Tracking Control of General Nonlinear Systems by a Direct Gradient Descent Method

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General nonlinear system is considered as a controlled object:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)), \quad \boldsymbol{y}(t) = \boldsymbol{h}(\boldsymbol{x}(t)) \tag{1}$$

where $\boldsymbol{x}(t) \in R^n$ is the state vector, $\boldsymbol{u}(t) \in R^r$ is the control input, $\boldsymbol{y}(t) \in R^m$ is the measured output, and we study the problem of output tracking so that $\boldsymbol{y}(t)$ tracks a desired output $\boldsymbol{y}_d(t)$ (the reference signal). Output tracking control (output regulation or servo mechanism) for nonlinear systems has intensively been investigated.

In the pioneering work of Isidori and Byrnes, nonlinear output regulation problem has been formulated and solved, in which the objective is to design a dynamic controller such that the closed-loop system is stable and the error approaches zero asymptotically. Supposing the reference signal $\boldsymbol{y}_d(t)$ to be generated by the exosystem

$$\dot{\boldsymbol{w}}(t) = \boldsymbol{s}(\boldsymbol{w}(t)), \quad \boldsymbol{y}_d(t) = \boldsymbol{q}(\boldsymbol{w}(t))$$

and considering the extended system

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)), \quad \boldsymbol{y}(t) = \boldsymbol{h}(\boldsymbol{x}(t)) \\ \dot{\boldsymbol{w}}(t) = \boldsymbol{s}(\boldsymbol{w}(t)), \quad \boldsymbol{y}_d(t) = \boldsymbol{q}(\boldsymbol{w}(t)) \\ \boldsymbol{e}(t) = \boldsymbol{y}_d(t) - \boldsymbol{y}(t) \end{cases} \tag{2}$$

they solved the output regulation problem by means of an error feedback controller (a dynamic controller)

$$\dot{\boldsymbol{z}}(t) = \boldsymbol{\eta}(\boldsymbol{z}(t), \boldsymbol{e}(t)), \quad \boldsymbol{u}(t) = \boldsymbol{\alpha}(\boldsymbol{z}(t)). \tag{3}$$

More precisely, the output regulation means that the unforced closed-loop system with $\boldsymbol{w} = \boldsymbol{0}$ is exponentially stable and that the forced closed-loop system (2)–(3) satisfies $\lim_{t\to\infty} \boldsymbol{e}(t) \to \boldsymbol{0}$ for any initial condition $(\boldsymbol{x}(0), \boldsymbol{z}(0), \boldsymbol{w}(0))$. Isidori and Byrnes derived a necessary condition for the output regulation, called the nonlinear regulator equation, using the center manifold theorem. Though the Isidori–Byrnes theory is precise and sophisticated, it requires many assumptions and, in order to synthesize a solution numerically, one has to solve the nonlinear regulator equation described by a system of nonlinear partial differential equations, which is difficult to solve as in the Hamilton–Jacobi–Bellman equation.

Direct gradient descent control was proposed by Shimizu et al., which directly manipulates control inputs so as to decrease a performance index such as the squared error from a desired equilibrium state based on the gradient of the performance index with respect to the control inputs. The gradient is derived from sensitivity equations. A similar method called speed gradient control was also proposed by Fradkov et al. In this paper we investigate output tracking control of nonlinear multivariable systems by use of the direct gradient descent method. Our main concern is the control of plants with relative degrees of more than one. The aim of our control is to modify $\boldsymbol{u}(t)$ so that a performance index $F(\boldsymbol{y}(t), \boldsymbol{u}(t))$ decreases. The problem is written as

$$\underset{\boldsymbol{u}(t)}{\text{decrease}} \; F(\boldsymbol{y}(t), \boldsymbol{u}(t)) \tag{4a}$$

$$\text{subj. to} \quad \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)), \quad \boldsymbol{x}(t_0) = \boldsymbol{x}_0 \tag{4b}$$

$$\boldsymbol{y}(t) = \boldsymbol{h}(\boldsymbol{x}(t)) \tag{4c}$$

To solve this problem, we prepare some fundamental results concerning the gradient of the performance index. We confine our attention in this section to the basic state feedback regulation case:

$$\underset{\boldsymbol{u}(t)}{\text{decrease}} \; F(\boldsymbol{x}(t), \boldsymbol{u}(t)) \tag{5a}$$

$$\text{subj. to} \quad \dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t)), \quad \boldsymbol{x}(0) = \boldsymbol{x}_0 \tag{5b}$$

For any continuous $\boldsymbol{u} : \boldsymbol{u}(t),\ t \geq 0$, system (5b) has a unique smooth solution $\boldsymbol{x} : \boldsymbol{x}(t),\ t \geq 0$. We denote the state trajectory $\boldsymbol{x}$ associated with a given $\boldsymbol{u}$ by $\boldsymbol{x}(\boldsymbol{u})$, whose value at t will be denoted by $\boldsymbol{x}(t; \boldsymbol{u})$. Then, for an arbitrarily fixed t, let us define a functional ϕ^t by

$$\phi^t[\boldsymbol{u}] := F(\boldsymbol{x}(t; \boldsymbol{u}), \boldsymbol{u}(t)). \tag{6}$$

Under appropriate assumptions, the *gradient* of the objective $F(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))$ with respect to $\boldsymbol{u}(t)$, denoted by $\nabla\phi^t[\boldsymbol{u}](t)$, is obtained as

$$\nabla\phi^t[\boldsymbol{u}](t) = \boldsymbol{f}_{\boldsymbol{u}}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))^T F_{\boldsymbol{x}}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))^T + F_{\boldsymbol{u}}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))^T. \tag{7}$$

As an on-line control law for problem (5), we apply the steepest descent method at each time $t \in [0,\infty)$ by using $\nabla\phi^t[\boldsymbol{u}](t)$. Namely, $\boldsymbol{u}(t)$ is modified by the direct gradient descent control algorithm

$$\dot{\boldsymbol{u}}(t) = -\mathcal{L}\nabla\phi^t[\boldsymbol{u}](t) \tag{8}$$

where $\mathcal{L} = \mathrm{diag}[\alpha_1,\alpha_2,\ldots,\alpha_r]$, $\alpha_i > 0$, is a proportional constant. Substituting (7) into (8) yields

$$\dot{\boldsymbol{u}}(t) = -\mathcal{L}\{\boldsymbol{f}_{\boldsymbol{u}}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))^T F_{\boldsymbol{x}}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))^T + F_{\boldsymbol{u}}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))^T\}. \tag{9}$$

Now we investigate an on-line tracking control for problem (4) based on the preliminary knowledge on state feedback regulation. Let us consider the case where the performance index is given in the quadratic form

$$F(\boldsymbol{y}(t),\boldsymbol{u}(t)) := (\boldsymbol{y}_d(t)-\boldsymbol{y}(t))^T Q(\boldsymbol{y}_d(t)-\boldsymbol{y}(t)) + (\boldsymbol{u}_d(t)-\boldsymbol{u}(t))^T R(\boldsymbol{u}_d(t)-\boldsymbol{u}(t)) \tag{10}$$

where $\boldsymbol{y}_d(t)$ is a desired output, $\boldsymbol{u}_d(t)$ is the corresponding control input, Q and R are diagonal positive definite matrices. In what follows, we consider the functional

$$\phi^t[\boldsymbol{u}] := F(\boldsymbol{y}(t;\boldsymbol{u}),\boldsymbol{u}(t)) \tag{11}$$

where $\boldsymbol{y}(t;\boldsymbol{u}) := \boldsymbol{h}(\boldsymbol{x}(t;\boldsymbol{u}))$. Applying (7), we obtain an expression of the gradient needed for the gradient descent tracking control as follows:

$$\begin{aligned}\nabla\phi^t[\boldsymbol{u}](t) &= \nabla\boldsymbol{x}(t;\boldsymbol{u})(t)\frac{\partial\boldsymbol{y}(t;\boldsymbol{u})}{\partial\boldsymbol{x}}^T F_{\boldsymbol{y}}(\boldsymbol{y}(t;\boldsymbol{u}),\boldsymbol{u}(t))^T + F_{\boldsymbol{u}}(\boldsymbol{y}(t;\boldsymbol{u}),\boldsymbol{u}(t))^T \\ &= -2\boldsymbol{f}_{\boldsymbol{u}}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))^T\frac{\partial\boldsymbol{h}(\boldsymbol{x}(t;\boldsymbol{u}))}{\partial\boldsymbol{x}}^T Q(\boldsymbol{y}_d(t)-\boldsymbol{y}(t;\boldsymbol{u})) - 2R(\boldsymbol{u}_d(t)-\boldsymbol{u}(t)).\end{aligned} \tag{12}$$

From this we realize that, for $y_i(t)$ with relative degree of more than 1, the error $(y_{id}(t)-y_i(t))$ cannot be evaluated at all in the calculation of $\nabla\phi^t[\boldsymbol{u}](t)$ since $\boldsymbol{f}_{\boldsymbol{u}}(\boldsymbol{x}(t),\boldsymbol{u}(t))^T\big(\partial h_i(\boldsymbol{x}(t))/\partial\boldsymbol{x}\big)^T = \boldsymbol{0}$. Therefore the error information on $y_i(t)$ is not used in modifying $\boldsymbol{u}(t)$ by the direct gradient descent control with (12), which implies that it is not always possible to accomplish the tracking control. In order to control those plants with higher relative degrees, it is essential to incorporate some information on higher order derivatives into the algorithm, and we consider the following performance index:

$$F\big(y_1^{(q_1-1)}(t),\ldots,y_m^{(q_m-1)}(t),\boldsymbol{u}(t)\big) := \sum_{i=1}^m \omega_i\big(y_{id}^{(q_i-1)}(t)-y_i^{(q_i-1)}(t)\big)^2 + (\boldsymbol{u}_d(t)-\boldsymbol{u}(t))^T R(\boldsymbol{u}_d(t)-\boldsymbol{u}(t)) \tag{13}$$

where $y_{id}^{(q_i-1)}(t)$ denotes the (q_i-1)-th order derivative of the i-th component $y_{id}(t)$ of the desired output. (q_i is the relative degree of $y_i(t)$.) We again use the same notation

$$\phi^t[\boldsymbol{u}] := F\big(y_1^{(q_1-1)}(t;\boldsymbol{u}),\ldots,y_m^{(q_m-1)}(t;\boldsymbol{u}),\boldsymbol{u}(t)\big). \tag{14}$$

The gradient $\nabla\phi^t[\boldsymbol{u}](t)$ is then given by

$$\begin{aligned}\nabla\phi^t[\boldsymbol{u}](t) &= \sum_{i=1}^m \nabla\boldsymbol{x}(t;\boldsymbol{u})(t)\frac{\partial y_i^{(q_i-1)}(t;\boldsymbol{u})}{\partial\boldsymbol{x}}^T F_{y_i^{(q_i-1)}} + F_{\boldsymbol{u}}^T \\ &= -2\sum_{i=1}^m \omega_i\boldsymbol{f}_{\boldsymbol{u}}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))^T\frac{\partial\alpha_i^{q_i-1}(\boldsymbol{x}(t;\boldsymbol{u}))}{\partial\boldsymbol{x}}^T\big(y_{id}^{(q_i-1)}(t)-y_i^{(q_i-1)}(t;\boldsymbol{u})\big) - 2R(\boldsymbol{u}_d(t)-\boldsymbol{u}(t))\end{aligned} \tag{15}$$

where we eliminated the arguments of F for simplicity. Hence the direct gradient descent control is given as follows:

$$\begin{aligned}\dot{\boldsymbol{u}}(t) &= -\alpha\nabla\phi^t[\boldsymbol{u}](t) \\ &= 2\alpha\Big[\sum_{i=1}^m \omega_i\boldsymbol{f}_{\boldsymbol{u}}(\boldsymbol{x}(t;\boldsymbol{u}),\boldsymbol{u}(t))^T\frac{\partial\alpha_i^{q_i-1}(\boldsymbol{x}(t;\boldsymbol{u}))}{\partial\boldsymbol{x}}^T\big(y_{id}^{(q_i-1)}(t)-y_i^{(q_i-1)}(t;\boldsymbol{u})\big) + R(\boldsymbol{u}_d(t)-\boldsymbol{u}(t))\Big].\end{aligned} \tag{16}$$

The proposed method is an on-line implementation and can be executed in a very simple and practical manner. The effectiveness of the control method was confirmed by computer simulation, and very good performance was observed with various examples.