A Class of Quadratic Programs with Linear Complementarity Constraints

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Abstract. We consider a class of quadratic programs with linear complementarity constraints (QPLCC) which belong to mathematical programs with equilibrium constraints (MPEC). We investigate various stationary conditions and present new and strong necessary and sufficient conditions for global and local optimality. Furthermore, we propose a Newton-like method to find an M-stationary point in finite steps without MEPC linear independence constraint qualification.

Key words: Nonsmooth optimization, Newton-like method, stationary points, mathematical programs with equilibrium constraints.

1 Introduction

In this paper, we consider the following quadratic programs with linear complementarity constraints (QPLCC):

$$\begin{array}{ll} \min & \frac{1}{2}(y-y_d)^T H(y-y_d) + \frac{\alpha}{2}(u-u_d)^T M(u-u_d) \\ \text{s.t.} & Nu - Ay \geq 0, Nu - Ay - Dy \geq 0, \\ & (Nu-Ay)^T(Nu-Ay-Dy) = 0, \\ & Bu \leq b, \end{array} \tag{1.1}$$

where $y_d \in R^n, u_d, \in R^m, b \in R^l, H \in R^{n\times n}, M \in R^{m\times m}, A \in R^{n\times n}, B \in R^{l\times m}, D \in R^{n\times n}, N \in R^{n\times m}$ are given. We assume that H, M, A, D are symmetric positive definite, B has full row rank and D is diagonal. It is easy to verify that (1.1) is equivalent to the following quadratic programs with nonsmooth constraints

$$\begin{array}{ll} \min & \frac{1}{2}(y-y_d)^T H(y-y_d) + \frac{\alpha}{2}(u-u_d)^T M(u-u_d) \\ \text{s.t.} & Ay + D\max(0,y) = Nu, \\ & Bu \leq b, \end{array} \tag{1.2}$$

where $\max(0, y)$ denotes the vector in R^n with the ith component equal to $\max(0, y_i)$. Nonsmooth equations in the constraints (1.2) can be found in a finite difference approximation or a finite element approximation of equilibrium analysis of confined magnetohydrodynamics (MHD) plasmas [2, 3, 4, 6, 22], thin stretched membranes partially covered with water [16], reaction-diffusion problems [1], structural oscillation and pounding [5]. Moreover, the

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QPLCC (1.1) has many applications in data estimation in engineering and includes some inverse linear complementarity problems [10, 24] as special cases.

The QPLCC (1.1) belongs to the class of mathematical programs with complementarity constraints which are also called mathematical programs with equilibrium constraints (MPEC) (see the recent monograph on this subject [18, 19]):

$$\begin{array}{lll}(\text{MPEC}) & \min & f(z) \\ & \text{s.t.} & F(z) \geq 0, G(z) \geq 0, \\ & & F(z)^T G(z) = 0, \\ & & g(z) \leq 0,\end{array} \tag{1.3}$$

where $f : R^m \to R, F, G : R^m \to R^n, g : R^m \to R^l$. Although (1.3) looks like a nonlinear programming problem with equality and inequality constraints, it is well-known that the usual nonlinear programming constraint qualification such as Mangasarian-Fromovitz qualification (MFCQ) does not hold (see [28, Proposition]). The following alternatives to the classical Krush-Kuhn-Tucker (KKT) condition have been suggested recently (see e.g. [23, 27]).

Definition 1.1 *A feasible point z^* of MPEC is called weakly stationary if there exists $\lambda = (\lambda^g, \lambda^F, \lambda^G) \in R^{l+2n}$ such that the following condition hold:*

$$0 = \nabla f(z^*) + \sum_{i \in I_g} \lambda_i^g \nabla g_i(z^*) - \sum_{i=1}^{n} [\lambda_i^F \nabla F_i(z^*) + \lambda_i^G \nabla G_i(z^*)] \tag{1.4}$$

$$\lambda_{I_g}^g \geq 0, \lambda_i^F = 0 \text{ for } i \text{ s.t. } F_i(z^*) > 0, \lambda_i^G = 0 \text{ for } i \text{ s.t. } G_i(z^*) > 0 \tag{1.5}$$

where $I_g = I_g(z^*) = \{i : g_i(z^*) = 0\}$. A feasible point z^* of MPEC is called C-stationary, M-stationary, S-stationary respectively if it is weakly stationary and for all i s.t. $F_i(z^*) = G_i(z^*) = 0$ one has

$$\begin{array}{l}\lambda_i^F \lambda_i^G \geq 0; \\ \text{either } \lambda_i^F > 0, \lambda_i^G > 0 \text{ or } \lambda_i^F \lambda_i^G = 0; \\ \lambda_i^F \geq 0, \lambda_i^G \geq 0\end{array}$$

respectively. It is easy to see from the above definition that

S-stationary condition $\Rightarrow$ M-stationary condition $\Rightarrow$ C-stationary condition.

The S-stationary condition is known to be equivalent to the classical KKT condition for MPEC and hence is unlikely to hold at an optimal solution unless certain strong constraint qualification such as MPEC linear independence constraint qualification (MPEC LICQ) holds. The M-stationary condition which is based on the limiting (Mordukhovich) subdifferentials ([13, 14]), on the other hand, is much more likely to hold at a local optimal solution. In particular since all constraint functions in (1.1) are linear, an M-stationary condition always holds at any local optimal solution of (1.1) *without any constraint qualifications.* Furthermore from the results in section 2 (see Lemma 2.2 for an equivalent formulation of an M-stationary condition of (1.1)), one can see that the M-stationary condition is much sharper than the C-stationary condition for (1.1).

Due to the nonconvexity of the feasible region, a necessary condition for a general MPEC problem is normally not sufficient. There are almost no sufficient optimality conditions existing in the literature of MPEC (with exception of [27, Theorem 2.3]). However by using the special structure of (1.2) and technique in [2], we can address the issue regarding the solution y of the equation $Ay + D\max(0, y) = Nu$ as an implicit function of u and give a condition under which such a solution function $y(u)$ is differentiable (see Definition 3.2). Under this condition, we can then show that (y, u) is a local optimal solution of (1.1), if and only if there exist $s \in R^n$ and $t \in R^l$ such that

$$\begin{aligned} 0 &= H(y - y_d) + As + DE(y)s, \\ 0 &= \alpha M(u - u_d) - N^T s + B^T t, \\ 0 &= Ay + D\max(0, y) - Nu, \\ 0 &= \min(t, b - Bu) \end{aligned} \tag{1.6}$$

where $E(y)$ is a diagonal matrix whose diagonal elements are

$$E_{ii}(y) = \begin{cases} 1 & \text{if } y_i > 0 \\ 0 & \text{if } y_i < 0 \\ 0 \text{ or } 1 & \text{if } y_i = 0. \end{cases}$$

The necessary and sufficient condition for local optimality (1.6) only holds under the condition which ensures the solution function $y(u)$ is differentiable at the solution. It is worth noting that the differentiability of $\max(0, y)$ implies the differentiability of the solution function y, but the converse is not true. Let $\partial\max(0, y)$ denote the generalized Jacobian of the mapping $\max(0, y)$ ([7]). For each i, the function $\max(0, y_i)$ is a nonsmooth convex function and hence the Clarke generalized gradient coincides with the subgradient in the sense of convex analysis. It is then easy to see that if $\Lambda(y) \in \partial\max(0, y)$, then $\Lambda(y)$ is a diagonal matrix whose diagonal elements are

$$\Lambda_{ii}(y) = \begin{cases} 1 & \text{if } y_i > 0 \\ 0 & \text{if } y_i < 0 \\ a_i \in [0, 1] & \text{if } y_i = 0. \end{cases}$$

Applying the generalized Lagrange multiplier rule of Clarke ([7]) to the formulation (1.2), if (y, u) is a local optimal solution and a suitable constraint qualification holds then there exist $s \in R^n$, $t \in R^l$ and $\Lambda(y) \in \partial\max(0, y)$ such that

$$\begin{aligned} 0 &= H(y - y_d) + As + D\Lambda(y)s, \\ 0 &= \alpha M(u - u_d) - N^T s + B^T t, \\ 0 &= Ay + D\max(0, y) - Nu, \\ 0 &= \min(t, b - Bu). \end{aligned} \tag{1.7}$$

The above first order necessary optimality condition can be easily verified as a C-stationary condition for the formulation (1.1). Note that (1.7) becomes a sufficient condition for local optimality if the mapping $\max(0, y)$ is differentiable at y.

In this paper we present some new and strong necessary and sufficient conditions for global and local optimality for (1.1) (equivalently for (1.2)) without assumptions on differentiability of either the mapping $\max(0, y)$ or the solution function $y(u)$. Although the resulting optimality condition is not a semismooth equation, we can still propose a Newton-like method for finding certain M-stationary points. Furthermore, we show that this method can be used to find an M-stationary point in finite number of steps without MEPC-LICQ.

It is very interesting to observe that although the problems (1.1) and (1.2) are completely equivalent, using the MPEC formulation as in (1.1) to treat the mathematical program with nonsmooth equation constraints (1.2) can result in much sharper necessary optimality conditions (M-stationary condition or S-stationary condition instead of C-stationary condition) for problem (1.2). Conversely using the nonsmooth equation formulation (1.2), one can derive sufficient optimality conditions that would not otherwise be obtained by using the MPEC formulation (1.1). This technique can be applied to other problems where the constraint functions include a pointwise maximum of two functions.

The following notations are used in this paper. For a given $v \in R^n$, we define the index sets

$$\mathcal{J}(v) := \{\, i \,|\, v_i > 0\}, \qquad \mathcal{K}(v) := \{\, i \,|\, v_i = 0\}, \qquad \mathcal{L}(v) := \{\, i \,|\, v_i < 0\}.$$

For any matrix $G \in R^{m\times n}$, index sets $\mathcal{M} \subseteq R^m$ and $\mathcal{N} \subseteq R^n$, let $G_{\mathcal{M}}$ be the submatrix of G whose entries lie in the rows of G indexed by $\mathcal{M}$ and $G_{\mathcal{M}\mathcal{N}}$ be the submatrix of G whose entries lie in the rows and columns of G indexed by $\mathcal{M}$ and $\mathcal{N}$, respectively.

2 M-stationary points

Since all functions are linear in (1.1), by [27, Theorem 2.2] if (y, u) is a local optimal solution, then it must be an M-stationary point for the MPEC (1.1). In this section, we give sufficient conditions for an M-stationary point to become a global or a local optimal solution.

Lemma 2.1 *(y, u) is an M-stationary point for (1.1) if and only if it together with some $s', w \in R^n, t \in R^l$ satisfies*

$$\begin{array}{l} 0 = H(y - y_d) + Aw + (A + D)s', \\ 0 = \alpha M(u - u_d) - N^T w - N^T s' + B^T t, \\ 0 = Ay + D\max(0, y) - Nu, \\ 0 = \min(t, b - Bu), \\ w_i = 0, \quad \textit{if } i \in \mathcal{J}(y), \\ s'_i = 0, \quad \textit{if } i \in \mathcal{L}(y) \end{array} \tag{2.1}$$

and

$$\textit{either } \min(w_i, s'_i) > 0 \textit{ or } w_i s'_i = 0, \quad \textit{if } i \in \mathcal{K}(y). \tag{2.2}$$

We call (s', w, t) an M-multiplier.

Proof: By the definition of an M-stationary point for MEPC (see Defintion 1.1), (y, u) is an M-stationary point for (1.1) if and only if it together with some $s', w \in R^n, t \in R^l$ satisfies

$$\begin{array}{l} 0 = H(y - y_d) + Aw + (A + D)s', \\ 0 = \alpha M(u - u_d) - N^T w - N^T s' + B^T t, \\ 0 = \min(t, b - Bu), \end{array}$$

$$
\begin{aligned}
&\text{if } (Nu - Ay)_i > 0, \text{ then } w_i = 0,\\
&\text{if } (Nu - Ay - Dy)_i > 0, \text{ then } s'_i = 0,\\
&\text{if } (Nu - Ay - Dy)_i = (Nu - Ay)_i = 0, \text{ then either } \min(w_i, s'_i) > 0 \text{ or } w_i s'_i = 0.
\end{aligned}
$$

Moreover, since (y, u) is a feasible solution, we have

$$
\begin{aligned}
&(Nu - Ay)_i > 0 \quad \Longleftrightarrow y_i > 0,\\
&(Nu - Ay - Dy)_i > 0 \quad \Longleftrightarrow y_i < 0,\\
&(Nu - Ay)_i = (Nu - Ay - Dy)_i = 0 \quad \Longleftrightarrow y_i = 0.
\end{aligned}
$$

Consequently, there exist $s', w \in R^n, t \in R^l$ such that (2.1)-(2.2) hold. ∎

In the rest part of this paper, we say (y, u) is an M-stationary point for (1.1) if it together with some $s', w \in R^n, t \in R^l$ satisfies (2.1)-(2.2).

For a locally Lipschitzian function $G : R^n \to R^n$, Qi [20] studied the following generalized Jacobian of G,

$$
\partial_B G(y) = \left\{ \lim_{\substack{y^k \to y\\ y^k \in D_G}} \nabla G(y^k) \right\},
$$

where D_G is the set where G is differentiable. By the definition, $\partial_B G(y)$ is contained in the Clarke generalized Jacobian $\partial G(y)$ (see Clarke [7]). In particular, we have

$$
\partial G(y) = \mathrm{conv}\partial_B G(y),
$$

where $\mathrm{conv}C$ denotes the convex hull of the set C.

Let $a \in R^n$ be a fixed vector with $a_i \in \{0, 1\}$, and let $E(y)$ be an $n \times n$ diagonal matrix whose diagonal elements satisfy

$$
E_{ii}(y) = \begin{cases} 1 & \text{if } i \in \mathcal{J}(y)\\ a_i & \text{if } i \in \mathcal{K}(y)\\ 0 & \text{if } i \in \mathcal{L}(y). \end{cases}
$$

It is easy to see that the matrix $E(y)$ is a specific element in $\partial_B \max(0, y)$. In order to design a Newton-like method for finding an M-stationary point in the following result we reformulate the M-stationary condition as a system of equations. The equivalent formulation is also useful for proving the sufficient optimality condition.

Lemma 2.2 *(y, u) is an M-stationary point for (1.1) if and only if there exists $(s, t) \in R^n \times R^l$ such that*

$$
\begin{pmatrix} H(y - y_d) + As + D(E(y) + C(y, s))s\\ \alpha M(u - u_d) - N^T s + B^T t\\ Ay + D\max(0, y) - Nu\\ \min(t, b - Bu) \end{pmatrix} = 0 \tag{2.3}
$$

where $C(y, s) = diag(\mu_1, \mu_2, \ldots, \mu_n)$ and

$$
\mu_i \in \begin{cases} \{0\} & \text{if } i \in \mathcal{J}(y) \cup \mathcal{L}(y) \quad \text{or} \quad i \in \mathcal{K}(y) \cap \mathcal{K}(s)\\ [-a_i, 1 - a_i] & \text{if } i \in \mathcal{K}(y) \cap \mathcal{J}(s)\\ \{-a_i, 1 - a_i\}, & \text{if } i \in \mathcal{K}(y) \cap \mathcal{L}(s). \end{cases}
$$

Proof: Suppose that (y,u) is an M-stationary point of (1.1) together with some $(s',w,t) \in R^n \times R^n \times R^l$. Let

$$s = s' + w.$$

To show the first equation in (2.3), we only need to verify that

$$s' = (E(y) + C(y,s))s. \tag{2.4}$$

For $i \in \mathcal{J}(y) \cup \mathcal{L}(y)$, $E_{ii}(y)s_i = s'_i$ and (2.4) holds with $\mu_i = 0$.

Now we consider $i \in \mathcal{K}(y)$.

From the conditions of w_i and s'_i in the definition of an M-stationary point, $s_i = s'_i + w_i = 0$ implies $s'_i = w_i = 0$. Hence (2.4) holds with $\mu_i = 0$ for $i \in \mathcal{K}(y) \cap \mathcal{K}(s)$.

For $s_i \neq 0$, we let

$$\mu_i = \frac{(1-a_i)s'_i - a_i w_i}{s'_i + w_i}.$$

If $i \in \mathcal{K}(y) \cap \mathcal{J}(s)$, by the definition of an M-stationary point, $s_i = s'_i + w_i > 0$ implies $s'_i \geq \min(s'_i, w_i) \geq 0$. Hence we get $s'_i = (a_i + \mu_i)s_i$ with $\mu_i \in [-a_i, 1-a_i]$.

If $i \in \mathcal{K}(y) \cap \mathcal{L}(s)$, using the definition of an M-stationary point again, from $s'_i + w_i < 0$, we obtain $\max(s'_i, w_i) = 0$. Hence we get $s'_i = (a_i + \mu_i)s_i$ with $\mu_i \in \{-a_i, 1-a_i\}$.

Consequently, the first equation in (2.3) holds. The other three equations in (2.3) follow from the definition of an M-stationary point.

Conversely suppose that (y,u) along with $(s,t) \in R^n \times R^l$ satisfies (2.3). It is easy to see that

$$0 \leq E_{ii}(y) + \mu_i \leq 1, \qquad i = 1, \dots, n.$$

Rewrite the first two equations in (2.3) as the following:

$$\begin{aligned}
0 &= H(y - y_d) + A(I - E(y) - C(y,s))s + (A + D)(E(y) + C(y,s))s, \\
0 &= \alpha M(u - u_d) - N^T(I - E(y) - C(y,s))s - N^T(E(y) + C(y,s))s + B^T t.
\end{aligned}$$

Let $w = (I - E(y) - C(y,s))s$ and $s' = (E(y) + C(y,s))s$. Then we have

$$w_i = 0, \ i \in \mathcal{J}(y), \qquad s'_i = 0, \ i \in \mathcal{L}(y)$$

and

$$w_i s'_i = (1 - E_{ii}(y) - \mu_i)(E_{ii}(y) + \mu_i)s_i^2 \geq 0, \quad i = 1, 2, \dots, n.$$

Moreover

$$\begin{aligned}
&w_i = s'_i = 0, && \forall i \in \mathcal{K}(y) \cap \mathcal{K}(s), \\
&w_i, s'_i \in [0, \infty) && \forall i \in \mathcal{K}(y) \cap \mathcal{J}(s), \\
&w_i s'_i = 0 && \forall i \in \mathcal{K}(y) \cap \mathcal{L}(s).
\end{aligned}$$

That is, for any $i \in \mathcal{K}(y)$, either $\min(w_i, s'_i) > 0$ or $w_i s'_i = 0$. Hence (y,u) is an M-stationary point. ∎

In the following result, we provide conditions under which an M-stationary condition is sufficient for optimality.

Theorem 2.1 *Let (y^*, u^*) be an M-stationary point of (1.1). Then there exists (s^*, t^*) such that (2.3) holds. Moreover, the following statements hold.*

1. *(y^*, u^*) is the unique global optimal solution of (1.1), if $\mathcal{L}(s^*) = \emptyset$.*

2. *(y^*, u^*) is a local optimal solution of (1.1), if one of the following conditions holds.*

(i) $\mathcal{L}(y^*) \cap \mathcal{K}(s^*) = \emptyset$,

(ii) $((A + DE(y^*))^{-1}N)_{\mathcal{K}(y^*)} = 0$,

Proof: By Lemma 2.2, there exists (s^*, t^*) such that (2.3) holds. By the Taylor expansion for the quadratic objective function

$$f(y, z) = \frac{1}{2}(y - y_d)^T H(y - y_d) + \frac{\alpha}{2}(u - u_d)^T M(u - u_d),$$

we obtain that for any (y, u) satisfying the constraints in (1.2) and $(y, u) \neq (y^*, u^*)$,

$$\begin{aligned}
& f(y, u) - f(y^*, u^*) \\
= \; & (y - y^*)^T H(y^* - y_d) + \alpha(u - u^*)^T M(u^* - u_d) \\
& \quad + \frac{1}{2}(y - y^*)^T H(y - y^*) + \frac{\alpha}{2}(u - u^*)^T M(u - u^*) \\
> \; & (y - y^*)^T H(y^* - y_d) + \alpha(u - u^*)^T M(u^* - u_d) \\
\geq \; & (y - y^*)^T H(y^* - y_d) + \alpha(u - u^*)^T M(u^* - u_d) + (Bu - b)^T t^* \\
= \; & (y - y^*)^T H(y^* - y_d) + (u - u^*)^T (N^T s^* - B^T t^*) + (Bu - Bu^* + Bu^* - b)^T t^* \\
= \; & -(y - y^*)^T (A + D(E(y^*) + C(y^*, s^*)))s^* + (u - u^*)^T N^T s^* \\
= \; & -\left[(A + D(E(y^*) + C(y^*, s^*))(y - y^*) + N(u - u^*)\right]^T s^* \\
= \; & [D(E(y) - E(y^*))y - DC(y^*, s^*)(y - y^*)]^T s^*,
\end{aligned}$$

where the first strict inequality follows from the positive definiteness of matrices H and M, the second inequality uses $Bu \leq b$ and $t^* \geq 0$, the second equality uses (2.3), the third equality uses $(Bu^* - b)^T t^* = 0$ and (2.3), and the fifth equality uses $(A + DE(y))y = Nu$ and $(A + DE(y^*))y^* = Nu^*$. Now, we show the optimality by using the inequality

$$f(y, u) - f(y^*, u^*) \;\; > \;\; [D(E(y) - E(y^*))y - DC(y^*, s^*)(y - y^*)]^T s^*. \tag{2.5}$$

From the definition of $C(y^*, s^*)$, we have

$$\begin{aligned}
& [(D(E(y) - E(y^*))y)_i - \mu_i(D(y - y^*))_i] s_i^* \\
= & \begin{cases} D_{ii}(E_{ii}(y) - E_{ii}(y^*))y_i s_i^* & \text{if } i \in \mathcal{J}(y^*) \cup \mathcal{L}(y^*) \quad \text{or} \quad i \in \mathcal{K}(y^*) \cap \mathcal{K}(s^*) \\ D_{ii}(E_{ii}(y) - a_i - \mu_i)y_i s_i^* & \text{if } i \in \mathcal{K}(y^*) \cap (\mathcal{J}(s^*) \cup \mathcal{L}(s^*)). \end{cases}
\end{aligned}$$

By the definition of $E(y)$, we get

$$\begin{aligned}
(E(y) - E(y^*))_{ii} \geq 0 \quad & \text{if } y_i > 0, \\
(E(y) - E(y^*))_{ii} \leq 0 \quad & \text{if } y_i < 0
\end{aligned}$$

which implies that $(E(y) - E(y^*))_{ii} y_i \geq 0$. Hence we obtain

$$D(E(y) - E(y^*))y \geq 0. \tag{2.6}$$

Moreover since $\mu_i \in [-a_i, 1-a_i]$ we have

$$0 \le (E_{ii}(y) - a_i - \mu_i)y_i = \begin{cases} (1-a_i-\mu_i)y_i & \text{if } y_i > 0 \\ (-a_i-\mu_i)y_i & \text{if } y_i < 0 \\ -\mu_i y_i & \text{if } y_i = 0. \end{cases}$$

Therefore we have

$$(D(E(y) - E(y^*))y)_i - \mu_i(D(y-y^*))_i \ge 0, \qquad i = 1, 2, \ldots, n. \tag{2.7}$$

1. $\mathcal{L}(s^*) = \emptyset$ means that $s^* \ge 0$. From (2.5) and (2.7), (y^*, u^*) is the unique global solution.

2. (i) Since there is a neighborhood $\mathcal{N}$ of y^* such that $\mathcal{L}(y^*) \subseteq \mathcal{L}(y)$ and $\mathcal{J}(y^*) \subseteq \mathcal{J}(y)$ for all $y \in \mathcal{N}$, we have

$$(E(y) - E(y^*))_{\mathcal{L}(y^*)\cup\mathcal{J}(y^*)} = 0, \qquad \forall y \in \mathcal{N}. \tag{2.8}$$

Moreover from the definition of $C(y^*, s^*)$,

$$\mu_i = C_{ii}(y^*, s^*) = 0 \qquad \forall i \in \mathcal{L}(y^*) \cup \mathcal{J}(y^*). \tag{2.9}$$

The conclusion is obviously true for the case that $\mathcal{L}(s^*) = \emptyset$ or $\mathcal{K}(y^*) = \emptyset$. Suppose that $\mathcal{L}(s^*) \ne \emptyset$ and $\mathcal{K}(y^*) \ne \emptyset$. Then $\mathcal{L}(s^*) \cap \mathcal{K}(y^*) = \emptyset$, implies that $s_i^* \ge 0$, for all $i \in \mathcal{K}(y^*)$. Hence from (2.5), (2.8) and (2.9), we find that (y^*, u^*) is a local optimal solution.

(ii) For any feasible point (y, u) of (1.2), we have

$$N(u-u^*) = (A + DE(y^*))(y - y^*) + D(E(y) - E(y^*))y. \tag{2.10}$$

Let $\mathcal{K} = \mathcal{K}(y^*)$. From (2.10) and the assumption $((A + DE(y^*))^{-1}N)_{\mathcal{K}} = 0$, we find for any feasible point of (1.2) such that $y \in \mathcal{N}$,

$$\begin{aligned} 0 &= \Big((A + DE(y^*))^{-1}N(u-u^*)\Big)_{\mathcal{K}} \\ &= (y-y^*)_{\mathcal{K}} + ((A + DE(y^*))^{-1}D(E(y) - E(y^*))y)_{\mathcal{K}} \\ &= y_{\mathcal{K}} + (A + DE(y^*))^{-1}_{\mathcal{K}\mathcal{K}}(D(E(y) - E(y^*)))_{\mathcal{K}\mathcal{K}}y_{\mathcal{K}} \\ &= (A + DE(y^*))^{-1}_{\mathcal{K}\mathcal{K}}((A + DE(y^*))_{\mathcal{K}\mathcal{K}} + (D(E(y) - E(y^*)))_{\mathcal{K}\mathcal{K}})y_{\mathcal{K}} \\ &= (A + DE(y^*))^{-1}_{\mathcal{K}\mathcal{K}}(A + DE(y))_{\mathcal{K}\mathcal{K}}y_{\mathcal{K}}, \end{aligned}$$

where we used for $y \in \mathcal{N}$

$$(E(y) - E(y^*))_{ii} = 0, \qquad i \in \mathcal{J}(y^*) \cup \mathcal{L}(y^*). \tag{2.11}$$

Since $A + DE(y)$ is a positive definite matrix, we find

$$y_{\mathcal{K}} = 0.$$

From (2.11) and $\mu_i = 0$ for $i \in \mathcal{J}(y^*) \cup \mathcal{L}(y^*)$, we obtain

$$D(E(y) - E(y^*))y - DC(y^*, s^*)(y - y^*) = 0.$$

The desired result follows from (2.5). ∎

Corollary 2.1 *Let (y^*, u^*) be an M-stationary point of (1.1) with an M-multiplier (s^*, w^*, t^*).*

(i) If $\mathcal{L}(w^* + s^*) = \emptyset$, then (y^*, u^*) is the unique global optimal solution of (1.1).

(ii) If $\mathcal{K}(y^*) \cap \mathcal{L}(w^* + s^*) = \emptyset$, then (y^*, u^*) is a local optimal solution of (1.1).

Proof: From the proof of Lemma 2.2, (y^*, u^*, s', t^*) with $s' = w^* + s^*$ satisfies (2.3). The results follow from Theorem 2.1. ∎

3 Other stationary points and constraint qualifications

In this section, we study relationship between M-stationary points and other stationary points. Moreover, we give constraint qualifications under which the various stationary conditions hold and study the conditions under which these stationary conditions provide sufficient conditions for local or global optimality.

For convenience we first summarize the C-,M-and S-stationary condition for problem (1.1) in the following definition based on the derivation of M-stationary condition in Lemma 2.1.

Definition 3.1 *A feasible solution of (1.1) (y, u) is a weak stationary point if it together with some $s', w \in R^n, t \in R^l$ satisfies*

$$\begin{aligned}
&0 = H(y - y_d) + Aw + (A + D)s',\\
&0 = \alpha M(u - u_d) - N^T w - N^T s' + B^T t,\\
&0 = Ay + D\max(0, y) - Nu,\\
&0 = \min(t, b - Bu),\\
&w_i = 0, \quad \textit{if } i \in \mathcal{J}(y),\\
&s'_i = 0, \quad \textit{if } i \in \mathcal{L}(y).
\end{aligned} \tag{3.1}$$

A feasible solution of (1.1) (y, u) is a C-, M-, S-stationary point respectively if it is a weak stationary point and if $i \in \mathcal{K}(y)$ then

$$\begin{aligned}
&w_i s'_i \geq 0;\\
&\textit{either } \min(w_i, s'_i) > 0 \textit{ or } w_i s'_i = 0;\\
&w_i \geq 0, s'_i \geq 0
\end{aligned}$$

respectively. We call (s', w, t) a C-, M- and S-multiplier respectively.

Let $z = (y, u, s, t) \in R^n \times R^m \times R^n \times R^l$ and

$$F(z) := \begin{pmatrix} H(y - y_d) + As + DE(y)s \\ \alpha M(u - u_d) - N^T s + B^T t \\ Ay + D\max(0, y) - Nu \\ \min(t, b - Bu) \end{pmatrix}.$$

We say that (y, u) is a KKT point of (1.1) if it together with some $(s, t) \in R^n \times R^l$ satisfies

$$F(z) = 0. \tag{3.2}$$

Note that the above concept of a KKT point differs from the one given in [2] in that if $y_i = 0$, then $E_{ii}(y) = 0$ or 1 instead of $E_{ii}(y) = 0$.

Choosing $C(y, u) = 0$ in Lemma 2.2, we find that a KKT point is an M-stationary point.

Proposition 3.1 *If (y, u) is a KKT point of (1.1), then (y, u) is an M-stationary point with an M-multiplier (s', w, t) such that $w_i s'_i = 0$ if $i \in \mathcal{K}(y)$.*

Since $Ay + D\max(0, y)$ is strongly monotone, for any u there is a unique solution y satisfying the constraint

$$Ay + D\max(0, y) - Nu = 0.$$

Moreover $E(y)y = \max(0, y)$. Hence we may define the solution function of the equation constraint

$$y(u) = (A + DE(y(u)))^{-1}Nu.$$

Definition 3.2 *We say that the nonsmooth equation constraint qualification (NECQ) holds at (y, u) if either $\mathcal{K}(y(u)) = \emptyset$ or*

$$((A + DE(y(u)))^{-1}N)_{\mathcal{K}(y(u))} = 0. \tag{3.3}$$

By Theorem 2.1 in [2], the NECQ is equivalent to the differentiability of the solution function $y(\cdot)$ at u.

Theorem 3.1 *Suppose that (y^*, u^*) satisfies NECQ. Then the following statements are equivalent.*

1. *(y^*, u^*) is a local optimal solution of (1.1).*

2. *(y^*, u^*) is an M-stationary point.*

3. *(y^*, u^*) is a KKT point of (1.1).* [3]

Proof: $1 \Rightarrow 2$ follows Lemma 2.1.

$2 \Rightarrow 1$ follows Theorem 2.1.

$3 \Rightarrow 2$ follows Proposition 3.1

Now we show $1 \Rightarrow 3$. Assume that (y^*, u^*) is a local optimal solution of (1.1). From the proof of Theorem 2.1, we have $E(y) = E(y^*)$ for all feasible points of (1.1) in a neighborhood of (y^*, u^*). Hence, we can write these feasible points as

$$(y(u), u) = ((A + DE(y^*))^{-1}Nu, u).$$

Moreover, in the neighborhood, the nonsmooth program

$$\begin{array}{ll} \min & \frac{1}{2}(y(u) - y_d)^T H(y(u) - y_d) + \frac{\alpha}{2}(u - u_d)^T M(u - u_d) \\ \text{s.t} & Bu \le b \end{array} \tag{3.4}$$

is convex and smooth and has u^* as a local optimal solution. Hence, the KKT condition of (3.4)

$$\begin{pmatrix} ((A + DE(y^*))^{-1}N)^T H(y(u^*) - y_d) + \alpha M(u^* - u_d) + B^T t^* \\ \min(t^*, b - Bu^*) \end{pmatrix} = 0$$

holds. Note that there is a unique solution y satisfying $Ay + DE(y)y = Nu^*$. We have $y^* = y(u^*)$. Let

$$s^* = -(A + DE(y^*))^{-1}H(y^* - y_d).$$

[3]In [2], the equivalent relation between statements 1 and 3 are proved with $a = 0$.

Then we obtain $F(y^*, u^*, s^*, t^*) = 0$ and hence (y^*, u^*) is a KKT point of (1.1). ∎

Let (y^*, u^*) be a local optimal solution of (1.1). Then for any index set $\nu \subseteq \mathcal{K}(y^*)$, it is easy to see that (y^*, u^*) is a local optimal solution of the subproblem:

$$
\begin{array}{lll}
(QPLCC)_\nu & \min & f(y,u) := \frac{1}{2}(y - y_d)^T H(y - y_d) + \frac{\alpha}{2}(u - u_d)^T M(u - u_d) \\
& \text{s.t.} & (Nu - Ay)_i = 0, \quad \forall i \in \mathcal{L}(y^*), \\
& & (Nu - Ay - Dy)_i = 0, \quad \forall i \in \mathcal{J}(y^*), \\
& & (Nu - Ay)_i \geq 0, \quad (Nu - Ay - Dy)_i = 0 \quad \forall i \in \nu, \\
& & (Nu - Ay)_i = 0, \quad (Nu - Ay - Dy)_i \geq 0 \quad \forall i \in \mathcal{K}(y^*) \backslash \nu, \\
& & Bu \leq b.
\end{array}
$$

Note that the above subproblem is a strictly convex quadratic problem with linear constraints and hence the optimal solution is unique and the KKT condition is necessary and sufficient for optimality.

Definition 3.3 *We say that (y^*, u^*) is a piecewise stationary point (P-stationary point) for (1.1) if the KKT condition for $(QPLCC)_\nu$ holds for each index set $\nu \subseteq \mathcal{K}(y^*)$. In another word, (y^*, u^*) is a P-stationary point for (1.1) if for each index set $\nu \subseteq \mathcal{K}(y^*)$ there exist some $s', w \in R^n, t \in R^l$ satisfies (2.1) and*

$$
w_i \geq 0 \quad i \in \nu, \qquad s'_i \geq 0 \quad i \in \mathcal{K}(y^*) \backslash \nu. \tag{3.5}
$$

(s', w, t) is called a P-multiplier.

Note that the concept of the P-stationarity is equivalent to the concept of the B-stationarity in the sense of Scheel and Scholtes [23] for MPEC (1.1). It is easy to see that a P-stationary point must be a weak stationary point and an S-stationary point must be a P-stationary point. But in general there are no relationships between a P-stationary point and C-,M-stationary points and KKT points.

We now provide a necessary and sufficient optimality condition for (1.1) in terms of P-stationary conditions.

Theorem 3.2 *If (y^*, u^*) is a local optimal solution of (1.1), then (y^*, u^*) is a P-stationary point. Conversely, a P-stationary point (y^*, u^*) is the unique minimizer of the objective function $f(y, u)$ over all $(y, u) \in \cup_{\nu \subseteq \mathcal{K}(y^*)} \mathcal{F}_\nu$ where $\mathcal{F}_\nu$ is the set of feasible solutions of the subproblem $(QPLCC)_\nu$. Moreover if $\mathcal{K}(y^*) = \{1, 2, \ldots, n\}$, then a P-stationary point is the unique global minimizer of (1.1).*

Proof: If (y^*, u^*) is a local optimal solution of (1.1), then from the discussion before the definition of a P-stationary point, for each index set $\nu \subseteq \mathcal{K}(y^*)$, (y^*, u^*) is the unique minimizer of $f(y, u)$ on $\mathcal{F}_\nu$ and hence a P-stationary point. Conversely assume that (y^*, u^*) is a P-stationary point. Then the KKT condition for minimizing $f(y, u)$ on $\mathcal{F}_\nu$ holds for each index set $\nu \subseteq \mathcal{K}(y^*)$ at (y^*, u^*). That is, (y^*, u^*) is the unique minimizer for minimizing $f(y, u)$ on $\mathcal{F}_\nu$ for each index set $\nu \subseteq \mathcal{K}(y^*)$ since the subproblem $(QPLCC)_\nu$ is a strictly convex quadratic program. Consequently (y^*, u^*) is the unique minimizer of $f(y, u)$ over all $(y, u) \in \cup_{\nu \subseteq \mathcal{K}(y^*)} \mathcal{F}_\nu$. In the case where $\mathcal{K}(y^*) = \{1, 2, \ldots, n\}$, $\cup_{\nu \subseteq \mathcal{K}(y^*)} \mathcal{F}_\nu$ is the feasible region of MPEC (1.1) and hence a P-stationary point is the unique global minimizer. ∎

The well-known MPEC LICQ for MPEC (1.1) has the following form.

Definition 3.4 *(MPEC LICQ) Let (y, u) be a feasible point of (1.1). Let*

$$\mathcal{I} = \mathcal{I}(u) = \{i \,|\, (Bu)_i = b_i\}$$

$\mathcal{J} = \mathcal{J}(y)$, $\mathcal{K} = \mathcal{K}(y)$, $\mathcal{L} = \mathcal{L}(y)$. We say that MPEC LICQ holds at (y, u) if the rows of the matrix

$$Q = \begin{pmatrix} B_{\mathcal{I}} & 0 \\ N_{\mathcal{J}\cup\mathcal{K}} & -(A+D)_{\mathcal{J}\cup\mathcal{K}} \\ N_{\mathcal{L}\cup\mathcal{K}} & -A_{\mathcal{L}\cup\mathcal{K}} \end{pmatrix}$$

are linearly independent.

Under the MPEC LICQ, each subproblem $(QPLCC)_\nu$ satisfies LICQ and hence each subproblem has a unique multiplier. By definition of a P-multiplier, the P-multiplier is unique and hence coincides with the S-multiplier. In fact for the P-multipliers to coincide with the S-multipliers, all we need are the uniqueness of the $\mathcal{K}(y)$ component of a multiplier (s, w). In general the partial MPEC LICQ ([26, 27]) is a weaker condition than the MPEC LICQ which guarantees the equivalence of a P-multiplier and an S-multiplier. Note that the partial MPEC LICQ for (1.1) has the following form.

$$\lambda_1^T Q_1 + \lambda_2^T Q_2 = 0 \quad \Longrightarrow \quad \lambda_2 = 0$$

where

$$Q_1 = \begin{pmatrix} B_{\mathcal{I}} & 0 \\ N_{\mathcal{J}} & -(A+D)_{\mathcal{J}} \\ N_{\mathcal{L}} & -A_{\mathcal{L}} \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} N_{\mathcal{K}} & -(A+D)_{\mathcal{K}} \\ N_{\mathcal{K}} & -A_{\mathcal{K}} \end{pmatrix}.$$

It is easy to prove that for (1.1), the partial MPEC LICQ coincides with MPEC LICQ if B has full row rank. Therefore the partial MPEC LICQ does not provide a weaker constraint qualification than the MPEC LICQ for the problem we study.

We now provide a necessary and sufficient optimality condition for (1.1) in terms of the S-stationary condition in the following theorem.

Theorem 3.3 *Let (y^*, u^*) be a local optimal solution of (1.1) and let the MPEC LICQ hold at (y^*, u^*), then (y^*, u^*) is an S-stationary point. Conversely let (y^*, u^*) be an S-stationary point, then (y^*, u^*) is a local optimal solution of (1.2). Moreover if either $\mathcal{K}(y^*) = \{1, 2, \ldots, n\}$ or there exists an S-multiplier (s^*, w^*, t^*) such that $\mathcal{L}(w^* + s^*) = \emptyset$ then (y^*, u^*) is the unique global optimal solution of (1.2).*

Proof: Since the MPEC LICQ at (y^*, u^*) implies that a P-stationary point is an S-stationary point, it follows from Theorem 3.2 that a local optimal solution of (1.1) is an S-stationary point if the MPEC LICQ holds. Conversely if (y^*, u^*) is an S-stationary point then it is a P-stationary point and an M-stationary point. By Theorem 3.2 and Corollary 2.1, it is a local optimal solution and moreover it is the unique global optimal solution if either $\mathcal{K}(y^*) = \{1, 2, \ldots, n\}$ or $\mathcal{L}(w^* + s^*) = \emptyset$. ∎

Definition 3.5 *(Strong MPEC LICQ) Let (y, u) be a feasible point of (1.1). We say the Strong MPEC LICQ holds at (y, u) if the rows of the matrix*

$$\tilde{A} := \begin{pmatrix} -(A+D)_{\mathcal{J}\cup\mathcal{K}} \\ -A_{\mathcal{L}\cup\mathcal{K}} \end{pmatrix}$$

are linearly independent.

Obviously the Strong MPEC LICQ is stronger than MPEC LICQ.

Lemma 3.1 *$\mathcal{K}(y) = \emptyset$ if and only if the Strong MPEC LICQ holds at (y, u).*

Proof: It is obvious that there is a permutation matrix P such that

$$P\tilde{A} = \begin{pmatrix} -A - \tilde{D} \\ -(A + D)_{\mathcal{K}} \end{pmatrix},$$

where $\tilde{D}$ is a diagonal matrix which satisfies $\tilde{D}_{\mathcal{L}\cup\mathcal{K}} = 0$, and $\tilde{D}_{\mathcal{J}} = D_{\mathcal{J}}$.

Since A is a symmetric positive definite matrix and all diagonal elements of $\tilde{D}$ are nonnegative, $A + \tilde{D}$ is a nonsingular matrix. Therefore $P\tilde{A}$ has full-row rank if and only if $\mathcal{K} = \emptyset$. We complete the proof.

∎

By definitions of various stationary points, under the strong MPEC LICQ, all concepts of stationary points including the C-stationary points, M-stationary points, S-stationary points, P-stationary points and KKT points coincide with weak stationary points. Hence the following theorem holds without further proof.

Theorem 3.4 *Let (y^*, u^*) be a feasible solution of (1.1) and $\mathcal{K}(y^*) = \emptyset$. Then (y^*, u^*) is a local optimal solution of (1.1) if and only if it is a weak stationary point.*

4 Semismooth Newton methods

In this section, we present a semismooth Newton method for (1.1) and show that this method can find an M-stationary point of (1.1) in one step from any initial point in a neighborhood of the solution. To simplify our discussion, we choose $a_i = 0$ in the definition of $E(y)$. The method can be easily extended to any $a_i \in \{0, 1\}$.

Since $E(y)$ is a discontinuous mapping and hence not semismooth, the semismooth Newton method can not be applied directly. For a fixed positive number ϵ, we consider

$$\phi_\epsilon(y_i) = \begin{cases} 1 & \text{if } y_i \geq 2\epsilon \\ (y_i - \epsilon)/\epsilon & \text{if } \epsilon < y_i < 2\epsilon \\ 0 & \textit{otherwise } y_i \leq \epsilon. \end{cases}$$

It is easy to find that ϕ_ϵ satisfies

$$\lim_{\epsilon \downarrow 0} \phi_\epsilon(y_i) =: \phi^\circ(y_i) = \begin{cases} 1 & \text{if } y_i > 0 \\ 0 & \text{otherwise } y_i \leq 0 \end{cases} \tag{4.1}$$

and

$$\phi^\circ(y_i) \in \partial \max(0, y_i).$$

Let $E_\epsilon(y)$ be an $n \times n$ diagonal matrix whose diagonal elements are

$$(E_\epsilon(y))_{ii} = \phi_\epsilon(y_i), \quad i = 1, 2, \ldots, n.$$

From (4.1), we have

$$\lim_{\epsilon \downarrow 0} E_\epsilon(y) = E(y). \tag{4.2}$$

Hence $E_\epsilon(y)$ is a continuous approximation of the discontinuous mapping $E(y)$. Replacing $E(y)$ by $E_\epsilon(y)$ in $F(z)$ gives

$$F_\epsilon(z) := \begin{pmatrix} H(y-y_d) + As + DE_\epsilon(y)s \\ \alpha M(u-u_d) - N^T s + B^T t \\ Ay + D\max(0,y) - Nu \\ \min(t, b - Bu). \end{pmatrix}.$$

Obviously, $F(z) = \lim_{\epsilon\downarrow 0} F_\epsilon(z)$. The function F_ϵ is a piecewise smooth function, and hence a semismooth function. We can apply a semismooth Newton method [6, 20, 21] to find a solution of $F_\epsilon(z) = 0$. Furthermore, the following lemma shows that for sufficiently small ϵ, the solution of $F_\epsilon(z) = 0$ defines an M-stationary point.

Lemma 4.1 *Let $z^* = (y^*, u^*, s^*, t^*)$ be a solution of $F_\epsilon(z) = 0$. Then (y^*, u^*) is an M-stationary point, if either $\mathcal{J}(y^*) = \emptyset$ or $2\epsilon \le \min\{y_i^*, i \in \mathcal{J}(y^*)\}$.*

Proof: Since z^* is a solution of $F_\epsilon(z) = 0$, we have

$$\begin{aligned} 0 &= H(y^* - y_d) + As^* + DE_\epsilon(y^*)s^*, \\ 0 &= \alpha M(u^* - u_d) - N^T s^* + B^T t^*. \end{aligned}$$

Rewrite the above system as the following equivalent system:

$$\begin{aligned} 0 &= H(y^* - y_d) + A(I - E_\epsilon(y^*))s^* + (A + D)E_\epsilon(y^*)s^*, \\ 0 &= \alpha M(u^* - u_d) - N^T(I - E_\epsilon(y^*))s^* - N^T E_\epsilon(y^*)s^* + B^T t^*. \end{aligned}$$

Let $w := (I - E_\epsilon(y^*))s^*$ and $s' := E_\epsilon(y^*)s^*$. Then we obtain the first two equations in the definition of an M-stationary point. Moreover, from $2\epsilon \le y_i^*, i \in \mathcal{J}(y^*)$, we have

$$\begin{aligned} y_i^* > 0 &\Longrightarrow (E_\epsilon)_{ii}(y^*) = 1 \Longrightarrow w_i = 0, s_i' = s_i^*, \\ y_i^* \le 0 &\Longrightarrow (E_\epsilon)_{ii}(y^*) = 0 \Longrightarrow w_i = s_i^*, s_i' = 0. \end{aligned}$$

By definition, (y^*, u^*) is an M-stationary point. ∎

We consider a semismooth Newton-like method

$$z^{k+1} = z^k - F_\epsilon^\circ(z^k)^{-1} F_\epsilon(z^k), \tag{4.3}$$

where

$$F_\epsilon^\circ(z) = \begin{pmatrix} H & 0 & A + DE_\epsilon(y) & 0 \\ 0 & \alpha M & -N^T & B^T \\ A + DE(y) & -N & 0 & 0 \\ 0 & (c(u,t) - I)B & 0 & c(u,t) \end{pmatrix}$$

and $c(u,t) \in R^{l\times l}$ is a diagonal matrix whose diagonal elements are

$$c_{ii}(u,t) = \begin{cases} 1 & \text{if } t_i < b_i - (Bu)_i \\ 0 & \text{otherwise} \end{cases} \qquad i = 1, 2, \ldots, l.$$

Now we show that Method (4.3) is well defined.

Definition 4.6 *[8] A matrix M is called a P-matrix if all its principal minors are positive.*

Lemma 4.2 *[15] M is a P-matrix if and only if $I-C+CM$ is nonsingular for any diagonal matrix C whose diagonal elements satisfy $0 \le C_{ii} \le 1$.*

Replacing $E_\epsilon(y)$ by $E(y)$ in $F_\epsilon^\circ(z)$ gives

$$F^\circ(z) = \begin{pmatrix} H & 0 & A+DE(y) & 0 \\ 0 & \alpha M & -N^T & B^T \\ A+DE(y) & -N & 0 & 0 \\ 0 & (c(u,t)-I)B & 0 & c(u,t) \end{pmatrix}$$

Lemma 4.3 *$F^\circ(\hat{z})$ is nonsingular for any $\hat{z} \in R^{2n+m+l}$ and $\|F^\circ(\hat{z})^{-1}\|$ is bounded in R^{2n+m+l}.*

Proof: For a fixed $\hat{z} \in R^{2n+m+l}$, let $G = A + DE(\hat{y})$ and $C = c(\hat{u}, \hat{t})$. If $z = (y, u, s, t)$ is a solution of $F^\circ(\hat{z})z = 0$, then we have

$$Hy + Gs = 0, \tag{4.4}$$

$$\alpha Mu - N^T s + B^T t = 0, \tag{4.5}$$

$$Gy - Nu = 0, \tag{4.6}$$

$$(C - I)Bu + Ct = 0. \tag{4.7}$$

From (4.4) and (4.6), we obtain

$$y = -H^{-1}Gs = G^{-1}Nu$$

Substituting it for s in (4.5), we get the following system of linear equations

$$\begin{pmatrix} \alpha M + N^T G^{-1} H G^{-1} N & B^T \\ (C-I)B & C \end{pmatrix} \begin{pmatrix} u \\ t \end{pmatrix} = 0. \tag{4.8}$$

Let $\hat{M} = (\alpha M + N^T G^{-1} H G^{-1} N)^{-1}$. Obviously, $\hat{M}$ is positive definite, and thus $(B\hat{M}B^T)^{-1}$ is a P-matrix. The Schur complement of the 2-block-matrix is

$$(I - C)B\hat{M}B^T + C = (I - C + C(B\hat{M}B^T)^{-1})B\hat{M}B^T.$$

By Lemma 4.2, $(I - C + C(B\hat{M}B^T)^{-1})$ is nonsingular. Therefore, from (4.8), we find that (u, t) is a zero vector, and thus $z = (y, u, s, t) = 0$. Since z and $\hat{z}$ are arbitrarily chosen, we claim that $F^\circ(\hat{z})$ is nonsingular for any $\hat{z}$.

Moreover, by the symmetric positive property of $(B\hat{M}B^T)^{-1}$ and Lemma 4.2, $\|((I - C)B\hat{M}B^T + C)^{-1}\|$ is bounded, that is, the inverse of the Schur complement is bounded. This implies that the inverse of the coefficient matrix of (4.8) is bounded, and hence the inverse $\|F^\circ(\hat{z})^{-1}\|$ is bounded in R^{2n+m+l}. ∎

Lemmas 4.3 and 4.2 ensure that we can choose ϵ such that $F_\epsilon^o(z^k)$ is nonsingular, that is, Method 4.3 is well defined.

A solution of $F(z) = 0$ is not necessarily an optimal solution of (1.1), but it must be an M-stationary point. Now we show that Method (4.3) can find an M-stationary point in its neighborhood in one step.

Theorem 4.1 *Let $z^* = (y^*, u^*, s^*, t^*)$ be a solution of $F(z) = 0$. Let*

$$r^* = \frac{1}{2}\min\{|t_i^* - b_i + (Bu^*)_i| \ : \quad i \in \mathcal{J}(t^* - b + Bu^*) \cup \mathcal{L}(t^* - b + Bu^*)\}$$

if $\mathcal{J}(t^* - b + Bu^*) \cup \mathcal{L}(t^* - b + Bu^*) = \emptyset$. Otherwise let r^* be a positive number. Let

$$\bar{r} = \min\{|y_i^*| \ : \ i \in \mathcal{J}(y^*)\}$$

if $\mathcal{J}(y^*) \neq \emptyset$. Otherwise, let $\bar{r}$ be a positive number. Then Method (4.3) with $\epsilon \leq \bar{r}/3$ finds z^* from any $z^0 \in \mathcal{S}_*$ in one step, where

$$\mathcal{S}_* = \{z \in R^{2n+m+l} \ : \ \|y - y^*\|_\infty \leq \epsilon, \|t - t^*\|_\infty < r^*, \ \|u - u^*\|_\infty < r^*/(\sqrt{m}\|B\|_2)\}.$$

Proof: We only need to show that for any $z \in \mathcal{S}_*$, it holds

$$F_\epsilon(z) + F_\epsilon^\circ(z)(z^* - z) = 0$$

that is,

$$\begin{pmatrix} H(y - y_d) + As + DE_\epsilon(y)s + H(y^* - y) + (A + DE_\epsilon(y))(s^* - s) \\ \alpha M(u - u_d) - N^T s + B^T t + \alpha M(u^* - u) - N^T(s^* - s) + B^T(t^* - t) \\ Ay + D\max(0, y) - Nu + (A + DE(y))(y^* - y) - N(u^* - u) \\ \min(t, b - Bu) + (c(u,t) - I)B(u^* - u) + c(u,t)(t^* - t) \end{pmatrix} = 0. \tag{4.9}$$

For any $z \in \mathcal{S}_*$, we have

$$|t_i - b_i + (Bu)_i - t_i^* + b_i - (Bu)_i^*| < r^* + \|B_i\|_2\|u - u^*\|_2 \leq 2r^*,$$

which implies

$$\mathcal{J}(t^* - b + Bu^*) \subseteq \mathcal{J}(t - b + Bu), \quad \mathcal{L}(t^* - b + Bu^*) \subseteq \mathcal{L}(t - b + Bu),$$

and

$$c_{ii}(u,t) = c_{ii}(u^*, t^*), \quad i \in \mathcal{J}(t^* - b + Bu^*) \cup \mathcal{L}(t^* - b + Bu^*). \tag{4.10}$$

Now we show (4.9) by using its block structure. We first show the last equality. From

$$\min(t, b - Bu) = c(u,t)t + (c(u,t) - I)(Bu - b)$$

we get the last equality

$$\begin{aligned} &\min(t, b - Bu) + (c(u,t) - I)(Bu^* - u) + c(u,t)(t^* - t) \\ &= c(u,t)t^* + (c(u,t) - I)(Bu^* - b) \\ &= c(u^*, t^*)t^* + (c(u^*, t^*) - I)(Bu^* - b) \\ &= \min(t^*, b - Bu^*) \\ &= 0, \end{aligned}$$

where the second equality uses (4.10), and $t_i^* = b_i - Bu_i^*, \quad i \in \mathcal{K}(t^* - b + Bu^*)$.

Now we show the first and third block equalities in (4.9). For any $z \in \mathcal{S}_*$, we have

$$\mathcal{J}(y^*) \subseteq \mathcal{J}(y), \quad \mathcal{L}(y^*) \subseteq \mathcal{L}(y) \tag{4.11}$$

$$y_i \geq 2\epsilon, \quad i \in \mathcal{J}(y^*)$$

and

$$y_i \leq \epsilon, \quad i \in \mathcal{K}(y^*) \cup \mathcal{L}(y^*).$$

By the definition of E_ϵ, this implies

$$(E_\epsilon(y) - E_\epsilon(y^*))_{ii} = 0, \quad i = 1, 2, \ldots, n. \tag{4.12}$$

Hence from $F(z^*) = 0$, we obtain the first block equality,

$$\begin{aligned}
&H(y - y_d) + As + DE_\epsilon(y)s + H(y^* - y) + (A + DE_\epsilon(y))(s^* - s) \\
&= H(y^* - y_d) + As^* + DE(y^*)s^* + D(E_\epsilon(y) - E(y^*))s^* \\
&= D(E_\epsilon(y) - E(y^*))s^* \\
&= 0.
\end{aligned}$$

Note that $E(y)y = \max(0, y)$. Moreover, from (4.11), we have $E(y)y^* = E(y^*)y^*$, for $z \in \mathcal{S}_*$. Hence we obtain the third block equality

$$\begin{aligned}
&Ay + D\max(0, y) - Nu + A(y^* - y) + DE(y)(y^* - y) - N(u^* - u) \\
&= Ay^* + DE(y)y^* - Nu^* \\
&= Ay^* + DE(y^*)y^* - Nu^* \\
&= Ay^* + D\max(0, y^*) - Nu^* \\
&= 0.
\end{aligned}$$

The second block equality in (4.9) holds obviously. The proof of the theorem is therefore completed. ∎

To illustrate the study of the quadratic programs with linear complementarity constraints (1.1), we consider the following example.

Example 4.1 Let $n = 2, m = 1, l = 1, M = \alpha = b = 1, H = D = I, B = 1$.

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

(1) For $y_d = (0, 1)$, $u_d = 1$, $(\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2})$ is a solution of $F(z) = 0$ and $(\frac{1}{2}, 0, \frac{1}{2})$ is a solution of (1.1). The NECQ holds at $(\frac{1}{2}, 0, \frac{1}{2})$, but the MPEC LICQ does not hold at $(\frac{1}{2}, 0, \frac{1}{2})$.

(2) For $y_d = (0, -3)$ and $u_d = 1$, $z = (0, 0, 0, -1, -2, 0)$ is a solution of $F(z) = 0$, but $(y_1, y_2, u) = (0, 0, 0)$ is not a solution of (1.1). Both the NECQ and MPEC LICQ do not hold at $(0, 0, 0)$.

(3) For $y_d = (0, 1), u_d = 0$, $(0, 0, 0)$ is a solution of (1.1). Both the NECQ and MPEC LICQ do not hold at $(0, 0, 0)$. It is an M-stationary point but it is not a KKT point. That is, there is no (s_1, s_2, t) such that $z = (0, 0, 0, s_1, s_2, t)$ is a solution of $F(z) = 0$.

(4) For $y_d = (0, -1/3), u_d = 0$, $(y^*, u^*) = (-5/105, -1/105, -1/35)$ is the unique global optimal solution of (1.1). Moreover, (2.3) holds with $t^* = 0$ and $s^* = (-0.0762, -0.2)$. This means that $\mathcal{L}(s^*) = \emptyset$ is not a necessary condition for Statement 1 of Theorem 2.1.

Now we show case (1) and Method 4.3 for case (1). From the constraint

$$Ay + \max(0, y) = Nu$$

y can be defined as a function of u,

$$y(u) = \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} u & u \geq 0 \\ \begin{pmatrix} 5/3 \\ 1/3 \end{pmatrix} u & u < 0. \end{cases}$$

The objective function can be written as

$$f(y(u), u) = \begin{cases} \frac{1}{2}(u^2 + 1) + \frac{1}{2}(u-1)^2 & u \geq 0 \\ \frac{1}{2}(\frac{25}{9}u^2 + (\frac{u}{3} - 1)^2) + \frac{1}{2}(u-1)^2 & u < 0. \end{cases}$$

By simple calculation, we find $(y^*, u^*) = (\frac{1}{2}, 0, \frac{1}{2})$ is the unique solution of (1.1). Moreover, it is easy to verify

$$z^* = (y^*, u^*, s^*, t^*) = (\frac{1}{2}, 0, \frac{1}{2}, 0, \frac{1}{2}, 0)$$

is a solution of $F(z) = 0$. The NECQ holds at $(\frac{1}{2}, 0, \frac{1}{2})$ since $y(\cdot)$ is differentiable at $u = \frac{1}{2}$. However, the MPEC LICQ does not hold at $(\frac{1}{2}, 0, \frac{1}{2})$, since the rows of the matrix

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 3 & -3 & 1 \\ -1 & 1 & -3 \\ 3 & -2 & 1 \end{pmatrix}.$$

are linearly dependent. Let $\epsilon \leq \frac{1}{6}$. Let

$$\mathcal{S}_* = \{z \,:\, \|y - y^*\|_\infty \leq \epsilon,\ |t| < \frac{1}{4},\ |u - \frac{1}{2}| < \frac{1}{4}\}.$$

Then for any $z \in \mathcal{S}_*$, we have

$$y_1 \geq \frac{1}{3} \geq 2\epsilon, \text{ and } y_2 \leq \epsilon$$

which implies that for any $z \in \mathcal{S}_*$, we have

$$E_\epsilon(y) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = E(y^*), \qquad E(y)y^* = E(y^*)y^*.$$

By straightforward calculation, we can get

$$F_\epsilon(z) + F_\epsilon^\circ(z)(z^* - z) = F(z^*) = 0.$$

Note that $E(y) = E(y^*)$ does not hold in $\mathcal{S}_*$. Moreover, since the MPEC LICQ does not hold, active methods [24] for MPEC cannot be applied to this example.

Similarly, we can show cases (2)-(4).

Final Remark In this paper we present new necessary and sufficient optimality conditions for the quadratic program with linear complementarity problems (1.1) by using the concepts of M-stationary points, S-stationary points and the equivalence with the nonsmooth equation formulation (1.2). Moreover, we propose a fast locally convergent method (4.3). This method can be combined with some global algorithms such as [11, 12, 25, 17] for MPEC to solve the the QPLCC more efficiently. Building the relation between the mathematical problems with nonsmooth constraints (1.2) and the QPLCC (1.1) is also interesting to the study of MPEC. Due to the nonconvexity of the feasible region in MPEC, the necessary conditions for a general MPEC problem are normally not sufficient. Therefore there are almost no sufficient conditions existing in the literature of MPEC. However by using the special structure of our problem, we have provided some strong and concrete sufficient conditions for global and local optimality.

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