Optimal Control of Nonlinear Systems

S. P. BANKS and T. CIMEN
Department of Automatic Control and Systems Engineering,
University of Sheffield, Mappin Street, Sheffield, S1 4DU, U.K.

Abstract

In recent papers [1], [2] we have applied a sequence of linear time-varying approximations to find feedback controllers for nonlinear systems. Thus, for the optimal control problem

$$\min\ J = \frac{1}{2}x^T(t_f)F(x(t_f))x(t_f) + \frac{1}{2}\int_0^{t_f}\left\{x^T(t)Q(x(t))x(t) + u^T(t)R(x(t))u(t)\right\}dt \qquad (1)$$

subject to the dynamics

$$\dot{x}(t) = A(x(t))x(t) + B(x(t))u(t), \qquad x(0) = x_0 \qquad (2)$$

we have introduced the sequence of approximations

$$\min\ J^{[i]} = \frac{1}{2}x^{[i]T}(t_f)F(x^{[i-1]}(t_f))x^{[i]}(t_f) + \frac{1}{2}\int_0^{t_f}\left\{x^{[i]T}(t)Q(x^{[i-1]}(t))x^{[i]}(t) + u^{[i]T}(t)R(x^{[i-1]}(t))u^{[i]}(t)\right\}dt \qquad (3)$$

subject to the dynamics

$$\dot{x}^{[i]}(t) = A(x^{[i-1]}(t))x^{[i]}(t) + B(x^{[i-1]}(t))u^{[i]}(t), \qquad x^{[i]}(0) = x_0 \qquad (4)$$

Applying the maximum principle to the problem (1), (2) we have the system of equations

$$\begin{aligned}
\dot{x} &= A(x)x - B(x)R^{-1}(x)B^T(x)\,l, \qquad x(0) = x_0\\
\dot{l} &= -Q(x)x - \frac{1}{2}x^T\frac{\partial Q(x)}{\partial x}x - l^T\frac{\partial A(x)}{\partial x}x\\
&\quad - \frac{1}{2}l^T B(x)R^{-1}(x)\frac{\partial R(x)}{\partial x}R^{-1}(x)B^T(x)\,l - A^T(x)\,l + l^T\frac{\partial B(x)}{\partial x}R^{-1}(x)B^T(x)\,l \qquad (5)\\
l(t_f) &= F(x(t_f))x(t_f) + \frac{1}{2}x^T(t_f)\frac{\partial F(x(t_f))}{\partial x}x(t_f)
\end{aligned}$$

which is the complete necessary condition. In this paper we shall examine the relation between this system and (3) to obtain conditions under which the solution in (5) is optimal. We shall do this by again analyzing the sequence of approximating linear two-point boundary-value problems:

$$\begin{aligned}
\dot{x}^{[i]}(t) &= A(x^{[i-1]}(t))x^{[i]}(t) - B(x^{[i-1]}(t))R^{-1}(x^{[i-1]}(t))B^T(x^{[i-1]}(t))\,l^{[i]}(t), \qquad x^{[i]}(0) = x_0\\
\dot{l}^{[i]}(t) &= Q(x^{[i-1]}(t))x^{[i]}(t) - \frac{1}{2}x^{[i-1]T}(t)\frac{\partial Q(x^{[i-1]}(t))}{\partial x}x^{[i]}(t) - l^{[i-1]T}(t)\frac{\partial A(x^{[i-1]}(t))}{\partial x}x^{[i]}(t)\\
&\quad - \frac{1}{2}l^{[i-1]T}(t)B(x^{[i-1]}(t))R^{-1}(x^{[i-1]}(t))\frac{\partial R(x^{[i-1]}(t))}{\partial x}R^{-1}(x^{[i-1]}(t))B^T(x^{[i-1]}(t))l^{[i]}(t)\\
&\quad - A^T(x^{[i-1]}(t))\,l^{[i]}(t) + l^{[i-1]T}(t)\frac{\partial B(x^{[i-1]}(t))}{\partial x}R^{-1}(x^{[i-1]}(t))B^T(x^{[i-1]}(t))\,l^{[i]}(t)\\
l^{[i]}(t_f) &= F(x^{[i-1]}(t_f))x^{[i]}(t_f) + \frac{1}{2}x^{[i-1]T}(t_f)\frac{\partial F(x^{[i-1]}(t_f))}{\partial x}x^{[i]}(t_f).
\end{aligned}$$

References:
[1] Banks, S. P., and Dinesh, K., 'Approximate Optimal Control and Stability of Nonlinear Finite and Infinite-Dimensional Systems', *Annals of Operations Research 98*, 19–44, 2000.
[2] Banks, S. P., 'Exact Boundary Controllability and Optimal Control for a Generalised Korteweg de Vries Equation, to appear in *International Journal of Nonlinear Analysis, Methods and Applications*.