Sparse Solutions of Linear Complementarity Problems

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Abstract

This paper considers the characterization and computation of sparse solutions and least-p-norm $(0 < p < 1)$ solutions of the linear complementarity problems LCP(q, M). We show that the number of non-zero entries of any least-p-norm solution of the LCP(q, M) is less than or equal to the rank of M for any arbitrary matrix M and any number $p \in (0, 1)$, and there is $\bar{p} \in (0, 1)$ such that all least-p-norm solutions for $p \in (0, \bar{p})$ are sparse solutions. Moreover, we provide conditions on M such that a sparse solution can be found by solving convex minimization. Applications to the problem of portfolio selection within the Markowitz mean-variance framework are discussed.

Keywords: Linear complementarity problem, sparse solution, nonconvex optimization, restricted isometry property.

MSC2010 Classification: 90C33, 90C26.

1 Introduction

Given an $n \times n$ matrix M and an n-dimensional vector q, the linear complementarity problem (LCP) is to find $x \in R^n$ such that

$$Mx + q \geq 0, \quad x \geq 0 \quad \text{and} \quad x^T(Mx + q) = 0.$$

We denote the problem by LCP(q, M), its solution set by SOL(q, M) and its feasible set by FEA$(q, M) = \{x \,|\, Mx + q \geq 0, x \geq 0\}$. The LCP has many applications in engineering and economics. Moreover, the LCP plays a key role in optimization theory and presents optimality conditions for constrained quadratic programs [10, 13].

The solution set SOL(q, M) often has an infinite number of solutions when it is nonempty. Finding a special solution in the solution set for different goals has a long and rich history. Most readers are familiar with the least norm solution, which is defined by

$$\begin{array}{ll} \min & \|x\|_2^2 \\ \text{s.t.} & x \in \text{SOL}(q, M). \end{array} \tag{1.1}$$

For the monotone LCP where M is positive semi-definite, it is known that the solution set SOL(q, M) is a convex polyhedra and has a unique least norm solution. Algorithms for finding

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the least norm solution of the monotone LCP have been studied extensively [10]. It is worth noting that some attractive interior point methods are developed to find a maximal solution that has the number of positive components in (x, s) with $s = Mx + q$ is maximal [19].

In this paper, we consider the sparsity of solutions of the LCP. We call $\bar{x}$ $\in$SOL(q, M) a **sparse solution** of the LCP(q, M) if $\bar{x}$ is a solution of the following optimization problem

$$\begin{array}{ll} \min & \|x\|_0 \\ \text{s.t.} & x \in \text{SOL}(q, M), \end{array} \tag{1.2}$$

where $\|x\|_0$ =number of nonzero components of x.

Sparse solutions of the Z-matrix LCP(q, M) have been studied [7, 10] and used in dynamic linear complementarity systems [8, 18]. A square matrix is called a Z-matrix if its off-diagonal entries are non-positive. A vector $\bar{x} \in$ SOL(q, M) is called a least element solution of the LCP(q, M), if $\bar{x} \leq x$ for all $x \in$ SOL(q, M). If M is a Z-matrix, and SOL$(q, M) \neq \emptyset$, then SOL(q, M) has a unique least element solution which is the unique sparse solution of the LCP(q, M) and the unique solution of the following linear program [7, 10]

$$\begin{array}{ll} \min & e^T x \\ \text{s.t.} & Mx + q \geq 0, \quad x \geq 0, \end{array} \tag{1.3}$$

where e is the vector whose all entries are one. In other words, if M is a Z-matrix, then the unique least ℓ_1 norm solution in the feasible set FEA(q, M) is the unique sparse solution in the solution set SOL(q, M). Moreover, if M is a positive semi-definite Z-matrix, then the least element solution is the least norm solution of the LCP(q, M). However, little theoretical results and algorithms are known for sparse solutions of the LCP(q, M) when M is not a Z-matrix.

The function $\|x\|_0$ is discontinuous and brings difficulties to analyze the models and algorithms. The ℓ_p $(0 < p < 1)$ norm

$$\|x\|_p^p = \sum_{i=1}^n |x_i|^p$$

has been used as a continuous approximation function to $\|x\|_0$ in sparse approximation and representation. The concavity of $\|x\|_p^p$ can provide desirable sparsity. Hence it is interesting to study the relation between the sparse solutions of (1.2) and solutions of the following optimization problem

$$\begin{array}{ll} \min & \|x\|_p^p \\ \text{s.t.} & x \in \text{SOL}(q, M). \end{array} \tag{1.4}$$

We call a solution of (1.4) a **least-p-norm solution**.

It is known that finding a sparse solution of a system of linear equations is NP-hard [1, 3, 4]. Recently, Ge et al show that finding a solution of

$$\begin{array}{ll} \min & \|x\|_p^p \\ \text{s.t.} & Ax = b, \, x \geq 0 \end{array} \tag{1.5}$$

for $0 < p < 1$ is also NP-hard [14], where $A \in R^{m \times n}, b \in R^m$. From [1, 3, 4, 14], we can say that finding a sparse solution and a least-p-norm solution of the LCP(q, M) is NP-hard, since we can construct M, q such that

$$S := \{x \,|\, Mx + q = 0, \, x \geq 0\} \subseteq \text{SOL}(q, M),$$

and solving $\min_{x \in S} \|x\|_0$ is NP-hard by using the argument in [14]. Sparse solutions of linear equations have been studied extensively in the last decades [1, 3, 4, 14, 16]. Candes and Tao give sufficient conditions on the coefficient matrix A such that a sparse solution of the system of linear equations can be found by ℓ_1 minimization.

In contrast with the fast development in sparse solutions of optimization and linear equations, sparse solutions of the LCP seem to lack theory and algorithms.

The aim of this paper is to present properties of the sparse solutions and least-p-norm solutions of the LCP(q, M) and computation methods for finding the sparse solutions. In particular, we show that the number of non-zero entries of any least-p-norm solution of the LCP(q, M) is less than or equal to the rank of M for any matrix M and any number $p \in (0, 1)$, and there is $\bar{p} \in (0, 1)$ such that all least-p-norm solutions of (1.4) for $p \in (0, \bar{p})$ are sparse solutions of (1.2). Moreover, we provide conditions on M such that a sparse solution can be found by solving convex minimization.

This paper is related to the problem of finding sparse solutions to quadratic programs. Due to the optimality conditions, sparse solutions of the LCP are sparse solutions of convex constrained quadratic programs, which have important applications in portfolio optimization. The classic Markowitz portfolio optimization is formulated as the following quadratic program [17]

$$\begin{array}{ll} \min & w^T C w \\ \text{s.t.} & e^T w = 1, \quad r^T w = \rho, \\ & w \geq 0 \end{array} \tag{1.6}$$

where C is the covariance matrix of the return on the assets in the portfolio, w is the vector of portfolio weights that represent the amount of capital to be invested in each asset, r is the vector of expected returns of the different assets and ρ is a given total return. Sparsity is important for investors who often want to select limited assets for their investment. However, finding a sparse solution of (1.6) is a challenging problem for which many approaches have been proposed such as penalty regularized optimization, mixed integer quadratic programs, quadratic programs with constraints $\|w\|_0 \leq k$ for a given integer k [2, 5]. Since C is a symmetric positive semi-definite matrix, problem (1.6) is equivalent to the LCP(q, M) with

$$M = \begin{pmatrix} C & -B^T \\ B & 0 \end{pmatrix}, \quad B = \begin{pmatrix} e^T \\ r^T \\ -e^T \\ -r^T \end{pmatrix}, \quad q = \begin{pmatrix} \mathbf{0} \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \quad x = \begin{pmatrix} w \\ y \end{pmatrix},$$

where y is the Lagrange multiplier. Hence, sparse solutions of the Markowitz mean-covariance portfolio optimization are closely related to sparse solutions of the LCP.

It is easy to see that $q \geq 0$ if and only if $x = 0$ is the unique least-p-norm solution and the unique sparse solution. To avoid the triviality, we assume that $x = 0$ is not a solution of the LCP(q, M). Moreover, we assume the solution set SOL(q, M) is nonempty.

In section 2, we will show the sparsity of solutions of (1.2) and (1.4) for an arbitrary matrix M. In section 3, we study the sparsity of solutions of (1.2) and (1.4) for a symmetric positive semi-definite matrix M. In section 4, we show that if M is positive semi-definite and $M + M^T$ satisfies restricted orthogonality [4], then a sparse solution of the LCP(q, M) can be found by solving convex minimization. In section 5, we propose a computation procedure for finding a sparse solution of convex quadratic programs by solving quadratic programs and linear programs.

For a solution $\tilde{x} \in$SOL(q, M), we define the following index set:

$$J \; = \; \{i : \; \tilde{x}_i > 0\}.$$

We define the diagonal matrix D whose diagonal elements are

$$D_{ii} = \begin{cases} 1, & i \in J \\ 0, & \text{otherwise.} \end{cases}$$

Let J^c denote the complementarity set of J and $|J|$ the number of elements of J. Let e denote the vector whose all entries are one.

2 Arbitrary matrix M

From [10, p.98, 144], the solution set SOL(q, M) of an arbitrary LCP(q, M) is a union of a finite number of convex polyhedra. Since a convex polyhedron has only finite many extreme points, there are only finite many extreme points in the solution set SOL(q, M). We say x is an extreme point of SOL(q, M) if x does not lie in any open line segment joining two points of SOL(q, M). In general, SOL(q, M) is not a convex set. If x is an extreme point of SOL(q, M), then x is an extreme point of a convex polyhedron.

Lemma 2.1 *All least-p-norm solutions of the LCP(q, M) are extreme points of SOL(q, M).*

Proof: Let $\widetilde{x}$ be a least-p-norm solution. Suppose there exist $y, z \in$ SOL(q, M) such that $\widetilde{x} = \lambda y + (1 - \lambda)z$ for some $0 < \lambda < 1$. Recall that t^p is strictly concave for $t \geq 0$. Then it follows

$$\|\widetilde{x}\|_p^p = \sum_{j=1}^{n} (\lambda y_j + (1 - \lambda) z_j)^p \geq \lambda \sum_{j=1}^{n} y_j^p + (1 - \lambda) \sum_{j=1}^{n} z_j^p = \lambda \|y\|_p^p + (1 - \lambda)\|z\|_p^p \geq \|\widetilde{x}\|_p^p,$$

where the last inequality uses that $\widetilde{x}$ is a least-p-norm solution. Furthermore, the above equalities hold if and only if $y = z = \widetilde{x}$, which indicates that $\widetilde{x}$ is an extreme point of SOL(q, M). ∎

Theorem 2.1 *Let $\widetilde{x}$ and $\bar{x}$ be a least-p-norm solution and a sparse solution of the LCP(q, M). Then $\|\widetilde{x}\|_0 \leq$ rank(M) for $p \in (0, 1)$. Moreover, there is a $\bar{p} \in (0, 1)$ such that $\|\widetilde{x}\|_0 = \|\bar{x}\|_0$ for all $p \in (0, \bar{p})$.*

Proof: Note that we can choose a permutation matrix $U \in R^{n \times n}$ such that

$$UDU^T = \begin{pmatrix} I_{J,J} & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad UMU^T = \begin{pmatrix} M_{J,J} & M_{J,J^c} \\ M_{J^c,J} & M_{J^c,J^c} \end{pmatrix}.$$

Thus

$$U(I - D + DM)U^T = \begin{pmatrix} M_{J,J} & M_{J,J^c} \\ 0 & I \end{pmatrix}. \tag{2.1}$$

Since the LCP(Uq, UM) and the LCP(q, M) are equivalent, without loss of generality, we assume $U = I$ in (2.1), $J = \{1, 2, \ldots, k\}$ and

$$M = \begin{pmatrix} M_{J,J} & M_{J,J^c} \\ M_{J^c,J} & M_{J^c,J^c} \end{pmatrix}, \quad \widetilde{x} = \begin{pmatrix} \widetilde{x}_J \\ 0 \end{pmatrix}, \quad q = \begin{pmatrix} q_J \\ q_{J^c} \end{pmatrix}.$$

Note that $\widetilde{x}_J > 0$. It follows $M_{J,J}\widetilde{x}_J + q_J = 0$. If $\text{rank}(M_{\cdot,J}) < |J|$, then there exists a nonzero vector $h \in R^{|J|}$ such that $M_{\cdot,J}h = 0$, i.e., $M_{J,J}h = 0$ and $M_{J^c,J}h = 0$. Furthermore, by $\widetilde{x}_J > 0$, we can choose a sufficiently small real positive number δ_0 such that for all $|\delta| \leq \delta_0$,

$$\begin{aligned}
&\widetilde{x}_J + \delta h > 0, \\
&M_{J,J}(\widetilde{x}_J + \delta h) + q_J = M_{J,J}\widetilde{x}_J + q_J = 0, \\
&M_{J^c,J}(\widetilde{x}_J + \delta h) + q_{J^c} = M_{J^c,J}\widetilde{x}_J + q_{J^c} \geq 0.
\end{aligned}$$

Hence $([\widetilde{x}_J + \delta h]^T, 0)^T \in R^n_+$ is also a solution of the LCP(q, M) for $|\delta| \leq \delta_0$. Notice that $\widetilde{x}$ is a least-p-norm solution of the LCP(q, M). It leads to

$$\|\widetilde{x}\|_p^p = \min_{t \in (-\delta_0, \delta_0)} \|([\widetilde{x}_J + th]^T, 0)^T\|_p^p =: f(t), \quad 0 < p < 1.$$

It is impossible since

$$f''(t) = p(p-1)\sum_{i=1}^{|J|}(\widetilde{x}_i + th)^{p-2}(h_i)^2 < 0.$$

Hence we have $\text{rank}(M_{\cdot,J}) \geq |J|$, which implies that $\|\widetilde{x}\|_0 \leq$rank(M).

Now we prove the second part of this theorem.

We show that there is a number $\bar{p} \in (0, 1)$ such that any least-p-norm solution $\widetilde{x}$ is a sparse solution for $p \in (0, \bar{p})$.

By Lemma 2.1, all least-p-norm solutions of the LCP(q, M) are extreme points of SOL(q, M) for $p \in (0, 1)$. Let $\{x^1, x^2, \ldots, x^m\}$ be the set of extreme points of SOL(q, M). Then we have for all $\bar{x}$

$$\|\bar{x}\|_p^p \geq \min\{\|x^1\|_p^p,\ \|x^2\|_p^p,\ \ldots,\ \|x^m\|_p^p\} = \|\widetilde{x}\|_p^p. \tag{2.2}$$

If there is not a number $\bar{p} \in (0, 1)$ such that any least-p-norm solution $\widetilde{x}$ is a sparse solution for $p \in (0, \bar{p})$, then there are a sequence $\{p_i\}, p_i > 0, p_i \to 0$ as $i \to \infty$ and a sequence $\{x^{j_i}\}$ of extreme points of SOL(q, M) such that x^{j_i} is a least-p-norm solution and

$$\|x^{j_i}\|_0 > \|\bar{x}\|_0. \tag{2.3}$$

Since there are only finite many extreme points in SOL(q, M), without loss of generality, we assume $x^{j_i} = x^j$. However, we cannot have (2.3), since (2.2) implies

$$\|\bar{x}\|_0 = \lim_{p_i \downarrow 0} \|\bar{x}\|_{p_i}^{p_i} \geq \lim_{p_i \downarrow 0} \|x^j\|_{p_i}^{p_i} = \|x^j\|_0.$$

Hence, the second part of this theorem holds. Moreover, this together with $\|\widetilde{x}\|_0 \leq$ rank(M) for $p \in (0, 1)$ implies $\|\bar{x}\|_0 \leq$ rank(M). We complete the proof.

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From Lemma 2.1 and Theorem 2.1, we can have the following corollary.

Corollary 2.1 *There is an extreme point $\bar{x}$ of SOL(q, M) such that $\bar{x}$ is a sparse solution of the LCP(q, M).*

We use the following example to explain Theorem 2.1.

Example 2.1 Consider the LCP(q, M) with

$$M = \begin{pmatrix} 1 & 3 & 0 \\ 1 & 3 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad q = \begin{pmatrix} -4 \\ -4 \\ -1 \end{pmatrix}.$$

The solution set: $\mathrm{SOL}(q,M)=S_1 \bigcup S_2$ where

$$S_1 = \Big\{(x_1,x_2,0)^T:\ x_1+3x_2=4,\ x_1>1, x_2\ge 0\Big\},\quad S_2=\Big\{(1,1,x_3)^T:\ x_3\ge 0\Big\}.$$

The sparse solution: $\bar{x}=(4,0,0)^T$.

The least-p-norm solutions: $\widetilde{x}=(1,1,0)^T$ for $p>\frac{1}{2}$; $\widetilde{x}=(1,1,0)^T$ or $\widetilde{x}=(4,0,0)^T$ for $p=\frac{1}{2}$; and $\widetilde{x}=(4,0,0)^T$ for $0<p<\frac{1}{2}$.

The number of non-zero components in the sparse solution and all least-p-norm solutions is one or two, which is less than or equal to $\mathrm{rank}(M)$=2. Moreover, $\|\widetilde{x}\|_0=\|\bar{x}\|_0$ for all least-p-norm solutions with $p\in(0,\frac{1}{2})$.

Let us consider other $\mathrm{LCP}(q,M)$ with

$$M=\begin{pmatrix} 1 & 3 & 0 & 0 & 1\\ 1 & 3 & 0 & 0 & 1\\ 1 & 0 & 0 & 0 & 1\\ 0 & 1 & 0 & 0 & 1\\ 3 & 3 & 0 & 0 & 1\end{pmatrix},\qquad q=\begin{pmatrix} -4\\ -4\\ -1\\ -1\\ -6\end{pmatrix}.$$

The solution set: $\mathrm{SOL}(q,M)=S_1\bigcup S_2\bigcup S_3$ where

$$S_1=\Big\{(1,1,x_3,x_4,0)^T:\ x_3\ge 0, x_4\ge 0\Big\},\qquad S_2=\Big\{(1,x_2,0,0,3-3x_2)^T:\ 0\le x_2\le 1\Big\}$$

$$S_3=\Big\{(0,0,x_3,x_4,6)^T:\ x_3\ge 0,x_4\ge 0\Big\}.$$

The sparse solution: $\bar{x}=(0,0,0,0,6)^T$.

The least-p-norm solutions: $\widetilde{x}=(1,1,0,0,0)^T$ for $p>\frac{1}{\log_2 6}$; $\widetilde{x}=(1,1,0,0,0)^T$ or $\widetilde{x}=(0,0,0,0,6)^T$ for $p=\frac{1}{\log_2 6}$; and $\widetilde{x}=(0,0,0,0,6)^T$ for $0<p<\frac{1}{\log_2 6}$.

The number of non-zero components in the sparse solution and all least-p-norm solutions is less than or equal to $\mathrm{rank}(M)$=2. Moreover, $\|\widetilde{x}\|_0=\|\bar{x}\|_0$ for all least-p-norm solutions with $p\in(0,\frac{1}{\log_2 6})$.

3 Symmetric positive semi-definite matrix M

If M is a positive semi-definite matrix and the feasible set $\mathrm{FEA}(q,M)$ is nonempty, then the solution set is nonempty and convex [10]. The convexity of the solution set with the following lemma provides more desirable properties of sparse solutions and least-p-norm solutions of the $\mathrm{LCP}(q,M)$.

Lemma 3.1 *[10, Theorem 3.1.7,Theorem 3.4.4] Let M be symmetric positive semi-definite. Then $Mx^1=Mx^2$ for any $x^1,x^2\in SOL(q,M)$ and the columns of $M_{\cdot,\alpha}$ are linear dependent for each index set α with $detM_{\alpha,\alpha}=0$.*

Theorem 3.1 *Suppose that M is symmetric positive semi-definite. Let $\bar{x}$ be a sparse solution of the $LCP(q,M)$. With the index set J and diagonal matrix D, the following statements hold.*

(i) $M_{J,J}$ is nonsingular;

(ii) $\bar{x}=-(I-D+DM)^{-1}Dq$;

(iii) $\|\bar{x}\|_1\le L\|q\|_1$,

$$\text{where } L=\max\Big\{\ \|M_{\alpha,\alpha}^{-1}\|_1:\ M_{\alpha,\alpha} \text{ is nonsingular for } \alpha\subseteq\{1,\dots,n\}\Big\};$$

(iv) There is no another solution $x\in\mathrm{SOL}(q,M)$ with $\alpha=\{i:\ \ x_i>0\}$ such that $\alpha\subseteq J$.

The statements also hold for any least-p-norm solution of the $LCP(q,M)$.

Proof: (i) Following the proof of Theorem 2.1, without loss of generality, we assume $U = I$ in (2.1), $J = \{1, 2, \ldots, k\}$ and

$$M = \begin{pmatrix} M_{J,J} & M_{J,J^c} \\ M_{J^c,J} & M_{J^c,J^c} \end{pmatrix}, \quad \bar{x} = \begin{pmatrix} \bar{x}_J \\ 0 \end{pmatrix}, \quad q = \begin{pmatrix} q_J \\ q_{J^c} \end{pmatrix}.$$

Note that $\bar{x}_J > 0$. It follows $M_{J,J}\bar{x}_J + q_J = 0$. If $M_{J,J}$ is singular, then from Lemma 3.1, the columns of $M_{\cdot,J}$ are linearly dependent, and there exists a nonzero vector $h \in R^{|J|}$ such that $M_{\cdot,J}h = 0$. Define

$$\tau = \min_{h_i \neq 0, 1 \leq i \leq |J|} \frac{\bar{x}_i}{|h_i|}.$$

It is easy to verify that

$$\bar{x}_J \pm \tau h \geq \bar{x}_J - \tau|h| \geq 0, \quad M_{J,J}(\bar{x}_J \pm \tau h) + q_J = 0, \quad M_{J^c,J}(\bar{x}_J \pm \tau h) + q_{J^c} \geq 0.$$

Hence $([\bar{x}_J \pm \tau h]^T, 0)^T \in R^n_+$ is also a solution of the LCP(q, M) but $\|\bar{x}_J - \tau h\|_0 < \|\bar{x}\|_0$ or $\|\bar{x}_J + \tau h\|_0 < \|\bar{x}\|_0$. Hence, these together imply that $M_{J,J}$ is nonsingular.

(ii) From the nonsingularity of $M_{J,J}$, expression (2.1) with $U = I$ implies that $I - D + DM$ is nonsingular and

$$(I - D + DM)^{-1}D = \begin{pmatrix} M_{J,J} & M_{J,J^c} \\ 0 & I \end{pmatrix}^{-1} D = \begin{pmatrix} M_{J,J}^{-1} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.1}$$

From $(I - D)\bar{x} + D(M\bar{x} + q) = 0$ and (3.1), we obtain the desired results.

(iii) From (3.1), we have

$$\|(I - D + DM)^{-1}D\|_1 \leq \max\left\{ \|M_{\alpha,\alpha}^{-1}\|_1 : \ M_{\alpha,\alpha} \text{ is nonsingular for } \alpha \subseteq \{1, \ldots, n\} \right\},$$

which together with (ii) implies (iii).

(iv) Assume that there is another solution $\hat{x} \in \mathrm{SOL}(q, M)$ with $\alpha = \{i : \ \hat{x}_i > 0\}$ such that $\alpha \subseteq J$. From the proof of (i), without loss of generality, assume

$$M = \begin{pmatrix} M_{\alpha,\alpha} & M_{\alpha,\beta} & M_{\alpha,J^c} \\ M_{\alpha,\beta} & M_{\beta,\beta} & M_{\beta,J^c} \\ M_{J^c,\alpha} & M_{J^c,\beta} & M_{J^c,J^c} \end{pmatrix}, \qquad q = \begin{pmatrix} q_\alpha \\ q_\beta \\ q_{J^c} \end{pmatrix}, \qquad J = \alpha \cup \beta.$$

It is easy to verify that both $\hat{x}_J$ and $\bar{x}_J$ are solutions of the LCP$(q_J, M_{J,J})$. However $M_{J,J}$ is nonsingular and positive definite, and the LCP$(q_J, M_{J,J})$ has a unique solution. This is a contradiction.

Using the same argument and the proof of Theorem 2.1, we can see the same statements hold for any least-p-norm solution $\widetilde{x}$ of SOL(q, M).

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Corollary 3.1 *All sparse solutions of the LCP(q, M) are extreme points of SOL(q, M) if M is symmetric positive semi-definite.*

Proof: Let $\bar{x}$ be a sparse solution. Assume that there exist $y, z \in \mathrm{SOL}(q, M)$ such that $\bar{x} = \lambda y + (1 - \lambda)z$ for some $0 < \lambda < 1$. Since $\bar{x}$ is a sparse solution, this means $\bar{x}, y, z$ have the same support sets. However, from (iv) of Theorem 3.1 the support sets of $\bar{x}$, y, z are same if and only if $\bar{x} = y = z$. This is a contradiction. Hence $\bar{x}$ is an extreme point of SOL(q, M). ∎

From Theorem 2.1 and Theorem 3.1, if M is symmetric positive semi-definite, the number of non-zero components of any sparse solution and least-p-norm solution of the LCP(q, M) is less than or equal to $\max\{\text{rank}(M_{\alpha,\alpha}) : \quad \alpha \subseteq \{1, 2, \ldots, n\}\}$. We use the following example to explain the sparsity.

Example 3.1 Consider the LCP(q, M) with

$$M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad q = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

It is easy to see $\max\{\text{rank}(M_{\alpha,\alpha}) : \quad \alpha \subseteq \{1, 2\}\} = 1$.

The solution set: $\text{SOL}(q, M) = \{(x_1, x_2)^T : \quad x_1 + x_2 = 1,\ x_1, x_2 \geq 0\}$.

The sparse solution: $\bar{x} = \{(1, 0)^T,\ (0, 1)^T\}$.

The least-p-norm solution: $\widetilde{x} = \{(1, 0)^T,\ (0, 1)^T\}$ for $0 < p < 1$.

For $p = 1$, each solution in SOL(q, M) is the least ℓ_1 norm solution. For $p > 1$, $(\frac{1}{2}, \frac{1}{2})^T$ is the least ℓ_p norm solution.

Let us consider other LCP(q, M) with

$$M = \begin{pmatrix} 5 & -1 & 1 \\ -1 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \qquad q = \begin{pmatrix} -4 \\ 0 \\ -2 \end{pmatrix},$$

and $\max\{\text{rank}(M_{\alpha,\alpha}) : \quad \alpha \subseteq \{1, 2, \ldots, 3\}\} = 2$.

The solution set: $\text{SOL}(q, M) = \{(x_1, x_2, x_3)^T : \quad x_1 = \lambda + (1 - \lambda)\frac{2}{3},\ x_2 = \lambda,\ x_3 = (1 - \lambda)\frac{2}{3},\ 0 \leq \lambda \leq 1\}$.

The sparse solution: $\bar{x} = \{(\frac{2}{3}, 0, \frac{2}{3})^T,\ (1, 1, 0)^T\}$.

The least-p-norm solution: $\widetilde{x} = \{(\frac{2}{3}, 0, \frac{2}{3})^T\}$ for $0 < p < 1$, which is also the least ℓ_1 norm solution and the least ℓ_p norm solution for $p \geq 1$.

Remark 1 The sparsity of solutions of the LCP(q, M) is sensitive with the data (q, M). Consider the following LCP(q, M) with a symmetric positive semi-definite matrix

$$M = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 1 & 0 \\ 3 & 0 & 9 \end{pmatrix}, \qquad q = \begin{pmatrix} -2 \\ -1 \\ -3 \end{pmatrix}.$$

The sparse solution is $\bar{x} = (1, 0, 0)^T$ with $\|\bar{x}\|_0 = 1$. However, for $q + \varepsilon e$ with $0 < \varepsilon < \frac{3}{2}$, the sparse solution is $\bar{x} = (1 - 2\varepsilon/3, 0, \varepsilon/9)^T$ with $\|\bar{x}\|_0 = 2$. It is known that the solvability of the monotone LCP(q, M) is stable for nonnegative noise in q, since the feasibility implies the solvability for the monotone LCP(q, M). However, the sparsity of solutions of the monotone LCP(q, M) can change with any small positive noise in q.

4 Computation of sparse solutions

In this section, we show that we can find a sparse solution of the LCP(q, M) where M is a positive semi-definite matrix by solving a convex quadratic program and a linear program if the matrix $M + M^T$ satisfies the **s-restricted isometry property (RIP)** and **s,s'-restricted orthogonality (RO)**. An $m \times n$ matrix A is said to satisfy the s-RIP with a restricted isometry constant δ_s if for every $m \times |\Lambda|$ submatrix A_Λ of A and for every vector $z \in R^{|\Lambda|}$ with $|\Lambda| \leq s$,

$$(1 - \delta_s)\|z\|_2^2 \leq \|A_\Lambda z\|_2^2 \leq (1 + \delta_s)\|z\|_2^2. \tag{4.1}$$

Moreover, A is said to satisfy the s,s'-RO with a restricted orthogonality constant $\theta_{s,s'}$ for $s+s' \le n$ if for all submatrices $A_\Lambda \in R^{m\times|\Lambda|}$, $A_{\Lambda'} \in R^{m\times|\Lambda'|}$ of A with $|\Lambda| \le s$, $|\Lambda'| \le s'$ and for all vectors $z \in R^{|\Lambda|}$, $z' \in R^{|\Lambda'|}$

$$|(A_\Lambda z, A_{\Lambda'} z')| \le \theta_{s,s'} \|z\|_2 \|z'\|_2 \tag{4.2}$$

holds for all disjoint sets Λ and Λ'.

The concepts of s-RIP and s,s'-RO were introduced by Candes and Tao [4] and are used in many applications of sparse representations.

The LCP(q, M) can be equivalently written as a quadratic program

$$\begin{array}{ll} \min & x^T M x + q^T x \\ \text{s.t.} & Mx + q \ge 0,\ x \ge 0 \end{array} \tag{4.3}$$

in the sense that x^* a solution of the LCP(q, M) if and only if x^* is an optimal solution of (4.3) with the optimal value of zero. If M is a positive semi-definite matrix, then (4.3) is a convex quadratic program.

From Theorem 3.1.7 in [10], the solution set SOL(q, M) for a positive semi-definite matrix M equals to

$$\text{SOL}(q, M) = \left\{ x \in R^n_+ \mid Mx + q \ge 0,\ (M + M^T)x = c,\ q^T x = \gamma \right\}, \tag{4.4}$$

where $c = (M + M^T)x^*$, $\gamma = q^T x^*$ and x^* is an arbitrary solution of the LCP(q, M).

We consider the following linear program

$$\begin{array}{ll} \min & e^T x \\ \text{s.t.} & Mx + q \ge 0,\ x \ge 0,\ (M + M^T)x = c,\ q^T x = \gamma. \end{array} \tag{4.5}$$

Theorem 4.1 *Suppose that M is positive semi-definite. Let $\hat{x}$ be a solution of the linear program (4.5) with $\|\hat{x}\|_0 \le s$.*

(i) If $(M + M^T)$ satisfies the RIP with a restricted isometry constant $\delta_{2s} < 1$, then $\hat{x}$ is the unique sparse solution of the LCP(q, M).

(ii) If $(M + M^T)$ satisfies the RIP and RO with

$$\delta_s + \theta_{s,s'} + \theta_{s,2s'} < 1, \tag{4.6}$$

then $\hat{x}$ is the unique solution of the linear program (4.5) and the unique sparse solution of the LCP(q, M).

Proof: (i) From (4.4), we know that $\hat{x}$ is a solution of the LCP(q, M). Assume on contradiction that there is a sparse solution of the LCP(q, M) such that $\bar{x} \ne \hat{x}$. Then $\|\bar{x}\|_0 \le \|\hat{x}\|_0 \le s$ and $(M+M^T)(\hat{x} - \bar{x}) = 0$. Let the support set of $\hat{x} - \bar{x}$ be K. Then $|K| \le 2s$. Hence $\|\hat{x} - \bar{x}\|_0 \le 2s$, which together with the RIP, yields

$$(1 - \delta_{2s})\|\hat{x} - \bar{x}\|_2^2 = (1 - \delta_{2s})\|(\hat{x} - \bar{x})_K\|_2^2$$
$$\le \|(M + M^T)_K (\hat{x} - \bar{x})_K\|_2^2 = \|(M + M^T)(\hat{x} - \bar{x})\|_2^2 = 0.$$

This is contradiction to $\hat{x} \ne \bar{x}$. Therefore $\hat{x}$ is the unique sparse solution of the LCP(q, M).

(ii) From Theorem 1.3 in [4], $\hat{x}$ is the unique solution of the following linear program

$$\begin{array}{ll} \min & \|x\|_1 \\ \text{s.t.} & (M+M^T)x = c. \end{array} \tag{4.7}$$

Since the convex feasible set of (4.5) is contained in the convex set $\{x \,|\, (M+M^T)x = c\}$, $\hat{x}$ is also the unique solution of the linear program (4.5).

From Lemma 1.1 in [4], the condition in (4.6) implies $\delta_{2s} < 1$. Hence, from (i) of this theorem, $\bar{x}$ is the unique sparse solution of the LCP(q, M). ∎

Corollary 4.1 *Suppose that M is symmetric positive semi-definite. Let $\hat{x}$ with $\|\hat{x}\|_0 \le s$ be a solution of the linear program*

$$\begin{array}{ll} \min & e^T x \\ \text{s.t.} & x \ge 0,\, Mx = c,\, q^T x = \gamma, \end{array} \tag{4.8}$$

where $c = Mx^*$, $\gamma = q^T x^*$ and x^* is an arbitrary solution of the LCP(q, M).

(i) If M satisfies the RIP with a restricted isometry constant $\delta_{2s} < 1$, then $\hat{x}$ is the unique sparse solution of the LCP(q, M).

(ii) If M satisfies the RIP and RO with (4.6) then $\hat{x}$ is the unique solution of the linear program (4.8) and the unique sparse solution of the LCP(q, M).

Proof: From Theorem 3.1.7 in [10], the solution set SOL(q, M) for a symmetric positive semi-definite matrix M equals to

$$\mathrm{SOL}(q, M) = \left\{ x \in R^n_+ \,|\, Mx = c,\, q^T x = \gamma \right\}. \tag{4.9}$$

Following the proof of Theorem 4.1, we can obtain the desirable results. ∎

Example 4.1 *Consider the LCP(q, M) with*

$$M = \begin{pmatrix} 0.4 & -0.3 & 0.1 \\ -0.3 & 0.3 & -0.3 \\ 0.1 & -0.3 & 0.7 \end{pmatrix}, \qquad q = \begin{pmatrix} -0.4 \\ 0.3 \\ -0.1 \end{pmatrix}.$$

The solution set is SOL$(q, M) = \left\{ (1,0,0)^T + \lambda(2,3,1)^T : \quad \lambda \ge 0 \right\}$.

The restricted isometry constants are $\delta_1 = 0.4901$ and $\delta_2 = 0.8421$. From (i) of Corollary 4.1, $\bar{x} = (1,0,0)^T$ is the unique sparse solution of the LCP(q, M).

Remark 2. For an $m \times n$ matrix A, the concept of Spark(A) is also often used in the study of sparse solutions, which is defined as the smallest possible number such that there exists a subgroup of columns from A that are linearly dependent [11].

Suppose that M is positive semi-definite. Let $\hat{x}$ be a solution of the LCP(q, M) with $\|\hat{x}\|_0 \le \frac{1}{2}\mathrm{Spark}(M + M^T)$. Then $\hat{x}$ is a sparse solution of the LCP(q, M). This statement can be shown as follows.

Suppose x' is another solution of the LCP(q, M), then from (4.4), $(M+M^T)(\bar{x} - x') = 0$, which implies $\|\bar{x} - x'\|_0 \ \ge \mathrm{Spark}(M + M^T)$ and

$$\|x'\|_0 \ge \mathrm{Spark}(M + M^T) - \|\hat{x}\|_0 \ge \frac{1}{2}\mathrm{Spark}(M + M^T) \ge \|\hat{x}\|_0.$$

Similarly, if $\|\hat{x}\|_0 < \frac{1}{2}\text{Spark}(M+M^T)$, then $\hat{x}$ is the unique sparse solution of the LCP(q, M).

For a symmetric positive semi-definite M, if $\hat{x}$ is a solution of the LCP(q, M) with $\|\hat{x}\|_0 \leq \frac{1}{2}\text{Spark}(M)$ then $\hat{x}$ is a sparse solution of the LCP(q, M). Moreover, the strict inequality implies the uniqueness.

Example 4.2 *Consider the LCP(q, M) with*

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \qquad q = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}.$$

The solution set: $SOL(q, M) = \left\{(x_1, x_2, x_3)^T : \; x_1 + x_2 + x_3 = 1, \quad x_1 \geq 0, x_2 \geq 0, x_3 \geq 0\right\}$.

Spark$(M) = 2$, and $(1, 0, 0)^T, (0, 1, 0)^T, (0, 0, 1)^T$ are sparse solutions of the LCP(q, M).

5 Sparse solutions of quadratic programs

In this final section, we apply theorems in the last sections to sparse solutions of the following quadratic program

$$\begin{array}{ll} \min & \frac{1}{2}z^T H z + c^T z \\ \text{s.t.} & Az \geq b \\ & z \geq 0 \end{array} \tag{5.1}$$

where $H \in R^{m\times m}$ is positive semi-definite, $c \in R^m, A \in R^{k\times m}, b \in R^k$. This quadratic program includes the Markowitz mean-covariance portfolio optimization problem (1.6) as a special case. Let S_{QP} be the solution set of (5.1). We say $\bar{z}$ is a sparse solution of the quadratic program (5.1) if

$$\|\bar{z}\|_0 = \min\{\, \|z\|_0 : \quad z \in S_{QP}\}.$$

The quadratic program (5.1) is equivalent to the LCP(q, M) with

$$M = \begin{pmatrix} H & -A^T \\ A & 0 \end{pmatrix}, \quad q = \begin{pmatrix} c \\ -b \end{pmatrix}, \quad x = \begin{pmatrix} z \\ y \end{pmatrix},$$

where $y \in R^k$ is the Lagrange multiplier. Note that $M + M^T = \begin{pmatrix} H & 0 \\ 0 & 0 \end{pmatrix}$. The solution set SOL$(q, M)$ equals to

$$\text{SOL}(q, M) = \left\{x \in R^n_+ \mid Mx + q \geq 0,\, Hz = w,\, q^T x = \gamma\right\},$$

where $w = Hz^*$, $\gamma = q^T x^*$ and $x^* = (z^*, y^*)$ is an arbitrary solution of the LCP(q, M).

We consider the following linear program

$$\begin{array}{ll} \min & e^T x \\ \text{s.t.} & Mx + q \geq 0,\, x \geq 0,\, Hz = w,\, q^T x = \gamma. \end{array} \tag{5.2}$$

Let $\hat{x} = (\hat{z}, \hat{y})$ be a solution of the linear program (5.2) with $\|\hat{z}\|_0 \leq s$. According to Theorem 4.1, we have the following statements.

(i) If H satisfies the RIP with a restricted isometry constant $\delta_{2s} < 1$, then $\hat{z}$ is the unique sparse solution of the quadratic program (5.1).

(ii) If H satisfies the RIP and RO with (4.6) then $\hat{z}$ is the unique sparse solution of the quadratic program (5.1) and all solutions $x^* = (y^*, z^*)$ of the linear program (5.2) have the same component $z^* = \hat{z}$.

Based on the statements above and (4.3), we propose the following procedure to find a sparse solution of (5.1).

1. Find a solution x^* of the LCP(q, M) by solving the following quadratic program

$$\begin{array}{ll} \min & z^T H z + c^T z - b^T y \\ \text{s.t.} & Az \geq b,\ Hz - A^T y \geq -c,\ z, y \geq 0. \end{array} \tag{5.3}$$

2. Find a solution of the linear program (5.2).

We use the following code in Matlab to generate a solution $z \in R^m$ with $\|z\|_0 = s$ of (5.1), a positive semi-definite matrix H, a matrix $A \in R^{k\times m}$, and vectors $c \in R^m, b \in R^k$.

```
k=fix(m/5); s=fix(m/3); z=zeros(m,1); P=randperm(m);
z(P(1:2*s+m/10))=abs(randn(2*s+m/10,1)); H=randn(m,m); H=H*diag(z)*H';
A=randn(k-1,m); A=[A;-ones(1,m)]; z=zeros(m,1);
z(P(1:s))=abs(randn(s,1)); b=A*z; c=-H*z.
```

For each m, k, s, we generated 100 independent test problems by the code. The convex quadratic program (5.3) and the linear program (5.2) are solved by the Matlab code `quadprog` and `linprog` with initial iterate x_0 =zeros$(n, 1)$. Preliminary numerical results are reported in Table 1. In the last line of Table 1, we report $\|z_{LP}\|_0$, $n_1; n_2$ where z_{LP} is the numerical solution of the linear program (5.2), n_1 is the average of $\|z_{LP}\|_0$ for the 100 test problems and n_2 is the number of test problems with $\|z_{LP}\| \leq s$.

Table 1: 100 independent tests for each (m, k, s)

m	80	90	100	110	120	130	140	150
k	16	18	20	22	24	26	28	30
rank(H)	60	69	76	83	92	99	106	115
s	26	30	33	36	40	43	46	50
$\|z_{LP}\|_0$	25.9; 100	30; 100	33; 99	35.9; 100	40.2;98	42.9;100	46.2;99	50.3; 99

The numerical testing is performed using MATLAB R2011b on a Lenovo PC (Intel Quad CPU Q9550, 2.83GHz, 4.00GB of RAM). The numerical results are encouraging for the study of sparse solutions of the LCP, although the matrix H generated by the Matlab code may not satisfy the RIP and RO conditions.

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