Error Bounds for P-matrix Linear Complementarity Problems and Their Applications

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Outline

Computational Global Error Bounds for P-matrix LCP
Math. Programming 2006 (with S. Xiang)

Perturbation Error Bounds for P-matrix LCP
SIAM J. Optimization 2007 (with S. Xiang)

Non-Lipschitzian NCP
Math. Comp. 2008 (with G. Alefeld)

Extended Vertical LCP
Comp. Optim. Appl. 2008 (with C. Zhang & N. Xiu)

Part I

Linear Complementarity Problem

to find a vector $x \in R^n$ such that

$$Mx + q \geq 0, \quad x \geq 0, \quad x^T(Mx + q) = 0,$$

where $M \in R^{n \times n}$ and $q \in R^n$. We denote this problem by LCP(M, q) and its solution by x^*.

M **is called**

P-matrix, if $\max_{1 \leq i \leq n} x_i(Mx)_i > 0$ for all $x \neq 0$;

M-matrix, if $M^{-1} \geq 0$, $M_{ij} \leq 0$ $(i \neq j)$ for $i, j = 1, 2, \ldots, n$;

H-matrix, if its comparison matrix is an M-matrix.

- Natural Residual:

$$r(x) := \min(x, Mx + q)$$

- Global error bound if there exists a constant τ such that

$$\|x - x^*\| \leq \tau \|r(x)\|, \quad \forall x \in R^n.$$

- Question: How to compute τ?

Mathias-Pang Error Bound(1990)

M is a P-matrix

$$\|x - x^*\|_\infty \le \frac{1 + \|M\|_\infty}{c(M)} \|r(x)\|_\infty,$$

for any $x \in R^n$, where

$$c(M) = \min_{\|x\|_\infty = 1} \left\{ \max_{1 \le i \le n} x_i (Mx)_i \right\}.$$

M is an H-matrix with positive diagonals

$$c(M) \ge \frac{(\min_i b_i)(\min_i (\tilde{M}^{-1} b)_i)}{(\max_j (\tilde{M}^{-1} b)_j)^2} =: \tilde{c}(M, b),$$

for any vector $b > 0$, where $\tilde{M}$ is the comparison matrix of M, that is

$$\tilde{M}_{ii} = M_{ii} \qquad \tilde{M}_{ij} = -|M_{ij}| \quad \text{for} \quad i \ne j.$$

$$\mu(b, M) := \frac{1 + \|M\|_\infty}{\tilde{c}(M, b)} \ge \frac{1 + \|M\|_\infty}{c(M)}$$

New Error Bound

M is a P-matrix,

$$\|x - x^*\|_p \le \max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_p \|r(x)\|_p,$$

where $D = \text{diag}(d_1, d_2, \ldots, d_n)$.

M is an H-matrix with positive diagonals,

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_p \le \|\tilde{M}^{-1} \max(\Lambda, I)\|_p$$

M is an M-matrix,

$$\max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_1 = \max_{v \in V} f(v).$$

$$V = \{v \,|\, M^T v \le e, v \ge 0\}$$

$$f(v) = \max_{1 \le i \le n} (e + v + M^T v)_i.$$

Comparison M is a P-matrix,

$$
\begin{aligned}
& \frac{1}{1+\|M\|_\infty}\|r(x)\|_\infty \quad \text{(Mathias-Pang)} \\
\leq & \frac{1}{\max(1,\|M\|_\infty)}\|r(x)\|_\infty \text{(Cottle-Pang-Stone)} \\
= & \frac{1}{\max\limits_{d\in[0,1]^n}\|I-D+DM\|_\infty}\|r(x)\|_\infty \\
\leq & \|x-x^*\|_\infty \\
\leq & \max_{d\in[0,1]^n}\|(I-D+DM)^{-1}\|_\infty\|r(x)\|_\infty \\
\leq & \frac{\max(1,\|M\|_\infty)}{c(M)}\|r(x)\|_\infty \\
= & \frac{1+\|M\|_\infty}{c(M)}\|r(x)\|_\infty - \frac{\min(1,\|M\|_\infty)}{c(M)}\|r(x)\|_\infty \\
\leq & \frac{1+\|M\|_\infty}{c(M)}\|r(x)\|_\infty \qquad \text{(Mathias-Pang).}
\end{aligned}
$$

M is an H-matrix with positive diagonals

$$
\begin{aligned}
&\|x - x^*\|_\infty \\
&\le \max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_\infty \|r(x)\|_\infty \\
&\le \|\tilde{M}^{-1} \max(\Lambda, I)\|_\infty \|r(x)\|_\infty \\
&\le (\mu(M,b) - \|\tilde{M}^{-1} \min(\Lambda, I)\|_\infty)\|r(x)\|_\infty \\
&\le \mu(M,b)\|r(x)\|_\infty \quad \text{(Mathias-Pang).}
\end{aligned}
$$

M is an M-matrix

$$
\begin{aligned}
&\|x - x^*\|_\infty \\
&\leq \|M^{-1}\max(\Lambda, I)\|_\infty \|r(x)\| \\
&\leq \left(\frac{1 + \|M\|_\infty}{c(M)} - \|M^{-1}\min(\Lambda, I)\|\right)\|r(x)\|_\infty \\
&\leq \frac{1 + \|M\|_\infty}{c(M)}\|r(x)\|_\infty \qquad \text{(Mathias-Pang)}
\end{aligned}
$$

Numerical Examples

Example 1.(P-matrix) Schäfer (2004)

$$M = \begin{pmatrix} 1 & -4 \\ 5 & 7 \end{pmatrix}.$$

M is a P-matrix but not an H-matrix.

New Error Bound

$$\max_{d \in [0,1]^2} \|(I - D + DM)^{-1}\|_\infty = 5,$$

Mathias-Pang Error Bound

$$\frac{1 + \|M\|_\infty}{c(M)} \geq 13.$$

Example 2.(H-matrix) Cottle(1992)

$$M = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \text{where} \quad |t| \geq 1.$$

New Error bound

$$\max_{d \in [0,1]^2} \|(I - D(I-M))^{-1}\|_p$$

$$= \max_{d_1 \in [0,1]} (1 + d_1|t|)$$

$$= \|\tilde{M}^{-1} \max(I, \Lambda)\|_p$$

$$= 1 + |t|, \qquad p = 1, \infty.$$

Mathias-Pang Error Bound

$$\frac{1 + \|M\|_\infty}{c(M)} \geq t^2(2 + |t|) = O(t^3).$$

Example 3. ($M-$matrix)

$$M = \begin{pmatrix} b+\alpha\sin(\frac{1}{n}) & c & & & \\ a & b+\alpha\sin(\frac{2}{n}) & c & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & c \\ & & & a & b+\alpha\sin(1) \end{pmatrix}$$

Table 1.

α	a	b	c	κ_1	$\Vert M^{-1}\max(\Lambda, I)\Vert_\infty$	$\mu(M,e)$
0	-1	2	-1	2.0100e4	4.0200e4	2.0201e7
n^{-2}	-1.5	2	-0.5	3.9920e2	7.8832e2	1.5536e6
n^{-2}	-1.5	2.2	-0.5	6.3910e0	1.0999e1	3.6557e2
1	-1.5	3.0	-1.5	2.4399e1	7.3936e1	1.8060e4

$n = 400,$

$$\kappa_1 = \max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_1$$

$$\frac{(\min_i e_i)(\min_i (M^{-1}e)_i)}{(\max_j (M^{-1}e)_j)^2} =: \tilde{c}(M, e),$$

$$\mu(M, e) := \frac{1 + \|M\|_\infty}{\tilde{c}(M, e)}$$

Part II

Perturbation Error Bounds for LCP

- x^* is the solution of $\mathrm{LCP}(M, q)$
- x is the solution of $\mathrm{LCP}(M + \triangle M, q + \triangle q)$
- Question: $\|x - x^*\| \leq ?$,

$$\frac{\|x - x^*\|}{\|x^*\|} \leq ?$$

Cottle-Pang-Stone (1992)

M is a P-matrix. The following statements hold:

(i) for any two vectors q and p in R^n,

$$\|x(M,q) - x(M,p)\|_\infty \le c(M)^{-1}\|q-p\|_\infty,$$

where

$$c(M) = \min_{\|x\|_\infty = 1} \left\{ \max_{1\le i\le n} x_i(Mx)_i \right\}.$$

(ii) for each vector $q \in R^n$, there exists a neighborhood $\mathcal{U}$ of the pair (M,q) and a constant $c_0 > 0$ such that for any $(A,b),(B,p) \in \mathcal{U}$, A, B are P-matrices and

$$\|x(A,b) - x(B,p)\|_\infty \le c_0(\|A-B\|_\infty + \|b-p\|_\infty).$$

Remark

The above constant $c(M)$ is difficult to compute, and c_0 is not specified. It is hard to use this result for verifying accuracy of a computed solution of the LCP when the data (M, q) contain errors.

New Perturbation Error Bounds

M is a P-matrix,

$$\beta_p(M) = \max_{d\in[0,1]^n} \|(I - D + DM)^{-1}D\|_p,$$

where D =diag$(d_1, d_2, \ldots, d_n)$.

Using the constant $\beta_p(M)$, we give perturbation bounds for M being a P-matrix as follows.

$$\|x(M,q) - x(M,p)\| \leq \beta_p(M)\|q - p\|,$$

$$\|x(A,b) - x(B,p)\| \leq \frac{\beta_p(M)^2\|(-p)_+\|\|A-B\|}{(1-\eta)^2} + \frac{\beta_p(M)\|b-p\|}{1-\eta},$$

and

$$\frac{\|x(M,q) - x(A,b)\|}{\|x(M,q)\|} \leq \frac{2\epsilon}{1-\eta}\beta_p(M)\|M\|$$

for $A, B \in \mathcal{M} := \{A \,|\, \beta_p(M)\|M - A\| \leq \eta < 1\}$, and

$$\|q - b\| \leq \epsilon\|(-q)_+\|.$$

- If M is a P-matrix, then for $\|\cdot\|_\infty$,

$$\beta_\infty(M) \le \frac{1}{c(M)}.$$

- M is an H-matrix with positive diagonals,

$$\beta_p(M) \le \|\tilde{M}^{-1}\|_p$$

- M is an M-matrix,

$$\beta_p(M) = \|M^{-1}\|_p.$$

- M is a symmetric positive definite matrix,

$$\beta_2(M) = \|M^{-1}\|_2.$$

M is a positive definite matrix

$$\|x(M,q)-x(M,p)\|_2 \le \|(\frac{M+M^T}{2})^{-1}\|_2\|q-p\|_2,$$

$$\|x(A,b)-x(B,p)\|_2 \le \frac{\|(\frac{M+M^T}{2})^{-1}\|_2^2\|(-p)_+\|_2\|A-B\|_2}{(1-\eta)^2} + \frac{\|(\frac{M+M^T}{2})^{-1}\|_2\|b-p\|_2}{1-\eta},$$

and

$$\frac{\|x(M,q)-x(A,b)\|_2}{\|x(M,q)\|_2} \le \frac{2\epsilon\|M\|_2}{1-\eta}\|(\frac{M+M^T}{2})^{-1}\|_2$$

for $A, B \in \mathcal{M} := \{A \mid \|(\frac{M+M^T}{2})^{-1}\|_2\|M-A\|_2 \le \eta < 1\}$.

Example 4

$$M = \begin{pmatrix} 1 & -t \\ 0 & t \end{pmatrix},$$

where $t \geq 1$.

New Error bound

$$\beta_\infty(M) = \max_{d \in [0,1]^2} \|(I - D + DM)^{-1} D\|_\infty = 2$$

Mathias-Pang Error Bound

$$\frac{1}{c(M)} \geq t \to \infty (t \to \infty).$$

Relative Perturbation Bounds for LCP

- **Linear systems**

  Suppose

$$\begin{aligned} Ax &= b, & A \in R^{n\times n}, 0 \neq b \in R^n \\ (A + \triangle A)y &= b + \triangle b, & \triangle A \in R^{n\times n}, \triangle b \in R^n \end{aligned}$$

with $\|\triangle A\| \leq \epsilon \|A\|$ and $\|\triangle b\| \leq \epsilon \|b\|$. If $\epsilon\kappa(A) = \eta < 1$ and A is nonsingular, then $A + \triangle A$ is nonsingular and

$$\frac{\|y - x\|}{\|x\|} \leq \frac{2\epsilon}{1-\eta}\kappa(A).$$

• P-Matrix LCP

Suppose
$$\min(x, Mx+q) = 0$$
$$M \in R^{n\times n}, \quad 0 \neq (-q)_+ \in R^n$$

$$\min(y, (M+\triangle M)y + q + \triangle q) = 0$$
$$\triangle M \in R^{n\times n}, \quad \triangle q \in R^n.$$

with
$$\|\triangle M\| \le \epsilon \|M\|$$

and
$$\|\triangle q\| \le \epsilon \max(\|(-q)_+\|, \|q\| - \|Mx+q\|).$$

If M is a P-matrix and $\epsilon\beta(M)\|M\| = \eta < 1$, then $M + \triangle M$ is a P-matrix and

$$\frac{\|y-x\|}{\|x\|} \le \frac{2\epsilon}{1-\eta}\beta(M)\|M\|.$$

- M is an H-matrix with positive diagonals, $\epsilon\kappa_\infty(\tilde{M}) = \eta < 1$, and

$$\|\triangle M\|_\infty \leq \epsilon\|\tilde{M}\|_\infty$$

and

$$\|\triangle q\|_\infty \leq \epsilon \max(\|(-q)_+\|_\infty, \|q\|_\infty - \|Mx + q\|_\infty)$$

then $M + \triangle M$ is an H-matrix with positive diagonals and

$$\frac{\|y - x\|_\infty}{\|x\|_\infty} \leq \frac{2\epsilon}{1-\eta}\kappa_\infty(\tilde{M}).$$

- M is a symmetric positive definite matrix, $\epsilon\kappa_2(M) = \eta < 1$, and

$$\|\triangle M\|_2 \leq \epsilon\|M\|_2$$

and

$$\|\triangle q\|_2 \leq \epsilon \max(\|(-q)_+\|_2, \|q\|_2 - \|Mx + q\|_2),$$

then $M + \triangle M$ is a P-matrix and

$$\frac{\|y - x\|_2}{\|x\|_2} \leq \frac{2\epsilon}{1-\eta}\kappa_2(M).$$

- M is a positive definite matrix,

$$\epsilon\kappa_2(\frac{M+M^T}{2}) = \eta < 1$$

and

$$\|\triangle M\|_2 \leq \epsilon\|\frac{M+M^T}{2}\|_2$$

and

$$\|\triangle q\|_2 \leq \epsilon \max(\|(-q)_+\|_2, \|q\|_2 - \|Mx+q\|_2) \cdot \frac{\|M+M^T\|_2}{2\|M\|_2},$$

then $M + \triangle M$ is a positive matrix, and

$$\frac{\|x-y\|_2}{\|x\|_2} \leq \frac{2\epsilon}{1-\eta}\kappa_2(\frac{M+M^T}{2}).$$

Remark

The above inequalities indicate that the constant $\beta(M)\|M\|$ is a measure of sensitivity of the solution $x(M,q)$ of the LCP(M,q). Moreover, it is interesting to see that the measure is expressed in the terms of the condition number of M, that is,

$$\kappa_p(M) := \|M^{-1}\|_p\|M\|_p = \beta_p(M)\|M\|_p$$

for M being an M-matrix with $p \geq 1$ and a symmetric positive definite matrix with $p = 2$. Hence, it makes connection between perturbation bounds of the LCP and perturbation bounds of the systems of linear equations in the Newton-type methods for solving the LCP.

Newton-type method

$$(I - D_k + D_k M)(x - x^k) = -r(x^k), \tag{1}$$

or

$$\begin{pmatrix} M & -I \\ I - D_k & D_k \end{pmatrix} \begin{pmatrix} x - x^k \\ y - y^k \end{pmatrix} = -F(x^k, y^k), \tag{2}$$

where D_k is a diagonal matrix whose diagonal elements are in $[0, 1]$.

Sensitivity of (**??**) and (**??**) will effect implementation of the methods and reliability of the computed solution.

Proposition

For any diagonal matrix D=diag(d) with $0 \leq d_i \leq 1$, $i = 1, 2, \ldots, n$, the following inequalities hold

$$\kappa_\infty \begin{pmatrix} M & -I \\ I-D & D \end{pmatrix} \geq \kappa_\infty(I - D + DM)$$

and

$$\kappa_p \begin{pmatrix} M & -I \\ I-D & D \end{pmatrix} \geq \frac{1}{2}\kappa_p(I - D + DM), \qquad p \geq 1.$$

$$K_p(M) := \max_{d\in[0,1]^n} \|(I - D + DM)^{-1}\|_p \cdot \|I - D + DM\|_p.$$

$$\hat{K}_p(M) := \max_{d\in[0,1]^n} \| \begin{pmatrix} M & -I \\ D & I-D \end{pmatrix}^{-1} \| \cdot \| \begin{pmatrix} M & -I \\ D & I-D \end{pmatrix} \|.$$

$\hat{K}_\infty(M) \geq K_\infty(M)$.

Example 5

Let $M = aI(a \geq 1)$,

$$\begin{aligned}
\hat{K}_\infty(M) &\geq \kappa_\infty \begin{pmatrix} M & -I \\ I & 0 \end{pmatrix} \\
&= \| \begin{pmatrix} aI & -I \\ I & 0 \end{pmatrix} \|_\infty \| \begin{pmatrix} aI & -I \\ I & 0 \end{pmatrix}^{-1} \|_\infty \\
&= (1+a) \| \begin{pmatrix} 0 & I \\ -I & aI \end{pmatrix} \|_\infty \\
&= (1+a)^2
\end{aligned}$$

and

$$\begin{aligned}
K_\infty(M) &= \max_{d \in [0,1]^n} \|(I - D + DM)^{-1}\|_\infty \cdot \|I - D + DM\|_\infty \\
&\leq \frac{\max_{0 \leq \xi \leq 1} |(1 + a\xi - \xi)|}{\min_{0 \leq \xi \leq 1} |(1 + a\xi - \xi)|} = a.
\end{aligned}$$

For large a, $\hat{K}_\infty(M) - K_\infty(M) \geq a^2 + a + 1$ is very large.

Connection with Sensitivity of LCP(M, q)

M being an M-matrix with $\|M\|_\infty \geq 1$

$$\kappa_\infty(M) \leq K_\infty(M) \leq \kappa_\infty(M)\|\max(\Lambda, I)\|_\infty.$$

The condition number $\kappa_\infty(M)$ is a measure of sensitivity of the solution of the system of linear equations for the worst case. Note that we have shown that $\kappa_\infty(M)$ is a measure of sensitivity of the solution of LCP. Hence we may suggest that if Λ is not large, then the LCP is well-conditioned if and only if the system of linear equations (**??**) at each step of the Newton method is well-conditioned.

Numerical Examples

- $\|r(x)\| \leq 10^{-14}$
- $macheps = 10^{-16}$

Example 6 (Free boundary problem for journal bearings(Bierlein, 1975).

$$m_{ij} = \begin{cases} -h^3_{i+\frac{1}{2}}, & j = i+1, \\ h^3_{i-\frac{1}{2}} + h^3_{i+\frac{1}{2}}, & j = i, \\ -h^3_{i-\frac{1}{2}}, & j = i-1, \\ 0, & \text{otherwise} \end{cases} \quad i, j = 1, \cdots, n$$

$$q_i = \delta(h_{i+\frac{1}{2}} - h_{i-\frac{1}{2}}), \qquad i = 1, 2, \cdots, n.$$

$\delta = \frac{2}{n+1}$, $\epsilon = 0.8$ and

$$h_{i-\frac{1}{2}} = \frac{1 + \epsilon \cos(\pi(i - \frac{1}{2})\delta)}{\sqrt{\pi}}, \qquad i = 1, 2, \cdots, n+1.$$

Let

$$\triangle M = \epsilon_M \begin{pmatrix} 2 & 1 & & & \\ 1 & 2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2 & 1 \\ & & & 1 & 2 \end{pmatrix}, \qquad \triangle q = \epsilon_q e.$$

Table 2. Perturbation bounds of Example 6 ($\mu = \max(M_{ii}) + 1$)

n	ϵ_M	ϵ_q	$\kappa_\infty(M)$	$\mu\kappa_\infty(M)$	$\|\triangle x\|_\infty$	bound
10	0	-0.001	498.0448	1.3085e3	0.1740	0.2201
	0.001	0.001	498.0448	1.3085e3	0.7768	2.1288
	-0.001	-0.001	498.0448	1.3085e3	1.2196	3.8864
100	0	-0.001	1.0216e5	3.1546e5	23.4828	24.6533
	1.0e-5	0.001	1.0216e5	3.1546e5	2.5249	24.6533
	-1.0e-5	-0.001	1.0216e5	3.1546e5	51.4563	76.8289
1000	0	-1.0e-5	1.0168e7	3.1465e7	23.0367	24.2724
	1.0e-7	1.0e-5	1.0168e7	3.1465e7	2.5224	24.2724
	-1.0e-7	-1.0e-5	1.0168e7	3.1465e7	49.7432	73.9102

Example 7 (Ahn,1983).

We consider a tridiagonal H-matrix

$$M = \begin{pmatrix} 4 & -2 & & & \\ 1 & 4 & -2 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & -2 \\ & & & 1 & 4 \end{pmatrix}, \qquad \text{and} \quad q = -4e.$$

Notice that M is well-conditioned for any n. From our analysis, LCP(M, q) is not sensitive to small changes in data. Let $\triangle M$ and $\triangle q$ be defined in Example 6.

Table 3. Perturbation analysis of Example 7

$$\beta(\tilde{M}) = \|\tilde{M}^{-1}\|_\infty, \quad \nu = \max(1, \|M\|_\infty)\|\tilde{M}^{-1}\max(\Lambda, I)\|_\infty$$

n	ϵ_M	ϵ_q	$\beta(\tilde{M})\|M\|_\infty$	ν	$\|\triangle x\|_\infty$	bound
10	0.0	-1.0e-3	6.7828	27.1216	4.0812e-4	9.6899e-4
	1.0e-3	1.0e-3	6.7828	27.1216	2.4000e-3	7.3000e-3
	-1.0e-3	-1.0e-3	6.7828	27.1216	2.4000e-3	7.3000e-3
100	0.0	-1.0e-3	7.0000	28.0000	4.0825e-4	1.0000e-3
	1.0e-5	1.0e-3	7.0000	28.0000	4.2838e-4	1.1000e-3
	-1.0e-5	-1.0e-3	7.0000	28.0000	4.2838e-4	1.1000e-3
1000	0.0	-1.0e-5	7.0000	28.0000	4.0825e-6	1.0000e-5
	1.0e-7	1.0e-5	7.0000	28.0000	4.2839e-6	1.0653e-5
	-1.0e-7	-1.0e-5	7.0000	28.0000	4.2839e-6	1.0653e-5
10000	0.0	-1.0e-5	7.0000	28.0000	4.0825e-6	1.0000e-5
	1.0e-7	1.0e-5	7.0000	28.0000	4.2839e-6	1.0653e-5
	-1.0e-7	-1.0e-5	7.0000	28.0000	4.2839e-6	1.0653e-5

Applications

Non-Lipschitizan NCP

$$F(x) \geq 0, \;\; x \geq 0, \;\; x^T F(x) = 0$$

Extended Vertical LCP

$$\min(M_0 x + q_0, M_1 x + q_1, \ldots\ldots, M_m x + q_m) = 0$$

Stochastic LCP

$$M(\omega)x + q(\omega) \geq 0, x \geq 0, x^T(M(\omega) + q(\omega)) = 0$$

$$\min_{x \geq 0} E\left\| \min(M(\omega)x + q(\omega), x \right\|$$