Numerical Simulation of Flows in Highly Heterogeneous Porous Media

R. Lazarov,
Y. Efendiev, J. Galvis, K. Shi, J. Willems

The Second International Conference on Engineering and Computational Mathematics (ECM2013)
Hong Kong, December 18, 2013

Thanks: NSF, KAUST, IAMCS, deal.II
Outline

1. Introduction
   - Motivation and Problem Formulation
   - State of the art

2. Numerical Upscaling of Brinkman equations
   - Single Grid Approximation of Brinkman equations
   - Subgrid Approximation and Its Performance
   - Numerical Experiments

3. Hybridizable discontinuous Galerkin method (HDG)
Outline

1. Introduction
   - Motivation and Problem Formulation
   - State of the art

2. Numerical Upscaling of Brinkman equations
   - Single Grid Approximation of Brinkman equations
   - Subgrid Approximation and Its Performance
   - Numerical Experiments

3. Hybridizable discontinuous Galerkin method (HDG)
Motivation: media at multiple scales

Figure: Porous media: real-life scale and macro scale
Motivation: open industrial foams – media of low solid fraction

Figure: Industrial foams on microstructure scale; porosity over 93%

Figure: Trabecular bone: macro- and real life scales
Computer Generated Distribution of the Heterogeneous Permeability

Figure: periodic + rand; rand + rand, SPE10 slice

Another dimension of difficulty add orthotropic materials.
Computer Generated Distribution of the Heterogeneous Permeability

Figure: SPE10 3 dimensional permeability distributions
Reconstruction of micro-structures from CT scans

Figure: 3-dimensional micro-structure of cathode consisting of lithium iron phosphate (green) and carbon (black) and its use in a schematic construction of a battery
Flows in porous media are modeled by linear Darcy law that relates the macroscopic pressure $p$ and velocity $u$:

$$\nabla p = -\mu \kappa^{-1} u, \quad \kappa \quad \text{permeability}, \quad \mu \quad \text{viscosity} \quad (1)$$

This model is generally used in hydrology, reservoir modeling, and industrial processes.

In real problems exhibiting multiscale behavior $\kappa$ is:

1. highly varying by changing orders of magnitude;
2. highly heterogeneous and often anisotropic;
3. could depend also on a stochastic variable, i.e. $\kappa(x, \omega)$, where $\omega$ is in a high dimensional space.
(2) To fit better the data for flows near wells a nonlinear correction was proposed by Forcheimer, $\beta > 0$,

$$\nabla p = -\mu \kappa^{-1} u - \beta |u| u, \quad \beta > 0 \text{ fitting parameter.} \quad (2)$$

(3) For flows in highly porous media Brinkman (1947) enhanced Darcy's law by adding dissipative term scaled by viscosity:

$$\nabla p = -\mu \kappa^{-1} u + \mu \Delta u. \quad (3)$$
(2) To fit better the data for flows near wells a nonlinear correction was proposed by Forcheimer, $\beta > 0$,

$$\nabla p = -\mu \kappa^{-1} u - \beta |u| u, \quad \beta > 0 \quad \text{fitting parameter.} \quad (2)$$

(3) For flows in highly porous media Brinkman (1947) enhanced Darcy’s law by adding dissipative term scaled by viscosity:

$$\nabla p = -\mu \kappa^{-1} u + \mu \Delta u. \quad (3)$$
(4) Another venue for a two-phase flow is a Richards model:

$$\nabla p = -\mu \kappa^{-1} u, \quad \text{where} \quad \kappa = k(x)\lambda(x, u)$$

(4)

with $k(x)$ heterogeneous function is the **intrinsic permeability**, while $\lambda(x, u)$ is a smooth function that varies moderately in both $x$ and $u$, related to the **relative permeability**.
Thus our model is the following boundary value problem:

\[-\tilde{\mu}\Delta u + \nabla p + \mu \kappa^{-1}u = f, \quad x \in \Omega\]
\[\nabla \cdot u = 0, \quad x \in \Omega\]
\[u = g, \quad x \in \Gamma\]

This system of generalized Stokes is often called Brinkman model. It contains both limits:

Darcy: \[\nabla p + \mu \kappa^{-1}u = f\]

Stokes: \[-\mu \Delta u + \nabla p = f\]

Following the Russian or French sources this could be considered also as fictitious domain method or penalty formulation.
Thus our model is the following boundary value problem:

\[-\mu \Delta u + \nabla p + \mu \kappa^{-1} u = f, \quad x \in \Omega\]
\[\nabla \cdot u = 0, \quad x \in \Omega\]
\[u = g, \quad x \in \Gamma\]

This system of generalized Stokes is often called Brinkman model. It contains both limits:

Darcy: \(\nabla p + \mu \kappa^{-1} u = f\)

Stokes: \(-\mu \Delta u + \nabla p = f\).

Following the Russian or French sources this could be considered also as fictitious domain method or penalty formulation.
Thus our model is the following boundary value problem:

\[-\tilde{\mu} \Delta u + \nabla p + \mu \kappa^{-1} u = f, \quad x \in \Omega\]
\[\nabla \cdot u = 0, \quad x \in \Omega\]
\[u = g, \quad x \in \Gamma\]

This system of generalized Stokes is often called Brinkman model. It contains both limits:

- **Darcy:** \(\nabla p + \mu \kappa^{-1} u = f\) and
- **Stokes:** \(-\mu \Delta u + \nabla p = f\).

Following the Russian or French sources this could be considered also as fictitious domain method or penalty formulation.
Darcy and Brinkman equations are introduced as macroscopic equations, **without direct link to the underlying microscopic behavior**. Allaire in 1991 studied **homogenization** of slow viscous fluid flows (with negligible no-slip effects on interface between the fluid and the solid obstacles) for periodic arrangements.

![Figure: Darcy vs. Brinkman](image-url)
What is a good method for such problems?

Our aim is development and study of numerical methods that address the following main issues of these classes of problem:

- Work well in both limits, Darcy and Brinkman;
- Work well for both linear and nonlinear highly heterogeneous problems;
- Can be used as stand alone numerical upscaling procedure;
- Can be used in construction of robust with respect to high variations of the permeability field preconditioners.
These problems are a small subclass of problems in large area of flows in porous media that include:

- single and multi-phase flows in porous media
- transport of species and reacting flows;
- reservoir modeling and simulation;
- working with multiple scales;
- share similarities with heat conduction and mass transfer.

Some of these complex problems were discussed in the Workshop 2 at this conference dedicated to Professor Mary Wheeler.
State of the Art: Contributions of Mary Wheeler

Mary Wheeler has contributed to this field in a way that only few researchers have done:
(1) She has brought in this filed some of the most innovative ideas and has authored and co-authored some of the most influential papers;
(2) Through her industrial affiliates she has introduced the new ideas and methods to the petroleum industry and has worked on various applications, in particular in multi-phase flows;
(3) She has trained, guided, and mentored a generation of numerical analysts and applied mathematicians who contributed further to the advancement of this field.
State of the Art: Available Numerical Methods and Tools

- **Numerical upscaling** as subgrid approximation methods:
  
  (1) standard FEM for Darcy: comprehensive exposition in the book of Efendiev & Hou, 2009,
  
  
  
State of the Art: Available Numerical Methods and Tools

- **Preconditioning techniques** based on coarse grid space with multiscale finite element functions:
  2. Using energy-minimizing coarse spaces: Xu & Zikatanov, 2004,
Objectives of our group led by Y. Efendiev:

1. Derive, study, implement, and test a **numerical upscaling** procedure for highly porous media that works well in both limits, Brinkman and Darcy;

2. go beyond the polynomial spaces, e.g. utilize spectral local solutions, snapshots, etc – viewed as **numerical model reduction** technique.

3. Design and study of **preconditioning techniques** for such problems for porous media of high contrast with targeted applications to oil/water reservoirs, filters, insulators, etc.
Outline

1. Introduction
   - Motivation and Problem Formulation
   - State of the art

2. Numerical Upscaling of Brinkman equations
   - Single Grid Approximation of Brinkman equations
   - Subgrid Approximation and Its Performance
   - Numerical Experiments

3. Hybridizable discontinuous Galerkin method (HDG)
Let us go back to the Brinkman equations:

\[- \tilde{\mu} \Delta u + \nabla p + \mu \kappa^{-1} u = f, \quad x \in \Omega\]
\[\nabla \cdot u = 0, \quad x \in \Omega\]
\[u = 0, \quad x \in \Gamma\]

Find \(u \in H_0^1(\Omega)^n := \mathcal{V}\) and \(p \in L_0^2(\Omega) := \mathcal{W}\) such that

\[\int_{\Omega} (\tilde{\mu} \nabla u : \nabla v + \frac{\mu}{\kappa} u \cdot v) dx + \int_{\Omega} p \nabla \cdot v dx = \int_{\Omega} f \cdot v dx \quad \forall v \in \mathcal{V}\]

\[\int_{\Omega} q \nabla \cdot u dx = 0 \quad \forall q \in L_0^2(\Omega)\]

This problem has unique solution \((u, p) \in \mathcal{V} \times \mathcal{W}\).
The finite element of Brezzi, Douglas, and Marini of degree 1

For this finite element we have:
- On a rectangle $T$ the polynomial space is characterized by

\[ \{ \mathbf{v} = P_1^2 + \text{span}\{ \text{curl}(x_1^2 x_2), \text{curl}(x_1 x_2^2) \} \}; \]

with dof

- $(\mathcal{V}_H, \mathcal{W}_H) \subset (H_0(\text{div}; \Omega), L_0^2(\Omega)) := \mathcal{V} \times \mathcal{W}$;
- Has a natural variant for $n = 3$. 
Since the tangential derivative along the internal edges will be in general discontinuous, i.e. \( \mathcal{V}_H \not\subseteq H^1_0(\Omega) \); therefore we will apply the discontinuous Galerkin method: Find \((u_H, p_H) \in (\mathcal{V}_H, \mathcal{W}_H)\) such that for all \((v_H, q_H) \in (\mathcal{V}_H, \mathcal{W}_H)\)

\[
\begin{align*}
    a(u_H, v_H) + b(v_H, p_H) &= F(v_H) \\
    b(u_H, q_H) &= 0.
\end{align*}
\] (5)

Because of the nonconformity of the FE spaces, the bilinear form \(a(u_H, v_H)\) has a special form given below.
Discretization of Wang and Ye 2007 on Single Grid

\[ b(v_H, p_H) := \int_{\Omega} p_H \nabla \cdot v_H \, dx \]

\[ F(v_H) := \int_{\Omega} f \cdot v_H \, dx \]

\[ a(u_H, v_H) := \sum_{T \in \mathcal{T}_H} \int_T (\tilde{\mu} \nabla u_H : \nabla v_H + \frac{\mu}{\kappa} u_H \cdot v_H) \, dx \]

\[ - \sum_{e \in \mathcal{E}_H} \int_e \tilde{\mu} \left( \{ u_H \} [v_H] + \{ v_H \} [u_H] - \frac{\alpha}{|e|} [u_H] [v_H] \right) \, ds \]

\[ \{ v \}|_e := \frac{1}{2} (n^+_e \cdot \nabla (v \cdot \tau^+_e))|_{e^+} + n^-_e \cdot \nabla (v \cdot \tau^-_e)|_{e^-} \]

\[ [v]|_e := v|_{e^+} \cdot \tau^+_e + v|_{e^-} \cdot \tau^-_e \]
Provided exact solution \((u, p)\) of Brinkman problem is in \((H^2(\Omega), H^1(\Omega))\) we have:

\[
\|p - p_H\|_{L^2(\Omega)} \leq CH(\|u\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)})
\]

\[
\|u - u_H\|_{L^2(\Omega)} \leq CH^2(\|u\|_{H^2(\Omega)} + \|p\|_{H^1(\Omega)}).
\]

There are two main issues in this setting:

1. This will be a very large saddle point system, if we want to resolve the finer scale of the heterogeneity by \(H\);
2. The system is very ill-conditioned due to the step-size \(H\) and the high variability of the permeability.
The idea is to approximate the boundary value problem by using two grids – coarse $\mathcal{T}_H$ and fine $\mathcal{T}_h$:

- The fine triangulation $\mathcal{T}_h$ is obtained by refinement of the coarse $\mathcal{T}_H$ and resolves the media heterogeneities;
- The desire is to formulate a global problem associated with the coarse mesh only.
The MSFEM for second order problems construct the space $S_H \subset H^1_0(\Omega)$ of piecewise polynomials over the coarse mesh $\mathcal{T}_H$ and the spaces $S_h \subset H^1_0(K)$ for each $K \in \mathcal{T}_h$. Then the fine-grid component $u_h$ is eliminated (via say static condensation) and we end up with a problem on the coarse grid component $u_H$.

Then the multiscale solution has the form $u_H + u_h$, where $u_H \in S_H$ and $u_h \in S_h$.

Unfortunately, this simple idea is not going to work for Brinkman system. We need to do better.
The approximation spaces use the two grids $\mathcal{T}_h$ and $\mathcal{T}_H$ and are constructed in the following way:

- $W_h$ and $V_h$ consist of bubbles on each $T \in \mathcal{T}_H$;
- the FE spaces on the composite mesh are $W_{H,h} = W_h \oplus W_H \subset L^2_0$, and $V_{H,h} = V_h \oplus V_H \subset H_0(\text{div})$

**Crucial properties:**

1. $\nabla \cdot V_h = W_h$ and $\nabla \cdot V_H = W_H$
2. $v_h \cdot n = 0$ on $\partial T$, $\forall v_h \in V_h$ and $\partial T \in \mathcal{T}_H$
3. $W_H \perp W_h$
Unique decomposition in components yields: find $u_H + u_h \in W_h \oplus W_H$, and $p_H + p_h \in V_h \oplus V_H$ such that

$$a(u_H + u_h, v_H + v_h) + b(v_H + v_h, p_H + p_h) = F(v_H + v_h),$$
$$b(u_H + u_h, q_H + q_h) = 0,$$

for all $v_H + v_h \in W_h \oplus W_H$ and $q_H + q_h \in V_h \oplus V_H$.

Our goal is to derive numerical upscaling procedure.
Splitting of the Solution

Equivalently, testing separately for \( \mathbf{v}_H, q_H \) and \( \mathbf{v}_h, q_h \), we get

\[
(*) \left\{ \begin{array}{lcl}
a (\mathbf{u}_H + \mathbf{u}_h, \mathbf{v}_H) + b (\mathbf{v}_H, \rho_H + \rho_h) &=& F(\mathbf{v}_H) \quad \forall \mathbf{v}_H \in \mathcal{V}_H \\
b (\mathbf{u}_H + \mathbf{u}_h, q_H) &=& 0 \quad \forall q_H \in \mathcal{W}_H
\end{array} \right.
\]

\[
(**) \left\{ \begin{array}{lcl}
a (\mathbf{u}_H + \mathbf{u}_h, \mathbf{v}_h) + b (\mathbf{v}_h, \rho_H + \rho_h) &=& F(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathcal{V}_h \\
b (\mathbf{u}_H + \mathbf{u}_h, q_h) &=& 0 \quad \forall q_h \in \mathcal{W}_h
\end{array} \right.
\]

Since

\[ b (\mathbf{v}, q) = (\nabla \cdot \mathbf{v}, q), \quad \nabla \cdot \mathcal{V}_h = \mathcal{W}_h, \quad \nabla \cdot \mathcal{V}_H = \mathcal{W}_H, \quad \mathcal{W}_H \perp \mathcal{W}_h. \]

If we knew \( \mathbf{u}_H \) then we could find easily \( \mathbf{u}_h \) from

\[ a (\mathbf{u}_h, \mathbf{v}_h) + b (\mathbf{v}_h, \rho_h) + b (\mathbf{u}_h, q_h) = F(\mathbf{v}_h) - a (\mathbf{u}_H, \mathbf{v}_h) \]
\[ a(u_h, v_h) + b(v_h, p_h) + b(u_h, q_h) = F(v_h) - a(u_H, v_h) \]

Decompose (**) further into

\[
\begin{align*}
\{ & \quad a(\delta_u(u_H), v_h) + b(v_h, \delta_p(u_H)) = -a(u_H, v_h) \quad \forall v_h \in \mathcal{V}_h \\
& \quad b(\delta_u(u_H), q_h) = 0 \quad \forall q_h \in \mathcal{W}_h \\
\} &\quad (**)
\end{align*}
\]

Note that

- \((\delta_u(u_H), \delta_p(u_H))\) is \textbf{linear} in \(u_H\)
- \((\bar{\delta}_u, \bar{\delta}_p)\) and \((\delta_u(u_H), \delta_p(u_H))\) can be computed \textbf{locally}
The Upscaled Equation

We have
\[(u_h, p_h) = (\bar{\delta}u + \delta u(u_H), \bar{\delta}p + \delta p(u_H)).\]

Putting all these together we obtain:

Symmetric Form of the Upscaled Equation

Thus we get
\[a(u_H + \delta u(u_H), v_H + \delta u(v_H)) + b(v_H, p_H) = F(v_H) - a(\bar{\delta}u, v_H),\]
\[b(u_H, q_H) = 0,\]

which involves only coarse-grid degrees of freedom.
Sub-grid Algorithm (numerical upscaling): Willems, 2009

1. Solve for the fine responses \((\overline{\delta u}, \overline{\delta p})\) and \((\delta u(\varphi_H), \delta p(\varphi_H))\) for coarse basis functions \(\varphi_H\) and each coarse cell.

2. Solve the upscaled equation for \((u_H, p_H)\).

3. Piece together the solutions to get
\[
(u_{H,h}, p_{H,h}) = (u_H, p_H) + (\delta u(u_H), \delta p(u_H)) + (\overline{\delta u}, \overline{\delta p}).
\]

For pure Darcy this reduces to the method of Arbogast, 2004.

The Question is: DOES IT WORK?
Sub-grid Algorithm (numerical upscaling): Willems, 2009

1. Solve for the fine responses \((\delta u, \delta p)\) and \((\delta u(\varphi_H), \delta p(\varphi_H))\) for coarse basis functions \(\varphi_H\) and each coarse cell.

2. Solve the upscaled equation for \((u_H, p_H)\).

3. Piece together the solutions to get

\[
(u_{H,h}, p_{H,h}) = (u_H, p_H) + (\delta u(u_H), \delta p(u_H)) + (\delta u, \delta p).
\]

For pure Darcy this reduces to the method of Arbogast, 2004.

The Question is: DOES IT WORK?
Sample Geometries

Periodic  Vuggy  SPE10
Periodic Geometry - Velocity

\[ \mathbf{u} = [1, 1] \text{ on } \partial \Omega, \quad \mathbf{f} = 0, \quad \tilde{\mu} = \mu = 1, \quad \kappa^{-1} = \begin{cases} 10^3 & \text{in the holes} \\ 10^{-2} & \text{elsewhere} \end{cases} \]

Reference velocity

Subgrid velocity

Relative \( L^2 \)-error: 3.69e-03.
Periodic Geometry - Pressure

Reference pressure

Subgrid pressure

Relative $L^2$-error: 1.77e-02.
Numerical Upscaling of Brinkman equations
Hybridizable discontinuous Galerkin method (HDG)

Single Grid Approximation of Brinkman equations
Subgrid Approximation and Its Performance
Numerical Experiments

SPE10 - Pure Subgrid for Brinkman

\[ \mathbf{u} = [1, 0] \text{ on } \partial \Omega, \ f = 0, \ \mu = 1 \times 10^{-2}, \ T_h : 128^2, \]
\[ \kappa^{-1} : \text{ranging from } 1 \times 10^5 \text{ in blue to } 1 \times 10^2 \text{ in red.} \]

Relative \( L^2 \)-error velocity error: 2.26e-01
$u = [1, 0]$ on $\partial \Omega$, $f = 0$, $\mu = 1 \times 10^{-2}$, $\kappa^{-1}$: as shown on the plot; $T_h : 128^2$. 

(a) Ref. solution  (b) $H = 1/16$.  (c) $H = 1/8$.  (d) $H = 1/4$.

Figure: Velocity component, $u_1$, for SPE10 on 3 coarse grids.
Vuggy media - Subgrid Brinkman

\[ u = [1, 0] \text{ on } \partial \Omega, \quad f = 0, \quad \mu = 1 \times 10^{-2}, \]
\[ \kappa^{-1} = 1 \times 10^3 \quad \mathcal{T}_h : 128^2. \]

Figure: Velocity component, \( u_1 \), for vuggy geometry.
Introduction
Numerical Upscaling of Brinkman equations
Hybridizable discontinuous Galerkin method (HDG)

SPE10 benchmark

Reference
Pure subgrid
1 iteration
5 iterations

R. Lazarov, Y. Efendiev, J. Galvis, K. Shi, J. Willems
Numerics for Flows in Heterogeneous Media
SPE10 benchmark: convergence of the iterations

Pure Darcy case for periodic geometry for different contrasts.

Brinkman case for SPE10 geometry for different contrasts.
How to fight the resonance error?

There are various techniques to overcome the resonance error:

1. Using boundary conditions based on reducing the operator to the boundaries of the coarse mesh $T_H$;
2. Oversampling;
3. Mortar finite element approximation;
4. Hybridizable DG FEM;
5. WG - weak Galerkin FEM shows great potential.
Outline

1. Introduction
   - Motivation and Problem Formulation
   - State of the art

2. Numerical Upscaling of Brinkman equations
   - Single Grid Approximation of Brinkman equations
   - Subgrid Approximation and Its Performance
   - Numerical Experiments

3. Hybridizable discontinuous Galerkin method (HDG)
This is the subject of our recent research with Efendiev and Shi. First, we discuss HDG for Darcy flow. Recall that we have two grids $\mathcal{T}_h$ and $\mathcal{T}_H$. We consider the mixed formulation of Darcy equation (with $\alpha(x) = \kappa(x)^{-1}$):

$$\alpha \mathbf{u} + \nabla p = 0 \quad x \in \Omega, \quad \nabla \cdot \mathbf{u} = f \quad x \in \Omega \quad p = g \quad x \in \partial\Omega.$$
Let $\mathcal{E}_H$, $\mathcal{E}_h$ denote the set of all edges of $\mathcal{T}_H$, $\mathcal{T}_h$, respectively. We define

- $\mathcal{E}_h^0 := \{ e_h \in \mathcal{E}_h \mid e_H \notin \partial K_H \}$ – the internal edges of $\mathcal{T}_h$
- $\partial \mathcal{T}_h := \bigcup_{K_h \in \mathcal{T}_h} \partial K_h$ – the edges the fine mesh $\mathcal{T}_h$
- $\partial \mathcal{T}_H := \bigcup_{K_H \in \mathcal{T}_H} \partial K_H$ – the edges of the coarse mesh $\mathcal{T}_H$
the HDG generates an approximation \((p_h, u_h, \hat{p}_{h,H})\) to \((p, u, p|_{\mathcal{E}_h})\) on the finite dimensional (often piece-wise polynomial but not necessarily) spaces defined on each \(K \in \mathcal{T}_h:\)

\[
V_h := \{ v \in L^2(\mathcal{T}_h) : v|_K \in V(K) \},
\]

\[
W_h := \{ w \in L^2(\mathcal{T}_h) : w|_K \in W(K) \},
\]

and \(M_{h,H} := M_0^h \oplus M_H,\) with

\[
M_h := \{ \mu \in L^2(\mathcal{E}_h) : \mu|_{F_h} \in M_h(F), F_h \in \mathcal{E}_h \},
\]

\[
M_0^h := \{ \mu \in M_h : \mu|_{F_h} = 0, F_h \subset F_H \in \mathcal{E}_H \},
\]

\[
M_H := \{ \mu \in L^2(\mathcal{E}_H) : \mu|_{F_H} \in M_H(F_H), F_H \in \mathcal{E}_H \}.
\]
Then we seek \((p_h, u_h, \hat{p}_{h,H}) \in W_h \times V_h \times M_{h,H}\) as the only solution of the following weak problem:

\[-(p_h, \nabla \cdot v)_{T_h} + (\alpha u_h, v)_{T_h} + \langle \hat{p}_{h,H}, v \cdot n \rangle_{\partial T_h} = 0, \quad \forall v \in V_h\]

\[-(u_h, \nabla w)_{T_h} + \langle \hat{u}_{h,H} \cdot n, w \rangle_{\partial T_h} = (f, w)_{T_h}, \quad \forall w \in W_h\]

\[\langle \hat{u}_{h,H} \cdot n, \mu \rangle_{\partial T_h} = 0, \quad \forall \mu \in M^0_{h,H}\]

\[\langle \hat{u}_{h,H}, \lambda \rangle_{\partial D} = \langle g, \lambda \rangle_{\partial D}, \quad \forall \lambda \in M^\partial_{h,H}\]

Here \(M^0_{h,H} = \{\mu \in M_{h,H} : \mu|_{\partial D} = 0\}, M^\partial_{h,H} = M_{h,H}|_{\partial D}\),

\((\eta, \zeta)_{T_h} := \sum_{K \in T_h} (\eta, \zeta)_K\) and

\[\hat{u}_{h,H} \cdot n = u_h \cdot n + \tau (p_h - \hat{p}_{h,H}) \quad \text{on} \quad \partial T_h.\]
HDG

We split into two sets of equations by testing separately for $\mu$ in $M^0_h$ and $M^0_H := \{ \mu \in M_H : \mu|_{\partial D} = 0 \}$:

$$\left\langle \hat{u}_{h,H} \cdot n, \mu \right\rangle_{\partial T_h} = 0, \forall \mu \in M^0_h \quad \& \quad \left\langle \hat{u}_{h,H} \cdot n, \mu \right\rangle_{\partial T_H} = 0, \forall \mu \in M^0_H.$$

On each cell $K \in \mathcal{T}_H$, given the boundary data of $\hat{p}_{h,H} = \xi_H$ for $\xi_H \in M_H(F)$, $F \in \partial K$, we find $(u_h, p_h, \hat{p}_{h,H})$ on $K = K_H$:

$$-(p_h, \nabla \cdot v)_K + (\alpha u_h, v)_K + \left\langle \hat{p}_{h,H}, v \cdot n \right\rangle_{\partial K} = 0,$$

$$-(u_h, \nabla w)_K + \left\langle \hat{u}_{h,H} \cdot n, w \right\rangle_{\partial K} = (f, w)_K,$$

$$\left\langle \hat{u}_{h,H} \cdot n, \mu \right\rangle_{\partial K} = 0$$

for all $(w, v, \mu) \in W_h|_K \times V_h|_K \times M^0_h|_K$. In fact the above local system is the regular HDG methods defined on a coarse cell $K$. 

R. Lazarov, Y. Efendiev, J. Galvis, K. Shi, J. Willems

Numerics for Flows in Heterogeneous Media
The numerical upscaling method

This HDG method defines a global mapping from $M_H$ to $W_h \times V_h \times M^0_H$. This implies that one possible implementation of the method would be to solve the local problems for all basis functions of $M^0_H$ and then solve the global system on the coarse mesh:

$$a(\xi_H, \mu) := \langle \hat{u}_{h,H}(\xi_H), \mu \rangle_{\partial T_H} = 0, \quad \forall \mu \in M^0_H,$$

$$\langle \xi_H, \lambda \rangle_{\partial D} = \langle g \cdot n, \lambda \rangle_{\partial D} \quad \forall \lambda \in M^\partial_H.$$  

(6)
This is quite similar to the mortar MS FEM, e.g. T Arbogast, G Pencheva, MF Wheeler, I Yotov, 2007. The main difference is that instead of special choice of the mortar space the stability is provided by the stabilization parameter (or often called operator) $\tau$ in the choice of the numerical flux

$$\hat{u}_{h,H} \cdot n = u_h \cdot n + \tau (p_h - \hat{p}_{h,H}) \quad \text{on} \quad \partial T_h.$$  

This choice often generates superconvergent schemes or allows more flexible choices of the approximation spaces, including locally generated solutions of some spectral problems.
HDG – current and future work

Now we are working in various directions that include:

- An implementation of this part
- Generating the numerical trace space by solving local spectral problems or traces of snapshot spaces
- Brinkman/Stokes problem and its applications;
- Theoretical justification of the method and optimal error estimates;
- Efficient solution methods for the resulting systems of linear equations;

**Most important:** this is a framework for novel model reduction technique for parameter dependent problems.
Thank you for your attention !!!