Nonnegative Tensor Factorization, Completely Positive Tensors and an Hierarchical Elimination Algorithm

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Abstract

Nonnegative tensor factorization has applications in statistics, computer vision, exploratory multiway data analysis and blind source separation. A symmetric nonnegative tensor, which has a symmetric nonnegative factorization, is called a completely positive (CP) tensor. The H-eigenvalues of a CP tensor are always nonnegative. When the order is even, the Z-eigenvalue of a CP tensor are all nonnegative. When the order is odd, a Z-eigenvector associated with a positive (negative) Z-eigenvalue of a CP tensor is always nonnegative (nonpositive). The entries of a CP tensor obey some dominance properties. The CP tensor cone and the copositive tensor cone of the same order are dual to each other. We introduce strongly symmetric tensors and show that a symmetric tensor has a symmetric binary decomposition if and only if it is strongly symmetric. Then we show that a strongly symmetric, hierarchically dominated nonnegative tensor is a CP tensor, and present a hierarchical elimination algorithm for checking this. Numerical examples are also given.

Key words: completely positive tensor, eigenvalues, dominance properties, copositive tensor, strongly symmetric tensor, hierarchical dominance, hierarchical elimination algorithm.

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1 Introduction

Nonnegative tensor factorization has applications in statistics, computer vision, exploratory multiway data analysis and blind source separation [2, 9]. As the research topic of nonnegative matrix factorization is closely related with the theory of completely positive (CP) matrices [1, 4, 5, 10], in this paper, we introduce completely positive (CP) tensors, study their spectral properties and other properties, demonstrate the dual relationship between the CP tensor cone and the copositive tensor cone, and give a checkable sufficient condition for CP tensors by showing a strongly symmetric, hierarchically dominated nonnegative tensor is a CP tensor.

Let \( A = (a_{i_1 \cdots i_m}) \) be a real \( m \)-th order \( n \)-dimensional tensor. Denote the set of all nonnegative vectors in \( \mathbb{R}^n \) by \( \mathbb{R}^n_+ \). For any vector \( u \in \mathbb{R}^n \), \( u^m \) is a rank-one \( m \)-th order symmetric \( n \)-dimensional tensor \( u^m = (u_{i_1} \cdots u_{i_m}) \). If

\[
A = \sum_{k=1}^{r} (u^{(k)})^m,
\]

where \( u^{(k)} \in \mathbb{R}^n_+ \) for \( k = 1, \cdots, r \), then \( A \) is called a completely positive (CP) tensor. The minimum value of \( r \) is called the CP rank of \( A \). The concepts of completely positive tensors and their CP ranks extend the concepts of completely positive matrices and their CP ranks [1, 4, 5, 10].

The eigenvalues of a CP matrix are always nonnegative. In the next section, after summarizing some necessary knowledge about eigenvalues of tensors, we prove that the H-eigenvalues of a CP tensor are always nonnegative. We further show that when the order \( m \) is even, the Z-eigenvalue of a CP tensor are all nonnegative, while when the order \( m \) is odd, a Z-eigenvector associated with a positive (negative) Z-eigenvalue of a CP tensor is always nonnegative (nonpositive).

In Section 3, we prove some dominance properties which the entries of a CP tensor must obey. These properties form some checkable necessary conditions for a CP tensor.

It is well-known that the CP matrix cone and the copositive matrix cone are dual to each other. Recently, motivated by the study of spectral hypergraph theory, Qi [8] introduced copositive tensors. In Section 4, we show that the CP tensor cone and the copositive tensor cone of the same order are dual to each other.

It is also well-known [5] that a diagonally dominated symmetric nonnegative matrix is a CP matrix. This forms a checkable condition for a CP matrix. To extend this result to CP tensors, in Section 5, we introduce strongly symmetric tensors and show that a symmetric tensor is strongly symmetric if and only if it has a symmetric binary decomposition. We present a hierarchical elimination algorithm for checking this.

In Section 6, we further define strongly symmetric, hierarchically dominated nonnegative
tensors and show that a strongly symmetric, hierarchically dominated nonnegative tensor is a CP tensor. We show that the hierarchical elimination algorithm given in Section 5 can be used to check this condition too.

Some numerical examples are given in Section 7.

Some final remarks are made in Section 8.

For a vector \( x \in \mathbb{R}^n \), denote \( \text{supp}(x) = \{ i : 1 \leq i \leq n, x_i \neq 0 \} \). For a finite set \( S \), \( |S| \) denotes its cardinality.

### 2 Eigenvalues of a CP Tensor

Let \( \mathcal{A} = (a_{i_1 \cdots i_m}) \) be a real \( m \)th order \( n \)-dimensional tensor, and \( x \in C^n \). Then

\[
\mathcal{A}x^m = \sum_{i_1, \cdots, i_m = 1}^{n} a_{i_1 \cdots i_m} x_{i_1} \cdots x_{i_m},
\]

and \( \mathcal{A}x^{m-1} \) is a vector in \( C^n \), with its \( i \)th component defined by

\[
(\mathcal{A}x^{m-1})_i = \sum_{i_2, \cdots, i_m = 1}^{n} a_{ii_2 \cdots i_m} x_{i_2} \cdots x_{i_m}.
\]

Let \( s \) be a positive integer. Then \( x^{[s]} \) is a vector in \( C^n \), with its \( i \)th component defined by \( x_i^s \). We say that \( \mathcal{A} \) is symmetric if its entries \( a_{i_1 \cdots i_m} \) are invariant for any permutation of the indices. If \( \mathcal{A}x^m \geq 0 \) for all \( x \in \mathbb{R}^n \), then we say that \( \mathcal{A} \) is positive semi-definite. Clearly, only when \( m \) is even, a nonzero tensor \( \mathcal{A} \) can be positive semi-definite.

The following definitions of eigenvalues, H-eigenvalues, E-eigenvalues and Z-eigenvalues were introduced in [7].

If \( x \in C^n, x \neq 0, \lambda \in C, x \) and \( \lambda \) satisfy

\[
\mathcal{A}x^{m-1} = \lambda x^{[m-1]},
\]

then we call \( \lambda \) an eigenvalue of \( \mathcal{A} \), and \( x \) its corresponding eigenvector. By \( (2) \), if \( \lambda \) is an eigenvalue of \( \mathcal{A} \) and \( x \) is its corresponding eigenvector, then

\[
\lambda = \frac{(\mathcal{A}x^{m-1})_j}{x^{m-1}_j},
\]

for some \( j \) with \( x_j \neq 0 \). In particular, if \( x \) is real, then \( \lambda \) is also real. In this case, we say that \( \lambda \) is an H-eigenvalue of \( \mathcal{A} \) and \( x \) is its corresponding H-eigenvector.

We say a complex number \( \lambda \) is an E-eigenvalue of \( \mathcal{A} \) if there exists a complex vector \( x \) such that

\[
\begin{align*}
Ax^{m-1} &= \lambda x, \\
x^T x &= 1.
\end{align*}
\]

(3)
In this case, we say that \( x \) is an E-eigenvector of the tensor \( A \) associated with the E-eigenvalue \( \lambda \). By (3), if \( \lambda \) is an E-eigenvalue of \( A \) and \( x \) is its E-corresponding eigenvector, then

\[
\lambda = Ax^m.
\]

Thus, if \( x \) is real, then \( \lambda \) is also real. In this case, we say that \( \lambda \) is an Z-eigenvalue of \( A \) and \( x \) is its corresponding Z-eigenvector.

By [7], we have the following proposition.

**Proposition 1** A real \( m \)th order \( n \)-dimensional tensor \( A \) has always Z-eigenvalues. If \( m \) is even, then \( A \) always has at least one H-eigenvalue. When \( m \) is even, \( A \) is positive semi-definite if and only if all of its H-eigenvalues (Z-eigenvalues) are nonnegative.

If the entries of a \( m \)th order \( n \)-dimensional tensor \( A = (a_{i_1 \cdots i_m}) \) are invariant under any permutation of their indices, then we say that \( A \) is symmetric. If all the entries of \( A \) are nonnegative, then we say that \( A \) is a nonnegative tensor. By (1), a CP tensor is a symmetric nonnegative tensor. By [11], a symmetric nonnegative tensor has at least one H-eigenvalue, which is the largest modulus of its eigenvalues.

We now have the following theorem on H-eigenvalues of a CP tensor.

**Theorem 1** Suppose that \( A = (a_{i_1 \cdots i_m}) \) is an \( m \)th order \( n \)-dimensional CP tensor, expressed by (1), with \( m \geq 2 \). Then the H-eigenvalues of \( A \) are always nonnegative.

**Proof.** First, assume that \( m \) is even. For any \( x \in \mathbb{R}^n \), we have

\[
Ax^m = \sum_{k=1}^{r} (u^{(k)})^m x^m = \sum_{k=1}^{r} \left( (u^{(k)})^\top x \right)^m \geq 0.
\]

Thus, \( A \) is positive semi-definite. By Proposition 1, all of the H-eigenvalues are nonnegative.

Now assume that \( m \) is odd. By the discussion before this theorem, \( A \) has at least one H-eigenvalue. Suppose that \( \lambda \) is an H-eigenvalue of \( A \), with an H-eigenvector \( x \). Then \( x \in \mathbb{R}^n, x \neq 0 \). By the definition of H-eigenvalue and H-eigenvector, we have

\[
\lambda x^{m-1} = Ax^{m-1} = \sum_{k=1}^{r} (u^{(k)})^m x^{m-1} = \sum_{k=1}^{r} \left( (u^{(k)})^\top x \right)^{m-1} u^{(k)} \geq 0.
\]

Thus, \( \lambda \geq 0 \). This completes the proof. \( \square \)

By (3), when \( m \) is odd, if \( \lambda \) is a Z-eigenvalue of a tensor \( A \) with a Z-eigenvector \( x \), then \(-\lambda \) is a Z-eigenvalue of a tensor \( A \) with a Z-eigenvector \(-x \). Hence, when \( m \) is odd, we cannot expect that the Z-eigenvalues of a CP tensor are always nonnegative. However, in this case, we may get strong properties of Z-eigenvectors.
Theorem 2 Suppose that $\mathcal{A} = (a_{i_1 \cdots i_m})$ is an $n$th order $n$-dimensional CP tensor, expressed by (1), with $m \geq 2$. When the order $m$ is even, the Z-eigenvalue of a CP tensor are all nonnegative. When the order $m$ is odd, a Z-eigenvector associated with a positive (negative) Z-eigenvalue of a CP tensor is always nonnegative (nonpositive).

\textbf{Proof.} The proof of the case that $m$ is even is similar to the first part of the proof of Theorem [1].

Now assume that $m$ is odd. Suppose that $\lambda$ is a Z-eigenvalue of $\mathcal{A}$, with an Z-eigenvector $x$. By the definition of Z-eigenvalue and Z-eigenvector, we have

$$\lambda x = \mathcal{A}x^{m-1} = \sum_{k=1}^{r} (u^{(k)})^m x^{m-1} = \sum_{k=1}^{r} [(u^{(k)})^T x]^{m-1} u^{(k)} \geq 0.$$ 

Thus, if $\lambda > 0$, then $x \geq 0$, and if $\lambda < 0$, then $x \leq 0$. This completes the proof. $\square$

3 Dominance Properties of a CP Tensor

Denote $\mathcal{I} = \{(i_1, \cdots, i_m) : 1 \leq i_k \leq n, k = 1, \cdots, m\}$. For $(i_1, \cdots, i_m) \in \mathcal{I}$, let $[(i_1, \cdots, i_m)]$ be the set of all the distinct members in $\{i_1, \cdots, i_m\}$. For example, $[(1, 1, 4, 5)] = \{1, 4, 5\}$.

Let $(i_1, \cdots, i_m), (j_1, \cdots, j_m) \in \mathcal{I}$. We say that $(i_1, \cdots, i_m)$ is dominated by $(j_1, \cdots, j_m)$, and denote $(i_1, \cdots, i_m) \preceq (j_1, \cdots, j_m)$ if $[(i_1, \cdots, i_m)] \subseteq [(j_1, \cdots, j_m)]$. We say that $(i_1, \cdots, i_m)$ is similar to $(j_1, \cdots, j_m)$, and denote $(i_1, \cdots, i_m) \simeq (j_1, \cdots, j_m)$ if $[(i_1, \cdots, i_m)] = [(j_1, \cdots, j_m)]$.

We have the following dominance property for a CP tensor.

Theorem 3 Suppose that $\mathcal{A} = (a_{i_1 \cdots i_m})$ is an $n$th order $n$-dimensional CP tensor, expressed by (1), with $m \geq 2$. If $(i_1, \cdots, i_m) \preceq (j_1, \cdots, j_m)$ and $a_{j_1 \cdots j_m} \neq 0$, then $a_{i_1 \cdots i_m} > 0$.

\textbf{Proof.} We have that

$$0 \neq a_{j_1 \cdots j_m} = \sum_{k=1}^{r} u_{j_1}^{(k)} \cdots u_{j_m}^{(k)}.$$ 

Since $u^{(k)} \in \mathbb{R}_+^n$ for $k = 1, \cdots, r$, at least for one $k = \bar{k}$, $u_{j_1}^{(k)} > 0$, $\cdots$, $u_{j_m}^{(k)} > 0$. Since $\{i_1, \cdots, i_m\} \subseteq \{j_1, \cdots, j_m\}$, this implies that $u_{i_1}^{(k)} > 0$, $\cdots$, $u_{i_m}^{(k)} > 0$. Therefore,

$$a_{i_1 \cdots i_m} = \sum_{k=1}^{r} u_{i_1}^{(k)} \cdots u_{i_m}^{(k)} > 0.$$ 

This completes the proof. $\square$

Corollary 1 Suppose that $\mathcal{A} = (a_{i_1 \cdots i_m})$ is an $n$th order $n$-dimensional CP tensor, expressed by (1), with $m \geq 2$. If $(i_1, \cdots, i_m) \simeq (j_1, \cdots, j_m)$, then $a_{j_1 \cdots j_m} = 0$ if and only if $a_{i_1 \cdots i_m} = 0$. 

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When $m = 2$, this property can be derived from the symmetric property of the matrix $A$. When $m > 2$, this property cannot be derived from the symmetric property of the tensor $A$. For example, for a third order CP tensor $A = (a_{ijk})$, we have $a_{iij} = a_{ijj}$ for all $i$ and $j$, satisfying $1 \leq i, j \leq n$. But this is not true for a general third order symmetric tensor. This motivates us to introduce strongly symmetric tensors in Section 5.

Suppose that $(j_1, \ldots, j_m) \in I$ and $I = \{(i^{(1)}_1, \ldots, i^{(1)}_m), \ldots, (i^{(s)}_1, \ldots, i^{(s)}_m)\} \subseteq I$. Assume that $(i^{(p)}_1, \ldots, i^{(p)}_m) \preceq (j_1, \ldots, j_m)$ for $p = 1, \ldots, s$, and for any index $i \in \{j_1, \ldots, j_m\}$, if it appears $t$ times in $\{j_1, \ldots, j_m\}$, then it appears in $I$ at times. Then we call $I$ an \textit{s-duplicate} of $(j_1, \ldots, j_m)$.

We have the following strong dominance property for a CP tensor.

\textbf{Theorem 4} Suppose that $A = (a_{i_1\cdots i_m})$ is an $m$th order $n$-dimensional CP tensor, expressed by \[\Box\], with $m \geq 2$. Assume that $I = \{(i^{(1)}_1, \ldots, i^{(1)}_m), \ldots, (i^{(s)}_1, \ldots, i^{(s)}_m)\}$ is an \textit{s-duplicate} of $(j_1, \cdots j_m) \in I$. Then

$$\frac{1}{s} \sum_{p=1}^{s} a_{i^{(p)}_1 \cdots i^{(p)}_m} \geq a_{j_1 \cdots j_m}.$$ 

\textbf{Proof.} We have that

$$\frac{1}{s} \sum_{p=1}^{s} a_{i^{(p)}_1 \cdots i^{(p)}_m} = \sum_{k=1}^{r} \frac{1}{s} \sum_{p=1}^{s} a_{u^{(k)}_{i^{(p)}_1, \ldots, i^{(p)}_m}} \geq \sum_{k=1}^{r} \left( \prod_{p=1}^{s} a_{u^{(k)}_{i^{(p)}_1, \ldots, i^{(p)}_m}} \right)^{\frac{1}{s}} = \sum_{k=1}^{r} a_{u^{(k)}_{j_1, \ldots, j_m}} = a_{j_1 \cdots j_m},$$

where the inequality is due to the fact that the geometric mean of some positive numbers is never greater than their arithmetic mean. This completes the proof. \qed

\textbf{Corollary 2} Suppose that $A = (a_{i_1\cdots i_m})$ is an $m$th order $n$-dimensional CP tensor, expressed by \[\Box\], with $m \geq 2$. Assume that $(j_1, \cdots j_m) \in I$. Then

$$\frac{1}{m} \sum_{p=1}^{m} a_{j_p \cdots j_p} \geq a_{j_1 \cdots j_m}.$$ 

\section{The CP Tensor Cone and the Copositive Tensor Cone}

Denote the set of all $m$th order $n$-dimensional CP tensors by $CP_{m,n}$. By \[\Box\], it is easy to see that $CP_{m,n}$ is a closed convex cone. Suppose that $B$ is a real $m$th order $n$th dimensional symmetric tensor. If for all $x \in \mathbb{R}_+^n$, we have $Bx^m \geq 0$, then $B$ is called a \textit{copositive tensor} \[\mathcal{S}\]. Denote the set of all $m$th order $n$-dimensional copositive tensors by $COP_{m,n}$. Then, it is also easy to see that $COP_{m,n}$ is a closed convex cone. When $m = 2$, a classical result is...
that the CP matrix cone and the copositive matrix cone are dual to each other. We now extend this result to the CP tensor cone and the copositive tensor cone.

Let $A = (a_{i_1 \cdots i_m})$ and $B = (b_{i_1 \cdots i_m})$ be two real $m$th order $n$-dimensional symmetric tensors. Their inner product is defined as

$$A \bullet B = \sum_{i_1, \cdots, i_m=1}^{n} a_{i_1 \cdots i_m} b_{i_1 \cdots i_m}.$$  

**Theorem 5** Let $m \geq 2$ and $n \geq 1$. Then $CP_{m,n}$ and $COP_{m,n}$ are dual to each other.

**Proof.** Suppose that $B$ is an $m$th order $n$-dimensional copositive tensor. For any $A \in CP_{m,n}$, by definition, we may assume that $A$ can be expressed by (1). Since $B$ is a copositive tensor, by definition, $B(u(k))^m \geq 0$, for $k = 1, \cdots, r$. Thus,

$$A \bullet B = \sum_{k=1}^{r} B(u(k))^m \geq 0.$$  

Thus, $B$ is in the dual cone of $CP_{m,n}$.

On the other hand, assume that $B$ is in the dual cone of $CP_{m,n}$. Let $x \in \mathbb{R}^n_+$. Then $x^m$ is an $m$ order $n$-dimensional CP tensor, i.e., $x \in CP_{m,n}$. We have $Bx^m = B \bullet x^m \geq 0$. This shows that $B$ is a copositive tensor.

Together, we see that $CP_{m,n}$ and $COP_{m,n}$ are dual to each other. $\square$

## 5 Strongly Symmetric Tensors

Suppose that $A = (a_{i_1 \cdots i_m})$ is a real $m$th order $n$-dimensional tensor. If for any $(i_1, \cdots, i_m) \sim (j_1, \cdots, j_m)$, $(i_1, \cdots, i_m), (j_1, \cdots, j_m) \in \mathcal{I}$, we have $a_{i_1 \cdots i_m} = a_{j_1 \cdots j_m}$, then we say that $A$ is a **strongly symmetric** tensor. Clearly, a strongly symmetric tensor is a symmetric tensor. It is also clear that a linear combination of strongly symmetric tensors is still a strongly symmetric tensor. Thus, the set of all real $m$th order $n$-dimensional strongly symmetric tensors is a linear space.

Let $A = (a_{i_1 \cdots i_m})$ be a real $m$th order $n$-dimensional symmetric tensor. If

$$A = \sum_{k=1}^{r} \alpha_k (v(k))^m,$$

where $\alpha_k$ are real numbers and $v(k)$ are binary vectors in $\mathbb{R}^n$ for $k = 1, \cdots, r$, then we say that $A$ has a **symmetric binary decomposition**, which is not a nonnegative tensor factorization, but a general symmetric tensor decomposition [3, 6].

It is easy to show the following proposition.
Proposition 2 Suppose that $\mathcal{A} = (a_{i_1 \ldots i_m})$ is a real $m$th order $n$-dimensional tensor with a symmetric binary decomposition. Then $\mathcal{A}$ is strongly symmetric.

Proof. Suppose that $\mathcal{A} = (a_{i_1 \ldots i_m})$ is expressed by \[4\]. Assume that $(i_1, \ldots, i_m) \sim (j_1, \ldots, j_m)$. Then
\[
a_{i_1 \ldots i_m} = \sum \{a_k : (i_1, \ldots, i_m) \preceq \text{supp} (v^{(k)})\} = \sum \{a_k : (j_1, \ldots, j_m) \preceq \text{supp} (v^{(k)})\} = a_{j_1 \ldots j_m}.
\]
This completes the proof.

For $k = 1, \ldots, m$, let
\[
\mathcal{I}_k = \{(i_1, \ldots, i_m) \in \mathcal{I} : |(i_1, \ldots, i_m)| = k\}.
\]
Then $\mathcal{I}_1, \ldots, \mathcal{I}_m$ form a partition of $\mathcal{I}$.

For $k = 1, \ldots, m$, let
\[
\mathcal{I}_{k+} = \{(i_1, \ldots, i_k, i_k, \ldots, i_k) \in \mathcal{I}_k : 1 \leq i_1 < i_2 \cdots < i_k \leq n\}.
\]
Then $\mathcal{I}_{k+}$ is the “representative” set of $\mathcal{I}_k$ in the sense that any member in $\mathcal{I}_k$ is similar to a member of $\mathcal{I}_{k+}$ and no two members in $\mathcal{I}_{k+}$ are similar.

Suppose that $\mathcal{A} = (a_{i_1 \ldots i_m})$ is a real $m$th order $n$-dimensional tensor. For $k = 1, \ldots, m$, let
\[
\mathcal{I}_{k+}(\mathcal{A}) = \{(i_1, \ldots, i_k, i_k, \ldots, i_k) \in \mathcal{I}_{k+} : a_{i_1 \ldots i_k i_k \ldots i_k} \neq 0\}.
\]

We now construct a hierarchical elimination algorithm to obtain symmetric binary decomposition of a strongly symmetric tensor. Suppose that $\mathcal{A} = (a_{i_1 \ldots i_m})$ is a real $m$th order $n$-dimensional strongly symmetric tensor.

Algorithm 1 Step 0. Let $k = 0$ and $\mathcal{A}^{(0)} = (a^{(0)}_{i_1 \ldots i_m})$ be defined by $\mathcal{A}^{(0)} = \mathcal{A}$.

Step 1. For any $e = (i_1, \ldots, i_{m-k}, \ldots, i_{m-k}) \in \mathcal{I}_{(m-k)+} (\mathcal{A}^{(k)})$, let $v^e \in \mathbb{R}^n_+$ be a binary vector such that $v^e_{i_1} = \cdots = v^e_{i_{m-k}} = 1$ and $v^e_i = 0$ if $i \not\in \{i_1, \ldots, i_{m-k}\}$.

Let $\mathcal{A}^{(k+1)} = (a^{(k+1)}_{i_1 \ldots i_m})$ be defined by
\[
\mathcal{A}^{(k+1)} = \mathcal{A}^{(k)} - \sum \left\{a^{(k)}_{i_1 \ldots i_{m-k} \ldots i_{m-k}} (v^e)^m : e = (i_1, \ldots, i_{m-k}, \ldots, i_{m-k}) \in \mathcal{I}_{(m-k)+} (\mathcal{A}^{(k)}) \right\}.
\]

Step 2. Let $k = k + 1$. If $k = m$, stop. Otherwise, go to Step 1.

Theorem 6 Suppose that $\mathcal{A} = (a_{i_1 \ldots i_m})$ is a real $m$th order $n$-dimensional strongly symmetric tensor. Then we have $\mathcal{A}^{(m)} = 0$ in Algorithm 1, i.e., we have
\[
\mathcal{A} = \sum_{k=0}^{m-1} \sum \left\{a^{(k)}_{i_1 \ldots i_{m-k} \ldots i_{m-k}} (v^e)^m : e = (i_1, \ldots, i_{m-k}, \ldots, i_{m-k}) \in \mathcal{I}_{(m-k)+} (\mathcal{A}^{(k)}) \right\}.
\]

Thus, a symmetric tensor has a symmetric binary decomposition if and only if it is strongly symmetric.
Proof. For \( k = 1, \ldots, m \), we now show by induction that \( A^{(k)} \) is strongly symmetric, and \( I_{(m-p)+}(A^{(k)}) = \emptyset \) for \( p = 0, \ldots, k - 1 \).

By Step 0 and the assumption, \( A^{(0)} \) is strongly symmetric.

For \( k = 0, \ldots, m - 1 \), assume that \( A^{(k)} \) is strongly symmetric, and \( I_{(m-p)+}(A^{(k)}) = \emptyset \) for \( p = 0, \ldots, k - 1 \) if \( k \geq 1 \). By (5) and Proposition 2, \( A^{(k+1)} \) is a linear combination of strongly symmetric tensors, thus also a strongly symmetric tensor. As in this iteration \(|\text{supp}(v^e)| = m - k \) for all \( v^e \) in (5), \( a_{i_1, \ldots, i_{m-k}, i_{m-k}}^{(k+1)} = a_{i_1, \ldots, i_{m}}^{(k)} = 0 \) if \(|[i_1, \ldots, i_{m}]| > m - k \). Thus, \( I_{(m-p)+}(A^{(k+1)}) = \emptyset \) for \( p = 0, \ldots, k - 1 \). By (5), we also have \( I_{(m-k)+}(A^{(k+1)}) = \emptyset \). The induction proof is completed.

This shows that \( A^{(m)} = 0 \). By this and (5), we have (6). Thus, a strongly symmetric tensor has a symmetric binary decomposition. By this and Proposition 2 the last conclusion also holds. This completes the proof. \( \square \)

6 Strongly Symmetric, Hierarchically Dominated Tensors

In (6), if all the coefficients \( a_{i_1, \ldots, i_{m-k}, i_{m-k}}^{(k)} \) are nonnegative, then \( A \) is a CP tensor. In this section, we explore a sufficient condition for this.

For \( p = 1, \ldots, m - 1 \), and \( q = 1, \ldots, m - p \), for any \( (i_1, \ldots, i_{p}, i_{p}, \ldots, i_{p}) \in \mathcal{I}_{p+} \), define

\[
\mathcal{J}_q(i_1, \ldots, i_{p}) = \{(j_1, \ldots, j_{p+q}, \ldots, j_{p+q}) \in \mathcal{I}_{(p+q)+} : (i_1, \ldots, i_{p}, \ldots, i_{p}) \preceq (j_1, \ldots, j_{p+q}, \ldots, j_{p+q})\}.
\]

An \( m \)th order \( n \)-dimensional strongly symmetric nonnegative tensor \( A = (a_{i_1, \ldots, i_{m}}) \) is said to be **hierarchically dominated** if for \( p = 1, \ldots, m - 1 \), and any \( (i_1, \ldots, i_{p}, i_{p}, \ldots, i_{p}) \in \mathcal{I}_{p+} \), we have

\[
a_{i_1, \ldots, i_{p}, \ldots, i_{p}} \geq \sum \{a_{j_1, \ldots, j_{p+1}, \ldots, j_{p+1}} : (j_1, \ldots, j_{p+1}, j_{p+1}, \ldots, j_{p+1}) \in \mathcal{J}_1(i_1, \ldots, i_{p})\}. \tag{7}
\]

Suppose that \( A \) is an \( m \)th order \( n \)-dimensional strongly symmetric, hierarchically dominated nonnegative tensor. By (7), for \( p = 1, \ldots, m - 2 \), and any \( (i_1, \ldots, i_{p}, i_{p}, \ldots, i_{p}) \in \mathcal{I}_{p+} \), we have

\[
a_{i_1, \ldots, i_{p}, \ldots, i_{p}} \geq \sum \{a_{l_1, \ldots, l_{p+2}, \ldots, j_{p+2}} : (l_1, \ldots, l_{p+2}, \ldots, j_{p+2}) \in \mathcal{J}_2(i_1, \ldots, i_{p})\}.
\]

Thus, by induction, we may prove the following proposition.
Proposition 3 Suppose that $\mathcal{A} = (a_{i_1\cdots i_m})$ is an $m$th order $n$-dimensional strongly symmetric, hierarchically dominated nonnegative tensor. Then for $p = 1, \cdots, m - 1$, and $q = 1, \cdots, m - p$, for any $(i_1, \cdots, i_p, i_{p+1}, \cdots, i_p) \in \mathcal{I}_{p+}$, we have

$$a_{i_1\cdots i_p\cdots i_p} \geq \sum \{a_{j_1\cdots j_{p+q}\cdots j_{p+q}} : (j_1, \cdots, j_{p+q}, \cdots, j_{p+q}) \in \mathcal{I}_q(i_1, \cdots, i_p)\}. \quad (8)$$

With this proposition, we may prove the following main theorem of this section.

Theorem 7 Suppose that $\mathcal{A} = (a_{i_1\cdots i_m})$ is an $m$th order $n$-dimensional strongly symmetric, hierarchically dominated nonnegative tensor. Then $\mathcal{A}^{(k)}$ are nonnegative for $k = 0, \cdots, m - 1$, in Algorithm 7. Thus, $\mathcal{A}$ is a CP tensor. Thus, a strongly symmetric, hierarchically dominated nonnegative tensor is a CP tensor.

Proof. For $k = 1, \cdots, m - 1$, we now show by induction that $\mathcal{A}^{(k)}$ is a strongly symmetric, hierarchically dominated nonnegative tensor. 

By Step 0 and the assumption, $\mathcal{A}^{(0)}$ is a strongly symmetric, hierarchically dominated nonnegative tensor.

For $k = 0, \cdots, m - 1$, assume that $\mathcal{A}^{(k)}$ is a strongly symmetric, hierarchically dominated nonnegative tensor. We now consider $\mathcal{A}^{(k+1)}$.

By the proof of Theorem 5, $\mathcal{A}^{(k+1)}$ is also strongly symmetric and for $p = 0, \cdots, k$, and any $(i_1, \cdots, i_m) \in \mathcal{I}_{(m-p)+}$, $a_{i_1\cdots i_m}^{(k+1)} = 0$. By strong symmetry of $\mathcal{A}^{(k+1)}$, for $p = 0, \cdots, k$, and any $(i_1, \cdots, i_m) \in \mathcal{I}_{m-p}$, $a_{i_1\cdots i_m}^{(k+1)} = 0$.

Now for $p = k + 1, \cdots, m - 1$, and any $(i_1, \cdots, i_m) \in \mathcal{I}_{(m-p)+}$, by (5),

$$a_{i_1\cdots i_{m-p}\cdots i_{m-p}}^{(k+1)} = a_{i_1\cdots i_{m-p}\cdots i_{m-p}}^{(k)} - \sum \{a_{i_1\cdots i_{m-k}\cdots i_{m-k}}^{(k)} : (l_1, \cdots, l_{m-k}, \cdots, l_{m-k}) \in \mathcal{I}_{m-k}(i_1, \cdots, i_{m-p})\}. \quad (9)$$

By Proposition 3, the right hand side of (6) is nonnegative. Thus, for $p = k + 1, \cdots, m - 1$, and any $(i_1, \cdots, i_m) \in \mathcal{I}_{(m-p)+}$, $a_{i_1\cdots i_{m-p}\cdots i_{m-p}}^{(k+1)} \geq 0$. By strong symmetry of $\mathcal{A}^{(k+1)}$, for $p = k + 1, \cdots, m - 1$, and any $(i_1, \cdots, i_m) \in \mathcal{I}_{m-p}$, $a_{i_1\cdots i_{m-p}\cdots i_{m-p}}^{(k+1)} \geq 0$. This shows that $\mathcal{A}^{(k+1)}$ is nonnegative.

Since $\mathcal{A}^{(k)} = (a_{i_1\cdots i_m}^{(k)})$ is hierarchically dominated, for $p = 1, \cdots, m - 1$, and any $(i_1, \cdots, i_{p+1}, \cdots, i_p) \in \mathcal{I}_{p+}$, we have

$$a_{i_1\cdots i_{p+1}\cdots i_p}^{(k)} \geq \sum \{a_{j_1\cdots j_{p+1}\cdots j_{p+1}}^{(k)} : (j_1, \cdots, j_{p+1}, \cdots, j_{p+1}) \in \mathcal{I}_1(i_1, \cdots, i_p)\}. \quad (10)$$

By (3), we have

$$a_{i_1\cdots i_p\cdots i_p}^{(k+1)} = a_{i_1\cdots i_p\cdots i_p}^{(k)} - \sum \{a_{i_1\cdots i_{m-k}\cdots i_{m-k}}^{(k)} : (l_1, \cdots, l_{m-k}, \cdots, l_{m-k}) \in \mathcal{I}_{m-p}(i_1, \cdots, i_{m-p})\}. \quad (11)$$
and

\[
\begin{align*}
&\sum_{(j_1, \cdots, j_{p+1})} a^{(k+1)}_{j_1 \cdots j_{p+1}} \\
=& \sum_{(j_1, \cdots, j_{p+1})} \{ a^{(k)}_{l_1 \cdots l_m-k \cdots l_m-k} : (l_1, \cdots, l_{m-k}, \cdots, l_{m-k}) \in J_{m-p-1-k}(j_1, \cdots, j_{p+1}) \} \tag{12}
\end{align*}
\]

Comparing (10), (11) and (12), for \(p = 1, \cdots, m - 1\), and any \((i_1, \cdots, i_p, \cdots, i_p) \in I_p\), we have

\[
a_{1 \cdots p \cdots p}^{(k+1)} \geq \sum \left\{ a_{j_1 \cdots j_{p+1}}^{(k+1)} : (j_1, \cdots, j_{p+1}) \in J_1(i_1, \cdots, i_p) \right\}.
\]

Thus, \(A^{(k+1)}\) is also hierarchically dominated. The induction proof is completed.

Hence, \(A\) is a CP tensor. Therefore, a strongly symmetric, hierarchically dominated nonnegative tensor is a CP tensor. This completes the proof. □

When \(m = 2\), Theorem 6 implies Kaykobad’s result [5].

**Corollary 3** Suppose that \(A = (a_{i_1 \cdots i_m})\) is a real \(m\)th order \(n\)-dimensional strongly symmetric, hierarchically dominated nonnegative tensor. Then the CP rank of \(A\) is not bigger than \(\sum_{k=0}^{m-1} \left( \begin{array}{c} n \\ m-k \end{array} \right) \).

**Proof.** By (6), the CP rank of \(A\) is not bigger than

\[
\sum_{k=0}^{m-1} \left| J_{(m-k)+}(A^{(k)}) \right| = \sum_{k=0}^{m-1} \left( \begin{array}{c} n \\ m-k \end{array} \right).
\]

This completes the proof. □

### 7 Numerical Examples

In this section, we present some strongly symmetric, hierarchically dominated nonnegative tensors with \(m = 3, n = 10\) and \(m = 4, n = 10\), and use Algorithm 1 to decompose them.

**Example** \(A\) is a strongly symmetric, hierarchically dominated nonnegative tensor. The entries of \(A\), whose index sets are not similar to the index sets of the entries defined below, are zero.

The \(m = 3, n = 10\) case:

1. \(A(1,1,1) = 1, A(2,2,2) = 5, A(3,3,3) = 3, A(4,4,4) = 2, A(5,5,5) = 4, A(6,6,6) = 2, A(7,7,7) = 2, A(8,8,8) = 2, A(9,9,9) = 5, A(10,10,10) = 4, A(1,5,5) = 1, A(2,3,3) = 1, A(2,6,6) = 1, A(2,8,8) = 1, A(3,4,4) = 1, A(3,5,5) = 1, A(4,5,5) = 1, A(5,9,9) = 1, A(6,9,9) = 1, A(7,9,9) = 1, A(7,10,10) = 1, A(8,10,10) = 1,
A(9, 10, 10) = 1, A(2, 6, 9) = 1, A(2, 8, 10) = 1, A(3, 4, 5) = 1, A(7, 9, 10) = 1.

(2) A(1, 1, 1) = 2, A(2, 2, 2) = 5, A(3, 3, 3) = 6, A(4, 4, 4) = 2, A(5, 5, 5) = 3, A(8, 8, 8) = 6, A(9, 9, 9) = 6, A(10, 10, 10) = 4, A(1, 5, 5) = 1, A(1, 10, 10) = 1, A(2, 3, 3) = 1, A(2, 8, 8) = 1, A(2, 9, 9) = 2, A(2, 10, 10) = 1, A(3, 4, 4) = 1, A(3, 8, 8) = 2, A(3, 9, 9) = 2, A(4, 8, 8) = 1, A(5, 8, 8) = 1, A(5, 10, 10) = 1, A(8, 9, 9) = 1, A(9, 10, 10) = 1, A(1, 5, 10) = 1, A(2, 3, 9) = 1, A(2, 9, 10) = 1, A(3, 4, 8) = 1, A(3, 8, 9) = 1.

(3) A(2, 2, 2) = 4, A(3, 3, 3) = 6, A(4, 4, 4) = 7, A(5, 5, 5) = 4, A(7, 7, 7) = 4, A(8, 8, 8) = 6, A(9, 9, 9) = 4, A(10, 10, 10) = 3, A(2, 3, 3) = 1, A(2, 4, 4) = 1, A(2, 5, 5) = 1, A(2, 8, 8) = 1, A(3, 4, 4) = 1, A(3, 5, 5) = 1, A(3, 7, 7) = 1, A(3, 8, 8) = 2, A(4, 5, 5) = 2, A(4, 7, 7) = 1, A(4, 9, 9) = 1, A(4, 10, 10) = 1, A(7, 8, 8) = 1, A(7, 9, 9) = 1, A(8, 9, 9) = 1, A(8, 10, 10) = 1, A(9, 10, 10) = 1, A(2, 3, 8) = 1, A(2, 4, 5) = 1, A(3, 4, 5) = 1, A(3, 7, 8) = 1, A(4, 7, 9) = 1, A(8, 9, 10) = 1.

The m = 4, n = 10 case:

(1) A(1, 1, 1, 1) = 1, A(2, 2, 2, 2) = 6, A(4, 4, 4, 4) = 6, A(5, 5, 5, 5) = 2, A(6, 6, 6, 6) = 3, A(7, 7, 7, 7) = 4, A(8, 8, 8, 8) = 8, A(9, 9, 9, 9) = 12, A(10, 10, 10, 10) = 4, A(1, 10, 10, 10) = 1, A(2, 4, 4, 4) = 2, A(2, 8, 8, 8) = 2, A(2, 9, 9, 9) = 2, A(4, 8, 8, 8) = 2, A(4, 9, 9, 9) = 2, A(5, 7, 7, 7) = 1, A(5, 9, 9, 9) = 1, A(6, 7, 7, 7) = 1, A(6, 9, 9, 9) = 1, A(6, 10, 10, 10) = 1, A(7, 9, 9, 9) = 2, A(8, 9, 9, 9) = 3, A(8, 10, 10, 10) = 1, A(9, 10, 10, 10) = 1, A(2, 4, 8, 8) = 1, A(2, 4, 9, 9) = 1, A(2, 8, 9, 9) = 1, A(4, 8, 9, 9) = 1, A(5, 7, 9, 9) = 1, A(6, 7, 9, 9) = 1, A(8, 9, 10, 10) = 1, A(2, 4, 8, 9) = 1.

(2) A(1, 1, 1, 1) = 9, A(2, 2, 2, 2) = 6, A(3, 3, 3, 3) = 8, A(4, 4, 4, 4) = 1, A(5, 5, 5, 5) = 1, A(6, 6, 6, 6) = 4, A(7, 7, 7, 7) = 6, A(8, 8, 8, 8) = 6, A(9, 9, 9, 9) = 9, A(10, 10, 10, 10) = 2, A(1, 2, 2, 2) = 1, A(1, 3, 3, 3) = 2, A(1, 5, 5, 5) = 1, A(1, 7, 7, 7) = 1, A(1, 8, 8, 8) = 2, A(1, 9, 9, 9) = 2, A(2, 3, 3, 3) = 1, A(2, 6, 6, 6) = 2, A(2, 7, 7, 7) = 2, A(3, 6, 6, 6) = 1, A(3, 8, 8, 8) = 2, A(3, 9, 9, 9) = 2, A(4, 9, 9, 9) = 1, A(6, 7, 7, 7) = 1, A(7, 9, 9, 9) = 1, A(7, 10, 10, 10) = 1, A(8, 9, 9, 9) = 2, A(9, 10, 10, 10) = 1, A(1, 2, 7, 7) = 1, A(1, 3, 8, 8) = 1, A(1, 3, 9, 9) = 1, A(1, 8, 9, 9) = 1, A(2, 3, 6, 6) = 1, A(2, 6, 7, 7) = 1, A(3, 8, 9, 9) = 1, A(7, 9, 10, 10) = 1, A(1, 3, 8, 9) = 1.

(3) A(1, 1, 1, 1) = 18, A(2, 2, 2, 2) = 6, A(3, 3, 3, 3) = 4, A(4, 4, 4, 4) = 2, A(5, 5, 5, 5) = 26, A(6, 6, 6, 6) = 18, A(8, 8, 8, 8) = 8, A(9, 9, 9, 9) = 24, A(10, 10, 10, 10) = 8, A(1, 5, 5, 5) = 6, A(1, 6, 6, 6) = 4, A(1, 8, 8, 8) = 2, A(1, 9, 9, 9) = 4, A(1, 10, 10, 10) = 2, A(2, 5, 5, 5) = 2, A(2, 6, 6, 6) = 2, A(2, 9, 9, 9) = 2, A(3, 4, 4, 4) = 1, A(3, 9, 9, 9) = 2, A(3, 10, 10, 10) = 1, A(4, 9, 9, 9) = 1, A(5, 6, 6, 6) = 6, A(5, 8, 8, 8) = 3, A(5, 9, 9, 9) = 7, A(5, 10, 10, 10) = 2,
8 Further Remarks

In this paper, we studied various properties of CP tensors, showed that a strongly symmetric, hierarchically dominated nonnegative tensor is a CP tensor, and presented a hierarchical elimination algorithm for checking this. These indicate that a rich theory for CP tensors can be established parallel to the theory of CP matrices [1, 4, 5, 10]. This theory will be a solid foundation for applications of nonnegative tensor factorization [2, 9]. Further research on topics such as CP ranks are needed.
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Table 4: $n = 10, m = 4$ (1)

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Table 6: $n = 10, m = 4$ (3)
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