# On the Lions & Stampacchia Theorem

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Let us consider a real Hilbert space $X$ with scalar product $\langle \cdot, \cdot \rangle$ and associated norm $\| \cdot \|$. We assume given a linear and continuous operator $A : X \rightarrow X$, (in short, $A \in \mathcal{L}(X)$), a closed and convex subset $K$ of $X$ and a fixed element $f \in X$.

By a variational inequality we mean the problem $\mathcal{V}(A, K, f)$ of finding $u \in K$ such that $\langle Au - f, v - u \rangle \geq 0$ for each $v \in K$.

Variational inequalities were introduced by **Fichera** in his analysis of the Signorini problem on the elastic equilibrium of a body under unilateral constraints (see for instance the survey by S. Mazzone).

G. Fichera, *Problemi elastostatici con vincoli unilaterali : il problema die Signorini con ambigue condizioni al contorno*, Mem. Accad. Naz. Lincei 8 (1964), p. 91-140.

S. Mazzone, *Variational analysis and applications*. Proceedings of the 38th Conference of the School of Mathematics "G. Stampacchia" in memory of Stampacchia and J.-L. Lions held in Erice, June 20–July 1, 2003. Edited by Franco Giannessi and Antonino Maugeri. Nonconvex Optimization and its Applications, **79**, Springer-Verlag, New York, 2005.

In their celebrated 1967 paper, **Lions and Stampacchia** extended Fichera's analysis to abstract variational inequalities associated to bilinear forms which are coercive or simply non negative in real Hilbert spaces as a tool for the study of partial differential elliptic and parabolic equations.

They had in view applications to problems with unilateral constraints in mechanics (for these problems we refer to the book by Duvaut & Lions for details).

The theory has since been expanded to include various applications in different areas such as economics, finance, optimization and game theory.

J.-L. Lions, G. Stampacchia, *Variational inequalities*, Comm. pure and appl. Math. **20** (1967), p 493 - 51.

G. Duvaut , J. L. Lions, *Les inéquations en mécanique et en physique*, Dunod, Paris, 1972.

**Theorem ($\oiint$) Lions & Stampacchia**

*For every bounded closed and convex set $K$ and $f \in X$ , the linear variational inequality $\mathcal{V}(A, K, f)$ admits at least one solution provided $A$ is* coercive, *that is*

$$\langle Au, u \rangle \geq a\|u\|^2 \text{ for every } u \in X \text{ and some } a > 0.$$

An important notion in the study of variational inequalities was provided by Brezis, who proved that the Lions & Stampacchia Theorem actually holds within the setting of reflexive Banach spaces, and for a very large class of (non-linear) operators, called *pseudo-monotone* operator.

Precisely, let $X$ be a reflexive Banach space with continuous dual $X^\star$. Let us denote by $\langle \cdot, \cdot \rangle$ the duality product between $X$ and $X^\star$ and by the symbol $\rightharpoonup$ the weak convergence on $X$. We say that $A$ is pseudo-monotone, if it is bounded and if it verifies

$$\langle Au, u - v\rangle \leq \liminf_n \langle Au_n, u_n - v\rangle \ \ \forall v \in X \tag{1}$$

whenever $\{u_n\}_{n\in\mathbb{N}^\star}$ is a sequence in $X$ such that

$$u_n \rightharpoonup u \ \text{ and } \ \limsup_n \langle Au_n, u_n - u\rangle \leq 0.$$

Monotone operators which are hemicontinuous are pseudomonotone

$T(x_0 + t_n y) \rightharpoonup T(x_0)$ as $t_n \to 0$ for all $y$

## Lemma

*Let $X$ be a real Hilbert space and $A \in \mathcal{L}(X)$. Then $A$ is pseudo-monotone if and only if*

$$[u_n \rightharpoonup 0] \Longrightarrow \left[\liminf_n \langle Au_n, u_n\rangle \geq 0\right]. \tag{2}$$

# Proof of the Lemma

Let $A$ be a $\mathcal{L}(X)$-pseudo-monotone operator, and $u_n \rightharpoonup 0$.

We only need to prove relation (2) when $\liminf_n \langle Au_n, u_n\rangle \leq 0$.

In this case we may also suppose, by taking, if necessary, a subsequence, that $\limsup_n \langle Au_n, u_n\rangle \leq 0$. Taking $v = 0$ in the definition of pseudomonotonicity we derive :

$$0 = \langle A0, 0 - 0\rangle \leq \liminf_n \langle Au_n, u_n\rangle ,$$

that is $0 = \liminf_n \langle Au_n, u_n\rangle$. Relation (2) holds accordingly for every $\mathcal{L}(X)$-pseudo-monotone operator.

Conversely take $A \in \mathcal{L}(X)$ such that relation (2) is verified. Pick $u_n \rightharpoonup 0$ such that

$$\limsup_n \langle Au_n, u_n - u \rangle \leq 0.$$

On one hand, as

$$\lim_n \langle Au, u_n - u \rangle = 0, \tag{3}$$

the previous relation implies that

$$\limsup_n \langle A(u_n - u), (u_n - u) \rangle \leq 0. \tag{4}$$

On the other hand, applying relation (2) to the sequence $\{u_n - u\}$ yields

$$\liminf_n \langle A(u_n - u), (u_n - u) \rangle \geq 0. \tag{5}$$

Combining relations (3), (4) and (5) we deduce that

$$\lim_n \langle Au_n, u_n - u \rangle = 0, \tag{6}$$

whenever $u_n \rightharpoonup u$ and $\limsup_n \langle Au_n, u_n - u \rangle \leq 0$.

# Proof of the Lemma

Any $\mathcal{L}(X)$-operator is also continuous with respect to the weak topology on $X$ ;
As $u_n \rightharpoonup u$ we deduce that

$$\lim_n \langle Au_n, w \rangle = \langle Au, w \rangle .$$

When applied for $w = u - v$, the previous relation shows that

$$\langle Au, u - v \rangle = \lim_n \langle Au_n, u - v \rangle = \liminf_n \langle Au_n, u - v \rangle , \tag{7}$$

for every sequence $u_n \rightharpoonup u$.

Summing up relations (6) and (7), we deduce that relation (1) holds whenever $u_n \rightharpoonup u$ and $\limsup_n \langle Au_n, u_n - u \rangle \leq 0$ ; in other words, the operator $A$ is pseudo-monotone.

# Remark

It is well known that, as long as non-linear operators are concerned, problem $\mathcal{V}(A, K, f)$ may admit solutions for every bounded closed and convex set $K$, even if the operator $A$ is not pseudo-monotone.

**In a real Hilbert setting, there exists a continuous and positively homogeneous operator which is not pseudo-monotone but for which the variational inequality $\mathcal{V}(A, K, f)$ has solutions provided $K$ is a closed and convex bounded set.**

## Example

Let $X$ be a separable Hilbert space with basis $\{b_i : i \in \mathbb{N}^\star\}$. As customary, for every real number $a$, let us set $a_+ = \max(a, 0)$ for the positive part of $a$. For every $i \in \mathbb{N}^\star$, let us define

$$A_i : X \to X,\ A_i(x) = -\left(3\langle x, b_i\rangle - 2\|x\|\right)_+ b_i,$$

and set $A(x) = \Sigma_{i=1}^{\infty} A_i(x)$. Then $A$ is a continuous and positively homogeneous mapping which fails to be pseudo-monotone, while the variational inequality $\mathcal{V}(A, K, f)$ admits solutions for every bounded closed and convex set $K$.

# Remark

Indeed, remark that any two sets from the family of open convex cones

$$K_i = \{x \in X : \ 3\langle x, b_i\rangle > 2\|x\|\}, \ i \in \mathbb{N}^\star$$

are disjoints.
Hence the operator the operator $A$ is well defined as at any point $x$, at most one among the values $A_i(x)$, $i \in \mathbb{N}^\star$, may be non-null.

As i each $A_i$ is continuous and positively homogeneous, so is $A$.

$A(b_i) = -b_i$, so

$$0 = \langle A0, 0 - 0\rangle > \liminf_i \langle Ab_i, b_i - 0\rangle = -1;$$

this inequality proves that relation (1) does not hold for $b_i$ instead of $u_i$, and 0 instead of $u$ and $v$.

Finally remark that

$$b_i \rightharpoonup 0 \text{ and } \limsup_i \langle Ab_i, b_i - 0\rangle = -1 \leq 0,$$

to infer that the operator $A$ is not pseudo-monotone.

# Remark

We need now to prove that the variational inequality $\mathcal{V}(A, K, f)$ has solutions for every bounded closed and convex set $K$.

**Let us consider first the case when the domain $K$ of the variational inequality is not entirely contained within one of the cones $K_i$.** As every convex set is also a connected set, and since $\{K_i : \ i \in \mathbb{N}^\star\}$ form a family of disjoint open sets, it follows that $K$ contains some point $\bar{x}$ which does not belong to any of the cones $K_i$. Accordingly, $A(\bar{x}) = 0$, fact which means that $\bar{x}$ is a solution of the problem $\mathcal{V}(A, K, f)$.

**Consider now the case of a bounded closed and convex set $K$ contained in some cone $K_p$, $p \in \mathbb{N}^\star$.** Then $A$ and $A_p$ coincide on the cone $K_p$, and thus on $K$.

Therefore, $A$ is pseudo-monotone.

The existence of a solution to problem $\mathcal{V}(A, K, f)$ is guarantied by Brezis' s theorem.

Indeed, remark that any two sets from the family of open convex cones

$$K_i = \{x \in X : \ 3\langle x, b_i\rangle > 2\|x\|\}, \ i \in \mathbb{N}^\star$$

are disjoints. This fact proves that the definition of the operator $A$ is meaningful, as at any point $x$, at most one among the values $A_i(x)$, $i \in \mathbb{N}^\star$, may be non-null.

$A$ is continuous and positively homogeneous, as is each of the the operators $A_i$.

$A$ is not pseudo-monotone. $A(b_i) = -b_i$, so

$$0 = \langle A0, 0-0\rangle > \liminf_i \langle Ab_i, b_i - 0\rangle = -1;$$

this inequality proves that relation (1) does not hold for $b_i$ instead of $u_i$, and 0 instead of $u$ and $v$. Finally let us remark that

$$b_i \rightharpoonup 0 \text{ and } \limsup_i \langle Ab_i, b_i - 0\rangle = -1 \le 0,$$

to infer that the operator $A$ is not pseudo-monotone.

$\mathcal{V}(A, K, f)$ has solutions for every bounded closed and convex set $K$.

Let us consider first the case when the domain $K$ of the variational inequality is not entirely contained within one of the cones $K_i$. As every convex set is also a connected set, and since $\{K_i : \; i \in \mathbb{N}^\star\}$ form a family of disjoint open sets, it follows that $K$ contains some point $x$ which does not belong to any of the cones $K_i$. Accordingly, $A(x) = 0$, fact which means that $x$ is a solution of the problem $\mathcal{V}(A, K, f)$.

Consider now the case of a bounded closed and convex set $K$ contained in the cone $K_p$ for some $p \in \mathbb{N}^\star$. Remarking that the operators $A$ and $A_p$ coincide on the cone $K_p$, and thus on $K$, we deduce that $A$ is pseudo-monotone. Accordingly, the existence of a solution to problem $\mathcal{V}(A, K, f)$ is garanteed in this case by Brezis' s theorem.

# Goal of the talk and main result

The aim of this note is to establish that, in the original *linear* setting of the Lions-Stampacchia Theorem, the pseudo-monotonicity of the operator $A$, which, in general, is only a sufficient condition for the existence of solutions for every bounded convex set $K$, becomes also a necessary one.

**Theorem ($\Upsilon$) -Main result**

*Let $X$ be a real Hilbert space and $A$ a linear and continuous operator. The following statements are equivalent.*

*$(i)$ $A$ is pseudo-monotone ;*
*$(ii)$ The variational inequality $\mathcal{V}(A, K, f)$ admits at least a solution for every bounded closed and convex set $K$ and $f \in X$.*

# A technical proposition

The following technical result proves that a $\mathcal{L}(X)$-operator is pseudo-monotone only if its restriction to some closed hyperplane of $X$ amounts the opposite of a monotone symmetric $\mathcal{L}(X)$-operator.

## Proposition

*Let $X$ be a real Hilbert space and suppose that the operator $A \in \mathcal{L}(X)$ is not pseudo-monotone. Then we can construct an infinite-dimensional and separable closed subspace $H$ of $X$ such that the restriction of $A$ to $H$ is both symmetric,*

$$\langle Au, v\rangle = \langle Av, u\rangle \quad \forall u, v \in H, \tag{8}$$

*and negatively defined,*

$$\langle Au, u\rangle \leq -\alpha\|u\|^2, \tag{9}$$

*for some $\alpha > 0$.*

# Sketch of the proof

$(i) \Rightarrow (ii)$ Brezis

$(ii) \Rightarrow (i)$ Let $A \in \mathcal{L}(X)$ be an operator which is not pseudo-monotone. According to the Proposition the bilinear and continuous form

$$\Theta : H \times H \to \mathbb{R}, \quad \Theta(x, y) = -\langle Ax, y\rangle$$

is symmetric and positively defined for some infinite-dimensional separable closed subspace $H$ of $X$.

Endow the vector space $H$ with the inner product $[x, y] = \Theta(x, y)$, and consider $B = \{b_i : \ i \in \mathbb{N}^\star\}$ a Hilbert basis of $(H, [\cdot, \cdot])$. As usually, if $x \in H$, let $x_i$ denote the $i$-th coordinate of $x$ with respect to $B$, $x_i = [x, b_i]$ for every $x \in H$, $i \in \mathbb{N}^\star$. We prove that the set $K$

$$K = \left\{ x \in H : \ x_i \geq \frac{1}{2^i} \text{ and } \sum_{i=1}^{\infty} \left(1 + \frac{1}{2^i}\right) x_i^2 \leq 2 \right\},$$

is a bounded closed and convex subset of $X$ such that the variational inequality $\mathcal{V}(A, K, 0)$ does not have solutions.