Some generalizations of Rockafellar's surjectivity theorem

J.E. Martínez-Legaz
Universitat Autònoma de Barcelona, Spain

The Second International Conference on Nonlinear Programming with Applications

April 7, 2008
Beijing, China

$X \neq \{0\}$ reflexive real Banach space, X^* its dual

$\langle \cdot, \cdot \rangle : X \times X^* \longrightarrow \mathbb{R}$ the duality pairing

$J = \partial \frac{1}{2} \|\cdot\|^2$, the duality mapping

For $x \in X, \quad J(x) = \left\{ x^* \in X^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2 \right\}.$

THEOREM. (Rockafellar, 1970). *Let $A : X \rightrightarrows X^*$ be a monotone operator. In order that A be maximal, it is necessary and sufficient that $R(A + J)$ be all of X^*.*

$$\begin{aligned} A &: X \rightrightarrows X^* \\ \varphi_A &: X \times X^* \longrightarrow \mathbb{R} \cup \{+\infty\} \\ \varphi_A(x, x^*) &= \langle x, x^* \rangle - \inf_{(y,y^*) \in Graph(A)} \langle x - y, x^* - y^* \rangle \end{aligned}$$

(Fitzpatrick, 1988)

If A is maximal monotone,

$$\begin{aligned} \varphi_A(x, x^*) &\geq \langle x, x^* \rangle \qquad \forall \ (x, x^*) \in X \times X^*. \\ \varphi_A(x, x^*) &= \langle x, x^* \rangle \qquad \Longleftrightarrow \qquad (x, x^*) \in Graph(A). \end{aligned}$$

φ_A is the smallest convex representation of A.

$$\sigma_A = cl\ conv\left(\langle\cdot,\cdot\rangle + \delta_{Graph(A)}\right)$$

(Burachik-Svaiter, 2002)

THEOREM. (ML-Svaiter, 2005). *Let* $A : X \rightrightarrows X^*$. *Then*

$$A \text{ is monotone} \qquad \Longleftrightarrow \qquad \sigma_A \geq \langle\cdot,\cdot\rangle\,.$$

Example: $Graph\,(A) = \{(0,0)\} \qquad \varphi_A \equiv 0$

If A is maximal monotone,
σ_A is the largest convex representation of A.

$$\begin{array}{rcll} \varphi_A\,(x,x^*) & = & \sigma_A^*\,(x^*,x) & \qquad \forall\ (x,x^*) \in X \times X^* \\ \sigma_A\,(x,x^*) & = & \varphi_A^*\,(x^*,x) & \qquad \forall\ (x,x^*) \in X \times X^* \end{array}$$

THEOREM. (Rockafellar, 1966). Suppose that $f, g : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ are l.s.c. proper convex functions.

If the domain of one of these functions contains an interior point of the domain of the other, then

$$\inf_{x \in X} \{f(x) + g(x)\} = \max_{x^* \in X^*} \{-f^*(x^*) - g^*(-x^*)\}.$$

THEOREM. (Simons, 1998)
Let $A : X \rightrightarrows X^*$ be monotone.

Then A is maximal monotone if and only if

$$Graph(A) + Graph(-J) = X \times X^*.$$

THEOREM. For every monotone operator $A : X \rightrightarrows X^*$, the following statements are equivalent:

(1) A is maximal monotone.

(2) $Graph\,(A) + Graph\,(-B) = X \times X^*$ for every maximal monotone operator $B : X \rightrightarrows X^*$ such that φ_B is finite-valued.

(3) There exists a maximal monotone operator $B : X \rightrightarrows X^*$ such that φ_B is finite-valued, $Graph\,(A) + Graph\,(-B) = X \times X^*$, and there exists $(p, p^*) \in Graph\,(B)$ such that

$$\begin{aligned} \langle p - y, p^* - y^* \rangle \;&>\; 0 \\ \forall \; (y, y^*) \;&\in\; Graph\,(B) \setminus \{(p, p^*)\}\,. \end{aligned}$$

Proof of (1) $\Longrightarrow$ (2).
Let $\left(x_0, x_0^*\right) \in X \times X^*$.
Define $A' : X \rightrightarrows X^*$ by

$$Graph\left(A'\right) := Graph\left(A\right) - \left(x_0, x_0^*\right)$$

and $h : X \times X^* \longrightarrow \mathbb{R} \cup \{+\infty\}$ by

$$h\left(x, x^*\right) := \varphi_B\left(-x, x^*\right).$$

$$\sigma_{A'}\left(x, x^*\right) + h\left(x, x^*\right) \geq \langle x, x^* \rangle + \langle -x, x^* \rangle = 0$$

There exists $(y, y^*) \in X \times X^*$ such that

$$\varphi_{A'}\left(y, y^*\right) + h^*\left(-y^*, -y\right) \leq 0$$

$$\begin{aligned} \varphi_{A'}\left(y, y^*\right) + h^*\left(-y^*, -y\right) &= \varphi_{A'}\left(y, y^*\right) + \sigma_B\left(-y, y^*\right) \\ &\geq \langle y, y^* \rangle + \langle -y, y^* \rangle = 0 \end{aligned}$$

$$\varphi_{A'}\left(y, y^*\right) = \langle y, y^* \rangle \text{ and } \sigma_B\left(-y, y^*\right) = \langle -y, y^* \rangle$$

$$\left(y, y^*\right) \in Graph\left(A'\right) \text{ and } \left(-y, y^*\right) \in Graph\left(B\right)$$

$$\begin{aligned} \left(x_0, x_0^*\right) &= \left(x_0, x_0^*\right) + \left(y, y^*\right) + \left(-y, -y^*\right) \\ &\in \left(x_0, x_0^*\right) + Graph\left(A'\right) + Graph\left(-B\right) \\ &= Graph\left(A\right) + Graph\left(-B\right). \end{aligned}$$

B satisfies the Brézis-Haraux condition if

$$\inf_{(y,y^*)\in Graph(B)} \langle x-y, x^*-y^*\rangle \;>\; -\infty$$
$$\forall\ (x,x^*) \;\in\; D(B)\times R(B)\,.$$

φ_B is finite-valued $\Longrightarrow$ B satisfies the B-H condition

THEOREM (Torralba, 1996).
For every maximal monotone operator $B : X \rightrightarrows X^*$, if $\alpha, \beta > 0$ and $(x, x^*) \in X \times X^*$ are such that

$$\inf_{(y,y^*)\in Graph(B)} \langle x-y, x^*-y^*\rangle \geq -\alpha\beta$$

then there exists $(z, z^*) \in Graph(B)$ such that

$$\|z-x\| \leq \alpha \text{ and } \|z^*-x^*\| \leq \beta.$$

COROLLARY. Let $B : X \rightrightarrows X^*$ be maximal monotone. If φ_B is finite-valued then $D(B)$ and $R(B)$ are dense.

A l.s.c. proper convex function $f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ is *supercoercive* if

$$\lim_{\|x\| \longrightarrow \infty} \frac{f(x)}{\|x\|} = +\infty.$$

f is *supercoercive* $\quad \Longleftrightarrow \quad$ f^* is finite-valued

COROLLARY.
Let $A : X \rightrightarrows X^*$ be a monotone operator and $f : X \longrightarrow \mathbb{R}$ be a supercoercive l.s.c. proper convex function.

If A is maximal monotone
then $Graph\,(A) + Graph\,(-\partial f) = X \times X^*$.

Conversely,
if $Graph\,(A) + Graph\,(-\partial f) = X \times X^*$ and
at least one of the functions f and f^* is Gateaux differentiable at some point
then A is maximal monotone.

$A : X \rightrightarrows X^*$ is *strictly monotone* if

for $x, y \in X$ with $x \neq y,\ x^* \in A(x)$ and $y^* \in A(y)$,
$$\langle x - y, x^* - y^* \rangle > 0.$$

LEMMA.
If $A : X \rightrightarrows X^*$ is monotone and
$B : X \rightrightarrows X^*$ is strictly monotone
then $(A + B)$ is strictly monotone and
hence $(A + B)^{-1}$ is single-valued on its domain.

COROLLARY.
Let $A : X \rightrightarrows X^*$ be a monotone operator and
$B : X \rightrightarrows X^*$ be a maximal monotone operator with finite-valued Fitzpatrick function φ_B.

If A is maximal monotone then $R(A + B) = X^*$.

Conversely, if B is single-valued and strictly monotone and $R(A + B) = X^*$ then A is maximal monotone.

THEOREM.
Let $A : X \rightrightarrows X^*$ be a monotone operator and
$f : X \longrightarrow \mathbb{R}$ be a l.s.c. proper convex function.

If f is Gateaux differentiable everywhere,
$R(A + \nabla f) = X^*$, and
∂f^* is single-valued on its domain
then A is maximal monotone.

Conversely, if A is maximal monotone and
f is supercoercive
then $R(A + \partial f) = X^*$.

PROPOSITION.
Let $A : X \rightrightarrows X^*$ be a monotone operator and
$f : X \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a l.s.c. proper convex function.

If $R\left(A\left(\cdot + w\right) + \partial f\right) = X^*$ for all $w \in X$ and
both f and f^* attain their unique global minima at the origin
then A is maximal monotone.

Conversely, if A is maximal monotone and
f is supercoercive
then $R\left(A\left(\cdot + w\right) + \partial f\right) = X^*$ for all $w \in X$.