A new stochastic equilibrium problem for two stage games

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Section 1

Motivation:
What is impact of risk on investment of a firm?

Investment and risk

Think of a market, for example wholesale electricity.

- Investment, e.g., in generation capacity, occurs ahead of production and trading.
- Production and sales may go on for decades. But financial instruments — like options to hedge against low prices (low revenue) in future — are often limited to 1 or 2 years at most. $\longrightarrow$ market is **incomplete** in that not all risks are traded

How can we model this from point of view of **firm**?

- A firm, even a bank, is necessarily risk averse.
- Corporate finance theory models the entire market (assuming liquid, complete markets etc) $\longrightarrow$ **risk neutral** probabilities. Whatever an individual's risk profile, it cannot beat the market; the market is **efficient**.
- Risk neutral prob. are **exogenous**— "read" from market data.

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Risk neutral probabilities are endogenous

Our approach cannot look like standard corporate finance

- Use game with N players rather than amorphous “market”
- Use coherent risk measures, e.g. $\mathrm{cV@R}$, to model risk aversion, not risk neutral expectations
- Players’ interactions in market determine their risk neutral probabilities ... which are **endogenous**.

Our model is rather simple, only two stages:

- Stage one is investment, stage two is production and sales.
- Main task is to establish **existence of equilibrium**
- Give some theory behind computational model & analysis of European electricity market [Ehrenmann-Smeers-07]
- Future: How do changes in financial products (completeness) affect equilibrium?

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Contents

Section 2

Notation motivation:
A deterministic two stage investment game

Deterministic spot market (game)

Simple **spot market** of N players & 1 commodity (electricity):

- Player i chooses production level $z_i \geq 0$ to max profit, given:
- production cost function $c_i(z_i)$
- production capacity $x_i \geq 0$
- spot market price $\lambda \geq 0$:

$$\max_{z_i} \ \lambda z_i - c_i(z_i) \quad \text{subject to} \quad 0 \leq z_i \leq x_i\,. \tag{1}$$

In **perfect competition**, λ comes out of market clearing condition, given total demand $d > 0$:

$$0 \leq \textstyle\sum_i z_i - d \quad \perp \quad \lambda \geq 0. \tag{2}$$

Note. Rewrite as min-max over $\lambda \geq 0$ and $z = (z_1, \ldots, z_N)$ to get unique equilibrium quantities z and price λ if

- each c_i is smooth & strictly convex with $c_i(0) = 0$
- total capacity exceeds demand: $\sum_i x_i > d$ ... **Slater CQ**

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First stage investment in capacity

The spot market is the second stage of the game.
In the first stage Player i invests in production capacity x_i:

- investment cost function $k_i(x_i)$
- capacity payment $\nu \geq 0$ ... to encourage investment

Switch from "max profit" to "min cost": Over both stages, Player i solves

$$\begin{array}{ll} \min_{x_i, z_i} & k_i(x_i) - \nu x_i + c_i(z_i) - \lambda z_i \\ \text{subject to} & 0 \leq x_i \\ & 0 \leq z_i \leq x_i. \end{array} \tag{3}$$

Capacity payment ν must clear the capacity market:

$$0 \leq \sum_i x_i - d \quad \perp \quad \nu \geq 0. \tag{4}$$

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The **two stage investment game** is described by (2)–(4).

Reduction to single stage game

Player i's optimization problem is equivalent to

$$\min_{x_i \geq 0} \; k_i(x_i) - \nu x_i + h_i(x_i, \lambda) \tag{5}$$

where **cost of capital** in scenario ω is

$$h_i(x_i, \lambda) \; = \; \inf_{z_i \in [0, x_i]} \{c_i(z_i) - \lambda z_i\}.$$

Assume $\exists$ unique spot market equilibrium $z(x)$, $\lambda(x)$ if $\sum_i x_i \geq d$.

Then $\qquad h_i(x_i, \lambda(x)) = c_i(z_i(x)) - \lambda(x) z_i(x)$.

But perfect competition $\Rightarrow$ Player i does not anticipate its effect on spot price λ . . . or other players' actions.

$\Rightarrow$ **Reduced (single stage) game** given by clearing capacity market (4), players' problems (5), and $\lambda = \lambda(x)$.

Notes. 1. (5) is <u>not an MPEC</u> . . . two stage game <u>not an EPEC!</u>

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Section 3

Coherent risk measures

Cost of future (random) outcomes

Think of $Y \in L_p(\Omega)$ as vector $(Y_\omega)_{\omega\in\Omega}$ of **future outcomes** or costs, each $Y_\omega \in \mathbb{R}$. Here $p \in (1,\infty]$,

$$\|Y\| = \left(\int_\Omega |Y_\omega|^p d\omega\right)^{1/p} \qquad \text{... where } d\omega \text{ means } dP(\omega)$$

So $\mathbb{1} = (1)_{\omega\in\Omega} \in L_p(\Omega)$, in fact $\int_\Omega d\omega = 1$.

What is cost (or value) of Y at the time of investment?

Simplest answer: use expectation under **probability measure**, defined as dual element $\Pi \in L_q(\Omega)$, where $1/p + 1/q = 1$, s.t.

$$\Pi_\omega \geq 0 \text{ a.e.} \qquad \text{and} \qquad \Pi\mathbb{1} = \int_\Omega \Pi_\omega dP(\omega) = 1.$$

Denote $\Pi Y = \int_\Omega \Pi_\omega Y_\omega dP(\omega)$ as expectation $\mathbb{E}_\Pi[Y]$.
$\longrightarrow$ investment game over two stage stochastic programs.

Alternative: **coherent risk measure** — risk function for short — $\rho : L_p(\Omega) \to \mathbb{R} \cup \infty$ is more general, recommended in finance, see [Artzner-et-al-99]. Risk functions include straight expectations.

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Coherent risk measures

In general, ρ is risk function if for all $Y, Y' \in L_p(\Omega)$ and $\alpha \in \mathbb{R}$:

- proper: $\rho(Y) > -\infty$
- sublinear: $\rho(Y + Y') \leq \rho(Y) + \rho(Y')$
- positive homogeneous: $\rho(\alpha Y) = \alpha\rho(Y)$ for $\alpha > 0$
- monotone: $\rho(Y) \leq \rho(Y')$ if $Y_\omega \leq Y'_\omega$ a.e.
- translation invariant: $\rho(Y + \alpha 1\!\!1) = \rho(Y) + \alpha$

To these we add the property of being lower semicontinuous (lsc).

Classical result [Hörmander 54]: ρ is proper, sublinear, positive homogeneous and lsc $\iff \rho = \sigma_D$, the support function of a nonempty, closed, convex set D in $L_q(\Omega)$, where

$$\sigma_D(Y) = \sup_{\zeta \in D} \zeta Y.$$

Refinement [Shapiro-Ruczinski-06]: ρ is lsc risk function $\iff$ $\rho = \sigma_D$ for nonempty, closed, convex set D of probability meas.

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Risk aversion vs risk neutrality

Example

Conditional value at risk, $\mathrm{cV@R}$, is one of best known risk functions [Rockafellar-Uryasev-02].
$\mathrm{cV@R}_{\Pi,\beta}(Y)$ is (roughly) Π-expectation of all but $\beta\%$ of lowest outcomes of Y.
Thus $\mathrm{cV@R}_{\Pi,\beta}(Y) \geq \mathbb{E}_{\Pi}[Y]$ — $\mathrm{cV@R}$is risk averse.

Generally say risk measure $\rho = \sigma_D$ is **risk neutral** if D is a singleton $\{\Pi\}$, and **risk averse** otherwise.

[Rucz.-Shap.-06, Example 7] shows that $\mathrm{cV@R}_{\Pi,\beta} = \sigma_D$ where D is set of probability measures Π' such that $\Pi'_\omega \leq \Pi_\omega/(1-\beta)$ a.e.

Note also that risk measures combine robust optimization with stochastic programming (c.f. Nemirovsky et al)

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Section 4

Stochastic two stage game

Subsection 4.1

Costing an uncertain future

Costing uncertain spot market outcomes

For each spot scenario ω, Player i's spot market cost is $G_{i\omega}(z_i, \lambda) = c_{i\omega}(z_i) - \lambda z_i$.

We may choose a different $z_i = Z_{i\omega}$ in each scenario, and the spot price may vary: $\lambda = \Lambda_\omega$.

Future outcomes are listed as $G_i(Z_i, \Lambda) = \big(G_{i\omega}(Z_{i\omega}, \Lambda_\omega)\big)_{\omega\in\Omega}$.

Player i minimizes investment & spot market cost:

$$\begin{array}{lll} \min\limits_{x_i, Z_i} & k_i(x_i) - \nu x_i + \rho_i\big(G_i(Z_i, \Lambda)\big) & \\ \text{subject to} & 0 \le x_i & (6) \\ & 0 \le Z_i \le x_i \mathbb{1} & \end{array}$$

Spot market clearing condition is given pointwise over scenarios,

$$0 \le \textstyle\sum_i Z_{i\omega} - D_\omega \quad \perp \quad \Lambda_\omega \ge 0 \quad \text{a.e.} \qquad (7)$$

where $D = (D_\omega)_\omega$ is vector of nonnegative demands.

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To be sure of meeting spot demand, assume D is bounded and let $d =$ sup of D ignoring zero measure sets

$$d \;=\; \text{ess sup}\, D_\omega \;<\; \infty.$$

Clearing capacity market is same as before,

$$0 \leq \sum_{i=1}^{N} x_i - d \quad \perp \quad \nu \geq 0. \tag{8}$$

Two stage stochastic game given by (6), (7) & (8)

Go to Reduced Stoch. Game

Reduction to single stage stochastic game

Cost of capital (or investment) in spot scenario ω is

$$H_{i\omega}(x_i, \lambda) \;=\; \inf_{z_i \in [0, x_i]} G_{i\omega}(z_i, \lambda) \tag{9}$$

Define vector of spot capital cost $H_i(x_i, \Lambda) = \big(H_{i\omega}(x_i, \Lambda_\omega)\big)_\omega$.

Assume (I), when $\sum_i x_i \geq d$, that each spot scenario ω has unique equilibrium $Z_\omega(x) \in \mathbb{R}^N$, $\Lambda_\omega(x) \geq 0$, with $Z(x), \Lambda(x) \in L_\infty(\Omega)$.

Proposition (Pointwise decomposition)

$Z_i(x)$ is a global solution of $\min_{0 \leq Z_i \leq x_i \mathbb{1}} \rho_i(G_i(Z_i, \Lambda(x)))$.

This is based on an **interchangeability** principle, see [Rucz.-Shap.-06, Proposition 4], c.f. [Rock.-Wets-88].

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Single stage stochastic game

Recall the game where each player has a two stage stochastic problem (6), plus clearing of spot & capacity markets (7) & (8).

Go to 2 Stage Stoch. Game

This reduces to the **single stage stochastic game**

$$\min_{x_i \geq 0} \quad k_i(x_i) - \nu x_i + \rho_i\big(H_i(x_i, \Lambda)\big)$$

with capacity market condition (8) and $\Lambda = \Lambda(x)$.
(As in simple deterministic game, Player i doesn't anticipate its effect on price or other players.)

Main point: have reduced infinite dimensional game to finite dimensions.

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Stochastic two stage game

Subsection 4.2

Hedging: A financial market in risk

Options trading to hedge against low prices

Players want to **hedge** against a low spot price.
So financial traders offer a list of **strike prices** $\Lambda^{O} = \left(\Lambda^{O}_{\theta}\right)_{\theta\in\Theta}$ where Θ is another measure space.

- Anyone can buy any amount of any **option** θ.
- If the spot price is λ, one unit of the option pays $(\Lambda^{O}_{\theta} - \lambda)_{+} = \max\left\{\Lambda^{O}_{\theta} - \lambda, 0\right\}$
- This is a hedge against price falling below Λ^{O}_{θ}.

Player i buys/sells $W_{i\theta}$ of option θ where $W_i = \left(W_{i\theta}\right)_{\theta} \in L_{p'}(\Theta)$ and $p' \in (1, \infty]$.
In spot scenario ω, given investments $\sum_i x_i \geq d$, Player i is paid

$$P^{O}_{\omega}(x)W_i = \int_{\Theta} (\Lambda^{O}_{\theta} - \Lambda_{\omega}(x))_{+} W_{i\theta} d\theta$$

This defines linear mapping $P^{O}(x) : L_{p'}(\Theta) \rightarrow L_{\infty}(\Omega)$.

Options trading to hedge against low prices

Players want to **hedge** against a low spot price.
So financial traders offer a list of **strike prices** $\Lambda^{\mathrm{O}} = \left(\Lambda^{\mathrm{O}}_\theta\right)_{\theta\in\Theta}$ where Θ is another measure space.

- Anyone can buy any amount of any **option** θ.
- If the spot price is λ, one unit of the option pays $(\Lambda^{\mathrm{O}}_\theta - \lambda)_+ = \max\left\{\Lambda^{\mathrm{O}}_\theta - \lambda, 0\right\}$
- This is a hedge against price falling below $\Lambda^{\mathrm{O}}_\theta$.

Player i buys/sells $W_{i\theta}$ of option θ where $W_i = \left(W_{i\theta}\right)_\theta \in L_{p'}(\Theta)$ and $p' \in (1, \infty]$.
In spot scenario ω, given investments $\sum_i x_i \geq d$, Player i is paid

$$P^{\mathrm{O}}_\omega(x)W_i = \int_\Theta (\Lambda^{\mathrm{O}}_\theta - \Lambda_\omega(x))_+ W_{i\theta} d\theta$$

This defines linear mapping $P^{\mathrm{O}}(x) : L_{p'}(\Theta) \to L_\infty(\Omega)$.

Pricing options

The cost (market price) for options is a nonnegative vector $C^{\mathrm{O}} = (C^{\mathrm{O}}_\theta)_\theta \in L_{q'}(\Theta)$ where $1/p' + 1/q' = 1$.

What determines the cost of options? Develop equilibrium conditions to find out!

First, Player i's (reduced) problem now looks like

$$\min_{x_i \geq 0,\, W_i \in L_{p'}(\Theta)} \quad k_i(x_i) - \nu x_i + C^{\mathrm{O}} W_i + \rho_i\big(H_i(x_i, \Lambda) - P^{\mathrm{O}} W_i\big) \tag{10}$$

where $\Lambda = \Lambda(x)$ and $P^{\mathrm{O}} = P^{\mathrm{O}}(x)$.
We also have conservation of cash:

$$\sum_{i=1}^{N} W_i = 0. \tag{11}$$

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Optimality conditions for Player i

Assume (II) smoothness of k_i and H_i with respect to x_i, Player i's optimality condition for (10) is

$$\Pi_i \in \partial\rho_i\big(H_i(x_i,\Lambda) - P^{\mathrm{O}}W_i\big) \tag{12}$$

$$0 \leq \nabla_{x_i}k_i(x_i) - \nu + \Pi_i\nabla_{x_i}H\big(x_i,\Lambda\big) \quad \perp \quad x_i \geq 0 \tag{13}$$

$$C^{\mathrm{O}} = \Pi_i P^{\mathrm{O}}. \tag{14}$$

Note each Π_i is a probability measure ... a **risk neutral** probability for Player i.

The full equilibrium conditions also require: clearing capacity market (8), $\Lambda = \Lambda(x)$, $P^{\mathrm{O}} = P^{\mathrm{O}}(x)$, and cash conservation (11).

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Section 5

Full stochastic equilibrium model

Existence of equilibrium

Assume (III) that

1. (a) all player's risk functions have form $\mathrm{cV@R}_{\Pi,\beta_i}$ for same Π but possibly different $\beta_i \in (0,1)$
   (b) $\Pi_\omega \geq \epsilon$ a.e. for some $\epsilon > 0$
2. some technical conditions, e.g., $P^{\mathrm{O}}(x) : L_{p'}(\Theta) \to L_\infty(\Omega)$ has closed range in $L_1(\Omega)$ whenever $\sum_i x_i \geq d$.

Assumption (III) part 1 requires overlap in sets D_i where $\rho_i = \sigma_{D_i}$: players' risk profiles are similar enough to avoid **arbitrage**.
(The conjecture holds if there are only finitely many options.)

Let $\mathcal{E}(x)$ be set of

$$\left(\Pi_1 \nabla_{x_1} H_1(x_1, \Lambda), \ldots, \Pi_N \nabla_{x_N} H_N(x_N, \Lambda)\right) \in \mathbb{R}^N$$

where $\Lambda = \Lambda(x)$, $P^{\mathrm{O}} = P^{\mathrm{O}}(x)$, and ((11)) & (12) hold.

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Existence of equilibria

Conjecture

The set mapping $\mathcal{E} : \mathbb{R}^N \to \mathbb{R}^N$ is upper semicontinuous and has nonempty, compact, convex values.

This allows us to convert the optimality conditions (12)-(14) and capacity market condition (8) into a standard fixed point setting. Main result becomes application of Kakutani's fixed point theorem.

Theorem

Let Assumptions (I), (II) and (III) hold. If each k_i is strongly convex then there exists a solution $x^* = (x_1^*, \ldots, x_N^*) \in \mathbb{R}^N$ of the reduced stochastic investment game with options trading.

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Extensions

Many technical/practical extensions possible

- Multiple time periods, c.f. [Ehrenmann-Smeers-07]
- several commodities: x_i lies in $\mathbb{R}^n$ not $\mathbb{R}$
- Player i's problem may directly depend on other players decisions x_{-i} or Z_{-i}
- nonsmooth (but still convex) objective functions and constraints
- More complex constraints under suitable constraint qualification

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. Second, writing $W = (W_1, \ldots, W_N)$, cost C^{O} is dual multiplier of above constraint in the **hedging problem**

$$\begin{array}{ll} \min\limits_{W} & \sum_{i=1}^{N} \rho_i\big(H_i(x_i, \Lambda) - P^{\mathrm{O}}(x) W_i\big) \\ \text{subject to} & (11) \end{array} \quad (15)$$

No arbitrage

Most technical part of analysis occurs here.