Wardrop's User Equilibrium Assignment under Stochastic Environment

Chao Zhang[a], Xiaojun Chen[b], Agachai Sumalee[*,c]

[a]*Department of Applied Mathematics, Beijing Jiaotong University, Beijing 100044, China*
[b]*Department of Applied Mathematics, The Hong Kong Polytechnic University, Hung Hom, Kowloon, China*
[c]*Department of Civil and Structural Engineering, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong, China*

Abstract

Various models of traffic assignment under stochastic environment have been proposed recently, mainly by assuming different travelers' behavior against uncertainties. This paper focuses on the expected residual minimization (ERM) model to provide a robust traffic assignment with an emphasis on the planner's perspective. The model is further extended to obtain a stochastic prediction of the traffic volumes by the technique of path choice approach. We show theoretically the existence and the robustness of the ERM solution. In addition, we employ an improved solution algorithm for solving the ERM model. Numerical experiments are carried out to illustrate the characteristics of the proposed model, by comparing with other existing models.

Key words: Wardrop's user equilibrium; Robust traffic assignment; Demand and supply uncertainty; Nonadditive cost; Expected residual minimization

---

1. Introduction

The main role of traffic or transportation model is to provide a forecast of future traffic state. The output from the model is often used in highway and public transport project design and evaluation. The current state of the art of traffic modeling involves various modeling paradigms ranging from the traditional four-step model, activity based model, to dynamic traffic assignment. An underlying structure of these modeling paradigms is the interaction between the demand and supply sides of the traffic system. The travel demand is normally defined as an origin-destination (OD) matrix or captured by a demand function. On the supply side, the performance of a road or highway is represented by a speed-flow function (in either static or dynamic framework). The forecast provided by the traffic model is then based on the equilibrium state between the demand and supply of travel which are deterministic inputs of the models.

[*]Corresponding author. Tel.: +852 3400 3963; fax: +852 2334 6389
Email address: ceasumal@polyu.edu.hk (Agachai Sumalee)

This paper focuses on the last stage of traffic model which is the traffic assignment. A concept which is widely adopted to define an equilibrium point of the traffic assignment is the Wardrop's user equilibrium (Wardrop, 1952). Under the Wardrop's equilibrium, travelers will only travel on the cheapest route in terms of his/her generalized nonadditive travel cost which may include travel time, out of pocket expense, etc.. Thus, at the equilibrium point no traveler can change his/her route unilaterally to reduce his/her own travel cost. The traditional approach of traffic assignment requires deterministic inputs of travel demand and supply (e.g. OD matrix and speed-flow relationship) in which the model will then provide a deterministic prediction of the future traffic condition (e.g. congestion level on each link in the next ten years). However, such long term forecasts often involve a high degree of uncertainty of the inputs (e.g. future travel demand in the next ten years). Thus, the validity of the project evaluation or any infrastructure design may also be subject to this uncertainty.

Recently, several transport network modeling approaches have been proposed to consider uncertainties from both demand and supply sides of the system. In particular, the concept of stochastic network (see e.g. Walting, 2002; Sumalee et al., 2009a) is developed to include the stochastic demand and supply characteristics into the traffic assignment model. The stochastic network framework takes the inputs of OD demand and/or road capacity as random variables. Walting (2002) proposed a framework of stochastic network model considering the stochastic travel demand which follows a stationary Poisson process and a probabilistic route choice model. Shao et al. (2006) and Sumalee et al. (2009b) proposed a similar model but used a normal distribution to represent the stochastic demand. Zhou and Chen (2008) on the other hand, adopted the log-normal stochastic demand in their model formulation. On the supply side, Lo and Tung (2003) introduced the stochastic link capacity, which is assumed to follow a uniform distribution, into the stochastic network model. Sumalee et al. (2009a) introduced both demand and supply uncertainties which are assumed to follow a log-normal distribution. In all cases, a key operational feature of the stochastic network model is the stochastic prediction of the equilibrium flows based on the stochastic inputs of OD demand and road capacity, i.e. equilibrium path and link flows will follow some statistical distributions. This, in some way, can be viewed as an attempt to consider uncertainties in the forecast of future traffic condition based on uncertain inputs.

However, all of the proposed stochastic network models mainly focus on the short-term uncertainty of the travel demand and supply. The definition of stochastic demand and supply, in fact, stems from the day-to-day variability of demand and supply in the network. For instance, it is evident that the number of travelers between each OD pair varies from day to day due to the intrinsic stochastic nature of travel demand and human behavior. Similarly, the road capacity may also change from day to day due to incidents or weather effect (Lam et al., 2008). The framework of stochastic network model emphasizes on capturing the effect of day-to-day uncertainties of travel condition on travelers in which a prediction of potential traffic states is made based upon these uncertainties. This model can be viewed as a user-oriented model.

On the other hand, from the planner's perspective the prediction of the future traffic condition should be ***robust*** against possible uncertainties of the future demand and supply.

A forecast of the traffic condition (in terms of traffic volumes and minimum travel costs) is considered robust if this forecast deviates as little as possible from the set of equilibrium states resulted from the traffic assignment with different potential OD matrices and capacities. Let S_1, S_2 and S_3 be the solution set of the Wardrop's user equilibrium assignment corresponding to the three different scenarios of the future demand, respectively. A forecasted traffic equilibrium $\hat{\mathbf{x}}$ is considered robust if the expected distance of $\hat{\mathbf{x}}$ to S_1, S_2 and S_3 is minimum. Thus, if $\hat{\mathbf{x}}$ is a robust prediction against the uncertain demand, any project evaluation and design based on $\hat{\mathbf{x}}$ should also be robust against anticipated uncertainty. This is probably one of the main concerns of the transport planners in using traffic model to evaluate or design a transport project.

Note that the notion of *robust Wardrop equilibrium* was used previously by Fernando and Nichlas (2006). However, their model focuses on the traveler's perspective in which the flow prediction is based on the Wardrop user equilibrium principle considering the worst-case of uncertain link travel times.

Failure to include uncertainty properly in a traffic assignment model may lead to a very expensive, or even fatal decision if the anticipated random variable is not realized. This paper focuses on the robust traffic assignment model with an emphasis on the planner's perspective in obtaining a robust prediction against the uncertain future demand and capacity. The model is based on the *expected residual minimization* (ERM) introduced by Chen and Fukushima (2005) for general stochastic nonlinear complementarity problem (SNCP). The ERM model uses optimization methods to find a robust prediction $\hat{\mathbf{x}}$ such that the expected distance from $\hat{\mathbf{x}}$ to the solution set S_ω of the NCP corresponding to each possible scenario ω is small, without solving the NCP for each scenario. Note that obtaining one solution of the NCP for a given scenario ω might be difficult due to the nonadditive travel cost. In addition, the whole solution set S_ω may also in general be nonsingleton and even nonconvex. Given a ERM solution, we also propose an approach to generate the statistical distribution of the traffic state following the method adopted in Sumalee et al. (2009a). The paper compares the prediction results of the proposed model with those from other stochastic network assignment models.

This paper is organized as follows. In the following section, we begin with the basic settings for traffic network under stochastic environment. We then focus on presenting the proposed expected residual minimization (ERM) model, together with the path choice proportion approach, and briefly review two other existing models for robust assignment in the stochastic network. We also analyze theoretically some properties including the existence and robustness of the solution. Section 3 describes how to apply the smoothing projected gradient (SPG) method proposed by Zhang and Chen (2009) for solving the ERM model. In Section 4, numerical examples on two small-size networks are provided to demonstrate that the ERM model and the SPG method is promising in providing a robust traffic assignment under uncertainties. We then conclude the paper in the final section.

2. Model formulation

2.1. Stochastic network framework

We consider a *strongly connected* network $[\mathcal{N}, \mathcal{A}]$, where $\mathcal{N}$ is the set of nodes and $\mathcal{A}$ is the set of links. We denote by K the set of all possible paths with cardinality $|K|$, and W the origin-destination (OD) movements with cardinality $|W|$.

Let K^r be a set of paths connecting the r^{th} OD, and $\Omega \subseteq R^m$ be the set of uncertain factors such as weather, accidents, etc. Let $\mathbf{Q}(\omega)$ be a demand vector with entries $Q^r(\omega)$ representing the stochastic travel demand on the r^{th} OD, and $\mathbf{C}(\omega)$ be a capacity vector with entries $C_a(\omega)$ denoting the stochastic capacity on link a, for uncertain factor $\omega \in \Omega$. We assume that the uncertain demand $\mathbf{Q}(\omega)$ is bounded for $\omega \in \Omega$ almost everywhere (a.e.), and the probability distributions of random vectors $\mathbf{Q}(\omega)$ and $\mathbf{C}(\omega)$ are known.

For a realization of random vectors $\mathbf{Q}(\omega)$ and $\mathbf{C}(\omega)$, $\omega \in \Omega$, an assignment of flows to all paths is denoted by the vector $\mathbf{F}(\omega)$, whose component $F_k^r(\omega)$ denotes the flow on the k^{th} path connecting the r^{th} OD, while an assignment of flows to all links is represented by the vector $\mathbf{V}(\omega)$ whose component $V_a(\omega)$ denotes the stochastic flow on link a. The relation between $\mathbf{F}(\omega)$ and $\mathbf{V}(\omega)$ is presented by

$$\mathbf{V}(\omega) = \Delta \mathbf{F}(\omega),$$

where $\Delta = (\delta_{a,k})$ is the link-path incidence matrix with entries $\delta_{a,k} = 1$ if link a is on path k, and $\delta_{a,k} = 0$ otherwise. A random unknown vector $\mathbf{U}(\omega)$ with components $U_r(\omega)$ representing the stochastic minimum travel cost for the r^{th} OD.

Let $\Gamma = (\gamma_{r,k})$ denote the OD-path incidence matrix with entries $\gamma_{r,k} = 1$ if path k connects the r^{th} OD, and $\gamma_{r,k} = 0$ otherwise. Thus each row of Γ is a nonzero vector since the network is strongly connected, and Γ has full row-rank since one path connects only one OD movement.

Given the path flow vector f, we know that the link flow vector $V = \Delta f$. The link travel time function $T(V, \omega)$ is a stochastic vector, and each of its entries $T_a(V, \omega)$ is assumed to follow a generalized Bureau of Public Roads (GBPR) function,

$$T_a(V, \omega) = t_a^0 \Big(1 + b_a \Big(\frac{V_a}{C_a(\omega)} \Big)^{n_a} \Big), \tag{1}$$

where t_a^0, b_a and n_a are given parameters. We employ the nonadditive path travel cost function $\Phi(f, \omega)$ extended from Gabriel and Bernstein (1997) by

$$\Phi(f, \omega) = \eta_1 \Delta^T T(\Delta f, \omega) + \Psi(\Delta^T T(\Delta f, \omega)) + \Lambda(f, \omega). \tag{2}$$

Here $\eta_1 > 0$ is the time-based operating costs factor, Ψ is the function converting time T to money, and Λ is the perturbed financial cost function. Various factors may cause the nonadditivity of the cost function such as the route-specific toll schemes and the nonlinear value of time.

Let us denote

$$x = \begin{pmatrix} f \\ u \end{pmatrix}, \quad G(x, \omega) = \begin{pmatrix} \Phi(f, \omega) - \Gamma^T u \\ \Gamma f - \mathbf{Q}(\omega) \end{pmatrix}, \tag{3}$$

where $x \in R^n$ is a deterministic vector with $n = |K| + |W|$. Here $f \in R^{|K|}$ is a path flow pattern and $u \in R^{|W|}$ is a travel cost vector corresponding to f.

For a fixed $\breve{\omega} \in \Omega$, the NCP formulation for Wardrop's user equilibrium, denoted by NCP($G(x, \breve{\omega})$), seeks $x \in R^n$ such that

$$x \geq 0, \quad G(x, \breve{\omega}) \geq 0, \quad x^T G(x, \breve{\omega}) = 0. \tag{4}$$

At any solution, it is known that u coincides with the vector of minimum OD travel costs corresponding to the equilibrium flow pattern f. Both path flows and the minimum OD travel costs are considered as decision variables, constituting useful information of the static Wardrop's user equilibrium for the planner. Moreover, the NCP($G(x, \breve{\omega})$) is equivalent to the system of nonlinear equations

$$\min(x, G(x, \breve{\omega})) = 0.$$

In the case that the travel demands and road capacities are endogenously considered to be random variables, we may not expect to find a traffic path flow pattern f and a related travel cost u such that they constitute a static user equilibrium under all realized demand and supply. That is, in general, there is no solution x satisfying the following SNCP

$$\min(x, G(x, \omega)) = 0 \quad \text{for all } \omega \in \Omega.$$

Hence it is meaningful to extend the static Wardrop's user equilibrium to a *user equilibrium under uncertainty* for the planner in the following two ways.

First, it should provide a deterministic equilibrium pattern for the planner, which deviates as little as possible from the set of equilibrium states under any possible scenario. Secondly, the planners can estimate the distribution of the random flow pattern, based on the vector of path choice proportions from the deterministic equilibrium pattern. From the above information, planner can then make a robust decision.

2.2. ERM model

We now explain the ERM model, which can meet the above two requirements for the traffic assignment under uncertainty mentioned earlier. The ERM model seeks a robust traffic assignment under stochastic environment by solving

$$\min_{x \in R^n_+} g(x) := E[\| \min(x, G(x, \omega)) \|^2], \tag{5}$$

where $E[\cdot]$ refers to the expectation operator and $\| \cdot \|$ denotes the Euclidean norm.

For any $\omega \in \Omega$, let S_ω denote the solution set of NCP($G(x, \omega)$) and $\text{dist}(x, S_\omega)$ represent the Euclidean distance function from the vector x to the solution set S_ω. We define the residual function $r_\omega(\cdot) : R^n \to R_+$ of NCP($G(x, \omega)$) by

$$r_\omega(x) := \| \min(x, G(x, \omega)) \|.$$

In the field of NCP, $r_\omega(\cdot) : R^n \to R_+$ is called a residual of NCP($G(x, \omega)$), if $r_\omega(x) = 0$ if and only if $x \in S_\omega$, i.e., $\text{dist}(x, S_\omega) = 0$. The objective function g plays the role of penalizing the vector $x \in R^n_+$ that is far away from solution set S_ω, $\omega \in \Omega$. In the next section, we will prove that by minimizing (5) the expected distance $E[\text{dist}(x_{ERM}, S_\omega)]$ tends to be small. Let $x_{ERM} = (f^T_{ERM}, u^T_{ERM})^T$ be the solution of (5) and S_{ERM} be the solution set of the ERM model. The planner can choose f_{ERM} as a robust path flow pattern and u_{ERM} as a robust minimum travel cost corresponding to f_{ERM} under uncertainties.

One may question if we can also get a robust solution by solving:

$$\min_{x \in R^n_+} E[\text{dist}(x, S_\omega)]$$

instead of the ERM. We argue that although the above minimization problem meets the robust requirement, it is impractical to compute $\text{dist}(x, S_\omega)$ since S_ω in general is not a singleton nor convex if nonadditive travel cost is adopted. Another possible alternative is to obtain a solution $x_\omega \in S_\omega$ for each scenario ω and solve

$$\min_{x \in R^n_+} \|x - x_\omega\|^2.$$

However there exists numerous scenarios ω when continuous distribution is used. To compute one solution for each scenario is computationally prohibitive. Furthermore, an arbitrary chosen $x_\omega \in S_\omega$ might lead to a bias prediction with respect to x_ω in the case that S_ω contains more than one element.

A stochastic traffic flow $\tilde{\mathbf{F}}_{ERM}(\omega)$ which can be derived from f_{ERM} has not been studied by Zhang and Chen (2008). Here we outline the proportion technique to obtain a stochastic traffic flow $\tilde{\mathbf{F}}(\omega)$ from a deterministic traffic assignment f.

We introduce the concept of vector of path choice proportions $\mathbf{p} = (p^r_k) \in R^{|K|}$ for a deterministic path flow pattern f, where K is the set of possible paths. The entry

$$p^r_k = \frac{f_k}{\sum_{j \in K^r} f_j} \tag{6}$$

is the proportion of flow on path $k \in K$ between the r^{th} OD. It is clear that

$$\sum_{k \in K^r} p^r_k = 1, \quad \text{for any } r^{\text{th}} \text{ OD movement.}$$

In a static setting of traffic network, if f lies in the set of Wardrop's user equilibria, the path choice proportion p^r_k, $k \in K^r$, determine the allocation of the given demand Q^r on the set of possible paths connecting the r^{th} OD. The random path flow can then be expressed as

$$\tilde{F}^r_k(\omega) = p^r_k Q^r(\omega), \tag{7}$$

and hence the demand conservation $\sum_{k \in K^r} \tilde{F}^r_k(\omega) = Q^r(\omega)$ holds for each fixed ω. The random path flow $\tilde{F}^r_k(\omega)$ follows the same type of distribution of $Q^r(\omega)$. We have

$$E[\tilde{F}^r_k(\omega)] = E[p^r_k Q^r(\omega)] = p^r_k E[Q^r(\omega)],$$

and

$$Var[\tilde{F}_k^r(\omega)] = Var[p_k^r Q^r(\omega)] = (p_k^r)^2 Var[Q^r(\omega)].$$

Note that variance is not conserved here but one can introduce covariance term, $cov(F_i, F_j)$, to ensure the variance conservation following Lam et al. (2008).

We can write $\tilde{\mathbf{F}}(\omega)$ in the form of the multiplication of matrices as

$$\tilde{\mathbf{F}}(\omega) = \text{diag}(\mathbf{p})\Gamma^T \mathbf{Q}(\omega),$$

where $\text{diag}(\mathbf{p})$ is the diagonal matrix satisfying $(\text{diag}(\mathbf{p}))_{ii} = \mathbf{p}_i$ for each i. Moreover,

$$E[\tilde{\mathbf{F}}(\omega)] = \text{diag}(\mathbf{p})\Gamma^T E[\mathbf{Q}(\omega)].$$

At follows, we briefly review two relevant models on traffic assignment under stochastic environment, which are also extended from the static Wardrop's user equilibrium, but in different manners compared with the ERM model.

• *Expected value (EV) model*

The EV model (see e.g. Walting, 2002; Clark and Walting, 2005; Sumalee et al., 2009b) assumes that to contend with random demand and supply, the travelers select paths to minimize their expected travel cost $\bar{\Phi}(f) = E[\Phi(f, \omega)]$. Thus the EV model reduces to the static Wardrop's user equilibrium NCP($E[G(x, \omega)]$),

$$\min(x, E[G(x, \omega)]) = 0. \tag{8}$$

Let us signify the solution of the EV model by $x_{EV} = (f_{EV}^T, u_{EV}^T)^T$. It is worth mentioning that f_{EV} in general is not unique, while the link flow vector v_{EV} is unique if $\bar{\Phi}$ is strictly monotone with respect to f.[1]

• *Best worst-case (BW) model*

The so-called BW model borrows the idea from robust optimization of stochastic programming by assuming that each user selects path to minimize the worst-case cost $\hat{\Phi}(f) = \max_{\omega\in\Omega} \Phi(f, \omega)$, and the worst-case demand $\hat{\mathbf{Q}} = \max_{\omega\in\Omega} \mathbf{Q}(\omega)$ will be assigned to the network. Denote

$$\hat{G}(x) = \begin{pmatrix} \hat{\Phi}(f) - \Gamma^T u \\ \Gamma f - \hat{\mathbf{Q}} \end{pmatrix}.$$

[1]We may also write the static User Equilibrium in the variational inequality (VI) form. The link flow vector $v_{EV} = \Delta f_{EV}$ is the static User equilibrium if f_{EV} satisfies

$$\bar{\Phi}(f_{EV})^T (f - f_{EV}) \geq 0, \quad \forall f \in \mathcal{F} = \{f \ : \ \Gamma f = E[Q(\omega)], \ f \geq 0\}.$$

This formulation ignores any travel time variability and presupposes that travelers consider only the deterministic mean path costs.

The BW model coincides to a static Wardrop's user equilibrium NCP($\hat{G}(x)$),

$$\min(x, \hat{G}(x)) = 0. \tag{9}$$

A special case of the BW model called a *robust Wardrop equilibrium* is proposed by Fernando and Nichlas (2006), where the path travel cost $\Phi(f, \omega) = \Delta^T T(V, \omega)$ is additive and only the link travel time

$$T_a(V, \omega) = l_a(V_a) + \omega_a \gamma_a$$

explicitly incorporates uncertainty. Here γ_a is an upper bound of possible deviation from the load-dependent nominal value $l_a(V_a)$, and the random vector $\omega = (\omega_a)$ belongs to

$$\Omega = \{\omega \ : \ \omega_a \in [0, 1], \ (\Delta^T \omega)_k \leq \delta \text{ for any link } a \text{ and path } k\},$$

corresponding to a given uncertainty budget δ.

The aforementioned three models coincide to the same NCP for Wardrop's user equilibrium if Ω is a singleton. Otherwise, the EV/BW model is equivalent to a certain NCP for Wardrop's user equilibrium, where EV takes the average and BW adopts the worst-case of stochastic demand and travel cost to deal with uncertainties. The ERM model, on the other hand, is a nonsmooth nonconvex optimization problem on R^n_+, which is not equivalent to a NCP.

From the computational point of view, the EV model, NCP($E[G(x, \omega)]$), is a standard NCP with smooth function $E[G(x, \omega)]$ in ordinary setting. The EV model can be solved efficiently if $E[G(x, \omega)]$ is monotone. However, the monotonicity of $E[G(x, \omega)]$ is in general not guaranteed due to the nonadditive cost. The BW model leads to computationally intensive method, because evaluating the function value $\hat{\Phi}(x)$ is not straightforward due to the maximum operator and numerous/continuous random vector $\omega \in \Omega$. We will illustrate the SPG method for solving the ERM model in Section 3.

2.3. Existence and robustness of ERM

In this subsection, we establish the existence and robustness of the solution provided by the ERM model for the traffic assignment under stochastic environment. We assume the following conditions on the travel cost function.

Assumption 1. For fixed $\omega \in \Omega$ almost everywhere (a.e.), the travel cost function $\Phi_k(f, \omega)$ on each path k is a nonnegative nondecreasing smooth function of flow f, and finite for any fixed f.

Assumption 1 holds in general for the travel cost function. The existence of ERM solution relates closely to the R_0-type property of function $G(x, \omega)$.

Definition 1. *(Zhang and Chen, 2008)* $G : R^n \times \Omega \to R^n$ *is called a stochastic* R_0 *function on* R^n_+ *if for every infinite sequence* $\{x^k\} \subseteq R^n_+$ *satisfying*

$$\lim_{k\to\infty} \|x^k\| = \infty, \qquad \limsup_{k\to\infty} \|(-G(x^k, \omega))_+\| < \infty \quad \text{for } \omega \in \Omega \text{ a.e.},$$

there is $i \in \{1, 2, \ldots, n\}$ *such that the probability* $\mathcal{P}\{\omega : \limsup_{k\to\infty} \min(x^k_i, G_i(x^k, \omega)) = \infty\} > 0.$

If $\Omega = \{\breve{\omega}\}$ is a singleton, the stochastic R_0 function $G(x, \breve{\omega})$ defined above reduces to the R_0 function, which constitutes an important class of NCP and guarantees the boundedness of $S_{\breve{\omega}}$. By Proposition 3.1 and Remark 3.1 in Zhang and Chen (2008), the stochastic function $G(x, \omega)$ is a stochastic R_0 function, and $G(x, \breve{\omega})$ for each fixed $\breve{\omega} \in \Omega$ a.e. is an R_0 function. We immediately have the following existence results.

Theorem 1. *Suppose that Assumption 1 holds. Then the solution set S_{ERM} of the ERM model (5) is nonempty and bounded.*

Lemma 1. *Suppose that Assumption 1 holds. Then for fixed $\omega \in \Omega$ a.e., the solution set S_ω is nonempty and bounded.*

PROOF. For fixed $\omega \in \Omega$ a.e., S_ω is nonempty by Theorem 5.3 in Aashtiant and Magnanti (1981), since the network is strongly connected, the demand $\mathbf{Q}(\omega)$ is bounded, and $\Phi_k(f, \omega)$ on each path k is a nonnegative continuous function of flow f by Assumption 1. Furthermore, $G(x, \omega)$ is an R_0 function indicates that the solution set S_ω is bounded.

Our main motivation to propose the ERM model for the traffic assignment under stochastic environment lies in that it might provide the planner a robust solution under any possible realization of random variables. Below, we show theoretically the robustness of the solution.

Assumption 2. For fixed $\omega \in \Omega$ a.e., the residual $r_\omega(x) = \| \min(x, G(x, \omega))\|$ is a local error bound for NCP$(G(x, \omega))$ in R^n_+.

Recall that $r_\omega(x)$ is a local error bound for NCP$(G(x, \omega))$ if there exists some constant $\tau_\omega > 0$ and $\epsilon_\omega > 0$ such that, for each $x \in R^n_+$, with $r_\omega(x) \leq \epsilon_\omega$,

$$\mathrm{dist}(x, S_\omega) \leq \tau_\omega r_\omega(x).$$

Assumption 2 holds if for fixed $\omega \in \Omega$ a.e., one of the following conditions holds:
1. The path travel cost function $\Phi_k(f, \omega)$ is linear of the path flow f for each path k.
2. (Chen, 2001, Theorem 3.1) $H_\omega(x) = \min(x, G(x, \omega))$ is BD-regular at all solutions in S_ω.

The locally Lipschitzian function $H_\omega(x)$ is said to be BD-regular at x if all elements in

$$\partial_B H_\omega(x) = \{\lim \nabla H_\omega(x^k) : x^k \to x,\ x^k \in D_{H_\omega}\}$$

are nonsingular, where D_{H_ω} is the set of points where H_ω is F-differentiable. We give a sufficient condition to guarantee the BD-regularity of $H_\omega(x)$.

Proposition 1. *$H_\omega(x)$ is BD-regular at $x^* = (f^{*T}, u^{*T})^T \in S_\omega$, if the principal submatrix $(\nabla\Phi(f^*, \omega))_{\gamma'\gamma'}$ is nonsingular for any index subset γ' of $\mathcal{K} = \{1, 2, \ldots, |K|\}$ satisfying $\gamma \subseteq \gamma' \subseteq \gamma \cup \beta$, where*

$$\begin{aligned} \beta &= \{i \in \mathcal{K} \ :\ f_i^* = 0 = G_i(x^*, \omega)\} \\ \gamma &= \{i \in \mathcal{K} \ :\ f_i^* > 0 = G_i(x^*, \omega)\}. \end{aligned}$$

Proof. Any element $V \in \partial_B H_\omega(x^*)$ can be expressed by

$$V = \begin{pmatrix} (\nabla\Phi(f^*,\omega))_{\gamma'\mathcal{K}} & (-\Gamma^T)_{\gamma'\mathcal{W}} \\ I_{\bar{\gamma}'\mathcal{K}} & O_{\bar{\gamma}'\mathcal{W}} \\ \Gamma_{\mathcal{W}\mathcal{K}} & O_{\mathcal{W}\mathcal{W}} \end{pmatrix},$$

for some subset γ'. Here $\mathcal{W} = \{1, 2, \ldots, |W|\}$, $\bar{\gamma}' = \mathcal{K} \setminus \gamma'$, and O refers to the zero matrix. Clearly we have

$$\det V = \det \begin{pmatrix} (\nabla\Phi(f^*,\omega))_{\gamma'\gamma'} & (-\Gamma^T)_{\gamma'\mathcal{W}} \\ \Gamma_{\mathcal{W}\gamma'} & O_{\mathcal{W}\mathcal{W}} \end{pmatrix}.$$

Note that $(\nabla\Phi(f^*,\omega))_{\gamma'\gamma'}$ is invertible from the assumption, and $\Gamma_{\mathcal{W}\gamma'}$ is of full row rank, since there is at least one path $k \in \gamma'$ connecting each OD movement. Hence $\det V \neq 0$ and V is nonsingular, which indicates that $H_\omega(x)$ is BD-regular at x^*.

Remark 1. The BD-regular condition required in Proposition 1 may be difficult to be verified in a general network case. Nevertheless, we can verify this condition under a more restricted setting. Consider a network in which there does not exist any unused path with minimum path travel time (i.e. $\beta = \emptyset$), the BD-regular condition can be ensured if:

- the non-additive path travel costs of the used path ($k \in \gamma$) is strongly dominated by the path flow on that path itself, i.e.,

$$\frac{\partial\Phi_k(f,\omega)}{\partial f_k} > \sum_{k\neq k'\in\gamma} \left|\frac{\partial\Phi_k(f,\omega)}{\partial f_{k'}}\right|, \quad \text{for any } k \in \gamma.$$

This condition indeed guarantees that the submatrix $(\nabla\Phi(f,\omega))_{\gamma\gamma}$ is a P-matrix which is nonsingular.

Lemma 2. *Suppose that Assumptions 1 and 2 hold. Then for fixed $\omega \in \Omega$ a.e., and any compact set $X \subset R^n_+$, there exists a positive constant τ_ω such that*

$$\mathrm{dist}(x, S_\omega) \leq \tau_\omega r_\omega(x), \quad \forall x \in X.$$

Proof. Assume on the contrary that the lemma is false. Then for fixed $\omega \in \Omega$ a.e. satisfying Assumption 2, and for each integer k, there exists an $x^k \in X$ such that

$$\mathrm{dist}(x^k, S_\omega) > k r_\omega(x^k).$$

Note that for fixed $\omega \in \Omega$ a.e., $r_\omega(x)$ is a local error bound for NCP$(G(x,\omega))$. Hence for such ω, there exist an integer $\bar{k} > 0$ and a scalar $\epsilon > 0$ such that $r_\omega(x^k) > \epsilon$ for all $k > \bar{k}$. This indicates that

$$\mathrm{dist}(x^k, S_\omega) \to \infty,$$

which is impossible since x^k contains in a compact set X, and S_ω is bounded. This completes the proof.

Theorem 2. *Suppose that $\Omega = \{\omega^1, \omega^2, \ldots, \omega^N\}$ is a finite set with the probability of each element to be positive, and Assumptions 1 and 2 hold. Then for any compact set $X \subset R^n_+$, there exists a positive constant τ such that*

$$E[\mathrm{dist}(x, S_\omega)] \leq \tau\sqrt{g(x)}.$$

Proof. According to Lemma 2, we know that for $\omega^j \in \Omega$ and any compact set $X \subset R^n_+$, there exists a positive constant τ_{ω^j} such that

$$\mathrm{dist}(x, S_{\omega^j}) \leq \tau_{\omega^j} r_{\omega^j}(x), \quad \forall x \in X.$$

Let $\tau = \max_j \tau_{\omega^j}$, and we get immediately

$$E[\mathrm{dist}(x, S_\omega)] \leq E[\tau\|\min(x, G(x,\omega))\|] \leq \tau\sqrt{g(x)}.$$

Remark 2. From the above theorem, we have

$$\begin{aligned} E[\mathrm{dist}(x_{ERM}, S_\omega)] &\leq \tau\sqrt{g(x_{ERM})} \\ &= \tau\sqrt{\min_{x\in R^n_+} g(x)}, \end{aligned}$$

which implies that the $E[\mathrm{dist}(x_{ERM}, S_\omega)]$ is likely to be small, and hence x_{ERM} of the ERM model can be considered as a robust solution, no matter what realization occurs.

3. Solution algorithm of the ERM model

The ERM model is in general a nonsmooth nonconvex minimization problem on R^n_+, with the objective function

$$g(x) = E[\|\min(x, G(x,\omega))\|^2] = E[H_\omega(x)^T H_\omega(x)].$$

The ERM model can be solved by the smoothing projected gradient (SPG) method (Zhang and Chen, 2009), which is extended from the classical projected gradient (PG) method.

Let $\tilde{g} : R^n \times R_+$ be a smoothing function of g, that is, $\tilde{g}(\cdot, \mu)$ is continuously differentiable in R^n for any $\mu \in R_{++}$, and for any $x \in R^n$,

$$\lim_{z\to x,\ \mu\downarrow 0} \tilde{g}(z, \mu) = g(x), \tag{10}$$

and $\{\lim_{z\to x,\ \mu\downarrow 0} \nabla_x \tilde{g}(z, \mu)\}$ is nonempty and bounded. Here $\nabla_x \tilde{g}(z, \mu)$ is the gradient of $\tilde{g}(\cdot, \mu)$ at point z. Let $[a]_+ = \max(a, 0)$ for any vector a. The SPG method is described in detail as follows.

Algorithm 1. *(SPG algorithm by Zhang and Chen (2009))*

Let $\hat{\varrho}$, ϱ_1 and ϱ_3 be positive constants, where $\varrho_1 << \varrho_3$. Let ϱ_2, σ, σ_1 and σ_2 be constants in $(0,1)$, where $\sigma_1 \leq \sigma_2$. Choose $x^0 \in R^n_+$ and $\mu_0 \in R_{++}$. For $k \geq 0$:

1. If $\|[x^k - \nabla_x \tilde{g}(x^k, \mu_k)]_+ - x^k\| = 0$, let $x^{k+1} = x^k$ and go to step 3. Otherwise, go to step 2.

2. (PG method)

Let $y^{0,k} = x^k$. For $j \geq 0$:

$$y^{j,k}(\alpha) = [y^{j,k} - \alpha \nabla_x \tilde{g}(y^{j,k}, \mu_k)]_+ \tag{11}$$

and $y^{j+1,k} = y^{j,k}(\alpha_{j,k})$ where $\alpha_{j,k}$ is chosen so that,

$$\tilde{g}(y^{j+1,k}, \mu_k) \leq \tilde{g}(y^{j,k}, \mu_k) + \sigma_1 (\nabla_x \tilde{g}(y^{j,k}, \mu_k), y^{j+1,k} - y^{j,k})$$

and

$$\varrho_3 \geq \alpha_{j,k} \geq \varrho_1, \quad \textit{or} \quad \alpha_{j,k} \geq \varrho_2 \bar{\alpha}_{j,k} > 0,$$

such that $\bar{y}^{j+1,k} = y^{j,k}(\bar{\alpha}_{j,k})$ satisfies

$$\tilde{g}(\bar{y}^{j+1,k}, \mu_k) > \tilde{g}(y^{j,k}, \mu_k) + \sigma_2 (\nabla_x \tilde{g}(y^{j,k}, \mu_k), \bar{y}^{j+1,k} - y^{j,k}). \tag{12}$$

If $\frac{\|y^{j+1,k} - y^{j,k}\|}{\alpha_{j,k}} < \hat{\varrho}\mu_k$, set $x^{k+1} = y^{j+1,k}$ and go to step 3.

3. Choose $\mu_{k+1} \leq \sigma \mu_k$.

Each iteration in the SPG algorithm only requires the calculation of the function value of the smoothing function $\tilde{g}$ and its gradient, without any computationally intensive operations. Note that the nonsmoothness of g comes essentially from the "min" operator. Hence its smoothing function $\tilde{g}$ can be constructed easily (see e.g. Chen and Ye, 1999, for reference). We provide a concrete smoothing function $\tilde{g}$ along with its gradient as an example.

Let $\rho(s)$ be the uniform density function

$$\rho(s) = \begin{cases} 1 & \text{if } -\frac{1}{2} \leq s \leq \frac{1}{2} \\ 0 & \text{otherwise.} \end{cases}$$

The Chen-Mangasarian family of smoothing approximation for the "min" operator

$$\min(a,b) = a - \max(0, a-b)$$

can be computed by

$$\begin{aligned} \phi(a,b,\mu) &= a - \int_{-\infty}^{\infty} \max(0, a-b-\mu s)\rho(s) ds \\ &= \begin{cases} b & \text{if } a-b \geq \frac{\mu}{2} \\ a - \frac{1}{2\mu}(a-b+\frac{\mu}{2})^2 & \text{if } -\frac{\mu}{2} < a-b < \frac{\mu}{2} \\ a & \text{if } a-b \leq -\frac{\mu}{2}. \end{cases} \end{aligned}$$

The smoothing function $\tilde{g}$ can then be defined by

$$\tilde{g}(x,\mu) = E[\tilde{H}_\omega(x,\mu)^T \tilde{H}_\omega(x,\mu)], \tag{13}$$

where $\tilde{H}_\omega : R^n \times R_{++} \to R^n$ is given by

$$\tilde{H}_\omega(x,\mu) = \begin{pmatrix} \phi(x_1, G_1(x,\omega), \mu) \\ \vdots \\ \phi(x_n, G_n(x,\omega), \mu) \end{pmatrix}.$$

The gradient of $\tilde{g}$ can be computed by

$$\nabla_x \tilde{g}(x,\mu) = 2E[\nabla_x \tilde{H}_\omega(x,\mu)\tilde{H}_\omega(x,\mu)],$$

where for each $i = 1, 2, \ldots, n$, the ith row of $\nabla_x \tilde{H}_\omega(x,\mu) \in R^{n\times n}$ is defined by

$$\begin{aligned}
&(\nabla_x \tilde{H}_\omega(x,\mu)^T)_{i.} \\
&= I_{i.} - (I_{i.} - \nabla_x G(x,\omega))_{i.} \int_{-\infty}^{\frac{x_i - G_i(x,\omega)}{\mu}} \rho(s)ds \\
&= \begin{cases} (\nabla_x G(x,\omega))_{i.} & \text{if } x_i - G_i(x,\omega) \geq \frac{\mu}{2} \\ I_{i.} - (I_{i.} - \nabla_x G(x,\omega))_{i.}(\frac{x_i - G_i(x)}{\mu} + \frac{1}{2}) & \text{if } -\frac{\mu}{2} < x_i - G_i(x,\omega) < \frac{\mu}{2} \\ I_{i.} & \text{if } x_i - G_i(x,\omega) \leq -\frac{\mu}{2}. \end{cases}
\end{aligned}$$

The SPG method is very easy to implement and attractive for large-scale problems. It is shown by Zhang and Chen (2009) that the SPG method is well-defined and globally convergent to a Clarke stationary point associated with $\tilde{g}$ under mild assumptions. It is worth pointing out that we may not obtain a global optimal point, or even a local minimizer. Nevertheless, if we choose the initial point of the SPG method good enough, e.g., to be a solution of the EV model, we may have a chance to get a more robust solution than that of the EV model.

4. Numerical results

We present our computational results in this section. The purpose of the numerical experiments is to illustrate the characteristics of the ERM model for the user equilibrium assignment under uncertainties in both demand and supply sides, compared with the static Wardrop's user equilibrium model and the EV model.

The semismooth Newton method (e.g. Luca et al., 1996) is adopted for solving the NCPs in order to get a solution x_{EV} of the EV model, and a solution x_ω of static Wardrop's user equilibrium under each scenario. The convergence of the semismooth Newton method for a fixed NCP, e.g., NCP($G(x,\omega)$) is evaluated by the residual

$$r_{2,\omega}(x) = 0.5(\sum_{i=1}^{n} \sqrt{x_i^2 + G_i(x,\omega)^2} - x_i - G_i(x,\omega))^2,$$

which should be close to zero to indicate convergence. Using x_{EV} as an initial point, we employ the SPG method in Algorithm 1 to obtain a solution x_{ERM} of the ERM model, with parameters

$$\mu_0 = 1, \quad \varrho_1 = \frac{1}{2}, \quad \varrho_2 = \frac{1}{4}, \quad \varrho_3 = 10^3, \quad \hat{\varrho} = 10^3, \quad \sigma = \frac{1}{2}, \quad \sigma_1 = \sigma_2 = 10^{-3} \text{ or } 10^{-2}.$$

We stop the SPG algorithm and set $x_{ERM} = x^k$ if $\|x^k - x^{k-1}\| \leq 10^{-12}$ or the total number of iterations exceeds a given maximum iteration which varies for different problems.

4.1. Example 1: a simple 5-link network

To demonstrate the properties of the ERM model, we first use a small tractable 5-link network shown in Fig. 1, which is subject to three discrete one-dimensional random demand vectors.

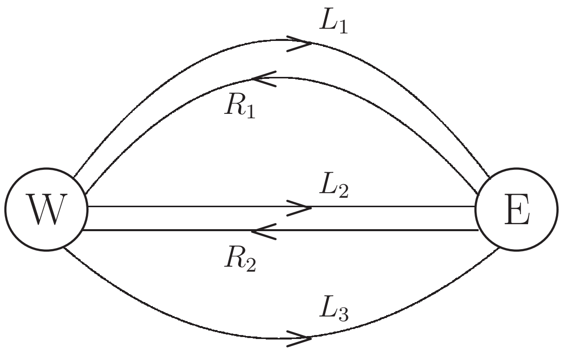

Fig. 1. The 5-link Network for Example 1

There are two two-way roads: a mountain road (L_1, R_1), a sea-side road (L_2, R_2), and one one-way ordinary road L_3 connecting the two cities West and East. The links L_1, L_2 and L_3 direct from West to East, and the links R_1 and R_2 are the returns. Let $\Omega = \{\omega^1, \omega^2, \omega^3\}$ with $\omega^1 = 0$, $\omega^2 = 1$, $\omega^3 = 2$ represent the set of different future scenarios, with probabilities $p_1 = \frac{1}{2}$, $p_2 = \frac{1}{4}$, $p_3 = \frac{1}{4}$, respectively.

The uncertainties in demand and supply sides are mainly due to different demand growth and supply change. The demand $\mathbf{Q}(\omega^1) = (260, 170)^T$ and $\mathbf{Q}(\omega^2) = \mathbf{Q}(\omega^3) = (160, 70)^T$, with the 1st and 2nd components for the demand connecting the OD pair – West to East, and the return, respectively. This example employs the GBPR function (1) with the parameters $n_a \equiv 1$ and $b_a = \frac{1}{2t_a^0}$, where t_a^0 and $C_a(\omega)^{-1}$ are listed in Table 1. We adopt the asymmetric path travel cost function (2) with parameter $\eta = 1$, $\Psi \equiv \mathbf{0}$, and

$$\Lambda(f, \omega) = \begin{pmatrix} 0 & 0 & 0 & 20\alpha(\omega) & 0 \\ 0 & 0 & 0 & 0 & 20\beta(\omega) \\ 0 & 0 & 0 & 0 & 0 \\ 8\alpha(\omega) & 0 & 0 & 0 & 0 \\ 0 & 4\beta(\omega) & 0 & 0 & 0 \end{pmatrix} f,$$

where $\alpha(\omega) = \frac{1}{2}\omega(\omega - 1)$ and $\beta(\omega) = \omega(2 - \omega)$. Here the asymmetric term Λ comes from the interaction of the correlated two-way roads.

Various deterministic assignment patterns from static Wardrop's user equilibrium, the EV, and the ERM models are listed in Table 2. The vector of path choice proportions $\mathbf{p} = (p_k^r)$ is important in the traffic assignment under uncertainty for the planner, which reflects the preference of choosing the possible paths, and plays essential role of generating stochastic traffic flow pattern $\tilde{\mathbf{F}}(\omega)$. We present vectors of path choice proportions in Table 3. We list in Table 4 the values $g(x)$, $E[\|x-x_\omega\|]$, $E[\|V-V_\omega\|]$, $E[\|u-u_\omega\|]$, and $E[\|\tilde{\mathbf{V}}(\omega)-V_\omega\|]$, which indicate the distance of a traffic assignment pattern under uncertainty to the static Wardrop's user equilibrium under each realization. Here the notations $x_\omega = (f_\omega^T, u_\omega^T)^T$, V refers to the link flow, and $\tilde{\mathbf{V}}(\omega) = \Delta\tilde{\mathbf{F}}(\omega)$ presents the stochastic link flow. In this example, it is easy to see that $V_\omega = f_\omega$ and $\tilde{\mathbf{V}}(\omega) = \tilde{\mathbf{F}}(\omega)$.

Table 1
Input network data

Parameter	Link No.				
	1 (L_1)	2 (L_2)	3 (L_3)	4 (R_1)	5 (R_2)
t_a^0	1000	950	1500	1000	1300
$C_a(\omega)^{-1}$	$10 + 40\alpha(\omega)$	$15 + 60\beta(\omega)$	10	$20 + 80\alpha(\omega)$	$25 + 100\beta(\omega)$

Table 2
Various deterministic traffic assignment patterns

Path flow (Link sequence)	traffic assignment patterns				
Travel cost (OD)	x_{ω^1}	x_{ω^2}	x_{ω^3}	x_{EV}	x_{ERM}
f_1 (L_1)	99.3	122.3	15.1	61.9	107.5
f_2 (L_2)	71.3	5.5	102.0	52.5	78.7
f_3 (L_3)	89.3	22.4	42.9	95.6	8.8
f_4 (R_1)	95.8	65.3	16.9	71.7	73.2
f_5 (R_2)	74.2	4.7	53.1	48.3	34.7
u_1 ($W \to E$)	596.7	1612.1	1714.7	1978.1	1540.9
u_2 ($E \to W$)	1057.8	1653.2	1964.2	2558.8	1733.3

Table 3
Various vectors of path choice proportions

Path flow	traffic assignment patterns				
Travel cost	x_{ω^1}	x_{ω^2}	x_{ω^3}	x_{EV}	x_{ERM}
p_1^1	0.5096	0.7651	0.0943	0.2949	0.5515
p_2^1	0.3654	0.0947	0.6373	0.2498	0.4035
p_3^1	0.1250	0.1401	0.2684	0.4553	0.0450
p_4^2	0.6340	0.9331	0.2410	0.5979	0.6786
p_5^2	0.3660	0.0669	0.7590	0.4021	0.3214

Table 4
Robust criteria of various traffic assignment patterns

Variouos criteria	traffic assignment patterns				
	x_{ω^1}	x_{ω^2}	x_{ω^3}	x_{EV}	x_{ERM}
g	2.05e4	3.06e5	7.94e5	1.50e6	1.15e4
$E[\|x - x_\omega\|]$	158.94	311.13	188.00	703.27	295.43
$E[\|V - V_\omega\|]$	64.36	92.93	113.51	108.72	76.89
$E[\|u - u_\omega\|]$	138.15	295.65	144.39	694.42	281.69
$E[\|\tilde{\mathbf{V}}(\omega) - V_\omega\|]$	39.48	99.18	120.98	99.39	55.42

From Tables 2 and 3, we can see that the traffic assignment patterns from the EV and ERM models are quite different. The EV model prefers L_3, the link with the highest free-flow travel time but no capacity variation. While the ERM model allocates little flow on L_3, and predicts lower travel cost compared to the EV model.

It is easy to check that the link travel time function $T(f, \omega)$ is strictly monotone for each scenario, which implies that $x_\omega = (f_\omega^T, u_\omega^T)^T = (V_\omega^T, u_\omega^T)^T$ is unique for each scenario, according to Theorem 6.2 of Aashtiant and Magnanti (1981). That is, $S_{\omega^j} = \{x_{\omega^j}\}$ is a singleton for $j = 1, 2, 3$ and

$$E[\|x - x_\omega\|] = E[\mathrm{dist}(x, S_\omega)].$$

Note that the path travel cost $\Phi_k(f, \omega)$ is a linear function of the path flow f for each k. There must exist a positive scalar τ such that

$$E[\mathrm{dist}(x, S_\omega)] \leq \tau \sqrt{\min_{x \in R_+^n} g(x)}.$$

We find from Table 4 that the SPG method reduces greatly the function value $g(x)$ at x_{ERM} from the initial point x_{EV}. The values of $E[\|x - x_\omega\|]$, $E[\|V - V_\omega\|]$, and $E[\|u - u_\omega\|]$ in Table 4 show that x_{ERM} has much smaller expected distance to scenarios than x_{EV} and therefore can be considered to be more robust. We also notice that from this example the stochastic link flow $\tilde{\mathbf{V}}(\omega)$ obtained from the deterministic flow pattern x_{ERM} is closer to V_ω under realizations than that from x_{EV}.

4.2. Example 2: the Nguyen and Dupuis network

We also illustrate and compare the models by the Nguyen and Dupuis network, which contains 13 nodes, 19 directed links, and 4 OD movements $1 \to 2$, $1 \to 3$, $4 \to 2$, and $4 \to 3$. The free-flow travel time t_a^0, and the mean of link capacity $E[C_a(\omega)]$ of the network are the same as which used by Yin et al. (2009).

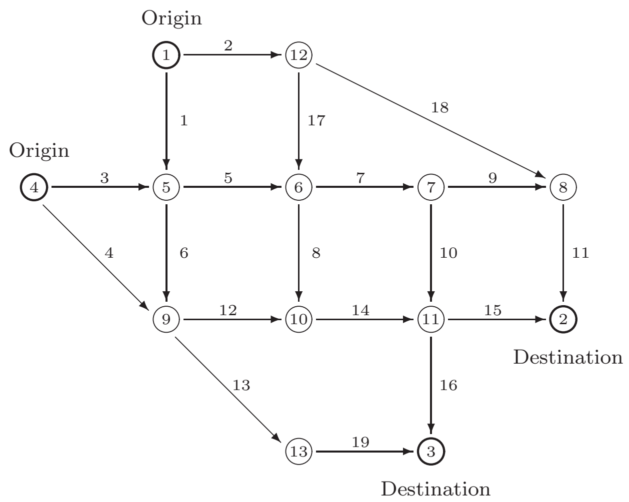

Fig. 2: Nguyen and Dupuis Network

Suppose the planner would like to forecast the robust traffic equilibrium pattern in the next ten years. According to the prediction of economic tendency, the demand vector (with the components following the order of OD movements $1 \rightarrow 2$, $1 \rightarrow 3$, $4 \rightarrow 2$, and $4 \rightarrow 3$) have three possible scenarios

$$\mathbf{Q}^1 = [800\ 800\ 1200\ 1200]^T, \quad \mathbf{Q}^2 = [400\ 1600\ 600\ 400]^T, \quad \mathbf{Q}^3 = [200\ 400\ 300\ 100]^T,$$

with probabilities $p_1 = \frac{1}{4}$, $p_2 = \frac{1}{4}$, and $p_3 = \frac{1}{2}$. Here $\mathbf{Q}^1$ and $\mathbf{Q}^2$ correspond to the optimistic predictions that the economy will flourish and a new port will be built at either destination 2 or 3, respectively. The demand $\mathbf{Q}^3$ corresponds to the pessimistic estimation of future economy.

The link capacity $C_a(\omega)$ follows a log-normal distribution $C_a(\omega) \sim LN(\mu_{c,a}, \sigma_{c,a})$. The probability density function of the log-normal distribution is

$$\Pr(C_a(\omega)|\mu_{c,a}, \sigma_{c,a}) = \frac{1}{C_a(\omega)\sigma_{c,a}\sqrt{2\pi}} \exp\Big(\frac{-(\ln C_a(\omega) - \mu_{c,a})^2}{2\sigma_{c,a}^2}\Big).$$

From the mean $E[C_a(\omega)]$ and the coefficient of variation $CV[C_a(\omega)]$ in Table 5, we can obtain the parameters $\mu_{c,a}$ and $\sigma_{c,a}$ by

$$\mu_{c,a} = \ln(E[C_a(\omega)]) - \frac{1}{2}\ln\Big(1 + (CV[C_a(\omega)])^2\Big) \quad \text{and} \quad \sigma_{c,a} = \sqrt{\ln\Big(1 + (CV[C_a(\omega)])^2\Big)}.$$

We choose for this example the GBPR link travel time function

$$T_a(V, \omega) = t_a^0\Big(1 + 0.15\big(\frac{V_a}{C_a(\omega)}\big)^4\Big),$$

and the nonadditive path travel cost function

$$\Phi_k(f,\omega) = \sum \delta_{a,k} T_a(\Delta f, \omega) + (\sum \delta_{a,k} T_a(\Delta f, \omega))^2 + \Lambda_k(f,\omega).$$

We consider three cases of stochastic environment for the Nguyen and Dupuis network. The coefficients of variation for $C_a(\omega)$ are listed in Table 6 for the three cases. In Cases 1 and 2, the path-specific cost $\Lambda_k(f,\omega) \equiv 0$ for all paths, while $\Lambda_k(f,\omega) = 200$ for paths $k = 1, 9, 14, 20$ and zero for other paths in Case 3.

Table 5
Link cost parameters

Link	Free-flow travel time t_a^0	Link capacity, C_a Mean	Coefficient of Variation Case 1	Case 2	Case 3
1	7.0	800	0.10,	0.30,	0.30
2	9.0	400	0.10,	0.10,	0.10
3	9.0	200	0.10,	0.10,	0.10
4	12.0	800	0.10,	0.30,	0.30
5	3.0	350	0.10,	0.10,	0.10
6	9.0	400	0.10,	0.10,	0.10
7	5.0	800	0.10,	0.10,	0.10
8	13.0	250	0.10,	0.10,	0.10
9	5.0	250	0.10,	0.10,	0.10
10	9.0	300	0.10,	0.10,	0.10
11	9.0	550	0.10,	0.30,	0.30
12	10.0	550	0.10,	0.30,	0.30
13	9.0	600	0.10,	0.10,	0.10
14	6.0	700	0.10,	0.30,	0.30
15	9.0	500	0.10,	0.30,	0.30
16	8.0	300	0.10,	0.10,	0.10
17	7.0	200	0.10,	0.10,	0.10
18	14.0	400	0.10,	0.10,	0.10
19	11.0	600	0.10,	0.10,	0.10

For the expectation operators appeared in EV and ERM, the Monte-Carlo method is employed to randomly generate $N = 1000$ samples of $(\mathbf{Q}(\omega^i), \mathbf{C}(\omega^i))$ for $i = 1, 2, \ldots, N$, where $\mathbf{Q}(\omega^i)$ are chosen from $\mathbf{Q}^1, \mathbf{Q}^2, \mathbf{Q}^3$ with the given probability, and each entry of $\mathbf{C}(\omega^i)$ follows the respective log-normal distribution independently. $G^N(x)$ and $g^N(x)$ are used to approximate $E[G(x,\omega)]$ and $g(x)$ in the EV and ERM model respectively by

$$G^N(x) := \frac{1}{N}\sum_{i=1}^N G(x,\omega^i), \quad g^N(x) := \frac{1}{N}\sum_{i=1}^N \|\min(x, G(x,\omega^i))\|^2.$$

We record the computational results for traffic assignment patterns x_{EV} and x_{ERM} in Table 6, and link flow patterns $v_{EV} = \Delta f_{EV}$ and $v_{ERM} = \Delta f_{ERM}$ in Table 7. Furthermore, we list some robust indicators in Table 8.

Table 6
Traffic assignment patterns of EV and ERM

Path flow (Link sequence)	x_{EV}			x_{ERM}		
Travel cost (OD)	Case 1	Case 2	Case 3	Case 1	Case 2	Case 3
f_1 $(2-18-11)$	365.9	382.1	364.5	276.1	212.9	203.3
f_2 $(1-5-7-9-11)$	34.1	17.9	35.5	34.1	29.0	57.3
f_3 $(1-5-7-10-15)$	0	0	0	0	0	31.8
f_4 $(1-5-8-14-15)$	0	0	0	0	0	21.1
f_5 $(1-6-12-14-15)$	0	0	0	0	0	0
f_6 $(2-17-7-9-11)$	0	0	0	30.2	42.0	23.7
f_7 $(2-17-7-10-15)$	0	0	0	5.5	10.5	9.0
f_8 $(2-17-8-14-15)$	0	0	0	0	2.4	0
f_9 $(4-12-14-15)$	550.2	531.3	529.6	483.2	366.0	214.1
f_{10} $(3-5-7-9-11)$	249.8	268.7	270.4	236.5	216.4	139.1
f_{11} $(3-5-7-10-15)$	0	0	0	0	22.8	128.4
f_{12} $(3-5-8-14-15)$	0	0	0	0	14.5	117.7
f_{13} $(3-6-12-14-15)$	0	0	0	0	0	65.3
f_{14} $(1-6-13-19)$	310.5	311.4	289.9	287.3	273.8	316.9
f_{15} $(1-5-7-10-16)$	289.5	288.6	310.1	262.0	226.2	85.0
f_{16} $(1-5-8-14-16)$	0	0	0	0	0.3	56.1
f_{17} $(1-6-12-14-16)$	0	0	0	0	0	3.7
f_{18} $(2-17-7-10-16)$	0	0	0	14.5	32.7	52.3
f_{19} $(2-17-8-14-16)$	0	0	0	7.7	24.7	33.2
f_{20} $(4-13-19)$	200.0	200.0	200.0	162.9	145.3	114.9
f_{21} $(4-12-14-16)$	0	0	0	0	0	0
f_{22} $(3-6-13-19)$	0	0	0	0	0	0
f_{23} $(3-5-7-10-16)$	0	0	0	0	0	41.6
f_{24} $(3-5-8-14-16)$	0	0	0	0	0	30.9
f_{25} $(3-6-12-14-16)$	0	0	0	0	0	0
u_1 (OD $1 \to 2$)	1449.6	1892.3	2053.6	1249.1	1169.4	1347.4
u_2 (OD $1 \to 3$)	1870.6	2435.8	2620.8	1656.3	1593.4	1821.5
u_3 (OD $4 \to 2$)	1540.6	1574.5	1750.0	1451.4	1433.9	1737.8
u_4 (OD $4 \to 3$)	1279.2	1430.6	1580.6	1182.6	1140.8	1322.3

Table 7
Link flow patterns of EV and ERM

Link	v_{EV}			v_{ERM}		
	Case 1	Case 2	Case 3	Case 1	Case 2	Case 3
1	634.1	617.9	635.5	583.4	529.3	571.8
2	365.9	382.1	364.5	334.1	325.1	321.4
3	249.8	268.7	270.4	236.5	253.6	523.0
4	750.2	731.3	729.6	646.2	511.3	329.0
5	573.4	575.2	616.1	532.6	509.1	709.0
6	310.5	311.4	289.9	287.3	273.8	385.9
7	573.4	575.2	616.1	582.9	579.5	568.1
8	0	0	0	7.7	41.8	259.0
9	284.0	286.6	305.9	300.8	287.4	220.1
10	289.5	288.6	310.1	282.1	292.1	348.0
11	649.8	668.7	670.4	576.9	500.2	423.4
12	550.2	531.3	529.6	483.2	366.0	283.1
13	510.5	511.4	489.9	450.2	419.1	431.8
14	550.2	531.3	529.6	490.9	407.9	542.1
15	550.2	531.3	529.6	488.7	416.1	587.4
16	289.5	288.6	310.1	284.3	283.9	302.7
17	0	0	0	57.9	112.3	118.2
18	365.9	382.1	364.5	276.1	212.9	203.3
19	510.5	511.4	489.9	450.2	419.1	431.8

Table 8
Robust indicators of various traffic assignment patterns for Example 2

Variouos criteria	x_{EV}			x_{ERM}		
	case 1	case 2	case 3	case 1	case 2	case 3
g	5.39e5	1.54e6	1.73e6	5.04e5	5.75e5	5.87e5
$E[\|V - V_\omega\|]$	1.19e3	1.19e3	1.19e3	1.18e3	1.17e3	1.31e3
$E[\|u - u_\omega\|]$	5.04e3	1.14e4	1.14e4	5.00e3	1.13e4	1.13e4
$E[\|\tilde{\mathbf{V}}(\omega) - V_\omega\|]$	468.0	489.6	555.5	468.4	515.3	906.3

We see from Table 6 that both traffic flow patterns f_{EV} and f_{ERM} depend heavily on paths $k = 1, 2, 9, 10, 14, 15, 20$. The difference lies in that in all the three cases f_{ERM} tends to employ more paths than f_{EV}, which alleviates the burden on the above heavily used paths. Moreover, the tendency is strengthened as the variation of link capacity increases in Case 2 and the path-specific cost is involved in Case 3. The ERM model suggests lower travel cost than the EV model. In Table 7, we find that the link flow pattern V_{EV} never uses links 8 and 17 in the three cases, which may due to the relatively high free-flow travel time and limited mean link capacity of the two links. In contrast, the link flow pattern V_{ERM} pays more attention on the two links as the variation of capacity increases and the path-specific cost is added in Case 2 and Case 3. We try to explain the above phenomenon as follows.

The EV model only considers the expected travel cost in deciding which path (eventually which link) to be used. The ERM model considers the weighted distance from the solution sets, that is, ERM considers both probability and magnitude (of flow and minimum cost). Under the ERM, the realization with a low probability may be highly influential on the solution due to the high value of demand and/or minimum travel cost. When the capacity variation is high (i.e. cases 2 and 3), the realization with a very low capacity and low probability may have the same level of impact on the solution of ERM compared to the realization with an average capacity but higher chance. This is due to the fact that ERM considers both the probability of the realization and the magnitude of the violation of the equilibrium condition of the ERM solution. In this case, the low capacity scenario may yield a high travel cost in which the violation of the UE condition under this realization may have a significant influence on the objective function of the ERM (despite its low probability). Thus, it is natural to observe the result in which the ERM spread flows on a larger set of paths.

Now we turn our attention to the data shown in Table 8. It is easy to see that the SPG method succeeds in decreasing the objective function $g(x)$ of the ERM model at x_{ERM} from the initial point x_{EV}. Before analyzing the expected distance, we obtain by direct computation that the Jacobian matrix of the path cost function $\Phi(f,\omega)$ is

$$\nabla\Phi(f,\omega) = (I + 2\mathrm{diag}(\Delta^T T(V,\omega)))\Delta^T \nabla T(V,\omega)\Delta,$$

where $\mathrm{diag}(\Delta^T T(V,\omega))$ is the diagonal matrix with ith diagonal element to be $(\Delta^T T(V,\omega))_i$ for $i = 1, 2, \ldots, |K|$, and $\nabla T(V,\omega)$ is the Jacobian matrix of the link cost function $T(V,\omega)$ with respect to the link flow vector V. Clearly $\nabla\Phi(f,\omega)$ is not a monotone matrix and hence $\Phi(f,\omega)$ is not monotone. Without monotonicity, S_ω may not be a singleton, and the algorithm for solving $\mathrm{NCP}(G(x,\omega))$ may lose efficiency for some realizations ω. The inefficiency of the semismooth Newton method is also verified in our computational experience.

Fortunately, the link flow patterns V_ω and the minimum travel cost u_ω are unique in the first two cases, since $\mathrm{NCP}(G(x,\omega))$ is equivalent to a monotone NCP with additive path travel cost function by Theorem 3.1 of Agdeppa et al. (2007). It is worth mentioning that we do not know whether V_ω is unique in Case 3 because of the nonzero path-specific cost $\Lambda(f,\omega)$. The expected distances $E[\|V - V_\omega\|]$ and $E[\|u - u_\omega\|]$ in the first two cases indicate that (V_{ERM}, u_{ERM}) is closer to (V_ω, u_ω) under different scenarios compared to (V_{EV}, u_{EV}). The ERM model thus provides more robust traffic assignment patterns under stochastic environment in the two cases. For this example, the stochastic link flow $\tilde{\mathbf{V}}(\omega)$ obtained from x_{EV} is a little bit closer than that from x_{ERM}. We just record but do not make any conclusion about the distances in Case 3, since we do not know whether (V_ω, u_ω) is unique or not in this case.

5. Conclusions and further studies

In this paper, we consider the Wardrop's user equilibrium assignment under stochastic environment. We focus on the ERM model, which is flexible to accommodate nonadditive path travel cost and endogenous uncertainties in both the demand and supply sides. By using the ERM model, a deterministic traffic assignment pattern is provided, as well as a stochastic traffic flow pattern by further employing the technique of path choice proportion. We show theoretically the existence and robustness of the solution obtained by the ERM model under some conditions.

Compared with the EV and BW models for traffic assignment under uncertainty, the robustness of x_{ERM} is provided theoretically for the first time in the sense that the expected distance $E[\mathrm{dist}(x_{ERM}, S_\omega)]$ tends to be small. We apply the SPG method for solving the ERM model. Numerical experiments on the two small-size examples show that the SPG method is effective. Moreover, we find that the traffic assignment patterns from the EV and ERM models are quite different, and the pattern from the ERM model is more robust than that from the EV model.

There exist some future extensions worth pursuing based on this paper. Firstly, mild assumptions to guarantee Assumption 2 are needed to make the robustness of ERM model more applicable. Secondly, it is an interesting task to modify the formulation of the objective function of the model to reduce the effect of the magnitude of violation from a realization with low probability on the final solution. Thirdly, it is also worthwhile to develop a more efficient SPG algorithm for real-sized network by taking into account of the special traffic network structure.

Acknowledgements

This work was supported in part by the Hong Kong Research Grant Council, the National Natural Science Foundation of China (70871008), and the National Basic Research Program of China (2010CB732501).

References

Aashtiant, H.Z., Magnanti, T.L., 1981, Equilibria on a congested transportation network, SIAM Journal on Algebraic and Discrete Methods 2, 213-226.

Agdeppa, R.P., Yamashita, N., Fukushima, M., 2007, The traffic equilibrium problem with nonadditive costs and its monotone mixed complementarity problem formulation, Transportation Research Part B 41, 862-874.

Aghassi M., Bertimas, D., 2006, Robust game theory, Mathematical Progamming, Series B 107, 231-273.

Chen, B., 2001, Error bounds for R_0-type and monotone nonlinear complementarity problem, Journal of Optimization Theory and Applications 108, 297-316.

Chen, X., Fukushima, M., 2005, Expected residual minimization method for stochastic linear complementariy problems, Mathematics of Operations Research 30(4), 1022-1038.

Chen, X., Ye, Y., 1999, On homotopy-smoothing methods for box-constrained variational inequalities, SIAM Journal on Control and Optimization 37, 589-616.

Chen, X., Zhang, C., Fukushima, M., 2009, Robust solution of monotone stochastic linear complementarity problems, Mathematical Programming, Series B 117, 51-80.

Clark S.D., Watling, D.P., 2005, Modelling network travel time reliablity under stochastic demand, Transportation Research Part B 39(2), 119-140.

Fenton, L.F., 1960, The sum of log-normal probability distributions in scatter transmission systems, IEEE Transactions on Communication Systems 8, 57-67.

Fernando O., Nichlàs, E.S., 2006, Robust Wardrop equilibrium, Technical Report, Department of Industrial and Systems Engineering, University of Southern California, illposed.usc.edu./˜ fordon/docs/rwe.pdf.

Ferris M.C., Pang J.S., 1997, Engineering and economic applications of complementarity problems, SIAM Review 39(4), 669-713.

Gabriel, S.A., Bernstein, D., 1997, The traffic equilibrium problem with nonadditive path costs, Transportation Science 31, 337-348.

Lam, W.H.K., Shao, H., Sumalee, A., 2008, Modeling impacts of adverse weather conditions on a road network with uncertainties in demand and supply, Transportation Research Part B 42(10), 890-910.

Lo, H.K., Chen, A., 2000, Traffic equilibrium problelm with route-specific costs: formulation and algorithms, Transportation Research Part B 34, 493-513.

Lo, H.K., Tung, Y.K., 2003, Network with degradable links: capacity analysis and design, Transportation Research Part B 37(4), 345-363.

Luca, D.T., Facchinei, F., Kanzow, C., 1996, A semismooth equation approach to the solution of nonlinear complementarity problems, Mathematical programming 75(3), 407-438.

Shao, H., Lam, W.H.K., Tam, M.L., 2006, A reliability-based stochastic traffic assignment model for network with multiple user classes under uncertainty in demand, Networks and Spatial Economics 6(3-4), 173-204.

Sumalee, A., Connors, R.D., Luathep, P., 2009a, Network equilibrium under cumulative prospect theory and endogenous stochastic demand and supply, in Lam etal (eds), Transportation and Traffic Theory 2009: Golden Jabilee, Springer.

Sumalee, A., Luathep, P., Lam, W.H.K, Connors, R.D., 2009b, Transport network capacity evaluation and design under demand uncertainty, Transportation Research Record, forthcoming.

Wardrop, J.G., 1952, Some theoretical aspects of road traffic research, Proceedings of Institution of Civil Engineers, Part II, 1, 325-378.

Watling, D.P., 2002, A second order stochastic network equilibrium model, I: Theoretical foundation, Transportation Science 36(2), 149-166.

Yin, Y.F., Madanat, S.M., Lu, X.Y., 2009, Robust improvement schemes for road networks under demand uncertainty, European Journal of Operations Research 198, 470-479.

Zhang, C., Chen, X., 2008, Stochastic nonlinear complementarity problem and applications to traffic equilibrium under uncertianty, Journal of Optimization Theory and Applications 137, 277-295.

Zhang, C., Chen, X., 2009, Smoothing projected gradient method and its application to stochastic linear complementarity problems, SIAM Journal on Optimization 20(2), 627-649.

Zhou, Z., Chen, A., 2008, Comparative analysis of three user equilibrium models under stochastic demand, Journal of Advanced Transportation 42(3), 239-263.