Error Bounds for Approximation in Chebyshev Points

Shuhuang Xiang[1], Xiaojun Chen[2] and Haiyong Wang[3]

Abstract. This paper improves error bounds for Gauss, Clenshaw-Curtis and Fejér's first quadrature by using new error estimates for polynomial interpolation in Chebyshev points. We also derive convergence rates of Chebyshev interpolation polynomials of the first and second kind for numerical evaluation of highly oscillatory integrals. Preliminary numerical results show that the improved error bounds are reasonably sharp.

Keywords. Chebyshev points, interpolation, error bound, oscillatory integral, numerical integration.

AMS subject classifications. 65D32, 65D30

1 Introduction

Polynomial approximation is used as the basic means of approximation in most areas of numerical analysis [7]. It is not only a powerful tool for the approximation of functions that are difficult to compute, but also an essential ingredient of numerical integration and approximate solution of differential and integral equations. It has been known that the Lagrange interpolation polynomial in the Chebyshev points of the first or second kind does not suffer from the Runge phenomenon ([19], pp. 146), which makes it much better than the interpolant in equally spaced points, and the accuracy of the approximation can improve remarkably fast when the number of interpolation points is increased [23, 29]. Polynomial interpolation using the Chebyshev points of the first and second kind has been studied in the field of numerical integration for the integral

$$I[f] = \int_{-1}^{1} f(x)dx. \tag{1.1}$$

Most discussions focus on implementation of a product-integration rule

$$I_n[f] = \sum_{k=0}^{n} w_k f(x_k), \tag{1.2}$$

where the weights w_k are determined by requiring the rule to be exact for any polynomial of degree $\leq n$. The corresponding rules are Fejér's first quadrature and Clenshaw-Curtis quadrature, respectively. Both have positive weights, and are guaranteed to converge for all continuous functions on $[-1, 1]$.

In almost every numerical analysis textbook, one can find the error estimate:

$$f(x) - L_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)\cdots(x - x_n), \tag{1.3}$$

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where L_n is the interpolation polynomial of f at $n+1$ distinct points $x_0, \ldots, x_n$. In this paper, we present new error estimates for polynomial interpolation in the Chebyshev points of the first and second kind, which is a direct extension of the results in [30]. Application of the new error estimate gives new error bounds for Gauss, Clenshaw-Curtis and Fejér's first quadrature for highly oscillatory integrals. Preliminary numerical results show that the proposed error bounds are reasonably sharp.

2 Error bounds for interpolant approximation in the Chebyshev points

Suppose that f is absolutely continuous on $[-1,1]$. Let p_n denote the interpolant of f of degree n in the Chebyshev points of the second kind

$$x_j = \cos\left(\frac{j\pi}{n}\right), \quad j = 0, 1, \ldots, n,$$

and q_n the interpolant in the Chebyshev points of the first kind

$$y_j = \cos\left(\frac{(2j+1)\pi}{2n+2}\right), \quad j = 0, 1, \ldots, n.$$

The Chebyshev series for f is defined as [26, 30]

$$f(x) = {\sum_{j=0}^{\infty}}' b_j T_j(x), \quad b_j = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_j(x)}{\sqrt{1-x^2}}dx, \tag{2.1}$$

where the prime denotes a sum whose first term is halved and $T_j(x) = \cos(j\cos^{-1}x)$ is the Chebyshev polynomial of degree j. From Boyd ([5], pp. 96), p_n and q_n can be expressed by

$$p_n(x) = {\sum_{j=0}^{n}}'' \widetilde{b}_j T_j(x), \qquad \widetilde{b}_j = \frac{2}{n}{\sum_{s=0}^{n}}'' f(x_s)T_j(x_s), \tag{2.2a}$$

$$q_n(x) = {\sum_{j=0}^{n}}' c_j T_j(x), \qquad c_j = \frac{2}{n+1}\sum_{s=0}^{n} f(y_s)T_j(y_s), \tag{2.2b}$$

where the double prime denotes a sum whose first and last terms are halved, and the coefficients $\widetilde{b}_j$ and c_j can be efficiently computed by FFT [7, 8, 30]. The Clenshaw-Curtis and Fejér's first quadrature formulae are defined by [5, 6, 7], respectively

$$I_n^{C\text{-}C}[f] = \int_{-1}^{1} p_n(x)dx, \qquad I_n^F[f] = \int_{-1}^{1} q_n(x)dx.$$

A MATLAB code for $I_n^{C\text{-}C}[f]$ can be found in [30]. Similarly, here is a MATLAB code for $I_n^F[f]$

```
function I=fejer(f,n)                             % (n+1)-pt Fejér's first quadrature of f
x=cos(pi*(2*(0:n)'+1)/(2*n+2));                   % Chebyshev points of the first kind
fx=feval(f,x)/(n+1);                              % f evaluated at these points
g=fft(fx([1:n+1 n+1:-1:1]));                      % FFT
hx=real(exp(2*i*pi*(0:2*n+1)/(4*n+4)).*g');       %
a=hx(1:n+1);a(1)=0.5*a(1);                        % Chebyshev coefficients
w=0*a';w(1:2:end)=2./(1-(0:2:n).^ 2);             % weight vector
I=a*w;                                            % the integral
```

A fast and accurate algorithm for computing the weights in (1.2) for the two quadrature rules in $O(n \log n)$ flops has been given by Waldvogel [31] and the corresponding interpolation polynomials can be computed efficiently by the barycentric Lagrange interpolation formula [4].

Based on the results recently developed by Trefethen [30] for Gauss and Clenshaw-Curtis quadrature, we consider new error estimates for approximation of f in the Chebyshev points.

Let $\|\cdot\|_T$ be the Chebyshev-weighted 1-norm defined by

$$\|u\|_T = \int_{-1}^{1} \frac{|u'(t)|}{\sqrt{1-t^2}} dt.$$

Lemma 2.1 *(i) (Trefethen [30]) If $f, f', \ldots, f^{(k-1)}$ are absolutely continuous on $[-1,1]$ and if $\|f^{(k)}\|_T = V_k < \infty$ for some $k \geq 0$, then for each $j \geq k+1$,*

$$|b_j| \leq \frac{2V_k}{\pi j(j-1)\cdots(j-k)}.$$

(ii) (Bernstein [3]) If f is analytic with $|f(z)| \leq M$ in the region bounded by the ellipse with foci ± 1 and major and minor semiaxis lengths summing to $\rho > 1$, then for each $j \geq 0$,

$$|b_j| \leq \frac{2M}{\rho^j}.$$

Lemma 2.2 *For any positive integers N and m, we have*

$$\sum_{j=N+1}^{\infty} \frac{1}{j(j+1)\cdots(j+m)} = \frac{1}{m(N+1)(N+2)\cdots(N+m)} \tag{2.3}$$

and

$$\sum_{j=N+1}^{\infty} \frac{1}{2j(2j+1)\cdots(2j+m)} \leq \frac{1}{2m(2N+1)(2N+2)\cdots(2N+m)}. \tag{2.4}$$

Proof: Since

$$\frac{1}{j(j+1)\cdots(j+m)} = \frac{1}{m}\left(\frac{1}{j(j+1)\cdots(j+m-1)} - \frac{1}{(j+1)(j+2)\cdots(j+m)}\right),$$

(2.3) follows directly from the sum of the above identity for $j = N+1, N+2, \ldots$.

The inequality (2.4) can be proved based on the fact that $\left\{\frac{1}{j(j+1)\cdots(j+m)}\right\}$ is monotonically decreasing about j and then

$$\begin{aligned}
&\sum_{j=N+1}^{\infty} \frac{1}{2j(2j+1)\cdots(2j+m)} \\
\leq\ & \frac{1}{2}\sum_{j=N+1}^{\infty}\left[\frac{1}{(2j-1)(2j)\cdots(2j+m-1)} + \frac{1}{2j(2j+1)\cdots(2j+m)}\right] \\
=\ & \frac{1}{2}\sum_{j=2N+1}^{\infty} \frac{1}{j(j+1)\cdots(j+m)} \\
=\ & \frac{1}{2m(2N+1)(2N+2)\cdots(2N+m)}.
\end{aligned}$$

∎

Theorem 2.1 *If $f, f', \ldots, f^{(k-1)}$ are absolutely continuous on $[-1,1]$ and if $\|f^{(k)}\|_T = V_k < \infty$ for some $k \geq 1$, then for each $n \geq k+1$,*

$$\frac{4V_k}{k\pi n(n-1)\cdots(n-k+1)} \geq \begin{cases} \|f-p_n\|_\infty, & (2.5a) \\ \|f-q_n\|_\infty. & (2.5b) \end{cases}$$

If f is analytic with $|f(z)| \leq M$ in the region bounded by the ellipse with foci ± 1 and major and minor semiaxis lengths summing to $\rho > 1$, then for each $n \geq 0$,

$$\|f-p_n\|_\infty \leq \frac{4M}{(\rho-1)\rho^n}, \quad \|f-q_n\|_\infty \leq \frac{4M}{(\rho-1)\rho^n}. \tag{2.6}$$

Proof: (2.1), (2.2a) and (2.2b) imply that for any $x \in [-1,1]$

$$\begin{aligned} \|f-p_n\|_\infty &\leq \sum_{j=0}^{n-1}{}' |b_j - \widetilde{b}_j| \|T_j\|_\infty + |b_n - \frac{\widetilde{b}_n}{2}| \|T_n\|_\infty + \sum_{j=n+1}^{\infty} |b_j| \|T_j\|_\infty \\ &= \sum_{j=0}^{n-1}{}' |b_j - \widetilde{b}_j| + |b_n - \frac{\widetilde{b}_n}{2}| + \sum_{j=n+1}^{\infty} |b_j| \end{aligned}$$

and

$$\begin{aligned} \|f-q_n\|_\infty &\leq \sum_{j=0}^{n}{}' |b_j - c_j| \|T_j\|_\infty + \sum_{j=n+1}^{\infty} |b_j| \|T_j\|_\infty \\ &= \sum_{j=0}^{n}{}' |b_j - c_j| + \sum_{j=n+1}^{\infty} |b_j|. \end{aligned}$$

Recalling (2.13.1.11) in [8] (also see Boyd [5], pp. 96)

$$\widetilde{b}_j - b_j = \sum_{\ell=1}^{\infty} (b_{2\ell n - j} + b_{2\ell n + j}), \quad j = 0, 1, \ldots, n,$$

we know that

$$\begin{cases} \dfrac{\widetilde{b}_0 - b_0}{2} = b_{2n} + b_{4n} + b_{6n} + b_{8n} \cdots, \\ \widetilde{b}_1 - b_1 = b_{2n-1} + b_{2n+1} + b_{4n-1} + b_{4n+1} + \cdots, \\ \cdots\cdots \\ \widetilde{b}_{n-2} - b_{n-2} = b_{n+2} + b_{3n-2} + b_{3n+2} + b_{5n-2} + \cdots, \\ \widetilde{b}_{n-1} - b_{n-1} = b_{n+1} + b_{3n-1} + b_{3n+1} + b_{5n-1} + \cdots, \\ \dfrac{\widetilde{b}_n}{2} - b_n = b_{3n} + b_{5n} + b_{7n} + b_{9n} \cdots \end{cases} \tag{2.7}$$

which gives

$$\sum_{j=0}^{n-1}{}' |b_j - \widetilde{b}_j| + |b_n - \frac{\widetilde{b}_n}{2}| \leq \sum_{j=n+1}^{\infty} |b_j|. \tag{2.8}$$

Therefore

$$\|f-p_n\|_\infty \leq \sum_{j=0}^{n-1}{}' |b_j - \widetilde{b}_j| + |b_n - \frac{\widetilde{b}_n}{2}| + \sum_{j=n+1}^{\infty} |b_j| \leq 2 \sum_{j=n+1}^{\infty} |b_j| \tag{2.9}$$

(see [5]). Similarly from (4.56) in [5], we find

$$c_j - b_j = \sum_{\ell=1}^{\infty} (-1)^\ell (b_{2\ell(n+1)-j} + b_{2\ell(n+1)+j}), \quad j = 0, 1, 2, \ldots,$$

which yields

$$\|f-q_n\|_\infty \le 2\sum_{j=n+1}^{\infty}|b_j| \tag{2.10}$$

(see [5]).

If $f, f', \ldots, f^{(k-1)}$ are absolutely continuous on $[-1,1]$ and $V_k<\infty$, it follows from Lemma 2.1 and Lemma 2.2 that for $n\ge k+1$

$$\sum_{j=n+1}^{\infty}|b_j| \le \frac{2V_k}{\pi}\sum_{j=n+1}^{\infty}\frac{1}{j(j-1)\cdots(j-k)} = \frac{2V_k}{k\pi n(n-1)\cdots(n+1-k)},$$

which together with (2.9) and (2.10) implies (2.5a) and (2.5b).

Similarly, if f is analytic with $|f(z)|\le M$ in the region bounded by the ellipse with foci ±1 and major and minor semiaxis lengths summing to $\rho>1$, from Lemma 2.1, we find

$$\sum_{j=n+1}^{\infty}|b_j| \le \sum_{j=n+1}^{\infty}\frac{2M}{\rho^j} = \frac{2M}{(\rho-1)\rho^n},$$

which together with (2.9) and (2.10) establishes (2.6). ∎

From the above estimate, the following theorem improves the error bounds given by Trefethen [30] for Gauss quadrature and Clenshaw-Curtis quadrature.

Theorem 2.2 *Suppose $f, f', \ldots, f^{(k-1)}$ are absolutely continuous on $[-1,1]$ and $\|f^{(k)}\|_T = V_k < \infty$ for some $k\ge 1$. Then*

$$\frac{32V_k}{15k\pi 2n(2n-1)\cdots(2n+1-k)} \ge \begin{cases} |I[f]-I_n^G[f]| & \text{for all } n\ge k/2, & (2.11a)\\ |I[f]-I_n^{C\text{-}C}[f]| & \text{for all sufficiently large } n, & (2.11b)\\ |I[f]-I_n^F[f]| & \text{for all sufficiently large } n, & (2.11c)\end{cases}$$

where $I[f]=\int_{-1}^{1}f(x)dx$, $I_n^G[f]$ is the Gauss quadrature with $n+1$ nodes, and "sufficiently large n" means $n>n_k$ for some n_k that depends on k but not f or V_k.

Proof: Following the proof of Theorem 5.1 in [30], the Gauss quadrature error can be estimated by

$$\begin{aligned}
|I[f]-I_n^G[f]| &= |\sum_{j=0}^{\infty}{}' b_j(I[T_j]-I_n^G[T_j])| \\
&\le \sum_{j=2n+2}^{\infty}|b_j||I[T_j]-I_n^G[T_j]| \\
&\le \frac{32}{15}\sum_{j=n+1}^{\infty}|b_{2j}| \\
&\le \frac{64V_k}{15\pi}\sum_{j=n+1}^{\infty}\frac{1}{2j(2j-1)\cdots(2j-k)} \quad \text{(Lemma 2.1 on } b_{2j}\text{)} \\
&\le \frac{32V_k}{15\pi}\sum_{j=2n+1}^{\infty}\frac{1}{j(j-1)\cdots(j-k)} \quad \text{(Lemma 2.2)} \\
&= \frac{32V_k}{15k\pi 2n(2n-1)\cdots(2n+1-k)},
\end{aligned}$$

where we use the estimate [30, Eq. (5.6)]

$$|I[T_j]-I_n^G[T_j]| \le \begin{cases} 32/15 & \text{if } j\ge 4 \text{ is even},\\ 0 & \text{if } j \text{ is odd}.\end{cases}$$

The error for $I_n^{C\text{-}C}[f]$ and $I_n^F[f]$ can be estimated based on the technique in [30] with the estimate

$$|I[T_j]-I_n[T_j]| \le \begin{cases} 72/35 & \text{if } j \text{ is even and } j\ge 6,\\ 0 & \text{if } j \text{ is odd,}\end{cases}$$

and

$$\begin{aligned}|I[f]-I_n[f]| &\le O(V_k n^{-k-2/3}) + \sum_{j=2n+2}^{\infty} |b_j||I[T_j]-I_n[T_j]|\\ &\le O(V_k n^{-k-2/3}) + \frac{72}{35}\sum_{j=n+1}^{\infty}|b_{2j}|\\ &\le O(V_k n^{-k-2/3}) + \frac{72V_k}{35k\pi 2n(2n-1)\cdots(2n+1-k)}.\end{aligned}$$

Since $\frac{72}{35k\pi 2n(2n-1)\cdots(2n+1-k)} = O(n^{-k})$, there exits an integer n_k depending on k but not f or V_k such that for $n > n_k$

$$O(V_k n^{-k-2/3}) + \frac{72V_k}{35k\pi 2n(2n-1)\cdots(2n+1-k)} \le \frac{32V_k}{15k\pi 2n(2n-1)\cdots(2n+1-k)}$$

and

$$|I[f]-I_n[f]| \le \frac{32V_k}{15k\pi 2n(2n-1)\cdots(2n+1-k)}.$$

Here I_n represents $I_n^{C\text{-}C}$ and I_n^F. ∎

Remark 1. Theorem 2.2 suggests that Gauss, Clenshaw-Curtis and Fejér's first quadrature are equally valuable and fundamental. Gauss quadrature is elegant and can be computed in $O(n^2)$ operations in [30]. Recently, Glaser, Liu and Rokhlin have reduced the cost to $O(n)$ operations [12]. The other two are simple and can be computed by FFT in $O(n\log n)$ operations. Let us consider the three quadrature formulae for $\int_{-1}^{1} x^{20}dx$ and $\int_{-1}^{1}\frac{1}{1+16x^2}dx$ respectively (Figure 1).

Remark 2. If f is analytic with $|f(z)|\le M$ in the region bounded by the ellipse with foci ±1 and major and minor semiaxis lengths summing to $\rho>1$, then for each $n\ge 0$, an estimate based on a contour integral for the interpolant q_n in the Chebyshev points of the first kind is given by ([7], pp. 391 and [19], pp. 149)

$$\|f-q_n\|_\infty \le \frac{2M(\rho+\rho^{-1})}{\rho^{n+1}(1-\rho^{-2n-2})(\rho+\rho^{-1}-2)}. \tag{2.12}$$

Comparing error bounds (2.6) and (2.12), the ratio of the former to the latter is less than $\frac{2\rho(\rho-1)}{\rho^2+1}$. For $1<\rho\le 1+\sqrt{2}$, the former is better, but for $\rho>1+\sqrt{2}$ the latter is better. For $\rho\approx 1$, the approximate error can be estimated by Theorem 2.1.

Remark 3. Comparing (2.11b) in Theorem 2.2 to (5.1) in Theorem 5.1 in [30], we see that the term

$$\frac{1}{(2n+1-k)^k}$$

in [30] is replaced by

$$\frac{1}{2n(2n-1)\cdots(2n+1-k)}$$

in this paper. For k fixed, the two error bounds have the same order with increasing n. However, for k close or equal to $2n$, the error bound (2.11b) is much smaller than that in [30] (see Table 1).

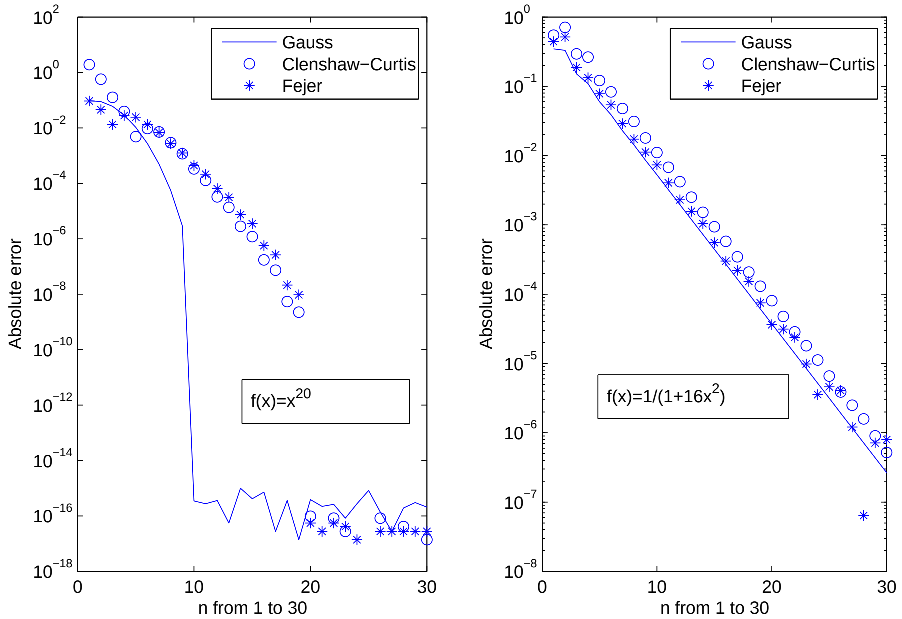

Figure 1: The absolute error for $\int_{-1}^{1} f(x)dx$ evaluated by Gauss, Clenshaw-Curtis and Fejér quadrature rules with n nodes: $f(x) = x^{20}$ or $f(x) = \dfrac{1}{1+16x^2}$.

Table 1: Comparison of (1) $:= \dfrac{1}{(2n+1-k)^k}$ and (2) $:= \dfrac{1}{2n(2n-1)\cdots(2n+1-k)}$.

n	k	(1)	(2)	n	k	(1)	(2)
5	1	0.100	0.100	20	1	0.250×10^{-1}	0.250×10^{-1}
	3	0.195×10^{-2}	0.139×10^{-2}		3	0.182×10^{-4}	0.169×10^{-4}
	5	0.129×10^{-3}	0.331×10^{-4}		20	0.359×10^{-26}	0.289×10^{-29}
	10	0.100×10^{1}	0.276×10^{-6}		40	0.100×10^{1}	0.123×10^{-47}
10	1	0.500×10^{1}	0.500×10^{1}	40	1	0.125×10^{-1}	0.125×10^{-1}
	3	0.171×10^{-3}	0.146×10^{-3}		3	0.211×10^{-5}	0.203×10^{-5}
	10	0.386×10^{-10}	0.149×10^{-11}		40	0.308×10^{-64}	0.114×10^{-70}
	20	0.100×10^{1}	0.411×10^{-18}		80	0.100×10^{1}	0.140×10^{-118}

Following the proof in Theorem 2.1, we can estimate the error bounds for the first and second derivatives of f.

Theorem 2.3 *If $f, f', \ldots, f^{(k-1)}$ are absolutely continuous on $[-1,1]$ and $\|f^{(k)}\|_T = V_k < \infty$ for some $k \geq 0$, then for each $n \geq k+1$, we have that for $k > 2$*

$$\frac{4(n+1)V_k}{n(k-2)\pi(n-2)(n-3)\cdots(n+1-k)} \geq \begin{cases} \|f' - p_n'\|_\infty, & (2.13a) \\ \|f' - q_n'\|_\infty, & (2.13b) \end{cases}$$

and for $k > 4$

$$\frac{4(n^2+n)V_k}{3(k-4)(n^2-5n+6)\pi(n-4)(n-5)\cdots(n+1-k)} \geq \begin{cases} \|f''-p_n''\|_\infty, & (2.14a) \\ \|f''-q_n''\|_\infty. & (2.14b) \end{cases}$$

Proof: From (2.1), (2.2a) and (2.2b), we have that for any $x \in [-1,1]$

$$\|f'-p_n'\|_\infty \leq \sum_{j=1}^{n-1} |b_j - \widetilde{b}_j| \|T_j'\|_\infty + |b_n - \frac{\widetilde{b}_n}{2}| \|T_n'\|_\infty + \sum_{j=n+1}^{\infty} |b_j| \|T_j'\|_\infty,$$

and

$$\|f''-p_n''\|_\infty \leq \sum_{j=1}^{n-1} |b_j - \widetilde{b}_j| \|T_j''\|_\infty + |b_n - \frac{\widetilde{b}_n}{2}| \|T_n''\|_\infty + \sum_{j=n+1}^{\infty} |b_j| \|T_j''\|_\infty.$$

Note that $T_j(x) = \cos(j\cos^{-1}(x))$ for $-1 \leq x \leq 1$ and

$$T_j'(x) = \frac{j\sin(j\cos^{-1}(x))}{\sqrt{1-x^2}} = \frac{j\sin(ju)}{\sin(u)}, \quad \|T_j'\|_\infty = j^2, \tag{2.15}$$

where $u = \cos^{-1}(x)$. Furthermore, from Equation (4.7.8) in Szegö [28],

$$\lim_{\lambda\to 0} \lambda^{-1} P_j^{(\lambda)}(x) = \frac{2}{j} T_j(x)$$

where $P_j^{(\lambda)}(x) = \dfrac{\Gamma\left(\lambda+\frac{1}{2}\right)\Gamma\left(j+2\lambda\right)}{\Gamma\left(2\lambda\right)\Gamma\left(j+\lambda+\frac{1}{2}\right)} P_j^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x)$, $P_j^{(\lambda-\frac{1}{2},\lambda-\frac{1}{2})}(x)$ is the Jacobi polynomial and $\Gamma(x)$ is the Gamma function, and, by differentiating with respect to x, one finds

$$\lim_{\lambda\to 0} \lambda^{-1} \frac{d}{dx} P_j^{(\lambda)}(x) = \frac{2}{j} T_j'(x),$$

while using Equation (4.7.14) in [28],

$$\lambda^{-1} \frac{d}{dx} P_j^{(\lambda)}(x) = 2\lambda P_{j-1}^{(\lambda+1)}(x), \tag{2.16}$$

we get

$$T_j'(x) = jP_{j-1}^{(1)}(x). \tag{2.17}$$

Differentiating (2.17) once more, and using again (2.16) (with $\lambda = 1$), we finally obtain

$$T_j''(x) = 2jP_{j-2}^{(2)}(x).$$

The latter gives

$$\|T_j''\|_\infty = 2j \max_{-1\leq x\leq 1} |P_{j-2}^{(2)}(x)|,$$

and inserting

$$\max_{-1\leq x\leq 1} |P_{j-2}^{(2)}(x)| = \frac{j(j-1)(j+1)}{3!}$$

(cf. Equation (7.33.1) in [28] with $n = j-2$ and $\lambda = 2$), it yields

$$\|T_j''\|_\infty = \frac{j^2(j-1)(j+1)}{3}. \tag{2.18}$$

Thus from (2.8), we see that

$$\sum_{j=1}^{n-1} |\widetilde{b}_j - b_j| j^2 + |b_n - \frac{\widetilde{b}_n}{2}| n^2 \leq {\sum_{j=0}^{n-1}}' |\widetilde{b}_j - b_j| n^2 + |b_n - \frac{\widetilde{b}_n}{2}| n^2 \leq n^2 \sum_{j=n+1}^{\infty} |b_j| \leq \sum_{j=n+1}^{\infty} |b_j| j^2$$

and

$$
\begin{aligned}
& \sum_{j=1}^{n-1} |\widetilde{b}_j - b_j| \frac{j^2(j+1)(j-1)}{3} + |b_n - \frac{\widetilde{b}_n}{2}| \frac{n^2(n+1)(n-1)}{3} \\
\leq \; & \frac{n^2(n+1)(n-1)}{3} \left(\sum_{j=0}^{n-1}{}' |\widetilde{b}_j - b_j| + |b_n - \frac{\widetilde{b}_n}{2}| \right) \\
\leq \; & \sum_{j=n+1}^{\infty} |b_j| \frac{j^2(j+1)(j-1)}{3}.
\end{aligned}
$$

Applying Lemma 2.1, Lemma 2.2, (2.15) and (2.18) gives that for $n \geq k+1$

$$
\begin{aligned}
& \|f' - p_n'\|_\infty \\
\leq \; & \sum_{j=1}^{n-1} |b_j - \widetilde{b}_j| \|T_j'\|_\infty + |b_n - \frac{\widetilde{b}_n}{2}| \|T_n'\|_\infty + \sum_{j=n+1}^{\infty} |b_j| \|T_j'\|_\infty \\
\leq \; & \sum_{j=1}^{n-1} |\widetilde{b}_j - b_j| j^2 + |b_n - \frac{\widetilde{b}_n}{2}| n^2 + \sum_{j=n+1}^{\infty} |b_j| j^2 \\
\leq \; & 2 \sum_{j=n+1}^{\infty} |b_j| j^2 \\
\leq \; & \sum_{j=n+1}^{\infty} \frac{4(n+1)V_k}{n\pi(j-2)(j-3)\cdots(j-k)} \\
= \; & \frac{4(n+1)V_k}{n(k-2)\pi(n-2)(n-3)\cdots(n+1-k)}
\end{aligned}
$$

and similarly

$$
\begin{aligned}
\|f'' - p_n''\|_\infty \;\leq\; & 2 \sum_{j=n+1}^{\infty} |b_j| \frac{j^2(j+1)(j-1)}{3} \\
\leq\; & \sum_{j=n+1}^{\infty} \frac{4(n^2+n)V_k}{3(n^2-5n+6)\pi(j-4)(j-5)\cdots(j-k)} \\
=\; & \frac{4(n^2+n)V_k}{3(n^2-5n+6)(k-4)\pi(n-4)(n-5)\cdots(n+1-k)}.
\end{aligned}
$$

The bounds (2.13b) and (2.14b) can be obtained by the same way. ∎

Theorem 2.4 *If f is analytic with $|f(z)| \leq M$ in the region bounded by the ellipse with foci ± 1 and major and minor semiaxis lengths summing to $\rho > 1$, then for each n,*

$$
\frac{4M(n^2 - (2n^2+2n-1)\rho + (n+1)^2\rho^2)}{(\rho-1)^3\rho^n} \geq \begin{cases} \|f' - p_n'\|_\infty, & (2.19a) \\ \|f' - q_n'\|_\infty, & (2.19b) \end{cases}
$$

and

$$
\frac{4M\left(n^4 - n^2 + 6(n^2+n-2)(n^2+n-1)\rho^2 + (n^2+2n)(n+1)^2\rho^4\right)}{3(\rho-1)^5\rho^{n-1}} \geq \begin{cases} \|f'' - p_n''\|_\infty, & (2.20a) \\ \|f'' - q_n''\|_\infty. & (2.20b) \end{cases}
$$

Proof: Using symbolic algebraic computation, such as MAPLE, and from Lemma 2.1 and the proof of Theorem 2.3, we have

$$
\|f' - p_n'\|_\infty \leq 2 \sum_{j=n+1}^{\infty} |b_j| \|T_j'\|_\infty \leq \sum_{j=n+1}^{\infty} \frac{4Mj^2}{\rho^j} = \frac{4M(n^2 - (2n^2+2n-1)\rho + (n+1)^2\rho^2)}{(\rho-1)^3\rho^n}
$$

and

$$
\begin{aligned}
& \|f''-p_n''\|_\infty \\
\leq\ & 2\sum_{j=n+1}^{\infty}|b_j|\|T_j''\|_\infty \\
\leq\ & \sum_{j=n+1}^{\infty}\frac{4Mj^2(j^2-1)}{3\rho^j} \\
=\ & \frac{4M}{3(\rho-1)^5\rho^{n-1}}\left[(n^4-n^2)-(4n^4+4n^3-10n^2+2n)\rho\right. \\
& \left.+6(n^2+n-2)(n^2+n-1)\rho^2-(4n^4+2n^3+2n^2-18n+12)\rho^3+(n+1)^2(n^2+2n)\rho^4\right] \\
<\ & \frac{4M}{3(\rho-1)^5\rho^{n-1}}\left[(n^4-n^2)+6(n^2+n-2)(n^2+n-1)\rho^2+(n+1)^2(n^2+2n)\rho^4\right].
\end{aligned}
$$

∎

3 Application to integration of $\int_{-1}^{1} f(x)e^{i\omega x^r}dx$

Sloan in [24] and Sloan and Smith in [25] considered the numerical evaluation of the integral

$$I[f]=\int_{-1}^{1}k(x)f(x)dx \tag{3.1}$$

with generalized *Fejér's first rule* [24] and *Clenshaw-Curtis rule* [1, 10, 11, 14, 17, 18, 21, 25] defined by

$$Q_n^F[f]=\int_{-1}^{1}k(x)q_n(x)dx,\qquad Q_n^{C\text{-}C}[f]=\int_{-1}^{1}k(x)p_n(x)dx, \tag{3.2}$$

where k is an absolutely integrable function and f is a suitably smooth function. If k satisfies

$$\int_{-1}^{1}|k(x)|^p dx<\infty$$

for some $p>1$, then

$$\lim_{n\to\infty}Q_n^F[f]=\lim_{n\to\infty}Q_n^{C\text{-}C}[f]=I[f]\ ([25]).$$

Moreover, Sloan and Smith in [26] presented practical implementation and computational error estimates (rather than rigorous error bounds) for the Clenshaw-Curtis integration method for $\int_{-1}^{1}k(x)f(x)dx$ with $k(x)\equiv 1$; $k(x)=|\lambda-x|^\alpha$ ($\alpha>-1$ and $|\lambda|\leq 1$); $k(x)=\cos\alpha x$; $k(x)=\sin\alpha x$.

The computation of $\int_a^b f(x)e^{i\omega g(x)}$ occurs in a wide range of practical problems and applications ranging from nonlinear optics to fluid dynamics, plasma transport, computerized tomography, celestial mechanics, computation of Schrödinger spectra, Bose-Einstein condensates...(cf. [16]). By a diffeomorphism transformation, the integral can be transferred into $\int_0^c w(x)e^{i\omega x^r}dx$. We refer the reader to [9, 32] for a detailed discussion. In this section, we consider the efficiency of Clenshaw-Curtis and Fejér's first quadrature (3.2) and use the results of Section 2 to present new error bounds for these two quadrature formulae when

$$I[f]=\int_{-1}^{1}f(x)e^{i\omega x^r}dx$$

where $\omega\geq 1$.

In Subsection 3.1, we consider x^r to be a real function, where r can be expressed by $r=\dfrac{p}{q}$, p and q are two relatively prime integers and q is odd. Then x^r is well-defined on $[-1,1]$. Moreover, if p is even then x^r is an even function; if p is odd then x^r is an odd function. In Subsection 3.2, we use the definition for x^r $(x \in [-1,0))$ in MATLAB, MAPLE and MATHEMATICA to evaluate x^r as follows

$$x^r = |x|^r e^{r\pi i}.$$

Therefore for each $x \in [-1,1]$, x^r is also well-defined.

3.1 $x^r = \sqrt[q]{x^p}$ where $r = \frac{p}{q}$, p and q are integers with q odd

In this subsection, we assume that $x^r \in [-1,1]$ is a real function and well-defined on $[-1,1]$. Then r is a rational number such that $r = \frac{p}{q}$ where p and q are two relatively prime integers, and q is odd. In this case, for $-1 \leq x < 0$, x^r is defined by

$$x^r = \begin{cases} |x|^r & \text{if } p \text{ is even,} \\ -|x|^r & \text{if } p \text{ is odd.} \end{cases}$$

From the results in [24, 25], we see that for any fixed ω, Clenshaw-Curtis and Fejér's first quadrature are convergent since $|e^{i\omega x^r}| \leq 1$.

Applying Theorem 2.1 directly implies that *if $f, f', \ldots, f^{(k-1)}$ are absolutely continuous on $[-1,1]$ and $\|f^{(k)}\|_T = V_k < \infty$ for some $k \geq 1$, then for each $n \geq k+1$, we have*

$$\frac{8V_k}{k\pi n(n-1)\cdots(n-k+1)} \geq \begin{cases} |I[f] - Q_n^{C\text{-}C}[f]|, & (3.3a) \\ |I[f] - Q_n^F[f]|. & (3.3b) \end{cases}$$

If f is analytic with $|f(z)| \leq M$ in the region bounded by the ellipse with foci ± 1 and major and minor semiaxis lengths summing to $\rho > 1$, then for each $n \geq 0$,

$$|I[f] - Q_n^{C\text{-}C}[f]| \leq \frac{8M}{(\rho-1)\rho^n}, \quad |I[f] - Q_n^F[f]| \leq \frac{8M}{(\rho-1)\rho^n}. \tag{3.4}$$

However, these error bounds are useless for sufficiently large values of ω since from the Riemann-Lebesgue lemma ([13], pp. 1101), $I[f] \to 0$ as $\omega \to \infty$.

Suppose that $p_n(x) = \sum_{j=0}^{n}{}'' \widetilde{b}_j T_j(x)$ and $q_n(x) = \sum_{j=0}^{n}{}' c_j T_j(x)$. In the following, we present a conversion algorithm from a finite Chebyshev series to a finite power series. These conversions can avoid solving a linear system with Vandemonde matrix which is ill-condition when n is large. Assume

$$p_n(x) = \sum_{m=0}^{n} a_m x^m, \qquad q_n(x) = \sum_{m=0}^{n} \widetilde{a}_m x^m.$$

Then from (2.16), (2.17a) and (2.17b) in [19], we have

$$a_m = \sum_{j=0}^{[\frac{n-m}{2}]} \gamma_j^{(m+2j)} \widetilde{b}_{m+2j}, \quad \widetilde{a}_m = \sum_{j=0}^{[\frac{n-m}{2}]} \gamma_j^{(m+2j)} c_{m+2j}, \quad m = 0, 1, \ldots, n,$$

where $\gamma_0^{(0)} = 1$ and

$$\gamma_j^{(\ell)} = (-1)^j 2^{\ell-2j-1} \frac{\ell}{\ell-j} \binom{\ell-j}{j}, \quad \ell \geq 1$$

(see [19]). Here, we take $\frac{1}{2}\widetilde{b}_0$, $\frac{1}{2}\widetilde{b}_n$ and $\frac{1}{2}c_0$ instead of $\widetilde{b}_0$, $\widetilde{b}_n$ and c_0 in the sums of computation of coefficients a_m and $\widetilde{a}_m$, respectively. Thus $Q_n^{C\text{-}C}[f]$ and $Q_n^F[f]$ can be rewritten as

$$Q_n^{C\text{-}C}[f] = \sum_{m=0}^{n} a_m I[x^m], \qquad Q_n^F[f] = \sum_{m=0}^{n} \widetilde{a}_m I[x^m], \tag{3.5}$$

where the moments $I[x^m] = \int_{-1}^{1} x^m e^{i\omega x^r}dx$ can be computed explicitly by the gamma function $\Gamma(z)$, the incomplete gamma function $\Gamma(\alpha, z)$ and the extended exponential integral $\mathrm{Ei}(a, z)$=$\mathrm{E}_a(z) = \int_1^\infty t^{-a}e^{-zt}dt = z^{a-1}\Gamma(1-a, z)$ ($a > 1$, $\Re(z) \geq 0$) ([2], pp. 228, pp. 260)

$$\begin{aligned} I[x^m] \quad &= \quad \frac{1}{r(-i\omega)^{(m+1)/r}}\left[\Gamma\left(\frac{m+1}{r}\right) - \Gamma\left(\frac{m+1}{r}, -i\omega\right)\right] + \frac{(-1)^m}{r(-i\cos(p\pi)\omega)^{(m+1)/r}} \cdot \\ &\quad\quad \left[\Gamma\left(\frac{m+1}{r}\right) - \Gamma\left(\frac{m+1}{r}, -i\cos(p\pi)\omega\right)\right](r > 0) \end{aligned}$$

(see [16, 32]);

$$\begin{aligned} I[x^m] \quad &= \quad \int_{-1}^{1} x^m e^{i\omega x^r}dx = \frac{1}{|r|}\int_1^\infty x^{-\frac{m+|r|+1}{|r|}}e^{i\omega x}dx + \frac{(-1)^m}{|r|}\int_1^\infty x^{-\frac{m+|r|+1}{|r|}}e^{i\omega\cos(p\pi)x}dx \\ &= \quad \frac{1}{|r|}\mathrm{Ei}\left(\frac{m+|r|+1}{|r|}, -i\omega\right) + \frac{(-1)^m}{|r|}\mathrm{Ei}\left(\frac{m+|r|+1}{|r|}, -i\omega\cos(p\pi)\right)(r < 0) \end{aligned}$$

(see [15]). Before we further discuss the error estimate, we first take $f(x) = \dfrac{1}{1+16x^2}$ as an example to illustrate the convergence of the quadrature error in the Chebyshev points and equispaced points for the highly oscillatory integrals, respectively. Figure 2 shows that equispaced points fail to approximate the integral $\int_{-1}^{1} f(x)e^{i\omega x^2}dx$ (in each panel, the upper dotted line corresponds to odd numbers of n, the lower dotted line to even numbers of n).

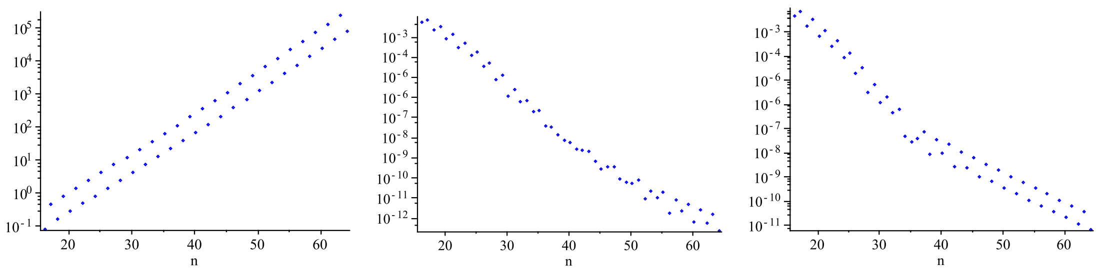

Figure 2: Convergence of quadrature errors of $\int_{-1}^{1} L_n(x)e^{i\omega x^2}dx$ using equispaced points (left), Chebyshev points of the second kind (middle), and the first kind (right) to interpolate $f(x)$, for $\int_{-1}^{1} \frac{1}{1+16x^2}e^{i\omega x^2}dx$. Here the number of interpolation nodes n ranges from 16 to 64 and the fixed frequency is $\omega = 20$.

3.1.1 The case where n is odd

Theorem 3.1 *Let $f \in C^2[-1, 1]$. Then for each odd number n*

$$|I[f] - Q_n^{C\text{-}C}[f]| \leq 2W_1(r, \omega)\|f' - p_n'\|_\infty, \tag{3.6a}$$

$$|I[f] - Q_n^F[f]| \leq 2W_1(r, \omega)(\|f - q_n\|_\infty + \|f' - q_n'\|_\infty), \tag{3.6b}$$

where $W_1(r, \omega) = \begin{cases} \dfrac{3}{|r|\omega} & \text{if } r < 0, \\ \dfrac{r+2}{r\omega^{\min(1,1/r)}} & \text{if } r > 0, \end{cases}$ *and $r = \dfrac{p}{q}$ is a nonzero rational number with q odd.*

We need the following lemmas to prove Theorem 3.1.

Lemma 3.1 *(van der Corput,[27]) Suppose that g is real-valued and smooth in (a,b) and that $|g^{(k)}| \geq 1$ for all $x \in (a,b)$ for a fixed value of k. Then*

$$|\int_a^b e^{i\omega g(x)} dx| \leq c(k)\omega^{-1/k}$$

holds when (i) $k \geq 2$, or (ii) $k = 1$ and g' is monotonic. Here $c(k) = 5 \cdot 2^{k-1} - 2$, which is independent of g and ω.

Lemma 3.2 *Let $r = \dfrac{p}{q}$ be a nonzero rational number with q odd. Then for any $x \in [-1,1]$, we have*

$$|\int_0^x e^{i\omega t^r} dt| \leq W_1(r,\omega). \tag{$*$}$$

Proof: Obviously, $(*)$ holds for $x = 0$, since $W_1(r,\omega)$ is positive. In the following, we assume $x \neq 0$.

In the case $r > 0$: For $x \in (0, \dfrac{1}{\omega^{1/r}}]$, we have

$$|\int_0^x e^{i\omega t^r} dt| \leq \int_0^{\omega^{-1/r}} dt \leq \omega^{-1/r}.$$

For $x \in [\dfrac{1}{\omega^{1/r}}, 1]$, by using the transformation $u = t^r$ and the triangle inequality it follows that

$$\begin{aligned}
|\int_0^x e^{i\omega t^r} dt| &\leq \int_0^{\omega^{-1/r}} dt + |\int_{\omega^{-1/r}}^x e^{i\omega t^r} dt| \\
&\leq \omega^{-1/r} + \frac{1}{r\omega} |\int_{\omega^{-1}}^{x^r} u^{(1-r)/r} de^{i\omega u}| \\
&\leq \omega^{-1/r} + \frac{1}{r\omega}(x^{1-r} + \omega^{1-1/r} + |\int_{\omega^{-1}}^{x^r} [u^{(1-r)/r}]' e^{i\omega u} du|) \\
&\leq \omega^{-1/r} + \frac{1}{r\omega}(x^{1-r} + \omega^{1-1/r} + \int_{\omega^{-1}}^{x^r} |[u^{(1-r)/r}]'| du) \\
&\leq \begin{cases} \omega^{-1/r} + \dfrac{2x^{1-r}}{r\omega} \leq \dfrac{r+2}{r\omega} & 0 < r \leq 1, \\ \omega^{-1/r} + \dfrac{2\omega^{1-1/r}}{r\omega} \leq \dfrac{2+r}{r\omega^{1/r}} & r > 1. \end{cases}
\end{aligned}$$

In the case $r < 0$: Since the derivative of x^r is monotonic in $(0,1]$ and

$$|(x^r)'| = |rx^{r-1}| \geq |r|,$$

by Lemma 3.1 we have

$$|\int_0^x e^{i\omega t^r} dt| = |\int_0^x e^{i|r|\omega \frac{t^r}{|r|}} dt| \leq \frac{3}{|r|\omega}.$$

Therefore for all $x \in [0,1]$ and $r \neq 0$

$$|\int_0^x e^{i\omega t^r} dt| \leq W_1(r,\omega). \tag{3.7}$$

In a similar way for all $x \in [-1,0)$, we obtain

$$|\int_x^0 e^{i\omega t^r} dt| \leq W_1(r,\omega). \tag{3.8}$$

∎

The following example shows the asymptotics of $M = \int_0^1 e^{i\omega x^r} dx$ (see Figure 3).

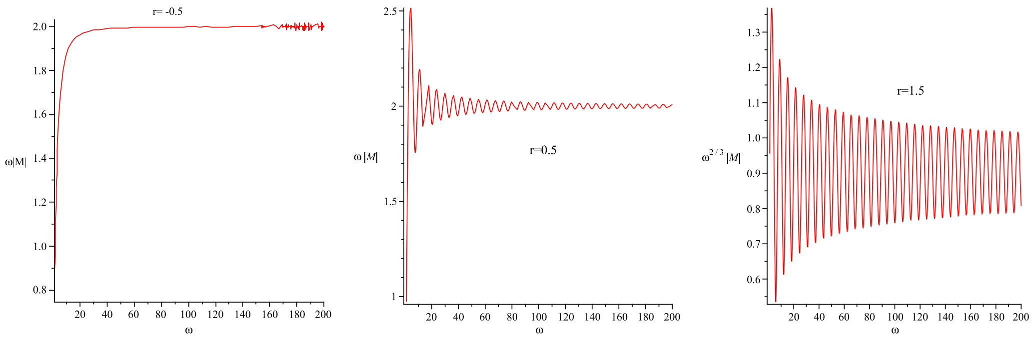

Figure 3: The absolute values of the moment $M = \int_0^1 e^{i\omega x^r}dx$ scaled by $\omega (r \le 1)$, $\omega^{1/r} (r > 1)$ for $r = -0.5, 0.5, 1.5$ respectively.

Lemma 3.3 *Let $r = \dfrac{p}{q}$ be a nonzero rational number with q odd. Then for every function $h \in C^1[-1,1]$,*

$$|\int_{-1}^{1} h(t)e^{i\omega t^r}dt| \le W_1(r,\omega)\left(|h(1)| + |h(-1)| + \int_{-1}^{1}|h'(t)|dt\right).$$

Proof: The integral $\int_{-1}^{1} h(t)e^{i\omega t^r}dt$ can be written as

$$\int_{-1}^{1} h(t)e^{i\omega t^r}dt = \int_0^1 h(t)F'(t)dt - \int_0^{-1} h(s)G'(s)ds$$

with

$$F(t) = \int_0^t e^{i\omega u^r}du, \quad G(s) = \int_0^s e^{i\omega u^r}du.$$

Integrating by parts we get

$$\int_{-1}^{1} h(t)e^{i\omega t^r}dt = h(1)F(1) - h(-1)G(-1) - \int_0^1 h'(t)F(t)dt + \int_0^{-1} h'(s)G(s)ds.$$

Applying (3.7) and (3.8) establishes the desired result. ∎

Proof of Theorem 3.1: From Lemma 3.3 and that $f(-1) - p_n(-1) = f(1) - p_n(1) = 0$, the error for Clenshaw-Curtis quadrature can be estimated by

$$|I[f] - Q_n^{C\text{-}C}[f]| \le W_1(r,\omega)\int_{-1}^{1}|f'(x) - p_n'(x)|dx \le 2W_1(r,\omega)\|f' - p_n'\|_\infty.$$

The error of $Q_n^F[f]$ can be estimated by

$$\begin{aligned} |I[f] - Q_n^F[f]| &\le W_1(r,\omega)\left(|f(1) - q_n(1)| + |f(-1) - q_n(-1)| + \int_{-1}^{1}|f'(x) - q_n'(x)|dx\right) \\ &\le 2W_1(r,\omega)(\|f - q_n\|_\infty + \|f' - q_n'\|_\infty). \end{aligned}$$

Based on Theorem 2.1, Theorem 2.3 and Theorem 2.4, we can easily compute upper bounds for the error bounds in Theorem 3.1. For example, if $f, f', \ldots, f^{(k-1)}$ are absolutely continuous on $[-1,1]$ and $\|f^{(k)}\|_T = V_k < \infty$ for some $k > 2$, then for each odd number n with $n \geq k+1$, we have

$$|I[f] - Q_n^{C\text{-}C}[f]| \leq \frac{8(n+1)V_k}{n(k-2)\pi(n-2)(n-3)\cdots(n+1-k)} W_1(r,\omega), \tag{3.9a}$$

$$|I[f] - Q_n^{F}[f]| \leq \frac{8(n+1)(1+\frac{k-2}{k(n^2-1)})V_k}{n(k-2)\pi(n-2)(n-3)\cdots(n+1-k)} W_1(r,\omega). \tag{3.9b}$$

3.1.2 The case where n is even

Lemma 3.4 *Let $r = \dfrac{p}{q}$ be a nonzero rational number with q odd. Then for every function $h \in C^1[-1,1]$,*

$$|\int_{-1}^{1} h(t)te^{i\omega t^r}dt| \leq W_2(r,\omega)\left(|h(1)| + |h(-1)| + \int_{-1}^{1} |h'(t)|dt\right).$$

where $W_2(r,\omega) = \begin{cases} \dfrac{3}{|r|\omega} & \text{if } r < 0, \\ \dfrac{r+4}{2r\omega^{\min(1,2/r)}} & \text{if } r > 0. \end{cases}$

Proof: The proof is similar to that of Lemma 3.2 and Lemma 3.3. Here we just show that for all $x \in [0,1]$

$$|\int_{0}^{x} te^{i\omega t^r}dt| \leq W_2(r,\omega).$$

Using $u = t^2$ we have

$$\int_0^x te^{i\omega t^r}dt = \frac{1}{2}\int_0^{x^2} e^{i\omega t^r}dt^2 = \frac{1}{2}\int_0^{x^2} e^{i\omega u^{r/2}}du.$$

This together with Lemma 3.2 establishes the desired result. ∎

Theorem 3.2 *Let $f(x) \in C^2[-1,1]$. Then for each even number n*

$$|I[f] - Q_n^{C\text{-}C}[f]| \leq 3W_2(r,\omega)\|f'' - p_n''\|_\infty, \tag{3.10a}$$

$$|I[f] - Q_n^{F}[f]| \leq W_2(r,\omega)(2\|f - q_n\|_\infty + 3\|f'' - q_n''\|_\infty), \tag{3.10b}$$

where $r = \dfrac{p}{q}$ is a rational number with p and q relatively prime and q odd.

Proof: Note that $f(1) - p_n(1) = f(-1) - p_n(-1) = f(0) - p_n(0) = 0$. It is easy to verify that for $x \in [-1,1]$

$$F(x) = \begin{cases} \dfrac{f(x) - p_n(x)}{x} & \text{if } x \neq 0, \\ f'(0) - p_n'(0) & \text{if } x = 0, \end{cases} \qquad F'(x) = \begin{cases} \left(\dfrac{f(x) - p_n(x)}{x}\right)' & \text{if } x \neq 0, \\ \dfrac{f''(0) - p_n''(0)}{2} & \text{if } x = 0, \end{cases}$$

and $f(x) - p_n(x) = xF(x)$.

From Lemma 3.4, the error for the Clenshaw-Curtis quadrature rule can be estimated by

$$|I[f] - Q_n^{C\text{-}C}[f]| = |\int_{-1}^{1} F(x)xe^{i\omega x^r}dx| \leq 2W_2(r,\omega)\|F'\|_\infty. \tag{3.11}$$

From the Maclaurin expansion of $f(x) - p_n(x)$, we see that

$$f(x) - p_n(x) = (f'(0) - p_n'(0))x + \frac{f''(\xi_1) - p_n''(\xi_1)}{2}x^2, \quad \xi_1 \in (0,x)$$

$$f'(x) - p_n'(x) = f'(0) - p_n'(0) + (f''(\xi_2) - p_n''(\xi_2))x, \quad \xi_2 \in (0, x),$$

and then

$$\frac{x(f'(x) - p_n'(x)) - (f(x) - p_n(x))}{x^2} = f''(\xi_2) - p_n''(\xi_2) - \frac{f''(\xi_1) - p_n''(\xi_1)}{2}$$

and

$$\|F'\|_\infty \le \max \left\{ \sup_{-1 \le x \le 1, x \ne 0} \left| \frac{x(f'(x) - p_n'(x)) - (f(x) - p_n(x))}{x^2} \right|, \frac{|f''(0) - p_n''(0)|}{2} \right\} \le \frac{3}{2} \|f'' - p_n''\|_\infty. \tag{3.12}$$

This together with (3.11) and (3.12) implies (3.10a).

The error of $Q_n^F[f]$ can be represented by Lemma 3.4 and (3.12) as

$$\begin{aligned} |I[f] - Q_n^F[f]| &= \left| \int_{-1}^1 F(x) x e^{i\omega x^r} dx \right| \\ &\le W_2(r, \omega) \left(|f(1) - q_n(1)| + |f(-1) - q_n(-1)| + 3\|f'' - q_n''\|_\infty \right) \\ &\le W_2(r, \omega)(2\|f - q_n\|_\infty + 3\|f'' - q_n''\|_\infty). \end{aligned}$$

∎

Remark 4. From Lemma 3.3 we see that

$$\int_{-1}^1 f(x) e^{i\omega x^r} dx = \begin{cases} O\left(\frac{1}{\omega}\right) & \text{if } r \le 1, \, r \ne 0, \\ O\left(\frac{1}{\omega^{1/r}}\right) & \text{if } r > 1. \end{cases}$$

In the case that n is even, the above error bound for $Q_n^{C-C}[f]$ in Theorem 3.2 can be improved as follows:

Theorem 3.3 *Let $f(x) \in C^2[-1, 1]$. Then for each even number n, $r \le 1 (r \ne 0)$ or $r = 2$, the error bound for Clenshaw-Curtis quadrature can be improved to*

$$I[f] - Q_n^{C\text{-}C}[f] = \begin{cases} O\left(\frac{1}{\omega^2}\right) & \text{if } r < 1, \\ O\left(\frac{1}{\omega^{1.5}}\right) & \text{if } r = 2. \end{cases} \tag{3.13}$$

In particular, for any integer n, $I[f] - Q_n^{C\text{-}C}[f] = O\left(\frac{1}{\omega^2}\right)$ for $r = 1$.

Proof: In the case that $0 < r < 1$ or $r = 2$: Since $f(1) - p_n(1) = f(-1) - p_n(-1) = f(0) - p_n(0) = 0$, and $\frac{f(x) - p_n(x)}{x^{r-1}}$ has a second derivative on $[-1, 1]$. Here we use the limit $\lim_{x \to 0} \frac{f(x) - p_n(x)}{x^{r-1}}$ to define the value $\frac{f(x) - p_n(x)}{x^{r-1}}$ at $x = 0$. We know that the limit always exists for $r < 1$ and $r = 2$. Thus, integrating by parts we have

$$I[f] - Q_n^{C\text{-}C}[f] = -\frac{1}{ir\omega} \int_{-1}^1 \left(\frac{f(x) - p_n(x)}{x^{r-1}} \right)' e^{i\omega x^r} dx,$$

which, together with Lemma 3.3, derives the desired result (3.13).

In the case that $r = 1$: Since $f(1) - p_n(1) = f(-1) - p_n(-1) = 0$ and

$$I[f] - Q_n^{C\text{-}C}[f] = -\frac{1}{i\omega} \int_{-1}^1 (f'(x) - p_n'(x)) e^{i\omega x} dx,$$

which, together with Lemma 3.3, derives the desired result. ∎

Remark 5. From Theorem 3.3, we see that for n even, Clenshaw-Curtis quadrature is more accurate for large values of ω than the corresponding Fejér's first quadrature (see Figure 4 and Table 2).

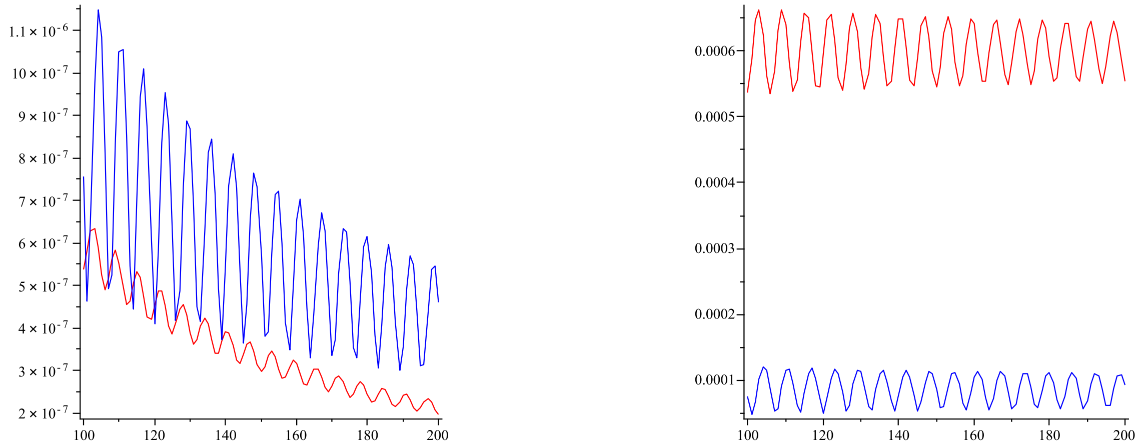

Figure 4: The error of the Clenshaw-Curtis method $Q_n^{C\text{-}C}[f]$ (left figure, bottom) and $Q_n^F[f]$ (left figure, top), and the error scaled by $\omega^{\frac{3}{2}}$ of $Q_n^{C\text{-}C}[f]$ (right figure, top) compared with the error scaled by ω of $Q_n^F[f]$ (right figure, bottom), for $I[f] = \int_{-1}^{1} \cos(x)e^{i\omega x^2}dx$. Here we choose $n = 4$ for both methods.

Table 2: Absolute errors in n-point approximations by $Q_n^{C\text{-}C}[f]$ and $Q_n^F[f]$ to the integral $I[f] = \int_{-1}^{1} e^x e^{i\omega x^2}dx$ with fixed frequency $\omega = 10000$.

n	10	11	12	13
$\lvert I[f] - Q_n^F[f]\rvert$	1.90×10^{-16}	1.84×10^{-14}	3.15×10^{-19}	2.51×10^{-17}
$\lvert I[f] - Q_n^{C\text{-}C}[f]\rvert$	3.65×10^{-17}	3.68×10^{-14}	6.08×10^{-20}	5.04×10^{-17}

Remark 6. Numerical results for $r = 2$ in Table 2 show that Clenshaw-Curtis or Fejér's first quadrature for n even can get higher accuracy than that for $n + 1$ for large values of ω. This is due to interpolation at $x = 0$ when n is even, and can also be seen by comparing the error bound (3.6a) and(3.6b) with $O(\omega^{-0.5})$ for n odd, (3.10b) with $O(\omega^{-1})$ (Fejér's first quadrature) and (3.13) with $O(\omega^{-1.5})$ (Clenshaw-Curtis quadrature) for n even.

Remark 7. From Theorems 3.1-3.3, Clenshaw-Curtis and Fejér's first quadrature for $r > 2$ have nearly the same accuracy (see Tables 3 and 4).

3.2 x^r is a complex function for $-1 \le x < 0$ defined by $x^r = |x|^r e^{r\pi i}$

By the definition of x^r on $[-1, 0)$, $e^{i\omega x^r}$ can be represented by

$$e^{i\omega x^r} = e^{-\omega|x|^r \sin(r\pi)} \cdot e^{\omega|x|^r \cos(r\pi)i}.$$

Table 3: Absolute errors in 8-point approximations by $Q_8^{C\text{-}C}[f]$ and $Q_8^F[f]$ to the integral $I[f] = \int_{-1}^{1} e^x e^{i\omega x^{15/7}} dx$.

ω	100	200	300	400	500
$\lvert I[f] - Q_8^F[f]\rvert$	1.13×10^{-9}	6.33×10^{-10}	4.57×10^{-10}	3.65×10^{-10}	3.05×10^{-10}
$\lvert I[f] - Q_8^{C\text{-}C}[f]\rvert$	2.23×10^{-9}	1.19×10^{-9}	8.21×10^{-10}	6.30×10^{-10}	5.13×10^{-10}

Table 4: Absolute errors in n-point approximations by $Q_n^{C\text{-}C}[f]$ and $Q_n^F[f]$ to the integral $I[f] = \int_{-1}^{1} e^x e^{i\omega x^{15/7}} dx$ with fixed frequency $\omega = 600$.

n	4	8	12	16	24
$\lvert I[f] - Q_n^F[f]\rvert$	7.47×10^{-6}	2.60×10^{-10}	1.32×10^{-15}	1.84×10^{-21}	2.26×10^{-34}
$\lvert I[f] - Q_n^{C\text{-}C}[f]\rvert$	1.07×10^{-5}	4.33×10^{-10}	2.33×10^{-15}	3.35×10^{-21}	4.26×10^{-34}

For $2k-1 < r < 2k$ $(k = 0, \pm 1, \pm 2, \ldots)$, $|\int_{-1}^{1} e^{i\omega x^r} dx|$ is drastically increased as ω tends to infinity. For example, let us consider $\int_{-1}^{1} e^{i\omega x^{1.01}} dx$ and $\int_{-1}^{1} e^{i\omega x^{\pi/2}} dx$:

ω	50	500	5000
$\int_{-1}^{1} e^{i\omega x^{1.01}} dx$	$-0.0296 + 0.0730i$	$-3.514 \times 10^3 - 1.263 \times 10^3 i$	$2.081 \times 10^{64} - 2.423 \times 10^{64} i$
$\int_{-1}^{1} e^{i\omega x^{\pi/2}} dx$	$-3.674 \times 10^{18} - 1.905 \times 10^{19} i$	$-7.959 \times 10^{208} - 9.089 \times 10^{207} i$	

Moreover, in this case, the former moment formula $I[x^m]$ is not valid since for the incomplete gamma function $\Gamma(z, \alpha)$ and the extended exponential integral $\mathrm{Ei}(z, \alpha)$, $\Re(z)$ should be nonnegative [2, 15, 16]. However, $\Re(-i\omega(-1)^r) = \sin(r\pi) < 0$. So in this subsection, we confine us to the case $2k \le r \le 2k+1$ $(k = 0, \pm 1, \pm 2, \ldots)$. Under this assumption, we see that

$$|e^{i\omega x^r}| = \begin{cases} e^{-\omega |x|^r \sin(r\pi)} (\le 1) & \text{if } -1 \le x < 0, \\ 1 & \text{if } 0 \le x \le 1. \end{cases}$$

From the results in [24, 25], we see that for any fixed ω, $Q_n^{C\text{-}C}[f]$ and $Q_n^F[f]$ are convergent and can be rewritten as

$$Q_n^{C\text{-}C}[f] = \sum_{m=0}^{n} a_m I[x^m], \qquad Q_n^F[f] = \sum_{m=0}^{n} \widetilde{a}_m I[x^m], \tag{3.14}$$

where the moments $I[x^m] = \int_{-1}^{1} x^m e^{i\omega x^r} dx$ can be computed explicitly by the gamma function $\Gamma(z)$, the incomplete gamma function $\Gamma(\alpha, z)$ and the extended exponential integral $\mathrm{Ei}(a, z)$ ([2], pp. 228, pp. 260)

$$\begin{aligned} I[x^m] \;\; = \;\; & \frac{1}{r(-i\omega)^{(m+1)/r}} \left[\Gamma\left(\frac{m+1}{r}\right) - \Gamma\left(\frac{m+1}{r}, -i\omega\right) \right] + \frac{(-1)^m}{r(\omega \sin(r\pi) - i\omega \cos(r\pi))^{(m+1)/r}} \cdot \\ & \left[\Gamma\left(\frac{m+1}{r}\right) - \Gamma\left(\frac{m+1}{r}, \omega \sin(r\pi) - i\omega \cos(r\pi)\right) \right] (r > 0) \end{aligned}$$

(see [16, 32]);

$$I[x^m] = \frac{1}{|r|} \mathrm{Ei}\left(\frac{m+|r|+1}{|r|}, -i\omega\right) + (-1)^m \frac{1}{|r|} \mathrm{Ei}\left(\frac{m+|r|+1}{|r|}, \omega \sin(r\pi) - i\omega \cos(r\pi)\right) (r < 0)$$

(see [15]).

In the case $r > 0$: From Lemma 3.2 for $x \in [0, 1]$ it follows that

$$|\int_0^x e^{i\omega t^r} dt| \le W_1(r, \omega).$$

For $x \in [-1, 0)$, we can apply the transformation $u = t^r$ to give the following

$$\begin{aligned}
|\int_0^x e^{i\omega t^r} dt| &= |\textstyle\int_0^{-x} e^{i(-1)^r \omega t^r} dt| \\
&\le \int_0^{\omega^{-1/r}} dt + |\int_{\omega^{-1/r}}^{-x} e^{i(-1)^r \omega t^r} dt| \\
&\le \omega^{-1/r} + \frac{1}{r\omega}|\int_{\omega^{-1}}^{|x|^r} u^{(1-r)/r} de^{i(-1)^r \omega u}| \\
&\le \omega^{-1/r} + \frac{1}{r\omega}(|x|^{1-r} + \omega^{1-1/r} + |\int_{\omega^{-1}}^{|x|^r} [u^{(1-r)/r}]' e^{i(-1)^r \omega u} du|) \\
&\le \omega^{-1/r} + \frac{1}{r\omega}(|x|^{1-r} + \omega^{1-1/r} + |\int_{\omega^{-1}}^{|x|^r} |[u^{(1-r)/r}]'| du) \\
&= \begin{cases} \omega^{-\frac{1}{r}} + \dfrac{2|x|^{1-r}}{r\omega} \le \dfrac{r+2}{r\omega} & \text{if } 0 < r \le 1 \\ \omega^{-\frac{1}{r}} + \dfrac{2\omega^{1-\frac{1}{r}}}{r\omega} = \dfrac{r+2}{r\omega^{\frac{1}{r}}} & \text{if } 1 < r \text{ with } 2k \le r \le 2k+1,\ k = 0, 1, \ldots, \end{cases} \\
&= W_1(r, \omega),
\end{aligned}$$

where the first inequality uses

$$|e^{i\omega x^r}| = e^{-|x|^r \omega \sin(r\pi)} \cdot |e^{i|x|^r \omega \cos(r\pi)}| \le 1.$$

In the case $r < 0$: similarly to the proof of Lemma 3.2 there follows

$$|\int_0^x e^{i\omega t^r} dt| \le W_1(r, \omega). \tag{3.15}$$

Therefore, all the estimates in last subsection are still satisfied.

Table 5: Absolute errors in 8-point approximations by $Q_8^{C\text{-}C}[f]$ and $Q_8^F[f]$ to the integral $I[f] = \int_{-1}^1 e^x e^{i\omega x^{15/7}} dx$.

ω	100	200	300	400	500
$\lvert I[f] - Q_8^F[f]\rvert$	2.34×10^{-10}	1.51×10^{-10}	1.13×10^{-10}	8.80×10^{-11}	6.97×10^{-11}
$\lvert I[f] - Q_8^{C\text{-}C}[f]\rvert$	4.55×10^{-10}	2.48×10^{-10}	1.71×10^{-10}	1.32×10^{-10}	1.08×10^{-10}

Table 6: Absolute errors in n-point approximations by $Q_n^{C\text{-}C}[f]$ and $Q_n^F[f]$ to the integral $I[f] = \int_{-1}^1 e^x e^{i\omega x^{15/7}} dx$ with fixed frequency $\omega = 600$.

n	4	8	12	16	24
$\lvert I[f] - Q_n^F[f]\rvert$	1.64×10^{-6}	5.52×10^{-11}	2.78×10^{-16}	3.85×10^{-22}	1.22×10^{-31}
$\lvert I[f] - Q_n^{C\text{-}C}[f]\rvert$	2.27×10^{-6}	9.10×10^{-11}	4.84×10^{-16}	6.85×10^{-22}	1.22×10^{-31}

Comparing with Table 3 and Table 4, we see that both quadrature formulae are efficient whenever x^r is a real or complex function under the given conditions.

Table 7: Approximation values in n-point Clenshaw-Curtis quadrature to $\int_{-1}^{1} \cos(x) e^{i\omega x^{\sqrt{5}}} dx$. Here we choose $\omega = 5000$.

n	Approximation value
4	$0.33240823207096 \times 10^{-1} + 0.19791422176985 \times 10^{-1}i$
8	$0.33240823400955 \times 10^{-1} + 0.19791417029304 \times 10^{-1}i$
16	$0.33240823400959 \times 10^{-1} + 0.19791417029175 \times 10^{-1}i$
24	$0.33240823400959 \times 10^{-1} + 0.19791417029175 \times 10^{-1}i$

4 Final remark

From the results in this paper, we see that polynomial interpolation with Chebyshev points of the first and second kind should perhaps be regarded as equally valuable and fundamental as each other. For most integrands, Gauss, Clenshaw-Curtis and Fejér's first quadrature are of approximately equal accuracy for $I[f] = \int_{-1}^{1} f(x)dx$, and if the integrand is entire then Gauss is twice as accurate [30]. For integration of $I[f] = \int_{-1}^{1} f(x) e^{i\omega x^r} dx$, Clenshaw-Curtis and Fejér's first quadrature extended by Kussmaul [17] and Sloan [24] are efficient, which avoid using derivative of f [24, 25], and the accuracy increases as the frequency increases.

The computation of the Chebyshev moments of the form $\int_{-1}^{1} T_j(x) e^{i\omega x} dx$, studied in [10, 20, 22], shares the great advantage of Clenshaw-Curtis and Fejér's first quadrature. In the future we will study the efficient computation of the Chebyshev moments $\int_{-1}^{1} T_j(x) e^{i\omega x^r} dx$.

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