Delineating nice classes of nonsmooth functions

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1 Why being afraid of nonsmooth analysis?

- abundance of concepts
- uncertainty of terminology
- lack of coherence in notation: $T^{C}(E,x)$ / $f^{0}(x,v)$
- feeling of unsecurity

2 Why being seduced by nonsmooth analysis?

- any sort of function or set can be treated
- new operations such as taking infima or suprema are no more out of reach
- the passages from functions to sets and to multimaps bring a unification of mathematics

3 Various forms of continuity and differentiability

Let $g : X_0 \to L(X,Y)$, where X, Y are n.v.s. and $X_0 \subset X$ is open, $x_0 \in X_0$

g is continuous at x_0 if $\|g(x) - g(x_0)\| \to 0$ as $x \to x_0$

g is pointwise continuous at x_0 if $x \to g(x).u$ is continuous at x_0, $\forall\ u \in X$

g is jointly continuous if $(x, u) \mapsto g(x).u$ is continuous.

g is directionally continuous if $\forall v \in S_X$, $g(x + tw) \to g(x)$ as $(w, t) \to (v, 0_+)$.

$\Longrightarrow$ Various forms of continuous differentiability when $g = f'$

4 Subdifferentials

$\mathcal{F}(X) \subset \mathcal{S}(X)$, the set of lsc functions $f : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$.

A *subdifferential* is here a correspondence $\partial : \mathcal{F}(X) \times X \rightrightarrows X^*$ satisfying:

- $0 \in \partial f(\overline{x})$ when $\overline{x}$ is a minimizer of $f \in \mathcal{F}(X)$.

Other conditions:

- (Exact mean value theorem) ∂ is *Lipschitz-valuable* on X if $\forall f \in \mathcal{L}(X)$, $\overline{x}, \overline{y} \in X$ $\exists w \in [\overline{x}, \overline{y}]$, $w^* \in \partial f(w)$ s.t. $f(\overline{y}) - f(\overline{x}) = \langle w^*, \overline{y} - \overline{x} \rangle$.

- (Fuzzy mean value theorem) ∂ is *valuable* on X if $\forall f \in \mathcal{F}(X)$, $\overline{x} \in \operatorname{dom} f$, $\overline{y} \in X \backslash \{\overline{x}\}$, $r \in \mathbb{R}$ s.t. $f(\overline{y}) \geq r$, there exist $u \in [\overline{x}, \overline{y})$ and sequences (u_n^*), $(u_n) \to u$ s.t. $u_n^* \in \partial f(u_n)$, $(f(u_n)) \to f(u)$, $\liminf_n \langle u_n^*, \overline{y} - \overline{x} \rangle \geq r - f(\overline{x})$

4.1 Some subdifferentials

- The *firm* (or *Fréchet*) subdifferential:

$$x^* \in \partial^F f(x) \;\Leftrightarrow\; f(x+u) \geq f(x) + \langle x^*, u\rangle - o(\|u\|).$$

- The *p-proximal subdifferential*: $o(\|u\|) = c\,\|u\|^p$ for some $c > 0$.

- The *limiting subdifferential* associated to a subdifferential ∂ :

$$\partial^L f(x) := w^* - \limsup_{(u,f(u))\to(x,f(x))} \partial f(u),$$

  where the w^*-limsup is the set of cluster points of bounded sequences (u_i^*) with $u_i^* \in \partial f(u_i)$, $(u_i) \to x$, $(f(u_i)) \to f(x)$.

- *Subdifferentials associated with directional derivatives* via

$$\partial^{(\cdot)} f(x) := \{x^* \in X^* : x^*(u) \leq f^{(\cdot)}(x,u) \forall u \in X\}.$$

4.2 Directional derivatives

- The (lower) *directional derivative* (or lower Hadamard derivative) of f

$$f^D(x,v) := \liminf_{(t,w)\to(0_+,v)} \frac{f(x+tw)-f(x)}{t}.$$

- The *Clarke–Rockafellar* derivative

$$f^C(x,v) := \inf_{r>0} \limsup_{\substack{(t,y)\to(0_+,x)\\ f(y)\to f(x)}} \inf_{w\in B(v,r)} \frac{f(y+tw)-f(y)}{t}.$$

- The *moderate* derivative of Michel-Penot

4.3 Tangent cones

- The directional or contingent *tangent cone* $T^D(E,e)$ to $E \subset X$ at $e \in \mathrm{cl}(E)$ is the set of $v \in X$ s.t. there exist $(t_n) \to 0_+$, $(e_n) \xrightarrow{E} e$ with $((e_n - e)/t_n) \to v$.
- The *firm tangent cone* $T^F(E,e)$ to $E \subset X$ at $e \in \mathrm{cl}(E)$ is the set of $v^{**} \in X^{**}$ s.t. there exist $(t_n) \to 0_+$, $(e_n) \xrightarrow{E} e$ with $((e_n - e)/t_n) \to v^{**}$ in (X^{**}, σ^{**}).
- The *Clarke tangent cone* $T^C(E,e)$ to E at e is the set of $v \in X$ s.t. for any sequence $(e_n) \xrightarrow{E} e$ there exist $(t_n) \to 0_+$, (y_n) in E s.t. $((y_n - e_n)/t_n) \to v$.

Proposition 1

$$\liminf_{e(\in E)\to\overline{e}} T^D(E,e) \subset T^C(E,\overline{e}) \subset \liminf_{e(\in E)\to\overline{e}} T^F(E,e).$$

4.4 Normal cones

To a subdifferential ∂ is associated a notion of *normal cone* to E at $e \in E$:

$$N(E, e) := \mathbb{R}_+ \partial \iota_E(e),$$

where ι_E is the *indicator function* of E ($\iota_E(x) = 0$ if $x \in E$, $+\infty$ otherwise).

Conversely, a normal cone notion N yields a notion of subdifferential ∂ :

$$\partial f(x) := \{x^* \in X^* : (x^*, -1) \in N(E_f, x_f)\}$$

where $E_f := \{(x, r) \in X \times \mathbb{R} : r \geq f(x)\}$ and $x_f := (x, f(x))$.

Examples:

- The *normal cone* $N^D(E, x)$ to E at $x \in \mathrm{cl}(E)$ is the polar cone to $T^D(E, x)$.
- The *Clarke normal cone* $N^C(E, x)$ to E at x is the polar to $T^C(E, x)$.
- The *firm normal cone* (or Fréchet normal cone) to E at x is given by

$$x^* \in N^F(E, x) \Leftrightarrow \langle x^*, u - x \rangle \leq o(\|u - x\|) \ u \in E.$$

Proposition 2 $N^F(E,x) = (T^F(E,x))^0$.

Proposition 3 $N^{LF}(E,x) \subset N^C(E,x)$.

Theorem 4 *Let E be a closed subset of an Asplund space X and let $x \in E$. Then,*

$$N^C(E,x) = \overline{\mathrm{co}}^*(N^{LF}(E,x)) := \overline{\mathrm{co}}^*(\limsup_{e \xrightarrow{E} x} N^F(E,e)).$$

If X is a reflexive Banach space one has

$$T^C(E,x) = \liminf_{e \xrightarrow{E} x} T^F(E,e) = \liminf_{e \xrightarrow{E} x} \overline{\mathrm{co}}(T^F(E,e)).$$

5 Sleekness and regularity

$E \subset X$ is *Clarke regular at* $\overline{x}$ if $T^D(E,\overline{x}) = T^C(E,\overline{x})$.

$E \subset X$ is *sleek at* $\overline{x}$ if $T^D(E,\overline{x}) = \liminf_{x(\in E)\to\overline{x}} T^D(E,x)$.

Characterizations:

$E \subset X$ is sleek at $\overline{x} \iff$ for all $\overline{v} \in T^D(E,\overline{x})$ there exists $v : E \to X$ continuous at $\overline{x}$ such that $v(\overline{x}) = \overline{v}$ and $v(x) \in T^D(E,x)$ for all $x \in E$.

E is sleek at $\overline{x} \iff T^D(E,\cdot)$ is lower semicontinuous at $\overline{x}$ on E.

Proposition 5 *(a) $E \subset X$ is Clarke regular at $\overline{x} \iff N^D(E,\overline{x}) = N^C(E,\overline{x})$.*
(b) E is sleek at $\overline{x} \implies E$ is regular at $\overline{x}$

Proof: (a) take polar cones.
(b) $T^C(E,\overline{x}) \subset T^D(E,\overline{x}) \subset \liminf_{x(\in E)\to\overline{x}} T^D(E,x) \subset T^C(E,\overline{x})$.

$f : X \to \overline{\mathbb{R}}$ is sleek at $\overline{x}$ if $E_f := \operatorname{epi} f$ is sleek at $\overline{x}_f := (\overline{x}, f(\overline{x}))$.
$f : X \to \overline{\mathbb{R}}$ is regular at $\overline{x}$ if $E_f := \operatorname{epi} f$ is regular at $\overline{x}_f$

6 Softness and regularity

Proposition 6 *If E is a closed subset of an Asplund space and if $\overline{x} \in E$, then, among the following assertions, one has the implications (a)⇒(b)⇔(c)⇔(d):*

(a) E is sleek at $\overline{x}$:

$$T^D(E,\overline{x}) \subset \liminf_{x \xrightarrow{E} \overline{x}} T^D(E,x);$$

(b) E is soft at $\overline{x}$:

$$N^L(E,\overline{x}) = N^D(E,\overline{x});$$

(c) E is Clarke regular at $\overline{x}$:

$$N^C(E,\overline{x}) = N^D(E,\overline{x}), T^C(E,\overline{x}) = T^D(E,\overline{x});$$

(d) E is pseudo-sleek at $\overline{x}$:

$$T^D(E,\overline{x}) \subset \liminf_{x \xrightarrow{E} \overline{x}} T^D(E,x)^{00}.$$

If moreover E has the cone property (= is epi-Lipschitz) around $\overline{x}$, all these properties are equivalent.

7 Firm softness and firm regularity

(a) E is *firmly tangentially regular at* $\overline{x}$ if $T^F(E,\overline{x}) = T^C(E,\overline{x})$

(b) E is *firmly (normally) regular at* $\overline{x}$ if $N^F(E,\overline{x}) = N^C(E,\overline{x})$

(c) E is *firmly soft at* $\overline{x}$ if $N^{LF}(E,\overline{x}) = N^F(E,\overline{x})$:
$(x_i^*) \to^* x^*$, bounded, $x_i^* \in N^F(E,x_i)$ for all $i \Longrightarrow x^* \in N^F(E,\overline{x})$

(d) E is *firmly metrically soft* at $\overline{x}$ if d_E is firmly soft at $\overline{x}$

(e) E is *firmly sleek at* $\overline{x}$ if $T^F(E,\overline{x}) = \liminf_{x(\in E)\to\overline{x}} T^F(E,x)$

f is *firmly soft at* $\overline{x}$ if its epigraph is firmly soft at $\overline{x}_f := (\overline{x}, f(\overline{x}))$.

Theorem 7 *(a)$\Longrightarrow$(b) and (a)$\Longrightarrow$(e)$\Longrightarrow$(c)$\Longleftrightarrow$(d)*

Theorem 8 *If X is Asplund (a)$\Longrightarrow$(b)$\Longleftrightarrow$(c)$\Longleftrightarrow$(d)*

Theorem 9 *If X is reflexive (a), (b), (c), (d), (e) are equivalent.*

8 Some favorable classes of functions

- **tangentially convex** functions introduced by Pshenichnyi, Janin, P...
- **d.c.** functions and **t.d.s.** functions: Tuy, Lethi, Tao, Caprari-P...
- **partially smooth** functions: Lewis
- **semismooth** functions introduced by Mifflin and Ngai-P in the lsc case
- **lower $\mathbf{C}^k$** functions studied by Spingarn, Rockafellar, P, Daniilidis-Malick...
- **amenable** functions introduced by Rockafellar and used by Poliquin
- **prox-regular** functions introduced by Poliquin-Rockafellar and extended to the infinite dimensional case by Bernard-Thibault
- **primal lower nice** functions introduced by Poliquin and studied by Bernard, Marcellin, Thibault...
- **piecewise $\mathbf{C}^1$** functions studied by Kojima, Pallaschke, Ralph, Scholtes...
- **sums of convex and $\mathbf{C}^1$** functions used by Michel, Szulkin...

- p-**paraconvex** functions studied by Rolewicz and Bougeard-P-Pommellet, Canino, Castellani-Pappalardo, Duda-Zajicek, Jourani, Ngai-P, P, P-Volle...

- for $p = 2$ these functions are also called **semiconvex** (Lasry-Lions, Attouch-Azé, Cannarsa-Sinestrari), subsmooth (Aussel-Daniilidis-Thibault), property ω (Colombo-Goncharov), weakly convex (Vial), lower-C^2 (Spingarn...)

- (p, q)-**convex** functions introduced by De Giorgi-Marino-Tosques and studied by Canino, Degiovanni, and their co-authors

- **approximately convex** functions introduced by Ngai-Luc-Théra and studied by Aussel-Daniilidis-Thibault, Colombo-Goncharov, Daniilidis-Georgiev, Ngai-P,....

- **approximately starshaped functions** P, Ngai-P

Definition 10 *A function $f \in \mathcal{F}(X)$ finite at a, is said to be approximately starshaped at a if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $x \in B(a, \delta)$, $t \in [0, 1]$, one has*

$$f((1-t)a + tx) \leq (1-t)f(a) + tf(x) + \varepsilon t(1-t) \|x - a\| .$$

Directional versions of the concepts...

9 Differences of convex functions

Theorem 11 *If $f = g - h$ with g finite at $\overline{x} \in \operatorname{core}(\operatorname{dom} h)$, g approximately starshaped at $\overline{x}$ and h approximately convex at $\overline{x}$, then*

$$\partial^D f(\overline{x}) = \partial^D g(\overline{x}) \boxminus \partial^D h(\overline{x}),$$

where for two subsets A, B of X^* one sets

$$A \boxminus B = \{x^* : B + x^* \subset A\}.$$

10 Can one generalize the notion of function of class C^1?

Among many different means:

f is said to be *equi-subdifferentiable at* a if $\partial^D f(a) \neq \varnothing$ and if for every $\varepsilon > 0$ one can find $\delta > 0$ s.t. for all $x \in B(a, \delta)$, $a^* \in \partial^D f(a)$

$$\langle a^*, x - a \rangle \leq f(x) - f(a) + \varepsilon \|x - a\| .$$

The function f is said to be *continuously* ∂*-subdifferentiable at* a if for all $\varepsilon > 0$ one can find some $\delta > 0$ such that for all $x \in B(a, \delta)$ with $|f(x) - f(a)| < \delta$, $x^* \in \partial f(x)$, $y \in B(x, \delta)$ one has

$$\langle x^*, y - x \rangle \leq f(y) - f(x) + \varepsilon \|y - x\| .$$

Proposition 12 *If f is continuously ∂-subdifferentiable at a then f is ∂-soft at $a : \partial^L f(a) = \partial^F f(a)$.*

Proposition 13 *If f is continuous around a, approximately starshaped at a with $\partial^D f(a) \neq \varnothing$, then f is equi-∂^D-subdifferentiable at a and $\partial^D f(a) = \partial^F f(a)$.*

11 Approximately convex functions

Definition 14 *(Ngai-Luc-Théra) A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is said to be approximately convex at $x_0 \in X$ if for any $\varepsilon > 0$ there exists $\delta > 0$ s.t. $\forall x, y \in B(x_0, \delta)$, $t \in [0,1]$ one has*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon t(1-t) \|x - y\| .$$

This class has interesting stability properties (cf Ngai-Luc-Théra):

Proposition 15 *The set of approximately convex functions around $x_0 \in X$ is a convex cone containing the functions which are strictly differentiable at x_0.*

It is stable under finite suprema. Moreover, if $f = h \circ g$, where $g : X \to Y$ is strictly differentiable at x_0 and $h : Y \to \mathbb{R}_\infty$ is approximately convex around $g(x_0)$, then f is approximately convex around x_0.

It can be shown that approximately convex functions retain some of the nice properties of convex functions. In particular they are continuous on segments contained in their domains and have radial derivatives.

12 Approximate convexity versus approximate monotonicity

Characterizations of approximate convexity ?
Previous results for the Lipschitz case: Daniilidis-Georgiev, Aussel-Daniilidis-Thibault

In order to look for characterizations, we need the following notion.

Definition 16 *(Spingarn) A multimap $M : X \rightrightarrows X^*$ is said to be approximately monotone around $\overline{x}$ on $E \subset X$ if for any $\varepsilon > 0$ there exists some $\delta > 0$ such that for any $x_1, x_2 \in E \cap B(\overline{x}, \delta)$, $x_1^* \in M(x_1)$, $x_2^* \in M(x_2)$ one has*

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -\varepsilon \left\| x_1 - x_2 \right\| .$$

For $E = X$ one says that M is approximately monotone around $\overline{x}$.

Theorem 17 *Let $x_0 \in \operatorname{dom} f$, f lsc. Suppose $\partial f \subset \partial^C f$. Then, among the following assertions, one has the implications (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d).*
If moreover ∂ is valuable on X, all these assertions are equivalent.

(a) f is approximately convex around x_0;

(b) $\forall \varepsilon > 0$ $\exists \rho > 0$ s.t. $\forall x \in B(x_0, \rho)$, $\forall v \in B(0, \rho)$ one has

$$f^C(x, v) \leq f(x + v) - f(x) + \varepsilon \|v\| ;$$

(c) $\forall \varepsilon > 0$ $\exists \rho > 0$ s.t. $\forall x \in B(x_0, \rho)$, $\forall x^* \in \partial f(x)$, $\forall v \in \overline{B}(0, \rho)$

$$\langle x^*, v \rangle \leq f(x + v) - f(x) + \varepsilon \|v\| ;$$

(d) ∂f is approximately monotone around x_0.

Corollary 18 *The preceding assertions (a), (b), (c), (d) are equivalent when*

(i) X is arbitrary and $\partial = \partial^C$, the Clarke subdifferential;

(ii) X is an Asplund space and $\partial = \partial^F$ or $\partial = \partial^D$.

13 Approximate convexity of sets

Definition 19 *$E \subset X$ is said to be approximately convex around $\overline{x}$ if d_E is approximately convex around $\overline{x}$.*

Theorem 20 *Let ∂ be a subdifferential s.t. $\partial f \subset \partial^C f \ \forall f \in \mathcal{L}(X)$. Among the following assertions, one has the implications (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Leftrightarrow$(d)$\Leftrightarrow$(e). If moreover ∂ is Lipschitz-valuable on X, all these assertions are equivalent.*

(a) E is approximately convex around $\overline{x}$;

(b) $\forall \varepsilon > 0 \ \exists \rho > 0$ s.t. $\forall x \in B(\overline{x}, \rho)$, $v \in B(0, \rho)$ one has

$$d^C(x, v) \leq d_E(x + v) - d_E(x) + \varepsilon \|v\|;$$

(c) $\forall \varepsilon > 0 \ \exists \rho > 0$ s.t. $\forall x \in B(\overline{x}, \rho)$, $x^* \in \partial d_E(x)$, $v \in \rho \overline{B}_X$ one has

$$\langle x^*, v \rangle \leq d_E(x + v) - d_E(x) + \varepsilon \|v\|;$$

(d) ∂d_E is approximately monotone around $\overline{x}$;

(e) $\forall \varepsilon > 0 \ \exists \sigma > 0$ s.t. $\forall x, y \in B(\overline{y}, \sigma)$, $x^* \in \partial d_E(x)$ one has

$$d_E(x) + \langle x^*, y - x \rangle \leq d_E(y) + \varepsilon \|y - x\|.$$

Corollary 21 *If E is approximately convex around $\overline{x}$ then d_E is firmly (Clarke) regular at $\overline{x}$.*

14 Intrinsic approximate convexity

Definition 22 *$E \subset X$ is intrinsically approximately convex around $\overline{x} \in E$ if for any $\varepsilon > 0$ there exists $\rho > 0$ s.t. for any $x_1, x_2 \in E \cap B(\overline{x}, \rho)$, $t \in [0,1]$, one has*

$$d_E((1-t)x_1 + tx_2) \leq \varepsilon t(1-t) \|x_1 - x_2\| .$$

This definition does not depend on the choice of the norm (up to equivalence).

Theorem 23 *Let $E \subset X$ and let ∂ be a subdifferential such that $\partial f \subset \partial^C f$ for $f \in \mathcal{L}(X)$. Then, among the following assertions, one has (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Leftrightarrow$(d)$\Leftarrow$(e). When X is a Lipschitz ∂-subdifferentiability space one has (e)$\Rightarrow$(a).*

(a) E is intrinsically approximately convex around $\overline{x}$;

(b) $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x, x' \in E \cap B(\overline{x}, \delta)$, one has

$$d^C(x, x' - x) \leq \varepsilon \|x - x'\| ;$$

(c) $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x, x' \in E \cap B(\overline{x}, \delta)$, $x^* \in \partial d_E(x)$, one has

$$\langle x^*, x' - x \rangle \leq \varepsilon \|x - x'\| ;$$

(d) $\partial d_E(\cdot)$ is approximately monotone around $\overline{x}$ on E;

(e) $\forall \varepsilon > 0\ \exists \sigma > 0$ s.t. $\forall w \in B(\overline{x}, \sigma)$, $x \in E \cap B(\overline{x}, \sigma)$, $w^* \in \partial d_E(w)$ one has

$$d_E(w) + \langle w^*, w - x \rangle \leq \varepsilon \|w - x\| .$$

Corollary 24 *If E is intrinsically approximately convex around $\overline{x}$, then for any subdifferential ∂ s.t. $\partial^F \subset \partial \subset \partial^C$ one has $\partial^F d_E(\overline{x}) = \partial d_E(\overline{x}) = \partial^C d_E(\overline{x})$.*

Some specializations:

Corollary 25 *If X is Asplund and $\partial = \partial^F$ these assertions are equivalent to*
(f) $\forall \varepsilon > 0\ \exists \delta > 0$ *s.t.* $\forall x, x' \in E \cap B(\overline{x}, \delta)$, $x^* \in N^F(E, x)$ *one has*

$$\langle x^*, x' - x \rangle \leq \varepsilon \|x^*\| \|x - x'\| ;$$

(g) $\forall \varepsilon > 0\ \exists \delta > 0$ *s.t.* $\forall x_i \in E \cap B(\overline{x}, \delta)$, $x_i^* \in N^F(E, x_i)$, $i = 1, 2$, *one has*

$$\langle x_1^* - x_2^*, x_2 - x_1 \rangle \geq -\varepsilon \max \left(\|x_1^*\| , \|x_2^*\| \right) \|x_1 - x_2\| . \tag{1}$$

Corollary 26 *If X is an Asplund space and $\partial = \partial^{LF}$ then all these assertions are equivalent.*

Corollary 27 *If $\partial = \partial^C$, among the assertions of the preceding Theorem, the following implications hold: (e)$\Rightarrow$(a)$\Rightarrow$(b)$\Leftrightarrow$(c)$\Leftrightarrow$(d).*
If X is an Asplund space, all these assertions are equivalent.

15 Approximately convex sets and functions

We endow $X := W \times \mathbb{R}$ with a *product norm*, i.e. a norm such that the projections and the insertions $w \mapsto (w, 0)$ and $r \mapsto (0, r)$ are nonexpansive.

Proposition 28 *Let W be a n.v.s. and let $f : W \to \mathbb{R}_\infty$ be a l.s.c. function which is approximately convex around $\overline{w} \in W$. Then, for any $\overline{r} \geq f(\overline{w})$ the epigraph E of f is intrinsically approximately convex around $\overline{x} := (\overline{w}, \overline{r})$.*

A kind of converse:

Theorem 29 *Let W be a Banach space and let $f : W \to \mathbb{R}$ be a function which is locally Lipschitzian around $\overline{w} \in W$ and such that the epigraph E of f is an intrinsically approximately convex subset of $X := W \times \mathbb{R}$ around $\overline{x} := (\overline{w}, f(\overline{w}))$. Then f is an approximately convex function around $\overline{w}$.*

Proposition 30 *Let $f : W \to \mathbb{R}$ be a function which is Lipschitz with rate $c > 0$ on some ball $B(\overline{w}, \rho)$. Suppose $X := W \times \mathbb{R}$ is endowed with the norm given by $\|(w, r)\| = c\|w\| + |r|$. If f is approximately convex around $\overline{w}$, then, for any $\overline{r} \geq f(\overline{w})$, the epigraph E of f is approximately convex around $\overline{x} := (\overline{w}, \overline{r})$.*

Recall that E *satisfies the cone property around* $\overline{x}$ if there exist $r, \rho > 0$ and $u \in S_X$ such that for every $x \in E \cap B(\overline{x}, \rho)$, $v \in B(u, r)$, $t \in (0, r)$ one has $x + tv \in E$.

Corollary 31 *Suppose* E *satisfies the cone property around* $\overline{x}$*. Then* E *is intrinsically approximately convex around* $\overline{x}$ *if, and only if, it is approximately convex around* $\overline{x}$ *for some compatible norm on* X*.*

Let us turn to sublevel sets.

Proposition 32 *Let* X *be an Asplund space and let* $f : X \to \mathbb{R}$ *be a continuous function. Suppose* f *is approximately convex around* $\overline{x} \in S := \{x \in X : f(x) \leq 0\}$ *and there exist* $c > 0$, $r > 0$ *such that* $\|x^*\| \geq c$ *for all* $x \in (X \backslash S) \cap B(\overline{x}, r)$ *and all* $x^* \in \partial^F f(x)$*. Then* S *is intrinsically approximately convex around* $\overline{x}$*.*

16 Approximately convex sets and projections

Lemma 33 *Let X be uniformly smooth and let $E \subset X$. Then $-d_E$ is firmly (Clarke) regular at any $w \in X \setminus E : \partial^C(-d_E) = \partial^F(-d_E)$.*

Theorem 34 *Suppose that the norm of X is differentiable on $X \setminus \{0\}$. Let $E \subset X$ and let U be an open subset of X. Consider the following assertions:*

(a) Each $w \in U$ has a unique metric projection $P_E(w)$ in E and $P_E(\cdot)$ is continuous on $U \setminus E$.

(b) $d_E(\cdot)$ is continuously differentiable on $U \setminus E$.

(c) $d_E(\cdot)$ is approximately convex on $U \setminus E$.

Then, one has (a)⇒(b)⇒(c). If X is uniformly Fréchet smooth, then (a)⇒(b)⇔(c). If, in addition, X is strictly convex and the norm of X has the Kadec-Klee property, then (a)⇔(b)⇔(c).

17 p-paraconvexity and p-paramonotonicity

Definition 35 *Given some $p \geq 1$, a function $f : X \to \mathbb{R} \cup \{+\infty\}$ on a n.v.s. X is said to be p-paraconvex around $\overline{x} \in \operatorname{dom} f := f^{-1}(\mathbb{R})$ if there exist $c, \delta > 0$ s.t. for any $x, y \in B(\overline{x}, \delta)$ and any $t \in [0, 1]$ one has*

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + ct(1-t) \|x - y\|^p .$$

Definition 36 *A multimapping $M : X \rightrightarrows X^*$ is said to be p-paramonotone around $\overline{x}$ on a subset E of X if there exist some $m, \delta > 0$ s.t. for any $x_1, x_2 \in E \cap B(\overline{x}, \delta)$, $x_1^* \in M(x_1)$, $x_2^* \in M(x_2)$ one has*

$$\langle x_1^* - x_2^*, x_1 - x_2 \rangle \geq -m \|x_1 - x_2\|^p .$$

For $E = X$ one simply says that M is p-paramonotone around $\overline{x}$.

18 Generic differentiability

It is a deep and famous result of D. Preiss that any locally Lipschitzian function on an Asplund space is Fréchet differentiable at the points of a dense subset.

However, it is not known whether this set is a $\mathcal{G}_\delta$, i.e. a countable intersection of open subsets.

We need an important characterization of Asplund spaces.

Lemma 37 *X is an Asplund space if and only if X^* has the Radon-Nikodým property, i.e. if every nonempty bounded subset A of X^* admits weak* slices of arbitrary small diameter.*

18.1 Generic differentiability of approx. convex functions

Theorem 38 *Let $f : U \to \mathbb{R}$ be a lower semicontinuous, approximately convex function on an open subset U of an Asplund space. Then f is Fréchet differentiable on a dense $\mathcal{G}_\delta$-subset of U.*

18.2 An extension to regular functions

Theorem 39 *Let U be an open subset of an Asplund space X and let $f : U \to \mathbb{R}$ be a locally Lipschitzian regular function. Then f is Fréchet differentiable at each point of a dense $\mathcal{G}_\delta$-subset of U.*