0.5	5	0.36	418	354	353



θ	N	CPU	F^*	F^*_{grav}	F^*_{iter}
1.0	5	0.20	429	423	415