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Stochastic Nonlinear Complementarity Problem and Applications to Traffic Equilibrium under Uncertainty

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Abstract The expected residual minimization (ERM) formulation for the stochastic nonlinear complementarity problem (SNCP) is studied in this paper. We show that the involved function is a stochastic R_0 function if and only if the objective function in the ERM formulation is coercive under a mild assumption. Moreover, we model the traffic equilibrium problem (TEP) under uncertainty as SNCP and show that the objective function in the ERM formulation is a stochastic R_0 function. Numerical experiments show that the ERM-SNCP model for TEP under uncertainty has various desirable properties.

Keywords Stochastic nonlinear complementarity problem · Expected residual minimization · Traffic equilibrium problem under uncertainty · Stochastic R_0 function

1 Introduction

In this paper, we consider the *stochastic nonlinear complementarity problem*

$$x \geq 0, \quad F(x,\omega) \geq 0, \quad x^T F(x,\omega) = 0, \quad \omega \in \Omega, \tag{1}$$

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where $\omega \in \Omega \subseteq R^m$ is a random vector with given probability distribution $\mathcal{P}$ and $F : R^n \times \Omega \to R^n$ is a given vector-valued function. We denote problem (1) by SNCP$(F(x,\omega))$.

If Ω is a singleton, SNCP$(F(x,\omega))$ reduces to the intensively studied *nonlinear complementarity problem*; see the comprehensive books ([1] and [2]) for theoretical analysis, numerical algorithms and applications especially in economics and engineering. In reality, due to stochastic factors, the function value of F depends not only on the variables x, but also on random vectors. The SNCP provides a framework for modeling of equilibria under uncertainty as a special case of stochastic variational inequalities. Recently, Lin and Fukushima [3] reformulated the SNCP as a stochastic mathematical programming problem with equilibrium constraints. When F is an affine function of x for any $\omega \in \Omega$,

$$F(x,\omega) = M(\omega)x + q(\omega), \quad \omega \in \Omega, \tag{2}$$

where $M(\omega) \in R^{n\times n}$ and $q(\omega) \in R^n$, the SNCP$(F(x,\omega))$ reduces to the *stochastic linear complementarity problem* (SLCP), denoted by SLCP$(M(\omega), q(\omega))$, which has been studied recently in [4–6].

The *expected value* (EV) formulation introduced in [7] and the *expected residual minimization* (ERM) introduced in [4] are two deterministic formulations for the SNCP. The EV formulation is to solve a single nonlinear complementarity problem NCP$(E[F(x,\omega)])$. The ERM formulation is to minimize the expected residual of the NCP$(F(x,\omega))$ for all $\omega \in \Omega$. A version of the ERM formulation using NCP functions is to find an optimal solution of

$$\min_{x\in R^n_+} f(x) := E[\|\Phi(x,\omega)\|^2], \tag{3}$$

where

$$\Phi(x,\omega) = (\phi(F_1(x,\omega), x_1), \dots, \phi(F_n(x,\omega), x_n)),$$

and $\phi : R^2 \to R$ is an NCP function, which satisfies

$$\phi(a,b) = 0 \quad \Longleftrightarrow \quad a \geq 0,\ b \geq 0,\ ab = 0.$$

Many NCP functions have been studied for solving nonlinear complementarity problems [2]. In this paper, we study the ERM formulation (3) for SNCP with the following three NCP functions:

(i) The min function $\phi_1(a,b) = \min(a,b)$.
(ii) The FB function $\phi_2(a,b) = a + b - \sqrt{a^2+b^2}$.
(iii) The penalized FB function $\phi_3(a,b) = \lambda\phi_2(a,b) + (1-\lambda)a_+b_+$, $\lambda \in (0,1)$.

It is known ([5] and [8]) that there exist positive constants c_1, c_2, c_3 such that

$$c_1|\phi_1(a,b)| \leq c_2|\phi_2(a,b)| \leq c_3|\phi_1(a,b)| \leq |\phi_3(a,b)|. \tag{4}$$

The above relation indicates that ϕ_1 and ϕ_2 have the same growth rate, and the growth rate of ϕ_3 is no less than that of ϕ_1 and ϕ_2. In the following, we use f_i to distinct f defined by ϕ_i for $i = 1, 2, 3$, and f when we study their common properties.

In this paper, we study the solution set of the ERM formulation (3) for the SNCP. In particular, we define a *stochastic* R_0 *function* and show that F is a stochastic R_0 function if and only if the objective function f_1 in the ERM formulation (3) for the SNCP($F(x,\omega)$) is coercive, i.e., $f_1(x) \to \infty$ as $\|x\| \to \infty$, under a mild assumption. Moreover, we model the traffic equilibrium problem (TEP) under uncertainty as SNCP and show that the involved function F is a stochastic R_0 function. Our numerical experiments show that a solution of the ERM formulation has high reliability and delivered rate.

The NCP model with effective algorithms for static TEP based on the Wardrop equilibrium principle [9] has been widely studied ([2] and [10–12]). On the other hand, disruptive events such as uncertain demands, adverse weather, road construction, traffic accidents, landslides, earthquakes, may disrupt greatly one static equilibrium of a network. Recently, Fernando and Nichlàs [13] address this problem and extend the Wardrop equilibrium principle to TEP under uncertainty by defining a *robust Wardrop equilibrium* (RWE). Their equilibria is supposed to be robust in the sense that it has optimal worst-case cost, which is different with the robustness of SNCP c.f. [5].

The remainder of this paper is organized as follows: In Sect. 2, we introduce the concepts of a stochastic R_0 function, and equicoercivity. We show that under the assumption that F is equicoercive, F being a stochastic R_0 function is a necessary and sufficient condition for the coercivity of f_1 in the ERM formulation. In Sect. 3, we model the TEP under uncertainty as a stochastic R_0 function NCP. In Sect. 4, we report numerical results of the ERM formulation and the EV formulation for TEP under uncertainty.

We will use the following notations. $\langle l, u\rangle$ represents the set $\{l, l+1, \dots, u\}$ for natural numbers l and u with $l < u$, $z_+ = \max(z, 0)$ for any given vector z, $|\mathcal{S}|$ denotes the cardinality of a given finite set $\mathcal{S}$, and $\|\cdot\|$ refers to the Euclidean norm. Given a set $\Omega \subseteq R^m$ of random vectors, let suppΩ be the support set of Ω. For a given subset $\hat{\Omega} \subseteq \Omega$ and a function $s : \Omega \to R_+$, we use $E_{\hat{\Omega}}[s(\omega)]$ to represent $E[s(\omega)1_{\{\omega\in\hat{\Omega}\}}]$ for simplicity, where $1_{\{\omega\in\hat{\Omega}\}}$ is the indicator function of the set $\hat{\Omega}$, which is equal to 1 if $\omega \in \hat{\Omega}$ and 0 if $\omega \in \Omega \setminus \hat{\Omega}$. Throughout the paper, we suppose the following assumption holds:

Assumption A1 $F(\cdot, \omega)$ is a continuous function for $\omega \in \Omega$ a.e. and $E[\|F(x,\omega)\|^2] < \infty$ at any $x \in R^n_+$.

Remark 1.1 Note that $\|\min(x, F(x,\omega))\| \le \|F(x,\omega)\|$ for any $x \in R^n_+$. It is easy to verify that $E[\|F(x,\omega)\|^2] < \infty$ at any $x \in R^n_+$ implies that $f(x) < \infty$ at any $x \in R^n_+$. Moreover, from Proposition 1 in Chap. 2 [14], if there exists a function $z(\omega)$ such that $\|F(x,\omega)\|^2 \le z(\omega)$ a.e. for all x in a neighborhood of $\hat{x}$, and $E[z(\omega)] < \infty$, then f is continuous at $\hat{x}$ under Assumption A1.

2 Solution Set of ERM for SNCP

In this section we investigate solvability of the ERM formulation for the SNCP. We define a stochastic R_0 function. Under the assumption that F is equicoercive, we prove that the involved function being a stochastic R_0 function is a necessary and sufficient condition for the coercivity of the objective function in the ERM formulation.

The solution set of the ERM formulation for the SLCP has been studied in [4–6]. Some results depending on the special affine construction of $F(x,\omega)$ in the SLCP cannot be simply generalized to the SNCP. For instance, Lemma 2.2 in [5] states that the ERM formulation for the SLCP($M(\omega), q(\omega)$) defined by the 'min' function always has a solution when Ω is composed of finite elements. However, the following example tells us that we do not have the same result for the SNCP($F(x,\omega)$).

Example 2.1 Let $F(x,\omega) = (\frac{1}{2} - \frac{3}{2}\omega)e^{-\frac{\omega}{2}x} - \omega$ where $\omega \in \Omega = \{\omega^1, \omega^2\}$. Here, $\omega^1 = 0$, $\omega^2 = 1$, and $\mathcal{P}\{\omega^1\} = \mathcal{P}\{\omega^2\} = \frac{1}{2}$. Then the objective function in the ERM formulation for SNCP($F(x,\omega)$) defined by ϕ_1 is

$$\begin{aligned} f_1(x) &= \frac{1}{2}\left\|\min\left(x, \frac{1}{2}\right)\right\|^2 + \frac{1}{2}\|\min(x, -e^{-\frac{x}{2}} - 1)\|^2 \\ &= \begin{cases} \frac{1}{2}x^2 + \frac{1}{2}(e^{-x} + 1 + 2e^{-\frac{x}{2}}) & x \in [0, \frac{1}{2}], \\ \frac{1}{8} + \frac{1}{2}(e^{-x} + 1 + 2e^{-\frac{x}{2}}) & x \in (\frac{1}{2}, \infty). \end{cases} \end{aligned}$$

It is easy to find that, for $x \in [0, \frac{1}{2}]$,

$$f_1(x) \geq \frac{1}{2}(e^{-\frac{1}{2}} + 1 + 2e^{-\frac{1}{4}}) = \frac{1}{2} + \frac{1}{2\sqrt{e}} + e^{-\frac{1}{4}} > \frac{5}{8},$$

and for $x \in (\frac{1}{2}, \infty)$, $f_1(x)$ is strictly decreasing and tending to $\frac{5}{8}$ as x tends to ∞. Hence, the ERM formulation defined by the 'min' function has no solution. Moreover, we have $E[F(x,\omega)] = -\frac{1}{4} - \frac{1}{2}e^{-\frac{x}{2}} < 0$ for any x. Thus, the EV formulation NCP($E[F(x,\omega)]$) has no solution.

However, for any $\lambda \in (0,1)$ and $x \in R_+$,

$$f_3(x) \geq \frac{1}{2}\left[\lambda\left(x + \frac{1}{2} - \sqrt{x^2 + \frac{1}{4}}\right) + \frac{1}{2}(1-\lambda)x\right]^2 \geq \frac{1}{8}(1-\lambda)^2 x^2,$$

which is coercive, and hence ERM formulation defined by ϕ_3 has a nonempty and bounded solution set.

2.1 Stochastic R_0 Function

The R_0 property relates closely to the boundedness of level sets in the literature of the complementarity problem. For NCP(G), $G: R^n \to R^n$ is an R_0 function if and only if the function $\|\min(x, G(x))\|^2$ is coercive.

Definition 2.1 (See [2]) The function $G: R^n \to R^n$ is called an R_0 function on a set $\mathcal{D} \subseteq R^n$ if, for every infinite sequence $\{x^k\} \subseteq \mathcal{D}$ satisfying

$$\lim_{k\to\infty} \|x^k\| = \infty, \qquad \limsup_{k\to\infty} \|(-x^k)_+\| < \infty, \qquad \limsup_{k\to\infty} \|(-G(x^k))_+\| < \infty, \quad (5)$$

there exists $i \in \langle 1, n\rangle$ such that $\limsup_{k\to\infty} \min(x_i^k, G_i(x^k)) = \infty$.

Now, we define a stochastic R_0 function.

Definition 2.2 $F : R^n \times \Omega \to R^n$ is called a stochastic R_0 function on a set $\mathcal{D} \subseteq R^n$ if, for every infinite sequence $\{x^k\} \subseteq \mathcal{D}$ satisfying

$$\begin{aligned} &\lim_{k\to\infty} \|x^k\| = \infty, \qquad \limsup_{k\to\infty} \|(-x^k)_+\| < \infty, \\ &\limsup_{k\to\infty} \|(-F(x^k,\omega))_+\| < \infty \quad \text{a.e.} \end{aligned} \tag{6}$$

there exists $i \in \langle 1, n\rangle$ such that $\mathcal{P}\{\omega : \limsup_{k\to\infty} \min(x_i^k, F_i(x^k,\omega)) = \infty\} > 0$.

If Ω is a singleton, Definition 2.2 reduces to Definition 2.1.

Definition 2.3 We say $F : R^n \times \Omega \to R^n$ is equicoercive on $\mathcal{D} \subseteq R^n$ if, for any $\{x^k\} \subseteq \mathcal{D}$ satisfying $\|x^k\| \to \infty$, the existence of $\{\omega^k\} \subseteq \operatorname{supp}\Omega$ with $\lim_{k\to\infty} F_i(x^k,\omega^k) = \infty$ $(\lim_{k\to\infty}(-F_i(x^k,\omega^k))_+ = \infty)$ for some $i \in \langle 1, n\rangle$ implies that there exists $\{x^{k_j}\} \subseteq \{x^k\}$ such that

$$\mathcal{P}\{\omega : \lim_{k_j\to\infty} F_i(x^{k_j},\omega) = \infty\} > 0\big(\mathcal{P}\{\omega : \lim_{k_j\to\infty} (-F_i(x^{k_j},\omega))_+ = \infty\} > 0\big).$$

Proposition 2.1 *$F : R^n \times \Omega \to R^n$ is equicoercive if Ω is a compact set and there exist constants $L > 0$ and $\delta > 0$ such that, if $\|\omega^1 - \omega^2\| < \delta$, then*

$$\|F(x,\omega^1) - F(x,\omega^2)\| < L, \quad \textit{for any } x \in R^n_+.$$

Proof We only consider the case that $\{x^k\} \subseteq \mathcal{D}$ with $\|x^k\| \to \infty$, and $\{\omega^k\} \subseteq \operatorname{supp}\Omega$ satisfy $\lim_{k\to\infty} F_i(x^k,\omega^k) = \infty$. For the case $\lim_{k\to\infty}(-F_i(x^k,\omega^k))_+ = \infty$, it can be proved in the similar way.

Since Ω is a compact set and $\operatorname{supp}\Omega$ is a closed set, $\{\omega^k\}$ has an accumulation point $\bar\omega \in \operatorname{supp}\Omega$. Let $\{\omega^{k_j}\} \subseteq \{\omega^k\}$ be a subsequence converging to $\bar\omega$. It is clear that there is $K > 0$ such that $\|\omega^{k_j} - \bar\omega\| < \delta$ for any $k_j \geq K$. Thus, for any $\|\omega - \bar\omega\| < \delta$ and $k_j \geq K$,

$$\begin{aligned} |F_i(x^{k_j},\omega) - F_i(x^{k_j},\omega^{k_j})| &\leq \|F(x^{k_j},\omega) - F(x^{k_j},\omega^{k_j})\| \\ &\leq \|F(x^{k_j},\omega) - F(x^{k_j},\bar\omega)\| \\ &\quad + \|F(x^{k_j},\bar\omega) - F(x^{k_j},\omega^{k_j})\| \\ &< 2L, \end{aligned}$$

which implies $\lim_{k_j\to\infty} F_i(x^{k_j},\omega) = \infty$. Hence,

$$\mathcal{P}\{\omega : \lim_{k_j\to\infty} F_i(x^{k_j},\omega) = \infty\} \geq \mathcal{P}\{\omega \in \Omega : \|\omega - \bar\omega\| < \delta\} > 0.$$

Therefore, F is equicoercive. $\square$

Remark 2.1 If Ω has only finite elements, or F is uniformly continuous with respect to $\omega \in \Omega$ on R^n_+, the condition of Proposition 2.1 holds.

From Definitions 2.1–2.3, we can easily get the following proposition.

Proposition 2.2 *Suppose that $F(\cdot, \bar{\omega})$ is an R_0 function on a set $\mathcal{D}$ for some $\bar{\omega} \in$* suppΩ, *and F is equicoercive on $\mathcal{D}$, then F is a stochastic R_0 function on $\mathcal{D}$.*

Proof Suppose that $\{x^k\} \subseteq \mathcal{D}$ satisfies (6). If $\limsup_{k\to\infty} \|(-F(x^k, \bar{\omega}))_+\| = \infty$, then there exist $i \in \langle 1, n\rangle$ and $\{x^{\tilde{k}_j}\} \subseteq \{x^k\}$ such that $\lim_{\tilde{k}_j\to\infty}(-F_i(x^{\tilde{k}_j}, \bar{\omega}))_+ = \infty$. By using the assumption that F is equicoercive, there exists $\{x^{k_j}\} \subseteq \{x^{\tilde{k}_j}\}$ such that

$$\mathcal{P}\{\omega : \lim_{k_j\to\infty} (-F_i(x^{k_j}, \omega))_+ = \infty\} > 0,$$

which contradicts to the fact that $\limsup_{k\to\infty} \|(-F(x^k, \omega))_+\| < \infty$ a.e. in (6). Hence $\limsup_{k\to\infty} \|(-F(x^k, \bar{\omega}))_+\| < \infty$. Since $F(\cdot, \bar{\omega})$ is an R_0 function, there exists $i \in \langle 1, n\rangle$ such that

$$\limsup_{k\to\infty} \min(x_i^k, F_i(x^k, \bar{\omega})) = \infty.$$

Using the assumption that F is equicoercive again, we obtain that

$$\mathcal{P}\{\omega : \limsup_{k\to\infty} \min(x_i^k, F_i(x^k, \omega)) = \infty\} > 0.$$

Therefore, F is a stochastic R_0 function on the set $\mathcal{D}$. $\square$

We use Example 3.2 in [15] to show that the assumption of equicoercivity in Proposition 2.2 cannot be omitted.

Example 2.2 Let $\omega \in \Omega = [-2, 2]$, where ω is uniformly distributed on Ω. Consider the function $F : R_+ \times \Omega \to R$ defined by

$$F(x, \omega) := \begin{cases} 2 + \omega, & \omega \in [-2, 0], \\ 2 - \omega, & \omega \in (0, 2], \end{cases}$$

for $x \in [0, 1]$ and

$$F(x, \omega) := \begin{cases} 2x + x^3\omega, & \omega \in [-\frac{2}{x^2}, -\frac{1}{x^2}], \\ x + x^3 + x^5\omega, & \omega \in (-\frac{1}{x^2}, 0], \\ x + x^3 - x^5\omega, & \omega \in (0, \frac{1}{x^2}], \\ 2x - x^3\omega, & \omega \in (\frac{1}{x^2}, \frac{2}{x^2}], \\ 0, & \omega \in [-2, -\frac{2}{x^2}) \cup (\frac{2}{x^2}, 2], \end{cases}$$

for $x \in (1, \infty)$. The function F is continuous on $R_+ \times \Omega$ and Assumption A1 holds. It is easy to check that $F(\cdot, 0)$ is an R_0 function on R_+, but F is not a stochastic R_0 function on R_+. Moreover, F is not equicoercive on R^n_+, as $\lim_{x^k \to \infty} F(x^k, 0) = \infty$, and $\mathcal{P}\{\omega : \limsup_{x^k \to \infty} F_i(x^k, \omega) = \infty\} = 0$.

The following example shows that the inverse of Proposition 2.2 does not hold. For a stochastic R_0 function F, even if F is equicoercive, it is not necessary to have that $F(\cdot, \tilde{\omega})$ is an R_0 function for some $\tilde{\omega} \in \operatorname{supp}\Omega$. Moreover, $E[F(\cdot, \omega)]$ is not necessary to be an R_0 function.

Example 2.3 Consider the function

$$F(x, \omega) = \big((-\omega)_+ e^{x_1},\ \omega_+ e^{x_2},\ \operatorname{sign}(\omega) x_3\big),$$

where $x = (x_1, x_2, x_3)$ and ω is uniformly distributed on $\Omega = [-1, 1]$. It is not difficult to show that Assumption A1 holds and F is equicoercive. For a fixed $\tilde{\omega} \le 0$ and a sequence $\{x^k\}$, where $x^k = (0, k, 0)$ for $k = 1, 2, \dots$, it is easy to verify that $\{x^k\}$ satisfies (5) with $G(x^k) = F(x^k, \tilde{\omega}) = (-\tilde{\omega}, 0, 0)$ and

$$\limsup_{k\to\infty} \min(x_i^k, G_i(x^k)) = 0, \quad \text{for } i = 1, 2, 3.$$

Similarly, for a fixed $\tilde{\omega} > 0$ and a sequence $\{x^k\}$ defined by $x^k = (k, 0, 0)$ for $k = 1, 2, \dots$, (5) holds with $G(x^k) = F(x^k, \tilde{\omega}) = (0, \tilde{\omega}, 0)$, and

$$\limsup_{k\to\infty} \min(x_i^k, G_i(x^k)) = 0, \quad \text{for } i = 1, 2, 3.$$

Thus, $F(\cdot, \omega)$ is not an R_0 function for any fixed $\omega \in [-1, 1]$. Moreover, we can show that $E[F(x, \omega)] = (e^{x_1}/4,\ e^{x_2}/4,\ 0)$ is not an R_0 function by using a sequence $\{x^k\}$ where $x^k = (0, 0, k)$ for $k = 1, 2, \dots$. However, for every infinite sequence $\{x^k\}$ satisfying (6), if $\limsup_{k\to\infty} x_1^k = \infty$, we have

$$\mathcal{P}\{\omega : \limsup_{k\to\infty} \min(x_1^k, F_1(x^k, \omega)) = \infty\} = \mathcal{P}\{\omega : \omega \in [-1, 0)\} = \frac{1}{2}.$$

If $\limsup_{k\to\infty} x_i^k = \infty$ where $i \in \langle 2, 3\rangle$, we have

$$\mathcal{P}\{\omega : \limsup_{k\to\infty} \min(x_i^k, F_i(x^k, \omega)) = \infty\} = \mathcal{P}\{\omega : \omega \in (0, 1]\} = \frac{1}{2}.$$

Therefore, F is a stochastic R_0 function.

We call $A \in R^{n\times n}$ an R_0 matrix [1] if

$$x \ge 0, \quad Ax \ge 0, \quad x^T A x = 0 \quad \Longrightarrow \quad x = 0.$$

We call $M(\cdot) : \Omega \to R^{n\times n}$ a stochastic R_0 matrix [6] if

$$x \ge 0, \quad M(\omega)x \ge 0, \quad x^T M(\omega) x = 0, \quad \text{a.e.} \quad \Longrightarrow \quad x = 0.$$

If Ω is a singleton, then $M(\omega)$ is an R_0 matrix. It is known that the R_0 function is a generalization of $Ax+b$ with A being an R_0 matrix [16]. Here we show that the stochastic R_0 function is a generalization of $M(\omega)x+q(\omega)$ with $M(\cdot)$ being a stochastic R_0 matrix.

Proposition 2.3 *Let F be an affine function of x for any $\omega \in \Omega$ defined by (2). Then, F is a stochastic R_0 function on R^n_+ if and only if $M(\cdot)$ is a stochastic R_0 matrix.*

Proof ('If' part) Suppose on the contrary that F is not a stochastic R_0 function on R^n_+, then there exists a sequence $\{x^k\} \subset R^n_+$ satisfying (6) in Definition 2.2, such that

$$\mathcal{P}\{\omega : \limsup_{k\to\infty} \min(x_i^k, F_i(x^k,\omega)) = \infty\} = 0, \quad \text{for all } i \in \langle 1, n\rangle.$$

Let x be any accumulation point of the bounded sequence $\{\frac{x^k}{\|x^k\|}\}$. Notice that $F(x^k,\omega) = M(\omega)x^k + q(\omega)$ for all $\omega \in \Omega$. We have

$$\|x\| = 1, \quad x \geq 0, \quad M(\omega)x \geq 0, \quad x^T M(\omega)x = 0, \quad \text{a.e.}$$

This contradicts $M(\cdot)$ being a stochastic R_0 matrix.

('Only if' part) Suppose on the contrary that $M(\cdot)$ is not a stochastic R_0 matrix, then there exists a vector $x \in R^n$ satisfying

$$0 \neq x \geq 0, \quad M(\omega)x \geq 0, \quad x^T M(\omega)x = 0 \quad \text{a.e.}$$

Note that $q_i(\omega) < \infty$ a.e. by using Assumption A1 that $E[\|F(0,\omega)\|^2] = E[\|q(\omega)\|^2] < \infty$. Define a sequence $\{x^k\}$ where $x^k = kx$ for $k = 1, 2, \ldots$. From $M(\omega)x^k \geq 0$, we have that $-F(x^k,\omega) = -(M(\omega)x^k + q(\omega)) \leq -q(\omega)$ for any k and

$$\limsup_{k\to\infty} \|(-F(x^k,\omega))_+\| \leq \|(-q(\omega))_+\| \leq \|q(\omega)\| < \infty, \quad \text{a.e.}$$

Hence, $\{x^k\} \subset R^n_+$ satisfies condition (6).

For an index $i \in \langle 1, n\rangle$ such that $x_i = 0$, we have

$$\min(x_i^k, F_i(x^k,\omega)) = \min(0, F_i(x^k,\omega)) \leq 0, \quad \omega \in \Omega.$$

For an index $i \in \langle 1, n\rangle$ such that $x_i > 0$, we have $(M(\omega)x)_i = 0$ a.e., which implies $F_i(x^k,\omega) = k(M(\omega)x)_i + q_i(\omega) = q_i(\omega)$ a.e. Therefore,

$$\mathcal{P}\{\omega : \limsup_{k\to\infty} \min(x_i^k, F_i(x^k,\omega)) = \infty\} = 0,$$

which contradicts F being a stochastic R_0 function. □

Now, we investigate the relation between F being a stochastic R_0 function and the coercivity of the objective function f_1 in the ERM formulation.

Theorem 2.1 *Suppose that F is equicoercive on R^n_+. Then, f_1 is coercive on R^n_+ if and only if F is a stochastic R_0 function on R^n_+.*

Proof (‘If’ part) Suppose on the contrary that f_1 is not coercive on R^n_+. Thus, there exists a sequence $\{x^k\} \subset R^n_+$ with $\|x^k\| \to \infty$ and a constant $a \in R_+$ such that

$$f_1(x^k) \le a, \quad \forall k.$$

First, consider the case that $\{x^k\}$ does not satisfy (6). Thus, there exists $i \in \langle 1, n\rangle$ such that $\mathcal{P}\{\omega : \limsup_{k\to\infty}(-F_i(x^k, \omega))_+ = \infty\} > 0$, and hence there are $\bar{\omega} \in \operatorname{supp}\Omega$ and a subsequence $\{x^{\tilde{k}_j}\} \subseteq \{x^k\}$ such that $\lim_{\tilde{k}_j\to\infty}(-F_i(x^{\tilde{k}_j}, \bar{\omega}))_+ = \infty$. By the assumption that F is equicoercive on R^n_+, there exists $\{x^{k_j}\} \subseteq \{x^{\tilde{k}_j}\}$ such that $\mathcal{P}\{\omega : \lim_{k_j\to\infty}(-F_i(x^{k_j}, \omega))_+ = \infty\} > 0$. Let

$$\Omega_1 := \{\omega : \lim_{k_j\to\infty} \min(x_i^{k_j}, F_i(x^{k_j}, \omega)) = -\infty\}.$$

Then, $\mathcal{P}\{\Omega_1\} > 0$. By the Fatou lemma [17],

$$E_{\Omega_1}[\liminf_{k_j\to\infty}(\min(x_i^{k_j}, F_i(x^{k_j}, \omega)))^2] \le \liminf_{k_j\to\infty} E_{\Omega_1}[(\min(x_i^{k_j}, F_i(x^{k_j}, \omega)))^2].$$

Since $\liminf_{k_j\to\infty}(\min(x_i^{k_j}, F_i(x^{k_j}, \omega)))^2 = \infty$ on Ω_1 and $\mathcal{P}\{\Omega_1\} > 0$, the left-hand side of the above inequality is infinite. Hence,

$$\liminf_{k_j\to\infty} E_{\Omega_1}[(\min(x_i^{k_j}, F_i(x^{k_j}, \omega)))^2] = \infty.$$

Moreover, it is easy to find

$$\begin{aligned} f_1(x^{k_j}) &= E[\|\Phi(x^{k_j}, F(x^{k_j}, \omega))\|^2] \\ &\ge E_{\Omega_1}[(\min(x_i^{k_j}, F_i(x^{k_j}, \omega)))^2] \to \infty, \quad \text{as } k_j \to \infty. \end{aligned}$$

This contradicts to the fact that $f_1(x^k) \le a$ for $\forall k$. Thus $\{x^k\} \subset R^n_+$ must satisfy (6). According to Definition 2.2, we choose an index $i \in \langle 1, n\rangle$ such that $\mathcal{P}\{\omega : \limsup_{k\to\infty} \min(x_i^k, F_i(x^k, \omega)) = \infty\} > 0$. Since F is equicoercive on R^n_+, we get that there exists $\{x^{k_j}\} \subseteq \{x^k\}$ such that $\mathcal{P}\{\omega : \lim_{k_j\to\infty} \min(x_i^{k_j}, F_i(x^{k_j}, \omega)) = \infty\} > 0$. Let

$$\Omega_2 := \{\omega : \lim_{k_j\to\infty} \min(x_i^{k_j}, F_i(x^{k_j}, \omega)) = \infty\}.$$

Then $\mathcal{P}\{\Omega_2\} > 0$. Again by using the Fatou lemma,

$$f_1(x^{k_j}) \ge E_{\Omega_2}[(\min(x_i^{k_j}, F_i(x^{k_j}, \omega)))^2] \to \infty, \quad \text{as } k_j \to \infty,$$

which is a contradiction to $f_1(x^k) \le a$ for $\forall k$.

(‘Only if’ part) Suppose on the contrary that F is not a stochastic R_0 function on R^n_+, then there exists a sequence $\{x^k\} \subset R^n_+$ satisfying (6), such that

$$\mathcal{P}\{\omega : \limsup_{k\to\infty} \min(x_i^k, F_i(x^k,\omega)) = \infty\} = 0, \quad \text{for any } i \in \langle 1, n\rangle. \tag{7}$$

We then declare that there must exist constants $\underline{c}$ and $\overline{c}$ such that, for any $i \in \langle 1, n\rangle$,

$$\underline{c} \le \min(x_i^k, F_i(x^k,\omega)) \le \overline{c}, \quad \forall \omega \in \operatorname{supp}\Omega. \tag{8}$$

Suppose on the contrary that (8) is not true; then, there exist $\{\omega^k\} \subseteq \operatorname{supp}\Omega$ and $\hat{i} \in \langle 1, n\rangle$ such that

$$\limsup_{k\to\infty}(-F_{\hat{i}}(x^k,\omega^k))_+ = \infty \quad \text{or} \quad \limsup_{k\to\infty} \min(x_{\hat{i}}^k, F_{\hat{i}}(x^k,\omega^k)) = \infty.$$

Hence there must exist a subsequence $\{x^{\tilde{k}_j}\} \subseteq \{x^k\}$ such that $\lim_{\tilde{k}_j\to\infty}(-F_{\hat{i}}(x^{\tilde{k}_j}, \omega^{\tilde{k}_j}))_+ = \infty$; or $\lim_{\tilde{k}_j\to\infty} \min(x_{\hat{i}}^{\tilde{k}_j}, F_{\hat{i}}(x^{\tilde{k}_j}, \omega^{\tilde{k}_j})) = \infty$. By the assumption that F is equicoercive on R^n_+, for the first case we know that there exists a subsequence $\{x^{k_j}\} \subseteq \{x^{\tilde{k}_j}\}$ such that $\mathcal{P}\{\omega : \lim_{k_j\to\infty}(-F_{\hat{i}}(x^{k_j},\omega))_+ = \infty\} > 0$, which contradicts to (6); For the second case, we know that $\mathcal{P}\{\omega : \lim_{k_j\to\infty} F_{\hat{i}}(x^{k_j},\omega) = \infty\} > 0$, which implies that

$$\mathcal{P}\{\omega : \limsup_{k\to\infty} \min(x_{\hat{i}}^k, F_{\hat{i}}(x^k,\omega)) = \infty\} > 0.$$

This contradicts (7). Therefore, (8) holds and we get

$$\begin{aligned} f_1(x^k) &= \sum_{i=1}^n E[(\min(x_i^k, F_i(x^k,\omega)))^2] \\ &= \sum_{i=1}^n E_{\operatorname{supp}\Omega}[(\min(x_i^k, F_i(x^k,\omega)))^2] \le n(\max\{|\overline{c}|, |\underline{c}|\})^2. \end{aligned}$$

Notice that the sequence $\{x^k\} \subset R^n_+$ satisfies (6) and the sequence $\{f_1(x^k)\}$ is bounded. This contradicts to the coercivity of f_1 on R^n_+. □

Remark 2.2 Following the proof of Theorem 2.1, we can see that if for every sequence $\{x^k\} \subset R^n_+$ satisfying $\lim_{k\to\infty} \|x^k\| = \infty$, there exists $i \in \langle 1, n\rangle$ and a subsequence $\{x^{k_j}\}$ such that

$$\mathcal{P}\{\omega : \lim_{k_j\to\infty}(-F_i(x^{k_j},\omega))_+ = \infty\} > 0, \quad \text{or}$$

$$\mathcal{P}\{\omega : \lim_{k_j\to\infty} \min(x_i^{k_j}, F_i(x^{k_j},\omega)) = \infty\} > 0,$$

then F is a stochastic R_0 function and f_1 is coercive on R^n_+.

Similar results for the coercivity of f defined by other NCP functions can be obtained by noticing their relations with ϕ_1. In particular, from (4), we have the following corollary.

Corollary 2.1

(i) *Suppose that F is equicoercive on R^n_+. Then, f_2 is coercive on R^n_+ if and only if F is a stochastic R_0 function on R^n_+.*
(ii) *If F is a stochastic R_0 function and equicoercive on R^n_+, then f_3 is coercive on R^n_+.*

From Theorem 2.1 and Corollary 2.1, we obtain immediately the following corollary.

Corollary 2.2 *If F is a stochastic R_0 function and equicoercive on R^n_+, then the solution set of (3) defined by ϕ_i, $i = 1, 2, 3$ is nonempty and bounded.*

3 ERM-SNCP Model for TEP under Uncertainty

Let $[\mathcal{N}, \mathcal{A}]$ represent a given transportation network, where $\mathcal{N}$ is the set of nodes, and $\mathcal{A}$ is the set of links. We use $\Omega \subseteq R^m$ to represent a set of random vectors. Each vector $\omega \in \Omega$, corresponding to one realization of stochastic factors such as weather, accidents, etc., is of given probability $\mathcal{P}$. For any realization $\omega \in \Omega$, let us denote

$\mathcal{I}$	the set of origin-destination (OD) pairs,
$\mathcal{R}_i$	the set of "available" routes, connecting OD pair i (which might, but not necessarily be all paths joining the OD pair),
$h_r(\omega)$	the flow on route r,
Δ	the link-route incidence matrix of the network,
Γ	the OD pair-route incidence matrix of the network,
$u_i(\omega)$	the shortest travel cost function for OD pair i,
$d_i(\omega)$	the demand function for OD pair i,
$C_r(h(\omega), \omega)$	the travel cost function for route r.

Moreover, let $\mathcal{R} = \bigcup_{i\in\mathcal{I}} \mathcal{R}_i$ and $u(\omega)$, $d(\omega)$, $h(\omega)$, $C(h(\omega), \omega)$ represent the vector composed of $u_i(\omega)$, $d_i(\omega)$, $h_r(\omega)$, $C_r(h(\omega), \omega)$ for $i \in \mathcal{I}, r \in \mathcal{R}$, respectively. It is clear that

$$u,\ d : \Omega \to R^{|\mathcal{I}|}_+, \qquad h : \Omega \to R^{|\mathcal{R}|}_+, \qquad C : R^{|\mathcal{R}|}_+ \times \Omega \to R^{|\mathcal{R}|}_+.$$

Here, we suppose that the uncertain demand $d(\omega)$ is bounded for almost all $\omega \in \Omega$. We say that the network $[\mathcal{N}, \mathcal{A}]$ is *strongly connected* if for any OD pair $i \in \mathcal{I}$ there is at least one route joining the origin to the destination. Then each row of Γ is a nonzero vector. Moreover, since one route connects only one OD pair, Γ has full row-rank. The link-route incidence matrix Δ is deterministic for the given network.

In a congested network, drivers have the incentive to compete with each other for selecting the route with minimal travel cost, at a certain level of travel demand. The traffic equilibrium problem (TEP) has been used for transportation planning, which seeks for flow pattern with the equilibrium property that no driver may decrease his travel cost by unilaterally changing his route. It is the interaction between drivers that forms the stable flow pattern in the equilibrium state and such flow pattern is used by the administrator for predicting the traffic flow. For more details about TEP, we refer to [18].

The Wardrop equilibrium principle [9] for the genesis of the TEP states that in the equilibrium state, for any OD pair the travel cost on every used routes equals and any route needs higher travel cost will have no traffic flow. Application of the Wardrop equilibrium for the realization $\omega \in \Omega$ gives

$$\begin{aligned} C_r(h(\omega),\omega) - u_i(\omega) \geq 0, \quad & h_r(\omega) \geq 0, \\ (C_r(h(\omega),\omega) - u_i(\omega))h_r(\omega) = 0, \quad & i \in \mathcal{I}, r \in \mathcal{R}_i. \end{aligned} \tag{9}$$

Moreover, according to the demand conservation, we have

$$\sum_{r \in \mathcal{R}_i} h_r(\omega) - d_i(\omega) = 0,$$

which is equivalent to

$$\begin{aligned} \sum_{r \in \mathcal{R}_i} h_r(\omega) - d_i(\omega) \geq 0, \quad & u_i(\omega) \geq 0, \\ \Bigg(\sum_{r \in \mathcal{R}_i} h_r(\omega) - d_i(\omega)\Bigg) u_i(\omega) = 0, \quad & i \in \mathcal{I}, r \in \mathcal{R}_i, \end{aligned} \tag{10}$$

under some mild assumptions that would be expected to meet always in practice [10]. (9)–(10) is the NCP formulation of static TEP ([10] and [19]) for each fixed $\omega \in \Omega$. In particular, we can write (9)–(10) as

$$x_\omega \geq 0, \quad F(x_\omega,\omega) \geq 0, \quad x_\omega^T F(x_\omega,\omega) = 0, \quad \omega \in \Omega, \tag{11}$$

where

$$x_\omega = \begin{pmatrix} h(\omega) \\ u(\omega) \end{pmatrix}, \quad F(x_\omega,\omega) = \begin{pmatrix} C(h(\omega),\omega) - \Gamma^T u(\omega) \\ \Gamma h(\omega) - d(\omega) \end{pmatrix}.$$

The solution x_ω of (11) depends on an unknown realization ω, which can only be predicted such as weather. It is interesting for the administrator to find a reliable flow pattern that is not far from optimal flow pattern x_ω given by (11). Such flow pattern may help for future planning. In other words, we wish that there was a deterministic vector $x \in R^{|\mathcal{R}|+|\mathcal{I}|}$ satisfying the SNCP

$$x \geq 0, \quad F(x,\omega) \geq 0, \quad x^T F(x,\omega) = 0, \quad \omega \in \Omega, \tag{12}$$

where

$$x = \begin{pmatrix} h \\ u \end{pmatrix}, \quad F(x,\omega) = \begin{pmatrix} C(h,\omega) - \Gamma^T u \\ \Gamma h - d(\omega) \end{pmatrix}. \tag{13}$$

However, in general, we can not find such vector x that is the equilibria for any random vector $\omega \in \Omega$. We have to consider a deterministic formulation of (12) such as the EV formulation NCP($E[F(x,\omega)]$) and the ERM formulation (3). The ERM formulation provides a solution $x^* = (h^*, u^*)$ that minimizes expected violation of the equilibrium (9)–(10), and represents the most likely equilibrium flow pattern h^* and travel cost u^* before we know the realization of uncertain factors. In general, we do not have $x^* = x_\omega$ for all $\omega \in \Omega$. The violation of x^* to (9)–(10) is natural, which means x^* has error to x_ω for some $\omega \in \Omega$.

In what follows, we let v_a be the travel flow on link a, and v be the link travel flow vector with components $v_a, a \in \mathcal{A}$. We use the function $t_a(v, \omega)$ to denote the travel time on link a, and $t(v, \omega)$ for the link travel time vector with components $t_a(v, \omega)$, $a \in \mathcal{A}$. Clearly, the link travel flow vector v and the route travel flow vector h have the following relationship:

$$v = \Delta h.$$

It is pointed out in [19] that in many cases the travel cost function is *nonadditive*, which may rise from a variety of transportation polices, nonlinear valuation of travel time, etc. In this paper, we add random factors ω to the general nonadditive travel cost function suggested in [19] as

$$C(h, \omega) = \eta_1 \Delta^T t(\Delta h, \omega) + g(\Delta^T t(\Delta h, \omega)) + \Lambda(h, \omega), \tag{14}$$

where $\eta_1 > 0$ is the time-based operating costs factor, $g : R_+^{|\mathcal{R}|} \to R_+^{|\mathcal{R}|}$ is the translation function converting time t to money, and Λ is the perturbed financial cost function (e.g., distance-based operating costs such as maintenance). We call (14) the perturbed general nonadditive travel cost function. We suppose the following assumption on the travel time function and the travel cost function holds.

Assumption A2 There exists a subset $\hat{\Omega} \subseteq \Omega$ with $\mathcal{P}\{\hat{\Omega}\} > 0$, such that, for any $\omega \in \hat{\Omega}$,

(i) the travel cost function $C_r(h, \omega)$ on each route is a nondecreasing function of flow h, and finite for any fixed h;
(ii) the travel time function $t_a(v, \omega)$ on each link is a nondecreasing function of flow v, finite for any fixed v, and coercive with flow on the link v_a, i.e., $t_a(v, \omega) \to \infty$ if $v_a \to \infty$.

Assumption A2 holds in various perturbed travel cost and travel time functions used in practice. For instance, let the perturbed travel cost function be

$$C(h, \omega) = (\Delta^T t(\Delta h, \omega))^2,$$

and let the travel time function t be

$$t_a(v, \omega) := (K(\omega)v + k(\omega))_a, \quad a \in \mathcal{A},$$

where $K(\omega) \in R_+^{|\mathcal{A}| \times |\mathcal{A}|}$ has positive diagonal elements and $k(\omega) \in R_+^{|\mathcal{A}|}$ for any $\omega \in \Omega$. For a fixed ω, this is the simple affine travel time function used in [11], where it is said that $K(\omega)$ is in general a positive semi-definite matrix.

Proposition 3.1 *Suppose that the network* $[\mathcal{N},\mathcal{A}]$ *is strongly connected and that Assumption A2 holds; then, F in (13) is a stochastic R_0 function on R^n_+.*

Proof For any infinite sequence $\{x^k\}\subset R^n_+$ satisfying (6), let us choose a subsequence $\{x^{k_j}\}\subseteq\{x^k\}$ such that $x_l^{k_j}\to\infty$ as $k_j\to\infty$ for some $l\in\langle 1,n\rangle$. Recall that $n=|\mathcal{R}|+|\mathcal{I}|$.

If $l\in\langle 1,|\mathcal{R}|\rangle$, we have $h_l^{k_j}\to\infty$ as $k_j\to\infty$. Notice that $\Delta_{al}=1$ for any link a on route l, thus $(\Delta h^{k_j})_a\ge h_l^{k_j}\to\infty$ as $k_j\to\infty$. This indicates that $t_a(\Delta h^{k_j},\omega)\to\infty$ as $k_j\to\infty$ for $\hat{\Omega}\subseteq\Omega$ with $\mathcal{P}\{\hat{\Omega}\}>0$ by (ii) of Assumption A2. Hence, for any $\omega\in\hat{\Omega}$,

$$C_l(h^{k_j},\omega)\ge\eta_1(\Delta^T t(\Delta h^{k_j},\omega))_l\ge\eta_1\Delta_{al}t_a(\Delta h^{k_j},\omega)=\eta_1 t_a(\Delta h^{k_j},\omega)\to\infty,$$

as $k_j\to\infty$. If $\{(\Gamma^T u^{k_j})_l\}$ is bounded, then $F_l(x^{k_j},\omega)=(C(h^{k_j},\omega)-\Gamma^T u^{k_j})_l\to\infty$ as $k_j\to\infty$ for $\omega\in\hat{\Omega}$. From Definition 2.2, we find that F is a stochastic R_0 function, since

$$\mathcal{P}\{\omega:\lim_{k_j\to\infty}\min(x_l^{k_j},F_l(x^{k_j},\omega))=\infty\}\ge\mathcal{P}\{\hat{\Omega}\}>0.$$

Otherwise, we have $(\Gamma^T u^{k_j})_l\to\infty$ as $k_j\to\infty$. This implies the existence of $i\in\mathcal{I}$ such that $\Gamma_{il}=1$ and $u_i^{k_j}\to\infty$. Thus, for any $\omega\in\hat{\Omega}$,

$$(\Gamma h^{k_j}-d(\omega))_i\ge\Gamma_{il}h_l^{k_j}-d_i(\omega)\to\infty,\quad\text{as } k_j\to\infty.$$

Hence, F is a stochastic R_0 function by noticing the expression of F in (13) and

$$\mathcal{P}\{\omega:\lim_{k_j\to\infty}\min(u_i^{k_j},(\Gamma h^{k_j}-d(\omega))_i)=\infty\}\ge\mathcal{P}\{\hat{\Omega}\}>0.$$

Now, we consider $l\in\langle|\mathcal{R}|+1,n\rangle$ and $\{h^{k_j}\}$ is bounded. Then, we have $u_i^{k_j}\to\infty$ as $k_j\to\infty$ for some $i\in\mathcal{I}$. Since the network is strongly connected, there exists $\Gamma_{ir}=1$ for any route r connecting OD pair i. Thus, we get

$$(\Gamma^T u^{k_j})_r\ge\Gamma_{ir}u_i^{k_j}=u_i^{k_j}\to\infty\quad\text{as } k_j\to\infty.$$

Moreover, $\{C_r(h^{k_j},\omega)\}$ is bounded for $\omega\in\hat{\Omega}$ by using (i) of Assumption A2 and the fact that $\{h^{k_j}\}$ is bounded. Hence $(C(h^{k_j},\omega)-\Gamma^T u^{k_j})_r\to-\infty$ as $k_j\to\infty$ for $\omega\in\hat{\Omega}$. From the expression of F in (13), we get

$$\lim_{k_j\to\infty}\|(-F(x^{k_j},\omega))_+\|\ge\lim_{k_j\to\infty}|-(C(h^{k_j},\omega)-\Gamma^T u^{k_j})_r|=\infty\quad\text{for }\omega\in\hat{\Omega},$$

where $\mathcal{P}\{\hat{\Omega}\}>0$. This is impossible, since $\{x^k\}$ satisfies (6).

Hence, F is a stochastic R_0 function on R^n_+. □

Remark 3.1 It is easy to see that F in fact satisfies the condition in Remark 2.2. Hence the objective function f_1 is coercive on R^n_+, and the solution set of the ERM formulation for SNCP$(F(x,\omega))$ is nonempty and bounded.

4 Evaluation of the ERM-SNCP Model for TEP under Uncertainty

In this section, we report computational experiments that compare the proposed ERM-SNCP model with EV-SNCP model through a simple example of TEP under uncertainty. We begin with definitions of performance measure which evaluate the quality for a flow pattern such as reliability, unfairness, and total travel cost.

The reliability ([4–6] and [20]) concerns the safety of a flow pattern, that is, the probability to be feasible. For a flow pattern h, its reliability is defined by

$$\mathrm{rel}(h) := \mathcal{P}\{\omega : (\Gamma h - d(\omega))_i \geq 0,\ i = |\mathcal{R}| + 1, \dots, n\}. \tag{15}$$

Notice that $(\Gamma h - d(\omega))_i \geq 0$ manifests that the demand for OD pair $i \in \mathcal{I}$ and $\omega \in \Omega$ can be delivered in the traffic flow pattern h.

For a flow pattern h, the expected ratio of the delivered demand to the total demand of the system is defined by

$$\mathrm{dr}(h) := E\left[\frac{1}{|\mathcal{I}|}\sum_{i\in\mathcal{I}} \frac{\min((\Gamma h)_i, d_i(\omega))}{d_i(\omega)}\right]. \tag{16}$$

Clearly $0 \leq \mathrm{dr}(h) \leq 1$ and the nearer $\mathrm{dr}(h)$ is to 1, the more feasibility the solution earns in practice.

For each fixed $\omega \in \Omega$, the Wardrop equilibria reflects the fairness to all users with the same OD pair, since the travel cost for each used route connecting the same OD pair is equal and less than any unused route. However, for the uncertain case, the travel cost for any flow pattern connecting the same OD pair is not necessarily the same. For a fixed $\omega \in \Omega$, the *unfairness* of a feasible flow pattern for an OD pair $i \in \mathcal{I}$ [21] is measured by

$$C_i^{\mathrm{unfair}}(h, \omega) = \frac{C_i^{\max}(h, \omega)}{C_i^{\min}(h, \omega)},$$

where $C_i^{\max}(h, \omega)$ and $C_i^{\min}(h, \omega)$ are the largest and smallest travel cost of routes being used, which connect OD pair i. Thus, the expected unfairness of the decision for the whole system under uncertainty is defined by

$$\mathrm{unf}(h) := E\left[\frac{1}{|\mathcal{I}|}\sum_{i\in\mathcal{I}} C_i^{\mathrm{unfair}}(h, \omega)\right] = E\left[\frac{1}{|\mathcal{I}|}\sum_{i\in\mathcal{I}} \frac{C_i^{\max}(h, \omega)}{C_i^{\min}(h, \omega)}\right]. \tag{17}$$

For a flow pattern h, the corresponding expected travel cost is defined by

$$\mathrm{tc}(h) := E[h^T C(h, \omega)]. \tag{18}$$

We use a simple example to illustrate the ERM-SNCP model for the traffic equilibrium under uncertainty.

Example 4.1 The transportation network shown in Fig. 1 is adopted from [22], which has 13 nodes, 19 links and 4 OD pairs ($1 \to 2$, $1 \to 3$, $4 \to 2$, $4 \to 3$), with the network characters t_a^0 and c_a^0.

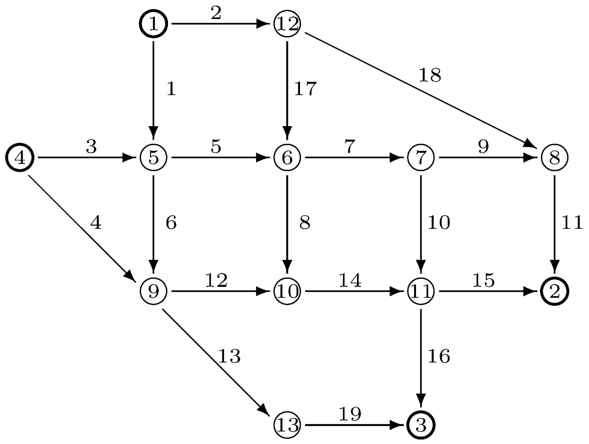

Fig. 1 An example network

We suppose that the perturbed travel cost function is defined as

$$C(h,\omega)=\Delta^T t(\Delta h,\omega),\quad \omega\in\Omega,$$

where the perturbed travel time function, derived from the Bureau of Public Road link travel time function (1964), can be written as

$$t_a(v,\omega):=t_a^0\left(1+0.15\left(\frac{v_a}{c_a(\omega)}\right)^4\right),\quad a\in\mathcal{A}.$$

Here $t_a^0>0$ is the travel time in the network without congestion, and $c_a(\omega)\ge 0$ represents perturbed link capacity with $\mathcal{P}\{\omega : c_a(\omega)>0\}>0$ for all $a\in\mathcal{A}$. For any fixed ω, it is a separable function, i.e., for each link, the travel time depends only on the travel flow and capacity of this link.

Case 1. Suppose that $c(\omega)\equiv c^0$ and $d(\omega)=\omega=(\omega_1,\omega_2,\omega_3,\omega_4)$, where $\omega_1,\omega_2,\omega_3$, ω_4 follow the independent truncated normal distributions, respectively,

$$\omega_1\sim 300\le N(400,2500)\le 500,\qquad \omega_2\sim 600\le N(800,2500)\le 1000,$$
$$\omega_3\sim 400\le N(600,2500)\le 800,\qquad \omega_4\sim 100\le N(200,900)\le 300.$$

Case 2. Based on case 1, we suppose that some great changes of capacity of the link $a=5$ may happen due to the weather and road condition, as

$$\mathcal{P}\left\{\omega : c_5(\omega)\equiv\frac{1}{4}c_5^0\right\}=\frac{1}{2},\qquad \mathcal{P}\{\omega : c_5(\omega)\equiv c_5^0\}=\frac{1}{2}.$$

Case 3. Based on case 1, we extend the range of ω_1,ω_2 as

$$\omega_1\sim 200\le N(400,2500)\le 600,\qquad \omega_2\sim 400\le N(800,2500)\le 1200.$$

Let x_{EV} and x_{ERM} be the solutions of the EV and the ERM formulations of the SNCP (12), respectively. In Table 1, we report the computation results for the performance measure (15)–(18) as well as the number of used routes $\mathrm{nr}(h)$. Here, a used

Table 1 Reliability, unfairness and total travel cost of x_{EV} and x_{ERM}

x_{EV}	Case 1	Case 2	Case 3
Reliability rel(h)	0.0623	0.0623	0.0626
Delivered rate dr(h)	93.24%	93.24%	91.22%
Unfairness unf(h)	1.25	1.56	1.25
Total travel cost tc(h)	7.93e + 4	8.47e + 4	7.93e + 4
Num. of used routes nr(h)	7	7	7
x_{ERM}	Case 1	Case 2	Case 3
Reliability rel(h)	0.5285	0.4586	0.5405
Delivered rate dr(h)	99.41%	99.14%	99.25%
Unfairness unf(h)	1.38	1.73	1.45
Total travel cost tc(h)	1.10e + 5	1.21e + 5	1.39e + 5
Num. of used routes nr(h)	19	16	19

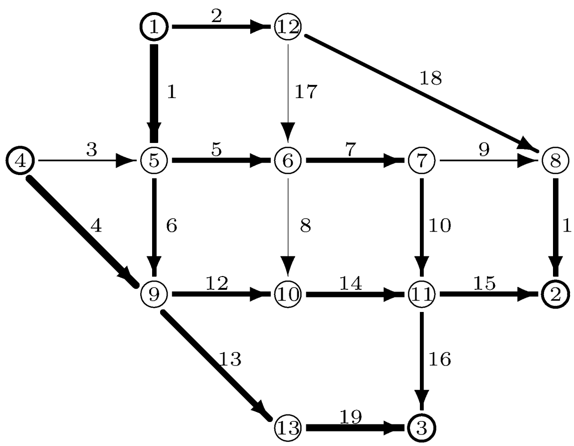

Fig. 2 Travel flow pattern of ERM-SNCP in case 1

route refers to the route that has flow $h_r \geq 0.0001$. The results are the average of 100 simulations for $\Omega = \{\omega^1, \omega^2, \ldots, \omega^{1000}\}$. The sample points were obtained by the Monte-Carlo method. Figures 2–4 show the travel flow pattern of the ERM-SNCP model for the three cases, respectively. Notice that the width of each link in Figs. 2–4 is proportional to the amount of travel flow on this link.

Preliminary numerical results of traffic equilibrium problems under uncertainty show that the flow pattern drawing from a solution x_{ERM} of the ERM formulation has higher reliability and delivered rate than the EV formulation. On the other hand, the EV-SNCP formulation has lower unfairness and total travel cost than the ERM formulation. This phenomenon can be explained as follows. The EV formulation seeks equilibria with the expected value of the travel cost function and travel demand. The ERM formulation minimizes the violation (residual) of the equilibrium for all $\omega \in \Omega$. Hence the ERM formulation has higher reliability and delivered rate than the EV for-

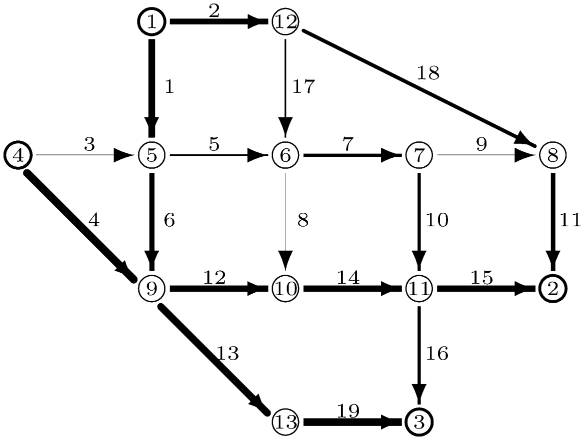

Fig. 3 Travel flow pattern of ERM-SNCP in case 2

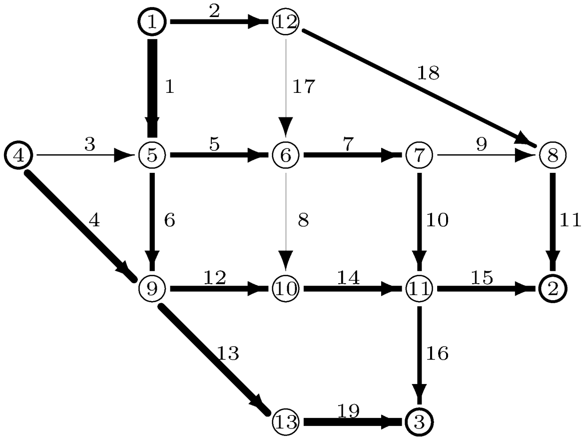

Fig. 4 Travel flow pattern of ERM-SNCP in case 3

mulation. Since the ERM flow pattern delivers much more vehicles, its cost is higher than the EV flow pattern. Moreover, the unfairness of each flow pattern is defined on the routes being used, and the ERM flow pattern uses more routes than the EV one. This makes ERM flow pattern has higher unfairness than the EV formulation. Therefore, the EV formulation is recommended to administrators who prefer low cost, and the ERM formulation is recommended to administrators who want a reliable travel flow pattern which minimizes the expected violation of the equilibrium.

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