Existence of Solutions to Underdetermined Equations and Spherical Designs

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Underdetermined Equations

$c : R^n \to R^m$ **is continuously differentiable,** $m < n$**.**

$$c(x) = 0 \tag{1}$$

Suppose

$$c(\hat{x}) \approx 0$$

and $c'(\hat{x})$ has full row rank, i.e

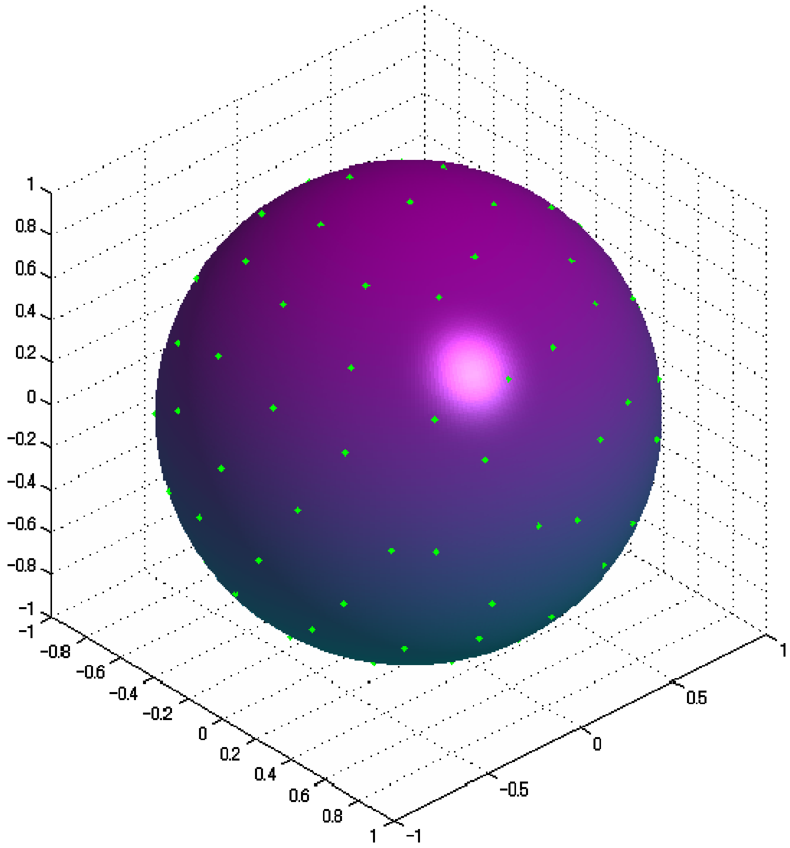
$$\text{rank}(c'(\hat{x})) = m.$$

We are interested in the existence of a solution of (1) in a neighborhood of $\hat{x}$.

Underdetermined equations arising in construction of spherical t-design on the unit sphere

$$\mathbf{S^2} = \{y \in R^3 \;:\; \|y\|_2 = 1\}$$

$\mathbf{P}_t$:polynomials in 3 variables, restricted to S^2.



A spherical t-design(Delsarte-Goethals 1977)
is a set of N points $\{y_1, y_2, \ldots, y_N\} \subset S^2$
such that

$$\int_{S^2} p(y)dy = \frac{4\pi}{N} \sum_{j=1}^{N} p(y_j)$$

for every polynomial $p \in \mathbf{P}_t$.

A quadrature rule, exact for all $p \in P_t$

The existence of a spherical t-design was proved (Seymour-Zaslavsky,1984).

Open Questions:

For Given t, number of points ?

how to find $\{y_1, y_2, \ldots, y_N\}$.

Our interest

What is the best choice of points on the sphere for interpolatory integration rule

The dimension of the space $\mathbf{P}_t$ is $(t+1)^2$.

Lower bound on the smallest number N_t^* of points required to give a spherical t-design are

$$N_t^* \geq \frac{(t+1)(t+3)}{4} \text{ if } t \text{ is odd}$$

$$N_t^* \geq \frac{(t+2)^2}{4} \text{ if } t \text{ is even.}$$

Our aim

Prove the existence of spherical t-designs with $(t+1)^2$ points and good conditioning interpolation matrix.

Main Results

1 Verify the existence of solutions to (1) using the Brouwer fixed point theorem.

2 Reformulate the problem finding a spherical t-design with $(t+1)^2$ points as underdetermined equations (1) with $m = (t+1)^2 - 1$ and $n = 2(t+1)^2 - 3$.

3 Provide error bounds of a computed spherical t-design to an exact spherical t-design.

4 Apply to high-order numerical integration on the sphere.

1. Verification method for $c(x) = 0$

$\hat{x}$: a computed solution of $c(x) = 0$.

$\mathcal{B}$: an index set $\{k_1, k_2, \ldots, k_m\}$ such that

$$c'_{\mathcal{B}}(\hat{x}) \in R^{m\times m} \text{ is nonsingular.}$$

$\mathcal{N} = \{1, 2, \ldots, n\}/\mathcal{B}$.

$X = \{\, x \mid \|x_{\mathcal{B}} - \hat{x}_{\mathcal{B}}\| \le r_1, \|x_{\mathcal{N}} - \hat{x}_{\mathcal{N}}\| \le r_2\}$.

Theorem 1 Suppose that $c : R^n \to R^m$ is continuously differentiable, $c'(\hat{x}) = m$ and

$$\|c'_{\mathcal{B}}(x) - c'_{\mathcal{B}}(\hat{x})\| \le K\|x - \hat{x}\|, \text{ for } x \in X.$$

Let

$$h := \|c'_{\mathcal{B}}(\hat{x})^{-1}\|(\frac{1}{2}K(r_1 + r_2)r_1 + \max_{x\in X}\|c'_{\mathcal{N}}(x)\|r_2).$$

- There is a solution of (1) in X if $\|c'_{\mathcal{B}}(\hat{x})^{-1}c(\hat{x})\| + h \le r_1$.
- There is no solution of (1) in X if $\|c'_{\mathcal{B}}(\hat{x})^{-1}c(\hat{x})\| - h > r_1$.

2. Reformulation

Given positive integer t

$$d_t = (t+1)^2 := \text{dim}\mathbf{P}_t$$

A set of points $Y = \{y_1, \ldots, y_{d_t}\} \subset S^2$

A spherical parametrization $\theta_j \in [0, \pi]$

and $\phi_j \in [0, 2\pi]$ of Y has

$$n = 2d_t - 3$$

variables as

$$y_j = \begin{bmatrix} \cos\phi_j \sin\theta_j \\ \sin\phi_j \sin\theta_j \\ \cos\theta_j \end{bmatrix},$$

$\theta_1 = 0, \phi_1 = 0$ and $\phi_2 = 0$ and

$$(\theta_2, \phi_3, \ldots, \phi_{d_t}, \theta_{d_t}) =: x = (x_1, x_2, \ldots, x_n).$$

2. Reformulation

Let

$$J_t(z) = \frac{1}{4\pi}\sum_{i=0}^{t}(2i+1)L_i(z), \qquad z \in [-1,1],$$

where $L_j : [-1,1] \to R$ is the Legendre polynomial.

The Gram matrix $G(x) \in R^{d_t \times d_t}$,

$$G_{i,j}(x) = J_t(y_i^T y_j) \qquad x \in R^{2d_t-3}$$

Let

$$E = \begin{pmatrix} 1 & -1 & 0 & \dots & 0 \\ 1 & 0 & -1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 1 & 0 & \dots & 0 & -1 \end{pmatrix} \qquad R^{(d_t-1)\times d_t}$$

and

$$e = (1,1,\dots,1)^T \qquad R^{d_t}$$

2. Reformulation

Theorem 2 Suppose that $G(x^*)$ is nonsingular. Then x^* corresponds to a spherical $t-$design with $(t+1)^2$ points if and only if x^* is a solution of

$$c(x) = EG(x)e = 0.$$

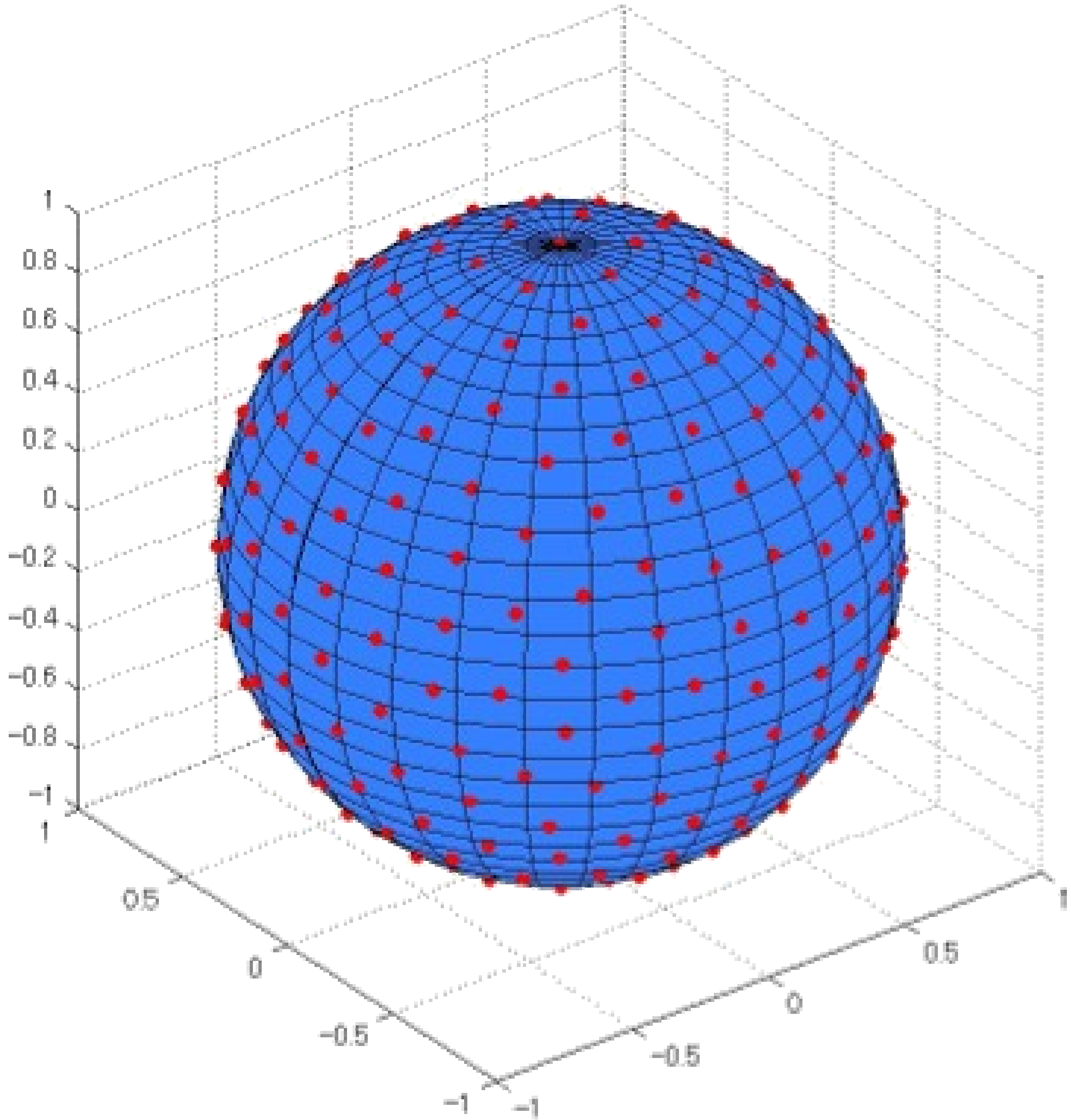


Image Spherical 16-design with 289 points

3.Error bounds for Computed Spherical Designs

Find spherical designs near the extremal points $\bar{Y}$:

$$\text{logdet} G(\bar{Y}) = \max_{Y \subset S^2} \text{logdet}(G(Y)).$$

$G(\bar{Y})$ is symmetric positive definite.

$$\text{logdet} G(\bar{Y}) \approx (\leq)\, (t+1)^2 \log(\frac{(t+1)^2}{4\pi})$$

Let

$\bar{x}$ correspond to the extremal points $\bar{Y}$

x^* correspond to the exact spherical designs Y^*

$\hat{x}$ correspond to the computed spherical designs $\hat{Y}$.

3.Error bounds for Computed Spherical Designs

Starting from $\bar{x}$, we find $\hat{x}$ such that

$G(\hat{x})$ is nonsingular, and

$$\rho := K\|c'_{\mathcal{B}}(\hat{x})^{-1}c(\hat{x})\|\|c'_{\mathcal{B}}(\hat{x})^{-1}\| \leq \frac{1}{2}$$

Then

$$\|\hat{Y} - Y^*\|_\infty \leq 2\|\hat{x} - x^*\|_\infty \leq 2r_1$$

where

$$r_1 = \frac{1 - \sqrt{1 - 2\rho}}{K\|c'_{\mathcal{B}}(\hat{x})^{-1}\|}.$$

4. Numerical integration on the sphere

The condition for the integration rule

$$Q_{d_t}(f) = \sum_{i=1}^{d_t} w_i f(\widehat{y}_i)$$

to be exact for all polynomials in $\mathbf{P}_t$ is that w is the solution of

$$G(\widehat{x})w = e.$$

Theorem 3. Suppose that $G(\widehat{x})$ is nonsingular. Let $w = G(\widehat{x})^{-1}e$. Then

$$\max_{1\le i\le d_t} |w_1 - w_i| \le \frac{4}{\|G(\widehat{x})e\|_\infty} \|G(\widehat{x})^{-1}\|_\infty \|c(\widehat{x})\|_\infty.$$

Our Numerical results

$$|w_i - \frac{4\pi}{d_t}| \le \max_{1\le i\le d_t} |w_1 - w_i| \le 10^{-13}$$

for $t \le 20$.

Weights are positive and almost same!

4. Numerical integration on the sphere

Sloan-Womersley (2004) The worst-case

for the interpolatory integration rule

$$\left|\int_{S^2} f(y)d(y) - \frac{4\pi}{d_t}\sum_{j=1}^{d_t} f(\hat{y}_j)\right| = 4\pi D(\hat{Y}) =: e(E_t),$$

where $D(\hat{Y})$ is the Cui-Freeden generalized

discrepancy for the points $\hat{Y}$

$$D(\hat{Y}) = \frac{1}{2\sqrt{\pi}d_t}\left[\sum_{j=1}^{d_t}\sum_{i=1}^{d_t}\left(1 - 2\log(1 + \sqrt{(1 - \hat{y}_i^T\hat{y}_j)/2})\right)\right]^{1/2}$$

Numerical results

$$D(\hat{Y}) < D(\bar{Y}) < D(CF(Y))$$

Table 1 :Extremal point $\bar{x}$, computed spherical design $\widehat{x}$,exact spherical design x^*, $\|\widehat{x} - x^*\| \leq r_1$, $x \in R^{2(t+1)^2-3}$

t	$\|c(\hat{x})\|$	logdet$G(\bar{x})$	logdet$G(\hat{x})$	r_1
2	4.4e-16	-3.21	-3.21	1.0e-15
3	2.6e-15	3.38	2.57	2.3e-15
4	7.3e-15	16.13	15.93	1.8e-14
5	7.5e-15	36.17	35.48	1.3e-14
6	2.6e-14	64.09	62.64	3.4e-14
7	6.0e-14	100.69	100.41	5.0e-14
8	1.9e-13	146.19	144.36	1.1e-13
9	4.5e-13	201.55	186.22	1.8e-13
10	8.0e-13	266.31	265.50	6.1e-11
11	2.7e-12	342.15	341.67	1.8e-13
12	6.0e-12	428.03	427.41	1.4e-12
13	1.4e-11	525.16	524.27	2.4e-11
14	3.7e-11	633.52	632.71	5.5e-11
15	9.6e-11	753.48	752.48	3.08e-11

Table 2: Worst case for the equal-weight rule E_t and generalized discrepency for Computed spherical designs

t	d_t	$e(E_t)$	$D(\hat{Y})$
2	9	0.349478	0.027811
3	16	0.229009	0.018239
4	25	0.162440	0.012927
5	36	0.123579	0.009834
6	49	0.098188	0.007814
7	64	0.079817	0.006352
8	81	0.067223	0.005349
9	100	0.058809	0.004680
10	121	0.049576	0.003945
11	144	0.043472	0.003459
12	169	0.038495	0.003063
13	196	0.034438	0.002741
14	225	0.031033	0.002469
15	256	0.028180	0.002242