A Bundle Method to Solve Multivalued Variational Inequalities

Salmon Geneviève   Strodiot Jean-Jacques   Nguyen Van Hien
Facultés Universitaires de Namur, Namur, Belgium

Let F be a monotone multivalued operator in a Hilbert space H, let C be a nonempty closed convex subset of H and let $p : H \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous proper convex function such that $C \subseteq \operatorname{dom} p \subseteq \operatorname{dom} F$. We consider the following general variational inequality problem:

$$(P)\quad \begin{cases} \text{find } x^* \in C \text{ and } r(x^*) \in F(x^*) \text{ such that, for all } x \in C; \\ \langle r(x^*), x - x^* \rangle + p(x) - p(x^*) \geq 0. \end{cases}$$

For solving problem (P), Cohen developed, several years ago, the auxiliary problem method. Let $K : H \to \mathbb{R}$ be an auxiliary function continuously differentiable and strongly convex and $\{\lambda_k\}_{k \in \mathbb{N}}$ be a sequence of positive numbers. The problem considered at iteration k is the following:

$$(P^k)\quad \begin{cases} \text{choose } r(x^k) \in F(x^k) \text{ and find } x^{k+1} \in C \text{ such that, for all } x \in C; \\ \langle r(x^k) + \lambda_k^{-1}(\nabla K(x^{k+1}) - \nabla K(x^k)), x - x^{k+1} \rangle + p(x) - p(x^{k+1}) \geq 0. \end{cases}$$

The strategy is to approximate, in the subproblems (P^k), the function p by a piecewise linear convex function p^k build step by step as in the bundle method in nonsmooth optimization. This makes the subproblems (P^k) more tractable. Moreover, to ensure the existence of subgradients at each iteration, we also introduce a barrier function in the subproblems. This function prevents the iterates to go outside the interior of the feasible domain C.

First, we set the conditions to be satisfied by the approximations p^k of p and we show how to build suitable approximations by means of a bundle strategy. In a second part, we prove the convergence of the general algorithm. We give conditions to ensure the boundedness of the sequence generated by the algorithm. Then we study the properties that a gap function must satisfy to obtain that each weak limit point of this sequence is a solution of the problem. In particular, we give existence theorems of such a gap function when the operator F is paramonotone, weakly closed and Lipschitz continuous on bounded subsets of its domain and when it is the subdifferential of a convex function. When it is strongly monotone, we obtain that the sequence generated by the algorithm strongly converges to the unique solution of the problem.