REGULARIZED MATHEMATICAL PROGRAMS WITH STOCHASTIC EQUILIBRIUM CONSTRAINTS: ESTIMATING STRUCTURAL DEMAND MODELS

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Abstract. The article considers a particular class of optimization problems involving set-valued stochastic equilibrium constraints. We develop a solution procedure that relies on an approximation scheme for the equilibrium constraints. Based on regularization, we replaces the approximated equilibrium constraints by those involving only single-valued Lipschitz continuous functions. In addition, sampling has the further effect of replacing the 'simplified' equilibrium constraints by more manageable ones obtained by implicitly discretizing the (given) probability measure so as to render the problem computationally tractable. Convergence is obtained by relying, in particular, on the graphical convergence of the approximated equilibrium constraints. The problem of estimating the characteristics of a demand model, a widely studied problem in micro-econometrics, serves both as motivation and illustration of the regularization and sampling procedure.

Key words. Stochastic equilibrium, monotone linear complementarity problem, graphical convergence, sample average approximation, regularization.

AMS subject classifications. 90C33, 90C15.

1. Introduction. Solving mathematical optimization involving equilibrium constraints is generally challenging and the design of solutions procedures to deal with such problems when the equilibrium constraints involve set-valued stochastic mappings brings along a new level of difficulty. Such problems arise from many important applications in economics, for example, the 'inverse' problem in micro-econometrics: given the prices and the decisions of the agents, is it possible to infer their utility functions?

This leads us to consider the following mathematical program with stochastic equilibrium constraints (MPSEC):

$$
\begin{array}{ll} \min_{x\in X} & \frac{1}{2}\langle x, Hx\rangle + \langle c, x\rangle \\ \text{subject to} & A_t\mathbb{E}[S_t(\boldsymbol{\xi}, x)] \ni b_t, \quad t = 1, \dots, T, \end{array} \tag{1.1}
$$

where $c \in \mathbb{R}^\nu$, $A_t \in \mathbb{R}^{m\times n}$, $b_t \in \mathbb{R}^m$, H is a positive semi-definite $\nu \times \nu$-matrix, $X \subseteq \mathbb{R}^\nu$ is a compact set, $\boldsymbol{\xi} : \Omega \to \Xi \subseteq \mathbb{R}^\ell$ is a random vector whose realizations are denoted by ξ and $(\Xi, \mathcal{F}, P)$ is the induced probability space. The potential consumers decisions, in market t, are described by the set-valued mappings

$$
S_t(\xi, x) = \operatorname{argmax}_s \{ \langle s, u_t(\xi, x)\rangle \mid \langle e, s\rangle \le 1,\ s \ge 0 \} \subseteq \mathbb{R}^n, \tag{1.2}
$$

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resulting from utility maximization where the coefficients of the objective functions, $u_t : \Xi \times \mathbb{R}^\nu \to \mathbb{R}^n$, are (given) continuous functions, $e = (1, \dots, 1) \in \mathbb{R}^n$ and

$$\mathbb{E}[S_t(\boldsymbol{\xi}, x)] = \{\mathbb{E}[s(\boldsymbol{\xi}, x)] \,|\, s(\boldsymbol{\xi}, x) \in S_t(\boldsymbol{\xi}, x),\ s(\cdot, x)\ P\text{-summable selection of } S_t(\cdot, x)\}$$

is the Aumann's (set-valued) expectation of $S_t(\,\cdot, x)$ [1] with respect to $\boldsymbol{\xi}$.

In the pure characteristics demand problem one seeks to estimate the parameters of the consumers' utility functions u_t [3, 11, 18]. The constraint $A_t\mathbb{E}[S_t(\boldsymbol{\xi}, x)] \ni b_t$, with A_t being the identity matrix, represents the market share equations and the jth component(s) of the solution(s) in $S_t(\xi, x)$ of (1.2) is the probability that the consumer purchases product j in market t given the environment ξ. The linear program (1.2) models the consumer's decision process, in market t, to acquire the product(s), that yield the highest utility given environment ξ. The variable x is split in two parts: $x^1 \in \mathbb{R}^{\nu_1}$ and $x^2 \in \mathbb{R}^{\nu_2}$. The vector x^1 describes the product characteristics or demand shocks that are observed by the providers (firms) and consumers but not explicitly identifiable in the data; the vector x^2 models the observed consumer's preferences or taste for any particular product based on its characteristics and price. The pure characteristics demand problem is to estimate x^2 while minimizing the errors associated with the demand shocks x^1. The objective function in this model has $c = 0$ and $H = \mathrm{diag}(H_1, H_2)$, where H_1 is a $\nu_1 \times \nu_1$ positive definite matrix and H_2 is the $\nu_2 \times \nu_2$ zero matrix.

There are quite a number of challenges one has to deal with to solve such a problem. To begin with the solution of (1.2), for any fixed (ξ, x) *is not necessarily unique*. In fact it is generally set-valued. Consider a simple example: $u_t(\xi, x) = (\xi_1 + x, \xi_2) \in \mathbb{R}^2$, where $\xi_1 \in \mathbb{R}$ and $\xi_2 > 0$, the solution set has the form

$$S_t(\xi, x) = \begin{cases} (1, 0) & x > \xi_2 - \xi_1, \\ \{(\alpha, 1 - \alpha) \,|\, \alpha \in [0, 1]\} & x = \xi_2 - \xi_1, \\ (0, 1) & x < \xi_2 - \xi_1. \end{cases}$$

One cannot find a single-valued function $s(\xi, x) \in S_t(\xi, x)$ which is continuous with respect to x. The use of a sample average approximation (SAA) scheme to approximate the market share equations as proposed in the present literature becomes intractable. Another major difficulty comes from the fact that all solution sets $S_t(\xi, x)$ for all $t = 1, \dots, T$ share the same x-variables.

Market share equations play an important role in demand estimation in economics [3, 11, 18]. The 'inverse' problem: to infer from consumers' choices their utility functions is a fundamental issue in economics. A scheme based on finding "nested fixed-points" has been proposed to obtain just approximating solutions but such an approach turned out to be computationally ineffective. Recently, Pang et al. [18] proposed a mathematical programming with linear complementarity constraints (MPLCC) approach for the pure characteristics demand model given a finite number of observations ξ^i, $i = 1, \dots, N$. Their approach provides a promising computational method to estimate the consumer utility provided that in all markets t and environments ξ, the (optimal) choice of each individual consumer turns out to be just one single product. Using their approach with a finite number of observations ξ^i, $i = 1, \dots, N$, and basic solutions, we can express (1.1)-(1.2) in terms of the following mathematical program with linear equilibrium constraints

$$\text{(1.3)} \qquad \begin{array}{ll} \min_{\,x \in X} & \frac{1}{2}\langle x, Hx\rangle + \langle c, x\rangle \\ \text{subject to} & A_t \frac{1}{N} \sum_{i=1}^N \hat{S}_t(\xi^i, x) \ni b_t, \quad t = 1, \dots, T,\ \xi^i \in \Xi,\ i = 1, \dots, N, \end{array}$$

where

$$\hat{S}_t(\xi, x) = \{\text{argmin}\|s\|_0 \,|\, s \in S_t(\xi, x)\},$$

here $\|s\|_0$ is the ℓ^0-norm which counts the number of nonzero entries in s. With the constraints $\langle e, s\rangle \le 1$ and $s \ge 0$, clearly, the linear program (1.2) always has a basic optimal solution $s(\xi, x)$ and any such basic optimal solution has just a single variable taking on the value 1 while all others are 0 when $\max_{1\le i\le n} u_i(\xi, x) > 0$. One could refer to a solution of this type, $s(\xi, x) \in \hat{S}_t(\xi, x)$, as a "sparse solution" of (1.2). However, the use of such 'sparse solutions' raises questions when there is, in fact, a multiplicity of optimal solutions. For example, when for $j = 1, 2, 3$, $(u_t(\xi, x))_j = \max_{1\le i\le n}(u_t(\xi, x))_i > 0$, why would the probability that a consumer purchases one of these three products be 1 and 0 for the others? Should the choice probability not be 1/3, for example, for each one of the three products? Other question arise about the consistency of the solutions of the MPLCC problem (1.3) to the given problem (1.1) as the sample size N goes to infinity.

Motivated by the MPLCC approach [18] and the preceding questions, we reformulate problem (1.1) as the following mathematical program with stochastic linear complementarity constraints (MPSLCC)

$$\begin{array}{rl} \min_{x\in X} & \frac{1}{2}\langle x, Hx\rangle + \langle c, x\rangle \\ \text{subject to} & A_t\mathbb{E}[S_t(\boldsymbol{\xi}, x)] \ni b_t, \quad t = 1, \ldots, T, \end{array} \tag{1.4}$$

where $S_t(\xi, x)$ consists of all the solutions to

$$\begin{array}{rclll} 0 & \le & s(\xi, x) \perp & -u_t(\xi, x) + \gamma(\xi, x)e & \ge 0 \\ 0 & \le & \gamma(\xi, x) \perp & 1 - \langle e, s(\xi, x)\rangle & \ge 0 \end{array} \tag{1.5}$$

for some $\gamma(\xi, x) \in \mathbb{R}$ or, equivalently, the linear complementarity problem (LCP):

$$0 \;\le\; \begin{pmatrix} s \\ \gamma \end{pmatrix} \;\perp\; M\begin{pmatrix} s \\ \gamma \end{pmatrix} + \begin{pmatrix} -u_t(\xi, x) \\ 1 \end{pmatrix} \;\ge 0 \tag{1.6}$$

with the positive semidefinite matrix

$$M = \begin{pmatrix} 0 & e \\ -e & 0 \end{pmatrix} \in \mathbb{R}^{(n+1)\times(n+1)}.$$

For fixed (t, ξ, x) and with $q_t(\xi, x) = (-u_t(\xi, x), 1) \in \mathbb{R}^{n+1}$, let us denote the complementarity problem (1.6) by $\text{LCP}(q_t(\xi, x), M)$ and by

$$S(q_t, M) = \{s \in \mathbb{R}^n \,\big|\, \text{for some } \gamma \ge 0, (s, \gamma) \text{ solves LCP}(q_t(\xi, x), M)\big\},$$

i.e., the solution set projected on the s-space[1].

Problem (1.4) can be considered as a stochastic mathematical program with equilibrium constraints (MPEC). The main difference with the stochastic MPEC studied in the literature, cf. [14, 16, 22] is that (1.4) involves constraints formulated in terms of Aumann's expected value of a set-valued mapping, determined by the solutions of

[1] To lighten up the notation, when no confusion is possible, we usually simply write q_t instead of the more precise, but cumbersome, $q_t(\xi, x)$.

an LCP, and thus the constraints cannot be explicitly characterized in terms of the usual KKT conditions.

It will be shown, cf. proof of Theorem 2.3, that the solution set $S(q_t, M)$ is bounded. With $M^\varepsilon = M + \varepsilon I, \varepsilon > 0$, it also implies that the LCP$(q_t(\xi, x), M^\varepsilon)$ has a *unique* solution, which is then denoted by $z_t^\varepsilon = (s_t^\varepsilon, \gamma_t^\varepsilon)$. It converges to the least norm solution of the LCP$(q_t(\xi, x), M)$ as $\varepsilon \downarrow 0$ [9, Theorem 5.6.2]. Moreover, for any fixed $\varepsilon > 0$, the function $q_t \mapsto z_t^\varepsilon$ is globally Lipschitz continuous and continuously differentiable at q_t if and only if the following nondegeneracy condition is satisfied: if for no j, $(z_t^\varepsilon)_j = 0 = (Mz_t^\varepsilon + q_t)_j$; see [7],[8, Lemma 2.1]. These appealing properties motivate us to consider a regularized version of MPSLCC: with z_t^ε being the unique solution of the regularized LCP$(q_t(\xi, x), M^\varepsilon)$

$$0 \le z \perp M^\varepsilon z + q_t(\xi, x) \ge 0, \qquad \text{where} \quad q_t(\xi, x) = \begin{pmatrix} -u_t(\xi, x) \\ 1 \end{pmatrix}, \tag{1.7}$$

the formulation of the regularized problem becomes,

$$\begin{array}{rl} \min_{x \in X} & \frac{1}{2}\langle x, Hx\rangle + \langle c, x\rangle \\ \text{subject to} & \|A_t \mathbb{E}[s_t^\varepsilon(\boldsymbol{\xi}, x)] - b_t\| \le r(\varepsilon), \qquad t = 1, \dots, T, \end{array} \tag{1.8}$$

and the SAA-version of the regularized MPSLCC

$$\begin{array}{rl} \min_{x \in X} & \frac{1}{2}\langle x, Hx\rangle + \langle c, x\rangle \\ \text{subject to} & \|A_t \frac{1}{N}\sum_{i=1}^N s_t^\varepsilon(\xi^i, x) - b_t\| \le \hat{r}(\varepsilon, N), \quad t = 1, \dots, T, \end{array} \tag{1.9}$$

where $r(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$, $\hat{r}(\varepsilon, N) \to r(\varepsilon)$ as $N \to \infty$ for any fixed $\varepsilon > 0$.

The advantage of working with (1.8) and (1.9) is that one can replace the set-valued mapping by a single valued function. Problem (1.9) is a mathematical program with a convex quadratic objective function and globally Lipschitz continuous inequality constraints. Moreover, we derive a closed form expression for z_t^ε; see Lemma 2.2.

The main contribution of this paper is to propose an efficient approach, via the SAA-version of the regularized MPSLCC, to find a solution of the mathematical program with stochastic equilibrium constraints (1.4) and to show that a sequence of solutions $\{x_N^\varepsilon\}$ of the SAA regularized stochastic MPSLCC (1.9) converges to a solution of the (given) problem (1.4) as $N \to \infty$ and $\varepsilon \downarrow 0$. In Section 2, we derive various properties of the solution functions s_t^ε and their convergence to the solution set $S_t(\xi, \bar{x})$ as $\varepsilon \downarrow 0$ and $x \to \bar{x}$. In particular, we provide a closed form for the solution functions which is used to prove the graphical convergence of the real-valued function s_t^ε to the set-valued mapping $S_t(\cdot)$. In Section 3, we prove the existence of solutions to the MPSLCC (1.4) and the SAA regularized MPSLCC (1.9). We show that any sequence of solutions of (1.9) has a cluster point as $\varepsilon \downarrow 0$ and $N \to \infty$, and that any such cluster point is a solution of the MPSLCC (1.4) (a.s.). In Section 4, we use the pure characteristics demand model to illustrate the MPSLCC (1.4) and the SAA-regularized method.

Throughout the paper, $\|\cdot\|$ stands for the ℓ_2 norm, e denotes the vector whose elements are all 1, $|\mathcal{J}|$ is the cardinality of a set $\mathcal{J}$ and u_+ denotes the vector whose components $(u_+)_i = \max(0, u_i)$, $i = 1, \cdots, n$. Vectors are not specifically identified as row or column vectors as the context always makes it clear how they are handled, inner products are denoted by $\langle \cdot, \cdot \rangle$.

2. Solution function of the regularized LCP. Let's now concern ourselves with the properties of the function s^ε_t generated as solution of $\mathrm{LCP}(q_t(\xi,x),M^\varepsilon)$.

LEMMA 2.1. *Problems (1.1) and (1.4) are equivalent.*

Proof. The LCP (1.5) consists exactly of the KKT-conditions for (1.2) with solution s_t if and only if $z_t=(s_t,\gamma_t)$ is a solution of (1.5) for some $\gamma_t \geq 0$. □

For simplicity, in the remainder of this section, we concentrate on the 'solution' function $z^\varepsilon=(s^\varepsilon,\gamma^\varepsilon)$ generated by the regularized linear complementarity problem $\mathrm{LCP}(q_t,M^\varepsilon)$ with $q_t:=(-u_t(\xi,x),1)\in \mathbb{R}^{n+1}$ for *fixed* t, ξ and x; in this section, we *drop* making reference to these quantities to simplify notations and the presentation. Our first aim will be to show that for given u, $s^\varepsilon(q)$ and $z^\varepsilon(q)$ are uniquely determined by a closed form expression. Note that $u_i=-q_i, i=1,\cdots,n$ and $q_{n+1}=1$, and we have

$$\|(-q)_+\|_1=-\sum_{q_i\leq 0} q_i=\sum_{u_i\geq 0} u_i=\|u_+\|_1.$$

LEMMA 2.2. *Given $\varepsilon>0$, the function z^ε is uniquely determined by the solution of the regularized LCP(q,M^ε) and is completely described as follows:*
Let $q_{k_1}\leq q_{k_2}\leq\cdots\leq q_{k_n}$, set

$$\alpha_j=-\sum\nolimits_{i=1}^{j} q_{k_i}+(j+\varepsilon^2)q_{k_j}-\varepsilon,\qquad j=1,\dots,n$$

$$\text{and } \mathcal{J}=\{j\,|\,\alpha_j\leq 0, j=1,\dots,n\},\quad J=|\mathcal{J}|,\quad \sigma=\sum_{i=1}^{J} q_{k_i}.$$

(a) If $\|(-q)_+\|_1\geq\varepsilon$, the solution $(s^\varepsilon,\gamma^\varepsilon)$ of the LCP(q,M^ε) has the form

$$s^\varepsilon_{k_j}=\begin{cases}\frac{\sigma-(J+\varepsilon^2)q_{k_j}+\varepsilon}{J\varepsilon+\varepsilon^3} & \text{if}\quad j\in\mathcal{J},\\ 0 & \text{if}\quad j\notin\mathcal{J},\end{cases}\qquad \gamma^\varepsilon=\frac{-\sigma-\varepsilon}{J+\varepsilon^2}. \tag{2.1}$$

(b) If $\|(-q)_+\|_1\leq\varepsilon$, the solution takes the form

$$\text{for } j=1,\dots,n,\qquad s^\varepsilon_{k_j}=\begin{cases}-q_{k_j}/\varepsilon & \text{if}\quad q_{k_j}<0,\\ 0 & \text{if}\quad q_{k_j}\geq 0,\end{cases}\qquad \gamma^\varepsilon=0. \tag{2.2}$$

(c) If $\|(-q)_+\|_1=\varepsilon$, $\sigma=-\|(-q)_+\|_1$ and $\mathcal{J}=\{j\,\big|\,q_{kj}\leq 0,\ j=1,\dots,n\}$, that is, formulas (2.1) and (2.2) coincide in this situation.

Moreover, in all cases, $\sum_{j=1}^n s^\varepsilon_j\leq 1+\varepsilon\|(-q)_+\|_1$.

Proof. Let $z^\varepsilon=(s^\varepsilon,\gamma^\varepsilon)$ be the solution (vector) of $\mathrm{LCP}(q,M^\varepsilon)$. Without loss of generality, assume $q_1\leq q_2\leq\cdots\leq q_n$, i.e., $k_j=j$.

(**a**) $\|(-q)_+\|_1\geq\varepsilon$: We show, first, that for $j\in\mathcal{J}$: $s^\varepsilon_j\geq 0$ and $(M^\varepsilon z^\varepsilon)_j+q_j=\varepsilon s^\varepsilon_j+\gamma^\varepsilon+q_j=0$. From $\alpha_j\leq 0$, $j\leq J$, $q_J-q_j\geq 0$, and $\alpha_J\leq 0$, we have

$$\begin{aligned}(J\varepsilon+\varepsilon^3)s^\varepsilon_j&=\sigma-(J+\varepsilon^2)q_j+\varepsilon+\alpha_J-\alpha_J=(J+\varepsilon^2)(q_J-q_j)-\alpha_J\geq 0\\ (J+\varepsilon^2)(\varepsilon s^\varepsilon_j+\gamma^\varepsilon+q_j)&=\sigma-(J+\varepsilon^2)q_j+\varepsilon-\sigma-\varepsilon+(J+\varepsilon^2)q_j=0.\end{aligned}$$

The next step is to show that $(M^\varepsilon z^\varepsilon)_j + q_j = \gamma^\varepsilon + q_j > 0$ when $j \notin \mathcal{J}$. By the definitions of $\mathcal{J}$ and J, one has $\alpha_{J+1} > 0$, $j \geq J+1$ and $q_j \geq q_{J+1}$. Hence,

$$
\begin{aligned}
(2.3)\qquad (J+\varepsilon^2)(\gamma^\varepsilon + q_j) &= -\sigma - \varepsilon + (J+\varepsilon^2)q_j - \alpha_{J+1} + \alpha_{J+1} \\
&= -\sigma - \varepsilon + (J+\varepsilon^2)q_j + \big(\sigma + q_{J+1} - (J+1+\varepsilon^2)q_{J+1} + \varepsilon\big) + \alpha_{J+1} \\
&= (J+\varepsilon^2)(q_j - q_{J+1}) + \alpha_{J+1} > 0.
\end{aligned}
$$

Finally, we show that $\gamma^\varepsilon \geq 0$ and $(M^\varepsilon z^\varepsilon)_{n+1} + 1 = 1 + \varepsilon\gamma^\varepsilon - \sum_{i=1}^n s_i^\varepsilon = 0$. Let $j_0 = \max\{j \,|\, q_j < 0\}$; such an index is guaranteed to exist since $\|(-q)_+\|_1 \geq \varepsilon$. Actually, we are going to establish that $q_j \leq 0$ for all $j \in \mathcal{J}$. Assume for the sake of contradiction that $q_j > 0$ for some $j \in \mathcal{J}$. Of course, then $j > j_0$ and

$$
\begin{aligned}
\alpha_j &= -\sum_{i=1}^{j_0} q_i - \varepsilon + \sum_{i=j_0+1}^{j} (q_j - q_i) + (j_0+\varepsilon^2)q_j \\
&= \|q_+\|_1 - \varepsilon + \sum_{i=j_0+1}^{j} (q_j - q_i) - (j_0+\varepsilon^2)q_j \geq 0,
\end{aligned}
$$

which contradicts the definition of $\mathcal{J}$. Hence, $q_j \leq 0$ for $j \in \mathcal{J}$, which together with $\|(-q)_+\|_1 \geq \varepsilon$ implies $0 < -\sigma \leq \|(-q)_+\|_1$.

Let's now show that $-\sigma \geq \varepsilon$. Note that $q_j \leq 0$, $j = 1, \ldots, J$ and $J \leq j_0$. If $j_0 = J$, then by definition of j_0, one has $-\sigma = \|(-q)_+\|_1 \geq \varepsilon$. If $j_0 > J$, from $q_{J+1} < 0$ and

$$
\begin{aligned}
\alpha_{J+1} &= -\textstyle\sum_{i=1}^J q_i - \varepsilon - q_{J+1} + q_{J+1} + (J+\varepsilon^2)q_{J+1} \\
&= -\sigma - \varepsilon + (J+\varepsilon^2)q_{J+1} \geq 0,
\end{aligned}
$$

and, thus, $-\sigma \geq \varepsilon$. Moreover, $\sum_{i=1}^J q_i = \sigma$ yields $\sum_{i=1}^n s_i^\varepsilon = (J - \varepsilon\sigma)/(J+\varepsilon^2)$ and

$$
(J\varepsilon + \varepsilon^3)\big(1 + \varepsilon\gamma^\varepsilon - \sum_{i=1}^n s_i^\varepsilon\big) = J\varepsilon + \varepsilon^3 + \varepsilon^2(-\sigma - \varepsilon) + \varepsilon^2\sigma - J\varepsilon = 0.
$$

Hence, the solution has the explicit form (2.1).

(**b**) $\|(-q)_+\|_1 \leq \varepsilon$: If $\|(-q)_+\|_1 = 0$, then $q \geq 0$ and (2.2) holds with $z^\varepsilon = 0$. If $\|(-q)_+\|_1 > 0$, then $j_0 \geq 1$. For $j \leq j_0$, $s_j^\varepsilon = -q_j/\varepsilon > 0$, $\gamma^\varepsilon = 0$ and

$$
(M^\varepsilon z^\varepsilon)_j + q_j = \varepsilon s_j^\varepsilon + \gamma^\varepsilon + q_j = -q_j + q_j = 0.
$$

For $j > j_0$, one has $q_j \geq 0$, $s_j^\varepsilon = 0$, $\gamma^\varepsilon = 0$ and $(M^\varepsilon z^\varepsilon)_j + q_j = \varepsilon s_j^\varepsilon + \gamma^\varepsilon + q_j \geq 0$, and for $j = n+1$, $\gamma^\varepsilon = 0$ and

$$
(M^\varepsilon z^\varepsilon)_{n+1} + 1 = 1 + \varepsilon\gamma^\varepsilon - \sum_{i=1}^n s_i^\varepsilon = 1 + \big(-\sum_{i=1}^{j_0} q_i\big)/\varepsilon \geq 0.
$$

(**c**) $\|(-q)_+\|_1 = \varepsilon$: For $j > j_0$,

$$
\alpha_j = -\sum_{i=1}^{j_0} q_i - \varepsilon + \sum_{i=j_0+1}^{j} q_i + (j+\varepsilon^2)q_j > 0,
$$

and for $j \le j_0$, $\alpha_j = -\sum_{i=1}^{j} q_i - \varepsilon + (j+\varepsilon^2)q_j \le 0$. Hence $\sigma = -\|(-q)_+\|_1$ and $\mathcal{J} = \{j \mid q_j \le 0\}$. Moreover, in this case

$$s_j^\varepsilon = \begin{cases} (\sigma - (J+\varepsilon^2)q_j + \varepsilon)/(J\varepsilon + \varepsilon^3) = -q_j/\varepsilon & \text{if} \quad j \in \mathcal{J}, \\ 0 & \text{if} \quad j \notin \mathcal{J}, \end{cases} \quad \gamma^\varepsilon = \frac{-\sigma - \varepsilon}{J + \varepsilon^2} = 0,$$

which implies that formulas (2.1) and (2.2) coincide.

Moreover, in case (a),

$$\sum_{i=1}^{n} s_i^\varepsilon \le 1 + (\varepsilon/J)\|(-q)_+\|_1 \le 1 + \varepsilon\|(-q)_+\|_1$$

and $\sum_{i=1}^{n} s_j^\varepsilon \le 1$ for (b) and (c) which completes the proof. $\square$

The following theorem shows that the unique solution $z^\varepsilon(q)$ of the regularized $\mathrm{LCP}(q, M^\varepsilon)$ is monotonically convergent, componentwise, to the least norm solution of the $\mathrm{LCP}(q, M)$ at rate $O(\varepsilon)$.

THEOREM 2.3. *Let $z^\varepsilon(q) = (s^\varepsilon(q), \gamma^\varepsilon(q))$ be the unique solution of the* $\mathrm{LCP}(q, M^\varepsilon)$ *and $z(q) = (s(q), \gamma(q))$ be the least norm solution of the* $\mathrm{LCP}(q, M)$. *Then for fixed q, we have $\lim_{\varepsilon \downarrow 0} \|z^\varepsilon(q) - z(q)\| = 0$. Moreover, there are positive constants $\bar{\varepsilon}, \kappa_1, \kappa_2$, such that for any $\varepsilon \in (0, \bar{\varepsilon})$,*

$$0 \le s^\varepsilon(q) - s(q) \le \kappa_1 \varepsilon e \quad \text{and} \quad 0 \le \gamma(q) - \gamma^\varepsilon(q) \le \kappa_2 \varepsilon. \tag{2.4}$$

Proof. We know that $s(q)$ is bounded since $\langle e, s(q)\rangle \le 1$ and $s(q) \ge 0$. When $\gamma(q) > 0$ from the complementarity conditions one must have $1 - \langle e, s(q)\rangle = 0$ which implies that there has to be an entry $s_j(q) > 0$ and $\gamma(q) + q_j = 0$. Hence, the solution set $\mathrm{SOL}(q, M)$ is bounded.

By [9, Theorem 3.1.8], we know that $z^\varepsilon(q)$ converges to the least norm solution $z(q)$ of $\mathrm{LCP}(q, M)$ as $\varepsilon \downarrow 0$ since the matrix M is positive semi-definite.

If $\|(-q)_+\|_1 = 0$, then $z^\varepsilon(q) = z(q) = 0$ for any $\varepsilon > 0$. Hence (2.4) holds for any $\varepsilon > 0$.

When $\|(-q)_+\|_1 > 0$, let

$$\sigma_1 = \min_{1 \le j \le n} q_j, \qquad \sigma_2 = \min\{0, \min_{\substack{1 \le j \le n \\ q_j \ne \sigma_1}} q_j\} \quad \text{and} \quad \bar{\varepsilon} := \min\{\frac{-\sigma_1 + \sigma_2}{1 - \sigma_2}, 1\}. \tag{2.5}$$

From $\|(-q)_+\|_1 > 0$, there is $\varepsilon_0 > 0$ such that $\|(-q)_+\|_1 \ge -\sigma_1 > \varepsilon_0$. Thus, for any $\varepsilon \in (0, \bar{\varepsilon})$, $\|(-q)_+\|_1 \ge \varepsilon$ and the solution $z^\varepsilon(q)$ has the explicit form (2.1) from Lemma 2.2 and $(-\sigma_1 + \sigma_2)/(1 - \sigma_2) \le \varepsilon_0$. Our next step is to show that $\alpha_j < 0$ if and only if $q_j = \sigma_1$ for $\varepsilon \in (0, \bar{\varepsilon})$ which implies $\mathcal{J} = \{j \mid q_j = \sigma_1,\ j = 1, \dots, n\}$ and $\sigma = J\sigma_1$.

Without loss of generality assume that $q_1 \le q_2 \le \cdots \le q_n$. If $q_j = \sigma_1$, then $\alpha_j = \varepsilon^2\sigma_1 - \varepsilon < 0$. Conversely, when $\alpha_j < 0$, from the definition of $\{\alpha_j\}$, one has

$$\alpha_{j+1} - \alpha_j = -q_{j+1} + (j + 1 + \varepsilon^2)q_{j+1} - (j + \varepsilon^2)q_j = (j + \varepsilon^2)(q_{j+1} - q_j) \ge 0. \tag{2.6}$$

Hence, it suffices to show that $\alpha_j \geq 0$ for $q_j \geq \sigma_2$. If $\varepsilon < 1 \leq (-\sigma_1 + \sigma_2)/(1 - \sigma_2)$, then $-\sigma_1 \geq 1 - 2\sigma_2$. For $q_j \geq \sigma_2$, remembering that $\sigma_2 \leq 0$,

$$\alpha_j \geq \sum_{i=1}^{j}(\sigma_2 - q_i) + \varepsilon^2\sigma_2 - \varepsilon \geq \sigma_2 - \sigma_1 + \varepsilon^2\sigma_2 - \varepsilon > -\sigma_1 - 1 + 2\sigma_2 \geq 0. \tag{2.7}$$

If $\varepsilon < (-\sigma_1 + \sigma_2)/(1 - \sigma_2) \leq 1$, then

$$\begin{aligned} \alpha_j &\geq \sum_{i=1}^{j}(\sigma_2 - q_i) + \varepsilon^2\sigma_2 - \varepsilon > \sigma_2 - \sigma_1 + (\frac{\sigma_2 - \sigma_1}{1 - \sigma_2})^2\sigma_2 - \frac{\sigma_2 - \sigma_1}{1 - \sigma_2} \\ &= \frac{\sigma_2 - \sigma_1}{1 - \sigma_2}(1 - \sigma_2 + \frac{\sigma_2 - \sigma_1}{1 - \sigma_2}\sigma_2 - 1) \geq 0. \end{aligned} \tag{2.8}$$

Hence, $\alpha_j < 0$ if and only if $q_j = \sigma_1$ for any $\varepsilon \in (0, \bar{\varepsilon})$. By Lemma 2.2, for $\varepsilon \in (0, \bar{\varepsilon})$, the solution $z^\varepsilon(q)$ of $\mathrm{LCP}(q, M^\varepsilon)$ has the form

$$s_j^\varepsilon(q) = \begin{cases} (1 - \varepsilon\sigma_1)/(J + \varepsilon^2) & \text{if} \quad j \in \mathcal{J}, \\ 0 & \text{if} \quad j \notin \mathcal{J}, \end{cases} \qquad \gamma^\varepsilon(q) = (-J\sigma_1 - \varepsilon)/(J + \varepsilon^2). \tag{2.9}$$

The least norm solution of the $\mathrm{LCP}(q, M)$ is the minimizer of the quadratic program

$$\min_{z \geq 0} \ \tfrac{1}{2}\|z\|^2 \ \text{subject to} \ \sum\nolimits_{j \in \mathcal{J}} z_j = 1, \ \ z_j = 0, \ j \notin \mathcal{J}, \ \ z_{n+1} = \gamma = -\sigma_1.$$

This least norm solution has the form (cf. the first order optimality conditions):

$$s_j(q) = \begin{cases} J^{-1} & \text{if} \quad j \in \mathcal{J}, \\ 0 & \text{if} \quad j \notin \mathcal{J}, \end{cases} \qquad \gamma(q) = -\sigma_1. \tag{2.10}$$

From (2.9) and (2.10), we easily see that

$$0 \leq s_j^\varepsilon(q) - s_j(q) \leq (-\sigma_1\varepsilon)/J^2, \quad \text{for } j = 1, \ldots, n,$$

and

$$0 \leq \gamma(q) - \gamma^\varepsilon(q) \leq (1 - \varepsilon\sigma_1)(\varepsilon/J) \leq (1 - \sigma_1)(\varepsilon/J).$$

Hence (2.4) holds with $\kappa_1 = (-\sigma_1)/J^2$ and $\kappa_2 = (1 - \sigma_1)/J$. □

THEOREM 2.4. *For any fixed q, if $\|(-q)_+\|_1 > 0$ one can find $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$, z^ε is differentiable at q. Moreover, if $\min_{1 \leq i \leq n} q_i = q_{i_1}$ is unique then there exists $\hat{\varepsilon} > 0$ and a neighborhood $\mathcal{N}_q$ of q such that for any $\varepsilon \in (0, \hat{\varepsilon})$, $q \mapsto z^\varepsilon(q)$ is linear on $\mathcal{N}_q$. When z^ε is differentiable at q, one has*

$$\nabla z^\varepsilon(q) = -(I - D + DM^\varepsilon)^{-1}D, \tag{2.11}$$

where D is a $n \times n$ diagonal matrix with diagonal entries

$$d_{ii} = \begin{cases} 1 & \text{if } z_i^\varepsilon(q) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. $\|(-q)_+\|_1 > 0$ means there is an $\varepsilon_0 > 0$ such that $\|(-q)_+\|_1 \geq -\sigma_1 > \varepsilon_0$. Consider $\varepsilon \in (0, \bar{\varepsilon})$ with $\bar{\varepsilon}$ as defined by (2.5). From (2.9),

$$z_j^\varepsilon(q) > 0, \quad \text{for} \quad j \in \mathcal{J} \cup \{n+1\}$$

and from (2.3),

$$(M^\varepsilon z^\varepsilon(q))_j + q_j = \gamma^\varepsilon(q) + q_j > 0, \quad \text{for} \quad j \notin \mathcal{J}.$$

Hence the strictly complementarity condition holds at $z^\varepsilon(q)$, that is, there is no j such that $z_j^\varepsilon(q) = (M^\varepsilon z^\varepsilon(q))_j + q_j = 0$. Differentiability of z^ε at q follows from [8, Lemma 2.1].

If there is a unique entry $q_{i_1} = \min_{1 \leq i \leq n} q_i$, then there is a neighborhood $\mathcal{N}_q$ of q such that for any $p \in \mathcal{N}_q$, $\{i \,|p_i = \min_{1 \leq j \leq n} p_j\} = \{i \,|q_i = \min_{1 \leq j \leq n} q_j\}$. Let

$$\hat{\sigma}_1 = \min_{p \in \mathcal{N}_q} \min_{1 \leq j \leq n} p_j, \qquad \hat{\sigma}_2 = \min\{0, \min_{p \in \mathcal{N}_q} \min_{\substack{1 \leq j \leq n \\ p_j \neq \sigma}} p_j\} \quad \text{and} \quad \hat{\varepsilon} = \min\{\frac{-\hat{\sigma}_1 + \hat{\sigma}_2}{1 - \hat{\sigma}_2}, 1\}.$$

Then for any $\varepsilon \in (0, \hat{\varepsilon})$, the strictly complementarity condition holds at $z^\varepsilon(p)$ for any $p \in \mathcal{N}_q$. Using [8, Lemma 2.1] again, we find that z^ε is differentiable at p and its derivative ∇z^ε is given in (2.11). Hence, z^ε is a linear mapping on $\mathcal{N}_q$. $\square$

Remark. For any fixed $\varepsilon > 0$, M^ε is positive definite. Hence for any q, the LCP(q, M^ε) has a unique solution $z^\varepsilon(q)$ which defines a globally Lipschitz continuous function z^ε on $\mathbb{R}^{n+1}$ [7, 9] with Lipschitz constant $1/\varepsilon$, cf. [8, Theorem 2.1]. The solution function $s^\varepsilon(q)$ can be viewed as a smoothing function of the indicator function $\mathbb{1}_{(0,\infty)}(u)$ for any $q = (-u, 1)$. To illustrate this, we consider the LCP (1.6) with $n = 1$. Then, the (first) s^ε component of the solution of LCP(q, M^ε) is

$$s^\varepsilon(q) = \begin{cases} (1 + \varepsilon u)/(1 + \varepsilon^2) & \text{if } u > \varepsilon, \\ u/\varepsilon & \text{if } u \in (0, \, \varepsilon], \\ 0 & \text{if } u \leq 0. \end{cases}$$

It is worth noting that for any fixed u

$$\mathbb{1}_{(0,\infty)}(u) = \lim_{\varepsilon \downarrow 0} s^\varepsilon(q) = \begin{cases} 1 & \text{if } u > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, the solution $z^\varepsilon(q)$ of the regularized LCP satisfies the set-valued constraints. In particular, when $n = 1$,

$$\text{(2.12)} \qquad \operatorname{Lim}_{u \to 0, \varepsilon \downarrow 0}\{s^\varepsilon(q)\} = [0, 1] \quad \text{and} \quad \lim_{u \downarrow 0, \varepsilon \downarrow 0} s^\varepsilon(q) = \begin{cases} 1 & \text{if } \varepsilon = o(|u|), \\ 0 & \text{if } |u| = o(\varepsilon). \end{cases}$$

where $\operatorname{Lim}_n C^n$ denotes the *set limit* of the sets C^n with value $C = \operatorname{Limsup} C^n = \operatorname{Liminf} C^n$. Continuous functions have been used to approximate the indicator function in the study of chance constraints [12, 17, 20]

$$\operatorname{Prob}\{c(\boldsymbol{\xi}, x) \leq 0\} = \mathbb{E}[\mathbb{1}_{(-\infty,0)} c(\boldsymbol{\xi}, x)] \leq \alpha,$$

where $c : \Xi \times \mathbb{R}^\nu \to \mathbb{R}$ and $\alpha \in (0, 1]$. However, these continuous approximation functions cannot easily be implemented in the vector-valued constraints case [11, 18],

$$\operatorname{Prob}\{c_j(\boldsymbol{\xi}, x) = \max_{1 \leq i \leq n} c_i(\boldsymbol{\xi}, x)\} = \mathbb{E}[\mathbb{1}_{\{\max_{1 \leq i \leq n} c_i(\boldsymbol{\xi}, x)\}} c_j(\boldsymbol{\xi}, x)] = b_j, \quad j = 1, \ldots, n,$$

$c_j : \Xi \times \mathbb{R}^\nu \to \mathbb{R}$ and $b_j \in (0, 1]$, $j = 1, \ldots, n$. The LCP approach and its solution $z^\varepsilon(q)$ of the regularized LCP has the ability to deal with vector-valued constraints.

3. Convergence analysis of the SAA regularized problems. In this section, we study the convergence of the SAA regularized method. The objective function of all three problems (1.4), (1.8) and (1.9) is $f(x) = \frac{1}{2}\langle x, Hx\rangle + \langle c, x\rangle$, their feasible sets,

$$\begin{aligned} D &= \{x \in X \mid A_t\mathbb{E}[S_t(\boldsymbol{\xi}, x)] \ni b_t, \quad t = 1, \dots, T\},\\ D^\varepsilon &= \{x \in X \mid \|A_t\mathbb{E}[s_t^\varepsilon(\boldsymbol{\xi}, x)] - b_t\| \le r(\varepsilon), \quad t = 1, \dots, T\},\\ D_N^\varepsilon &= \{x \in X \mid \|A_t \frac{1}{N}\sum\nolimits_{i=1}^N s_t^\varepsilon(\xi^i, x) - b_t\| \le \hat{r}(\varepsilon, N), \quad t = 1, \dots, T\}, \end{aligned}$$

and their solution sets,

$$X^* = \operatorname{argmin}_D f, \qquad X^\varepsilon = \operatorname{argmin}_{D^\varepsilon} f, \qquad X_N^\varepsilon = \operatorname{argmin}_{D_N^\varepsilon} f.$$

$\mathbb{B}_\varepsilon^o = \{\, y \mid \|y\| < \varepsilon\}$ will always denote an open ball centered at 0 with radius ε (in $\mathbb{R}^n$ or $\mathbb{R}^\nu$) and $\mathbb{B}_\varepsilon$ the corresponding closed ball.

In Section 3.1, we derive the convergence of X^ε to X^* as $\varepsilon \downarrow 0$, in Section 3.2 we obtain the convergence of X_N^ε to X^ε for any fixed $\varepsilon > 0$ as $N \to \infty$ and proceed to deduce the convergence of the solutions of the SAA regularized problems by showing the convergence of X_N^ε to X^* as $\varepsilon \downarrow 0$ and $N \to \infty$.

Denote by $d(v, U) = \inf_{u\in U} \|v - u\|$ the distance from v to a set $U \subseteq \mathbb{R}^n$ and for $U, V \subseteq \mathbb{R}^n$, the excess distance of the set U on V and the Pompeiu-Hausdorff distance between U and V by

$$\mathbb{e}(V, U) = \sup\nolimits_{v\in V} d(v, U) \text{ and } \mathbb{h}(U, V) = \max\big(\mathbb{e}(V, U), \mathbb{e}(U, V)\big).$$

3.1. Convergence of solution sets as $\varepsilon \to 0$. We consider the solutions sets of problems (1.4) and (1.8) For simplicity's sake, in this section and next one, we drop the index t and simply write A for A_t, S for S_t and so on. Moreover, we use $z^\varepsilon(q)$ and $z^\varepsilon(\xi, x)$ to denote $z^\varepsilon(q(\xi, x))$ and the same for their components s^ε and γ^ε.

Remember that the solution set $\{z^\varepsilon(\xi, x)\} = \mathrm{SOL}(q(\xi, x), M^\varepsilon)$ is a singleton and the solution set $Z^0(\xi, x) = \mathrm{SOL}(q(\xi, x), M)$ is convex and bounded for any (ξ, x). By Theorem 2.3, for every (ξ, x), one has

$$\lim_{\varepsilon\downarrow 0} \|z^\varepsilon(\xi, x) - \bar{z}^0(\xi, x)\| = 0,$$

where $\bar{z}^0(\xi, x)$ is the least-norm solution of the $\mathrm{LCP}(q(\xi, x), M)$, implying the pointwise convergence

$$\lim\nolimits_{\varepsilon\downarrow 0} d(z^\varepsilon(\xi, x), Z^0(\xi, x)) = 0. \tag{3.1}$$

However, from our Remark at the end of the previous section, we already know that for some particular choices of $\varepsilon_k \downarrow 0$, $x^k \to x$, $z^{\varepsilon_k}(\xi, x^k)$ may not converge to $\bar{z}^0$, the least-norm solution of $\mathrm{LCP}(q(\xi, x), M)$. Our *predominant motivation*, however, is to show that the solutions of the approximating problems converge to the solutions of the given problem and, in the process, establish the convergence of the feasible sets D^ε and solution sets X^ε to D and X^* respectively. To do this, we are naturally led to study the graphical, rather than the pointwise, convergence of the functions z^ε as $\varepsilon \downarrow 0$.

In first part of the arguments that follow, ξ remains fixed and thus it will be convenient to usually ignore the dependence on ξ of the functions u, q and the associated solutions functions $z^\varepsilon = (s^\varepsilon, \gamma^\varepsilon)$ and the solution set Z^0, only the dependence on the pair (x, ε) is relevant.

First, we review the definition of *graphical convergence* [19, Definition 5.32] of the function z^ε as $\varepsilon \downarrow 0$. Let $\mathbb{N} = \{1, 2 \dots\}$ be the set of natural numbers, $\mathcal{N}^\#_\infty = \{\text{all subsequences of } \mathbb{N}\}$ and $\mathcal{N}_\infty = \{\text{all indexes} \geq \text{ some } \bar{k}\}$. We use $(x^k, \varepsilon_k) \underset{N}{\rightarrow} (x, 0)$ to denote $\varepsilon_k \downarrow 0$ and $x^k \to x$ when $k \in N$.

DEFINITION 3.1. *For the mappings $z^\varepsilon : X \to \mathbb{R}^{n+1}$, the graphical outer limit, denoted by* g-limsup$_\varepsilon z^\varepsilon : X \rightrightarrows \mathbb{R}^{n+1}$ *is the mapping having as its graph the set* $\text{Limsup}_\varepsilon \, \text{gph}\, z^\varepsilon$*:*

$$\text{g-limsup}_\varepsilon\, z^\varepsilon(x) = \{z \mid \exists N \in \mathcal{N}^\#_\infty,\ (x^k, \varepsilon_k) \underset{N}{\rightarrow} (x, 0),\ z^{\varepsilon_k}(x^k) \underset{N}{\rightarrow} z\}.$$

The graphical inner limit, denoted by g-liminf$_\varepsilon z^\varepsilon$ *is the mapping having as its graph the set* $\text{Liminf}_\varepsilon \, \text{gph}\, z^\varepsilon$*:*

$$\text{g-liminf}_\varepsilon\, z^\varepsilon(x) = \{z \mid \exists\, N \in \mathcal{N}_\infty,\ (x^k, \varepsilon_k) \underset{N}{\rightarrow} (x, 0),\ z^{\varepsilon_k}(x^k) \underset{N}{\rightarrow} z\}.$$

If the outer and inner limits coincide, the graphical limit g-lim$_\varepsilon z^\varepsilon$ *exists and, thus,* $Z^0 = \text{g-lim}_\varepsilon z^\varepsilon$ *if and only if*

$$\text{g-limsup}_\varepsilon\, z^\varepsilon \subseteq Z^0 \subseteq \underset{\varepsilon}{\text{g-liminf}}\, z^\varepsilon$$

and one writes $z^\varepsilon \xrightarrow{g} Z^0$*; the mappings* z^ε *are said to* converge graphically *to* Z^0.

THEOREM 3.2. *For given ξ, if $x \mapsto u(\xi, x)$ is continuous on X then,* g-limsup$_\varepsilon s^\varepsilon \subseteq S^0$ *as well as* g-limsup$_\varepsilon z^\varepsilon \subset Z^0$. *If $u(\xi, \cdot)$ is also surjective, then the functions s^ε converge graphically to S^0, that is,*

$$s^\varepsilon \xrightarrow{g} S^0 \text{ as well as } z^\varepsilon \xrightarrow{g} Z^0. \tag{3.2}$$

Proof. Remember, throughout the proof that $\xi \in \Xi$ remains fixed. For a subsequence $N \in \mathcal{N}^\#_\infty$, let $(x^k, \varepsilon_k) \underset{N}{\rightarrow} (x, 0)$ with $\varepsilon_k \downarrow 0$, $u^k = u(\xi, x^k)$, $q^k = (-u^k, 1)$, $z^k = z^{\varepsilon_k}(q^k)$ and suppose $z^k \underset{N}{\rightarrow} z^0$; for any pair (ξ, x^k), the vector z^k is uniquely defined, cf. Lemma 2.2. Moreover, for any $\varepsilon_k > 0$, LCP(q^k, M^{ε_k}) has a unique solution $z^k(\xi, x^k) = (s^k, \gamma^{\varepsilon_k})$ that is also a unique solution of the system of (continuous) equations:

$$\text{Min}\ [\, z,\ M^{\varepsilon_k} z + q^k\,] = 0,$$

where "Min" has to be understood componentwise. Hence, with $M^k = M^{\varepsilon_k}$, one has

$$0 = \lim_k \text{Min}\ [z^k,\ M^k z^k + q^k] = \text{Min}\ [\lim_k z^k,\ \lim_k M^k z^k + q^k] = \text{Min}\ [z^0,\ M z^0 + q(\xi, x)]$$

which means $z^0 \in Z^0$ and consequently g-limsup$_k z^k \subseteq Z^0$ and, in particular, the same applies to the first n-entries of the z^k and z^0, i.e., $s^0 \in S^0 \supseteq \text{g-limsup}_k\{s^k\}$.

We now concern ourselves with the second assertion of the lemma. For $\tilde{x} \in X$, let $z(\tilde{x}) = \big(s(\tilde{x}), \gamma(\tilde{x})\big) \in Z^0(\xi, \tilde{x})$. We need to show that $z(\tilde{x}) \in \text{g-liminf}_\varepsilon\{z^\varepsilon\}$. The surjective property of $u(\xi,\cdot): X \to \mathbb{R}^n$ implies that for any $q^k = (-u^k, 1)$, there is $x^k \in X$ such that $q^k = \big(-u(\xi, x^k), 1\big)$. Let $\tilde q = (-u(\xi,x),1)$, $\tilde z = z(\tilde q)$ and show that

$$\exists\, N \in \mathcal{N}_\infty,\ (q^k, \varepsilon_k) \underset{N}{\to} (\tilde q, 0),\ \ z^{\varepsilon_k}(q^k) = z^k \underset{N}{\to} \tilde z = z(\tilde q). \tag{3.3}$$

Let $\eta_0 = \max_{1\le i\le n}(u_i(\xi,x) = -\tilde q_i)$. To prove (3.3), we examine all three cases: $\eta_0 > 0$, $\eta_0 = 0$ and $\eta_0 < 0$.

Case 1. $\eta_0 > 0$. Without loss of generality, assume

$$\tilde q_1 = \cdots = \tilde q_J < \tilde q_i,\ i = J+1,\dots,n, \quad \text{and} \quad s_1(\tilde q) \ge \ldots \ge s_J(\tilde q) \tag{3.4}$$

which implies

$$\sum_{i=1}^J s_i(\tilde q) = 1,\ \ s_i(\tilde q) \ge 0,\ i = 1,\dots,J, \quad s_i(\tilde q) = 0, \quad i = J+1,\dots,n,$$

and $\gamma(\tilde q) = \tilde q_1$. Choose a sequence $\varepsilon_k \downarrow 0$. Then, for some $\tilde k$,

$$\forall\, k \ge \tilde k, \quad J(\tilde q_{J+1} - \tilde q_1) + \varepsilon_k^2 \tilde q_{J+1} - \varepsilon_k > 0 \text{ and } -\tilde q_1 > \varepsilon_k, \tag{3.5}$$

which implies $\eta_0 > \varepsilon_k$. Let

$$\begin{aligned} q_i^k &= \tilde q_i - \lambda_i \varepsilon_k, \text{ with } \lambda_i = \big(J s_i(\tilde q) - 1\big)/J, \quad i = 1,\dots,J, \\ q_i^k &= \tilde q_i, i = J+1,\dots,n. \end{aligned}$$

From (3.4) and $q_i^k = \tilde q_i - \lambda_i \varepsilon_k$, one obtains

$$q_1^k \le \ldots \le q_J^k \le \tilde q_1 + \varepsilon_k J^{-1} \le -\eta_0 + \varepsilon_k < 0.$$

Note that, since $\sum_{i=1}^J \lambda_i = 0$,

$$\|(-q^k)_+\|_1 \ge \sum_{i=1}^J -q_i^k = -J\tilde q_1 + \varepsilon \sum_{i=1}^J \lambda_i = -J\tilde q_1 \ge \eta_0 > \varepsilon_k.$$

Now, apply Lemma 2.2(a) to obtain the solution $z^{\varepsilon_k}(q^k)$ for $k \ge \tilde k$.

Since $\tilde q_1 = \cdots = \tilde q_J = -\eta_0$, from $\varepsilon_k < \eta_0$, $\sum_{i=1}^J \lambda_i = 0$ and $-J^{-1} \le \lambda_J \le 0$,

$$\alpha_J^k = \varepsilon_k^2(\tilde q_1 - \lambda_J \varepsilon_k) - J\lambda_J \varepsilon_k - \varepsilon_k \le 0.$$

Moreover, from (3.5), one obtains

$$\alpha_{J+1}^k = J(\tilde q_{J+1} - \tilde q_1) + \varepsilon_k^2 q_{J+1} - \varepsilon_k > 0.$$

Using $\alpha_1^k \le \ldots \le \alpha_n^k$ for $k \ge \tilde k$, yields

$$\sigma^k = \sum_{i=1}^J q_i^k = J\tilde q_1 - \sum_{i=1}^J \lambda_i \varepsilon_k = J\tilde q_1, \quad k \ge \tilde k,$$

and for $k \ge \tilde k$,

$$s_i^{\varepsilon_k}(q^k) = \frac{J\tilde q_1 - (J+\varepsilon_k^2)(\tilde q_1 - \lambda_i \varepsilon_k) + \varepsilon_k}{J\varepsilon_k + \varepsilon_k^3} = \frac{J\lambda_i + \varepsilon_k(\tilde q_1 - \lambda_i \varepsilon_k) + 1}{J + \varepsilon_k^2},\ i = 1,\dots,J$$

and

$$s_i^{\varepsilon_k}(q^k) = 0, \quad i = J+1, \ldots, n.$$

When $k \to \infty$, $s_i^{\varepsilon_k}(q^k) \to \lambda_i + J^{-1} = s_i(\tilde{q})$, for $i = 1, \ldots, J$ $s_i^{\varepsilon_k}(q^k) \to 0$, for $i = J+1, \ldots, n$, and $\gamma^{\varepsilon_k}(q^k) = (\sigma^k - \varepsilon_k)/(J + \varepsilon_k^2) \to \tilde{q}_1 = \gamma(\tilde{q})$ i.e., $s^{\varepsilon_k}(q^k) \to s(\tilde{q})$ and $z^{\varepsilon_k}(q^k) \to z(\tilde{q})$.

Case 2. $\eta_0 = 0$. Without loss of generality, assume

$$0 = \tilde{q}_1 = ... = \tilde{q}_J < \tilde{q}_i,\, i = J+1, \ldots, n, \quad \text{and} \quad s_1(\tilde{q}) \geq \ldots \geq s_J(\tilde{q}) \tag{3.6}$$

which implies that

$$\sum_{i=1}^{J} s_i(\tilde{q}) \leq 1, s_i(\tilde{q}) \geq 0, \quad i = 1, \ldots, J \quad \text{and} \quad s_i(\tilde{q}) = 0, \quad i = J+1, \ldots, n.$$

Choose $\varepsilon_k \downarrow 0$ and let

$$q_i^k = -s_i(\tilde{q})\varepsilon_k, \quad i = 1, \ldots, J \quad \text{and} \quad q_i^k = \tilde{q}_i, \quad i = J+1, \ldots, n.$$

Since $\sum_{i=1}^{J} s_i(\tilde{q}) \leq 1$, one has $\sum_{i=1}^{J}(-q_i^k) = \varepsilon_k \sum_{i=1}^{J} s_i(\tilde{q}) \leq \varepsilon_k$ and $q_i^k = \tilde{q}_i > 0, i = J+1, \ldots, n$. Now, apply Lemma 2.2(b) to obtain the solution

$$s_i^{\varepsilon_k}(q^k) = (s_i(\tilde{q})\varepsilon_k)/\varepsilon_k, \quad i = 1, \ldots, J \quad \text{and} \quad s_i(\tilde{q}) = 0, \quad i = J+1, \ldots, n.$$

Obviously, when $k \to \infty$, $s_i^{\varepsilon_k}(q^k) \to s_i(\tilde{q})$, $i = 1, \ldots, n$, and $\gamma^{\varepsilon_k} = 0$, entailing $z^{\varepsilon_k}(q^k) \to z(\tilde{q})$.

Case 3. $\eta_0 < 0$. In this case $z(\tilde{q}) = z^{\varepsilon_k}(q^k) = 0$ for $q^k = \tilde{q}$ and $\varepsilon_k > 0$.

Together, cases 1-3 in the second part of the proof, yield $Z^0 \subseteq \mathrm{g} - \mathrm{liminf}_\varepsilon z^\varepsilon$.

Combining the two parts of the proof, yields $z^\varepsilon \xrightarrow{g} Z^0$ and $s^\varepsilon \xrightarrow{g} S^0$. □

Note that when the set $\{j \mid u_j(\xi, x) = \max_{1 \leq i \leq n} u_i(\xi, x))\}$ is a singleton, then both sets $\{s^\varepsilon(\xi, x)\}$ and $S^0(\xi, x)$ are singletons. In such a case, $\text{g-lim sup}_\varepsilon\, s^\varepsilon(\xi, x) = S^0(\xi, x)$.

Theorem 3.2 yields graphical convergence of the solution function of the regularized LCP problem to the solution set of the LCP with the constrains of (1.4). Note that the solution set of the LCP does not contain a Lipschitz continuous function with respect to x. Consequently, the results about the convergence analysis of the regularized stochastic MPEC [16] are not applicable in this instance. (1.4).

THEOREM 3.3. *Assume $u : \Xi \times X \to \mathbb{R}^n$ is continuous and bounded, then $\mathbb{e}(D^\varepsilon, D) \to 0$ as $\varepsilon \downarrow 0$.*

Proof. Let $z^\varepsilon : \Xi \times X \to \mathbb{R}^{n+1}$ be the single valued function and $Z : \Xi \times X \rightrightarrows \mathbb{R}^{n+1}$ be the set-valued function such that for any (ξ, x), $z^\varepsilon(\xi, x)$ is the unique solution of LCP$(q(\xi, x), M^\varepsilon)$ and $Z(\xi, x)$ is the solution set of LCP$(q(\xi, x), M)$. By Theorem 3.2, $\text{g-limsup}_\varepsilon\, z^\varepsilon \subset Z^0$.

Let $\varepsilon_k \downarrow 0$ and $x^k \in D^{\varepsilon_k}$. We establish that any cluster point, say $\bar{x}$, of $\{x^k\}$, is in D. From the boundedness of q and Lemma 2.2, we know that $s^{\varepsilon_k}(\xi, x^k)$ is bounded. Hence,

$$\lim_{\varepsilon_k \downarrow 0} \mathbb{E}[s^{\varepsilon_k}(\boldsymbol{\xi}, x^k)] = \mathbb{E}[\lim_{\varepsilon_k \downarrow 0} s^{\varepsilon_k}(\boldsymbol{\xi}, x^k)] \subseteq \mathbb{E}[S(\boldsymbol{\xi}, \bar{x})], \tag{3.7}$$

where the equality comes from the Dominated Convergence Theorem and the inclusion from Theorem 3.2. The sequence $\{\varepsilon_k\}$ being arbitrary, (3.7) implies

$$\text{g-limsup}_{\varepsilon}\, \mathbb{E}[s^{\varepsilon}(\boldsymbol{\xi}, x)] \subseteq \mathbb{E}[S(\boldsymbol{\xi}, x)].$$

From $x^k \in D^{\varepsilon_k}$, $x^k \in X$, $A\mathbb{E}[s^{\varepsilon_k}(\boldsymbol{\xi}, x^k)] + \mathbb{B}^o_{r(\varepsilon_k)} \ni b$, X compact and [19, Theorem 5.37], one has $\bar{x} \in X$ and $A\mathbb{E}[S(\boldsymbol{\xi}, \bar{x})] \ni b$, i.e., $\bar{x} \in D$ and $\mathbb{e}(D^{\varepsilon}, D) \to 0$ as $\varepsilon \downarrow 0$. □

Assumption 1. *For any $\delta > 0$, there exists an $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in [0, \bar{\varepsilon}]$, $D^{\varepsilon} \cap (X^* + \mathbb{B}_{\delta}) \neq \emptyset$.*

This is a standard assumption in parametric programming, see [5, Chapter 4].

Theorem 3.4. *Suppose Assumption 1 holds and q is bounded. Then we have*

$$\lim_{\varepsilon \downarrow 0} \min_{x \in D^{\varepsilon}} f(x) = \min_{x \in D} f(x) \tag{3.8}$$

$$\operatorname*{Limsup}_{\varepsilon \downarrow 0} \operatorname*{argmin}_{x \in D^{\varepsilon}} f(x) \subseteq \operatorname*{argmin}_{x \in D} f(x) \tag{3.9}$$

Proof. The objective function f is a quadratic convex function and independent of ε. We need only consider the limiting behavior of the feasible set D^{ε} as $\varepsilon \downarrow 0$.

Define a set-valued mapping $\mathcal{D} : [0, \bar{\varepsilon}] \rightrightarrows \mathbb{R}^n$ with $\mathcal{D}(\varepsilon) = D^{\varepsilon}$ and $\mathcal{D}(0) = D$. Since for every $\varepsilon \in [0, \bar{\varepsilon}]$, D^{ε} and D are closed, $\mathcal{D}$ is a closed-valued mapping. Moreover, by Theorem 3.3, $\mathcal{D}$ is outer semicontinuous or, equivalently [19, Theorem 5.7], gph$\mathcal{D}$ is closed.

Note that $D^{\varepsilon} \subseteq X$ for all $\varepsilon > 0$ and X is compact. Hence, from Assumption 1, we obtain the assertions (3.8) and (3.9) from [5, Proposition 4.4]. □

Theorem 3.4 means that under Assumption 1, the optimal value function $v_{\varepsilon} := \min_{x \in D^{\varepsilon}} f(x)$ is continuous at $\varepsilon = 0$ and the optimal solution set X^{ε} is outer semicontinuous at $\varepsilon = 0$. Assumption 1 is related to Robinson's constraint qualification and often used in perturbation analysis of optimization problem [5]. In the following, we present a sufficient condition for Assumption 1, and the existence of solutions of the MPSLCC (1.4), the regularized problem (1.8) and its associated SAA problem (1.9).

For a fixed feasible solution $\hat{x}$ of problem (1.4), let's define

$$\sigma_1(\xi) := \min_{1 \le j \le n} q_j(\xi, \hat{x}), \qquad \sigma_2(\xi) := \min\{0, \min_{\substack{1 \le j \le n \\ q_j(\xi, \hat{x}) \neq \sigma_1(\xi)}} q_j(\xi, \hat{x})\},$$

and

$$\Xi_{\varepsilon} := \{\xi \in \Xi \,|\, \sigma_2(\xi) - \sigma_1(\xi) \ge \varepsilon(1 + \tau_0) \text{ or } \sigma_1(\xi) \ge 0\},$$

where $\tau_0 := -\min_{\xi \in \Xi}\{\sigma_1(\xi), 0\}$. By the continuity of u, the functions σ_1, σ_2 and $\sigma_2 - \sigma_1$ are continuous on Ξ. Note that the measure $P(\Xi_0) = P(\Xi)$ and $P(\Xi_{\varepsilon})$ is continuous at $\varepsilon = 0$ when the density function is continuous or the support of $\boldsymbol{\xi}$ is finite, i.e., $|\Xi|$ is finite. Hence there is a continuous function $\tilde{r}$ on the interval $[0, \bar{\varepsilon}]$ for sufficiently small $\bar{\varepsilon} > 0$ such that

$$P(\Xi_{\varepsilon}) \ge 1 - \tilde{r}(\varepsilon), \quad \text{with} \quad \lim_{\varepsilon \downarrow 0} \tilde{r}(\varepsilon) = 0. \tag{3.10}$$

THEOREM 3.5. *Assume that there exists a feasible solution $\hat{x}$ of problem (1.4) such that $A\mathbb{E}[s(\boldsymbol{\xi},\hat{x})] = b$ and $z(\xi,\hat{x}) = \big(s(\xi,\hat{x}),\gamma(\xi,\hat{x})\big)$ is the least norm solution of the LCP$(q(\xi,\hat{x}),M)$. Then problems (1.4) and (1.8) are solvable with $r(\varepsilon) \geq n\|A\|(\tau_0\varepsilon + 2\tilde{r}(\varepsilon))$, where $\tau_0 = -\min_{\xi\in\Xi}\{\sigma_1(\xi),0\}$. Moreover,*

$$\min_{x\in D} f(x) \leq \liminf_{\varepsilon\downarrow 0} \min_{x\in D^\varepsilon} f(x). \tag{3.11}$$

If the feasible solution $\hat{x}$ is an optimal solution, then Assumption 1 holds.

Proof. For any $\xi \in \Xi$, the solution set of the LCP$(q(\xi,\hat{x}),M)$ is bounded. From Theorem 3.1.8 in [9], the solution $z^\varepsilon(\xi,\hat{x})$ of the LCP$(q(\xi,\hat{x}),M^\varepsilon)$ converges to the least norm solution of LCP$(q(\xi,\hat{x}),M)$ as $\varepsilon \downarrow 0$.

To show that $\hat{x} \in D^\varepsilon$, we first prove that for any $\varepsilon \in (0,1)$,

$$0 \leq s^\varepsilon(\xi,\hat{x}) - \bar{s}(\xi,\hat{x}) \leq (\tau_0\varepsilon)e, \quad \text{for} \quad \forall\, \xi \in \Xi_\varepsilon. \tag{3.12}$$

We prove (3.12) by consider two cases: $\sigma_1(\xi) < 0$ and $\sigma_1(\xi) \geq 0$.

Case 1. $\sigma_1(\xi) < 0$. Since $\sigma_1(\xi) < 0$, by definition of $\sigma_1(\xi)$ and $\sigma_2(\xi)$ above, one has

$$\varepsilon \leq \frac{\sigma_2(\xi) - \sigma_1(\xi)}{1+\tau_0} \leq \frac{\sigma_2(\xi) - \sigma_1(\xi)}{1-\sigma_2(\xi)}, \qquad \forall \xi \in \Xi_\varepsilon.$$

Let us define

$$\mathcal{J}(\xi) = \{j\,|q_j(\xi,\hat{x}) = \sigma_1(\xi), j = 1,\ldots,n\} \quad \text{and} \quad J(\xi) = |\mathcal{J}(\xi)|.$$

Following the proof of Theorem 2.3, cf. (2.5) and (2.9), we can show that the solution $z^\varepsilon(\xi,\hat{x})$ of LCP$(q(\xi,\hat{x}),M^\varepsilon)$ has the following form

$$s_j^\varepsilon(\xi,\hat{x}) = \begin{cases} \frac{1-\varepsilon\sigma_1(\xi)}{J(\xi)+\varepsilon^2} & \text{if} \quad j \in \mathcal{J}(\xi), \\ 0 & \text{if} \quad j \notin \mathcal{J}(\xi), \end{cases} \qquad \gamma^\varepsilon(\xi,\hat{x}) = \frac{-J(\xi)\sigma_1(\xi) - \varepsilon}{J(\xi)+\varepsilon^2}. \tag{3.13}$$

The least norm solution $\bar{s}(\xi,\hat{x}) = \operatorname{argmin}_{s\in S(\xi,\hat{x})}\|y\|$ are the first n-components of the least norm solution of the LCP$(q(\xi,\hat{x}),M^\varepsilon)$, which has the form

$$\bar{s}_j(\xi,\hat{x}) = \begin{cases} \frac{1}{J(\xi)}, & \text{if} \quad j \in \mathcal{J}(\xi), \\ 0, & \text{if} \quad j \notin \mathcal{J}(\xi), \end{cases} \qquad \bar{\gamma}(\xi,\hat{x}) = -\sigma_1(\xi).$$

Hence, for $\xi \in \Xi_\varepsilon$, we obtain

$$0 \leq s_i^\varepsilon(\xi,\hat{x}) - \bar{s}_i(\xi,\hat{x}) \leq \frac{1-\varepsilon\sigma_1(\xi)}{J(\xi)+\varepsilon^2} - \frac{1}{J(\xi)} \leq \frac{-\varepsilon\sigma_1(\xi)}{J(\xi)} \leq -\varepsilon\sigma_1(\xi) \leq \tau_0\varepsilon.$$

For **Case 2**, it is easy to show that the solution $z^\varepsilon(\xi,\hat{x})$ of LCP$(q(\xi,\hat{x}),M^\varepsilon)$ has the following form

$$z_j^\varepsilon(\xi,\hat{x}) = 0,\ j = 1,\ldots,n+1,$$

and it is just the least norm solution of LCP$(q(\xi,\hat{x}),M)$, which means $s_i^\varepsilon(\xi,\hat{x}) - \bar{s}_i(\xi,\hat{x}) = 0$.

Combining Cases 1 and 2, we have (3.12).

Now, for sufficiently small ε, we consider the expected value

$$
\begin{aligned}
&|\mathbb{E}[s^\varepsilon(\boldsymbol{\xi},\hat{x}) - \bar{s}(\boldsymbol{\xi},\hat{x})]| \\
&\quad = |\mathbb{E}[\mathbf{1}_{\{\boldsymbol{\xi}\in\Xi_\varepsilon\}}(s^\varepsilon(\boldsymbol{\xi},\hat{x}) - \bar{s}(\boldsymbol{\xi},\hat{x}))] + \mathbb{E}[\mathbf{1}_{\{\boldsymbol{\xi}\notin\Xi_\varepsilon\}}(s^\varepsilon(\boldsymbol{\xi},\hat{x}) - \bar{s}(\boldsymbol{\xi},\hat{x}))]| \\
&\quad \le \big(\tau_0\varepsilon + 2\tilde{r}(\varepsilon)\big)e,
\end{aligned}
$$

where the last inequality uses the explicit form $s^\varepsilon(\xi,\hat{x})$ in Lemma 2.2, (3.10) with $0 \le s^\varepsilon(\xi,\hat{x}) \le 2e, 0 \le \bar{s}(\xi,\hat{x}) \le e$ and

$$|s^\varepsilon(\xi,\hat{x}) - \bar{s}(\xi,\hat{x})| \le \max\{s^\varepsilon(\xi,\hat{x}), \bar{s}(\xi,\hat{x})\} \le 2e.$$

Hence, we have

$$\|A\mathbb{E}[s^\varepsilon(\boldsymbol{\xi},\hat{x})] - b\| = \|A\mathbb{E}[s^\varepsilon(\boldsymbol{\xi},\hat{x}) - \bar{s}(\boldsymbol{\xi},\hat{x})]\| \le \|A\|\|e\|(\tau_0\varepsilon + 2\tilde{r}(\varepsilon)) \le r(\varepsilon),$$

which implies that $\hat{x} \in D^\varepsilon$.

Since problems (1.4) and (1.8) have the same (continuous) objective function, the feasibility of the two problems implies their solvability.

If $\hat{x} \in X^*$, then Assumption 1 holds from $\hat{x} \in D^\varepsilon$. □

We will verify the conditions of Theorem 3.5 in Section 4.

Remark. The set-valued constraints in (1.2) can also be approximated by a sequence of equality constraints via regularized quadratic programs with unique solutions for fixed $\varepsilon > 0$:

$$\max_y \langle y, u\rangle + \varepsilon\langle y, y\rangle \qquad \text{subject to } \langle e, y\rangle \le 1,\ y \ge 0.$$

However, the solution of the KKT-conditions is not unique and the relevant results can't be derived from the 'regularized' system

$$\text{Min}\ [\, z,\ \hat{M}^\varepsilon z + q(\xi, x)\,] = 0,$$

where $\hat{M}^\varepsilon = \begin{pmatrix} \varepsilon I & e \\ -e & 0 \end{pmatrix}$ is positive semi-definite and I is $n \times n$ identity matrix. The novel idea in Theorem 3.2 is that we use the well-established theory for monotone LCP to derive the required properties of the regularized solution of z^ε and, in particular, its first n-components s^ε.

3.2. Convergence of the SAA-regularized problem. In this section, we consider the convergence of the solution set X_N^ε of (1.9) to the solution set X^* of (1.4) as $\varepsilon \downarrow 0$ and $N \to \infty$. First, we consider the convergence of the solution set X_N^ε of problem (1.9) to the solution set X^ε of problem (1.8) as $N \to \infty$ for a fixed $\varepsilon > 0$. Next, we use this convergence result together with Theorem 3.4 to obtain the convergence of X_N^ε to X^* as $\varepsilon \downarrow 0$ and $N \to \infty$.

Let v_ε and v_N^ε be the optimal values of problems (1.8) and (1.9).

Assumption 2. *There exists a measurable function $c : \Xi \to (0, +\infty)$ such that $\mathbb{E}[c(\boldsymbol{\xi})^2] < \infty$ and*

$$\|u(\xi, x) - u(\xi, \bar{x})\| \le c(\xi)\|x - \bar{x}\|$$

for all $x, \bar{x} \in X$ and P-almost every $\xi \in \Xi$.

PROPOSITION 3.6. *Let $\hat{r}(\varepsilon, N) := r(\varepsilon)+cN^{-\tau}$, where $\tau \in (0, \frac{1}{2})$ and c is a positive constant. Suppose that the samples are iid and Assumption 2 holds. Moreover, there exists η such that $\|(u(\xi, x))_+\|_1 \le \eta$ for $x \in X, \xi \in \Xi$. Then there exists an $\bar{\varepsilon} > 0$ such that the following statements hold for any $\varepsilon \in (0, \bar{\varepsilon}]$.*

(i) For N sufficiently large, $D^\varepsilon \subset D^\varepsilon_N$, almost surely;
(ii) For any $\Delta > 0$ there exists a sufficiently large N_Δ such that $\mathbb{h}(D^\varepsilon, D^\varepsilon_N) \le \Delta$ holds P-a.s. for $N \ge N_\Delta$;
(iii) $v^\varepsilon_N \to v_\varepsilon$ and $\mathbb{e}(X^\varepsilon_N, X^\varepsilon) \to 0$ P-a.s. as $N \to \infty$.

Proof. (i) Since X is a compact subset of $\mathbb{R}^\nu$, by the continuity of $u(\xi, \cdot)$, $z^\varepsilon(\xi, \cdot)$ is globally Lipschitz continuous on X for almost every $\xi \in \Xi$. Moreover, by Lemma 2.2, $\|z^\varepsilon(\cdot, \cdot)\| \le 1+\varepsilon\eta$. Then by the classical uniform law of large numbers ([21, Proposition 7, Section 6]), we have $\frac{1}{N}\sum_{i=1}^N s^\varepsilon(\xi^i, x) \to \mathbb{E}[s^\varepsilon(\boldsymbol{\xi}, x)]$ uniformly P-a.s. as $N \to \infty$.

By the Remark following Theorem 2.4 and by Assumption 2,

$$\|s^\varepsilon(\xi, x) - s^\varepsilon(\xi, \bar{x})\| \le \frac{1}{\varepsilon}\|u(\xi, x) - u(\xi, \bar{x})\| \le \frac{1}{\varepsilon}c(\xi)\|x - \bar{x}\|$$

and $\mathbb{E}[c(\xi)^2] < \infty$. Moreover, for all $\xi \in \Xi$, $s^\varepsilon(\xi, x)$ is uniformly bounded. Then the mean and variance of random variables $s^\varepsilon(\xi, x)$ are finite for all $x \in X$. By [21, Chapter 6] and the functional central limit theorem [2, Corollary 7.17],

$$\left\|\frac{1}{N}\sum_{i=1}^N s^\varepsilon(\xi^i, x) - \mathbb{E}[s^\varepsilon(\boldsymbol{\xi}, x)]\right\| = O_p(\frac{1}{\sqrt{N}}).$$

By Assumption 1, $D^\varepsilon \ne \emptyset$. Then for all $x \in D^\varepsilon$, there exists sufficiently large N_0, such that, when $N \ge N_0$,

$$\begin{aligned}
\|\frac{1}{N}A\sum_{i=1}^N s^\varepsilon(\xi^i, x) - b\| &\le \|\frac{1}{N}A\sum_{i=1}^N s^\varepsilon(\xi^i, x) - A\mathbb{E}[s^\varepsilon(\boldsymbol{\xi}, x)]\| + \|A\mathbb{E}[s^\varepsilon(\boldsymbol{\xi}, x)] - b\| \\
&\le cN^{-\tau} + r(\varepsilon) \\
&= \hat{r}(\varepsilon, N)
\end{aligned}$$

P-a.s. which implies that $x \in D^\varepsilon_N$ P-a.s.

(ii) Let $\Delta > 0$ and

$$\delta(\Delta) := \inf_{\{x \in X : d(x, D^\varepsilon) \ge \Delta\}} (||A\mathbb{E}[s^\varepsilon(\boldsymbol{\xi}, x)] - b|| - r(\varepsilon))_+.$$

By compactness of X and continuity of $s^\varepsilon(\xi, \cdot)$, one has $\delta(\Delta) > 0$.

Let N_Δ be sufficiently large such that

$$\sup_{x \in X}\left\|\frac{1}{N}A\sum_{i=1}^N s^\varepsilon(\xi^i, x) - A\mathbb{E}[s^\varepsilon(\boldsymbol{\xi}, x)]\right\| \le \frac{\delta(\Delta)}{2}$$

and $cN^{-\tau} < \frac{\delta(\Delta)}{2}$. For any point $x \in X$ with $d(x, D^\varepsilon) \ge \Delta$, one has

$$\|A\mathbb{E}[s^\varepsilon(\boldsymbol{\xi}, x)] - b\| \ge r(\varepsilon) + \delta(\Delta),$$

which implies

$$
\begin{aligned}
||\frac{1}{N}A\sum_{i=1}^{N}s^{\varepsilon}(\xi^{i},x)-b|| &\geq ||A\mathbb{E}[s^{\varepsilon}(\boldsymbol{\xi},x)]-b||-\|\frac{1}{N}A\sum_{i=1}^{N}s^{\varepsilon}(\xi^{i},x)-A\mathbb{E}[s^{\varepsilon}(\boldsymbol{\xi},x)]\| \\
&\geq r(\varepsilon)+\delta(\Delta)-\frac{\delta(\Delta)}{2} \\
&> r(\varepsilon)+cN^{-\tau} \\
&= \hat{r}(\varepsilon,N).
\end{aligned}
$$

This shows that $x\notin D_N^{\varepsilon}$. Hence for any $x\in D_N^{\varepsilon}$, $d(x,D^{\varepsilon})\leq\Delta$, which implies

$$
\mathbb{e}(D_N^{\varepsilon},D^{\varepsilon})\leq\Delta.
$$

Combining the above result with Part (i), one has $\mathbb{h}(D^{\varepsilon},D_N^{\varepsilon})\leq\Delta$ P-a.s. for $N\geq N_0$.

(iii) Since problems (1.8) and (1.9) have the same convex quadratic objective function, independent of ε and ξ, we only need to consider the limit behavior of the feasible set D_N^{ε} as $N\to\infty$. Let

$$
G_{\varepsilon}(x,N):=\|A\frac{1}{N}\sum_{i=1}^{N}s^{\varepsilon}(\xi^{i},x)-b\|-r(\varepsilon)-N^{-\tau}.
$$

Since $\frac{1}{N}\sum_{i=1}^{N}s^{\varepsilon}(\xi^{i},x)\to\mathbb{E}[s^{\varepsilon}(\boldsymbol{\xi},x)]$ uniformly on X P-a.s. as $N\to\infty$, $G_{\varepsilon}(x,N)\to\|A\mathbb{E}[s^{\varepsilon}(\boldsymbol{\xi},x)]-b\|-r(\varepsilon)$ as $N\to\infty$ uniformly on X P-a.s. and continuous with respect to $x\in X$. Hence for any $\bar{x}\in X$,

$$
\lim_{N\to\infty,x\to\bar{x}}G_{\varepsilon}(x,N)=\|A\mathbb{E}[s^{\varepsilon}(\boldsymbol{\xi},\bar{x})]-b\|-r(\varepsilon),
$$

which implies that D_N^{ε} with respect to N is closed P-a.s.for sufficiently large N. By part (i) of this proposition, $D^{\varepsilon}\subseteq X$ and $D_N^{\varepsilon}\subseteq X$ are nonempty for sufficiently large N. Moreover, from part (ii) of this proposition, we have $\mathbb{h}(D^{\varepsilon},D_N^{\varepsilon})\to 0$ which implies that for any neighborhood $\mathcal{V}_{X^{\varepsilon}}$ of X^{ε}, there exists a sufficiently large N_0 such that for all $N\geq N_0$, $\mathcal{V}_{X^{\varepsilon}}\cap D_N^{\varepsilon}\neq\emptyset$. Hence all conditions of [5, Proposition 4.4] are satisfied, and thus we derive $v_N^{\varepsilon}\to v_{\varepsilon}$ and $\mathbb{e}(X_N^{\varepsilon},X^{\varepsilon})\to 0$. □

The proof of part (ii) of Proposition 3.6 is motivated by the proof of [23, Lemma 4.2 (i)].

Now, we are ready to present the convergence of X_N^{ε} to X^* as $\varepsilon\downarrow 0$ and $N\to\infty$.

THEOREM 3.7. *Suppose the conditions of Theorem 3.4 and Proposition 3.6 hold. If the feasible set D is nonempty, then X_N^{ε} is nonempty and*

$$
\lim_{\varepsilon\downarrow 0}\lim_{N\to\infty}\mathbb{e}(X_N^{\varepsilon},X^*)=0 \quad P\text{-}a.s. \tag{3.14}
$$

Proof. Since,

$$
\begin{aligned}
\lim_{\varepsilon\downarrow 0}\lim_{N\to\infty}\mathbb{e}(X_N^{\varepsilon},X^*) &\leq \lim_{\varepsilon\downarrow 0}\lim_{N\to\infty}\mathbb{e}(X_N^{\varepsilon},X^{\varepsilon})+\lim_{\varepsilon\downarrow 0}\lim_{N\to\infty}\mathbb{e}(X_{\varepsilon},X^*) \\
&= \lim_{\varepsilon\downarrow 0}0+\lim_{\varepsilon\downarrow 0}\mathbb{e}(X_{\varepsilon},X^*) \quad P\text{-}a.s. \\
&= 0 \quad P\text{-}a.s.,
\end{aligned}
$$

the assertion now follows directly from Theorem 3.4 and Proposition 3.6. □

In general the two limits in (3.14) cannot be taken simultaneously.

4. The pure characteristics demand model. One important application of problem (1.1) is to estimate the parameters of the pure characteristics demand model proposed by Berry and Pakes [3]. Although the model has several advantages in describing consumers' preference and purchasing behavior, it faces serious challenges and difficulties in estimating some key parameters when relying on the generalized method of moments (GMM). Pang, Su and Lee [18] reformulated the GMM estimation problem of the pure characteristics demand model as a computationally tractable quadratic program with linear complementarity constraints; the reformulated GMM estimation problem can be thought as a special case of problem (1.1). To illustrate our SAA regularized approach and the convergence results established in §2 and §3, we consider an example of the pure characteristics demand model:

- T is the number of markets and n the number of products in each market.
- The utility function in market t is:

$$u_t(\xi, x) = c_t \beta(\xi_1, x_2, x_3) - \alpha(\xi_2, x_4) p_t + x_{1t}, \tag{4.1}$$

  where $c_t = (c_{1t}, \ldots, c_{nt}) \in \mathbb{R}^{n\times K}$, $c_{jt} \in \mathbb{R}^K$, $x_{1t} \in \mathbb{R}^n$, $x_2, x_3 \in \mathbb{R}^K$, $x_4 \in \mathbb{R}$. Let $x^1 = (x_{11}, \ldots, x_{1T}) \in \mathbb{R}^{nT}$, $x^2 = (x_2, x_3, x_4)$ and $x = (x^1, x^2) \in \mathbb{R}^\nu$. Here $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) : \Omega \to \Xi \subseteq \mathbb{R}^{K+1}$ represents a consumer (or, more precisely, a consumer's behavior), which is described as a random vector and $\boldsymbol{\xi}_1$, $\boldsymbol{\xi}_2$ are independent and normally distributed as in [18] and

$$\beta(\xi_1, x_2, x_3) = x_2 + x_3\xi_1 \quad \text{and} \quad \alpha(\xi_2, x_4) = \exp(x_4\xi_2).$$

  For product j in market t, we use $c_{jt} \in \mathbb{R}^K$ to denote the characteristics (K), $(p_t)_j \in \mathbb{R}$ denotes the observed price, and $(x_{1t})_j \in \mathbb{R}$ denotes the errors (demand shock) which is not available in the data. We assume that $X := \{x : \underline{x} \le x \le \bar{x}\}$ for given $\underline{x}$, $\bar{x} \in \mathbb{R}^\nu$ and $\nu = 2K + nT + 1$.
- Consumer ξ chooses to purchase product j in market t if and only if

$$(u_t(\xi, x))_j \ge \max_{1\le i\le n} \{(u_t(\xi, x))_i, 0\}.$$

- $b_t \in \mathbb{R}^n$ with $(b_t)_j$ the observed market share of product j in market t.

The GMM estimation problem is aimed at finding optimal parameters x by minimizing the model error $\|x^1, x^1\|^2$ subject to the generalized market share equations

$$\mathbb{E}[S_t(\boldsymbol{\xi}, x)] \ni b_t, \quad t = 1, \ldots, T,$$

which can be expressed as a quadratic program with stochastic equilibrium set-valued constraints in the following form

$$\begin{array}{rl} \min_{x\in X} & \frac{1}{2}\langle x^1, x^1\rangle \\ \text{subject to} & \mathbb{E}[S_t(\boldsymbol{\xi}, x)] \ni b_t, \quad t = 1, \ldots, T, \end{array} \tag{4.2}$$

where $S_t(\xi, x)$ consists of all solutions of the linear program:

$$\max_s \{\, \langle s, u_t(\xi, x)\rangle \,|\, \langle e, s\rangle \le 1,\ s \ge 0 \,\}.$$

Obviously, the GMM estimation problem (4.2) is a special case of problem (1.4). We can apply the SAA regularized method to handle the problem. The convergence results established in §3 are applicable. Specifically, the regularized problem of (4.2) is:

$$
\begin{array}{ll}
\min_{x\in X} & \frac{1}{2}\langle x^1, x^1\rangle \\
\text{subject to} & \|\mathbb{E}[s_t^\varepsilon(\boldsymbol{\xi}, x)] - b_t\| \le r(\varepsilon), \quad t = 1, \dots, T.
\end{array} \tag{4.3}
$$

Let $\{\xi^i = (\xi_1^i, \xi_2^i), i = 1, \dots, N\}$ be iid observations of $(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2)$. The SAA regularized problem then reads,

$$
\begin{array}{ll}
\min_{x\in X} & \frac{1}{2}\langle x^1, x^1\rangle \\
\text{subject to} & \|\frac{1}{N}\sum_{i=1}^N s_t^\varepsilon(\xi^i, x) - b_t\| \le \hat{r}(\varepsilon, N), \quad t = 1, \dots, T,
\end{array} \tag{4.4}
$$

where $\big(s_t^\varepsilon(\xi, x), \gamma_t^\varepsilon(\xi, x)\big)$ is the unique solution of LCP$(q_t(\xi, x), M^\varepsilon)$:

$$
0 \le \begin{pmatrix} s \\ \gamma \end{pmatrix} \perp M^\varepsilon \begin{pmatrix} s \\ \gamma \end{pmatrix} + \begin{pmatrix} -u_t(\xi, x) \\ 1 \end{pmatrix} \ge 0 \tag{4.5}
$$

for some $\gamma_t^\varepsilon(\xi, x) \in \mathbb{R}$.

In what follows, we consider several numerical examples based on this model. All the numerical examples were carried out in MATLAB 8.0 installed on a IBM Notebook PC with Windows 7 operating system, and Intel Core i5 processor. We use an SQP algorithm to solve the penalized form of problem (4.4) as follows,

$$
\min_{x\in X} \frac{1}{2}\langle x^1, x^1\rangle + \rho\sum_{t=1}^T (\|\frac{1}{N}\sum_{i=1}^N s_t^\varepsilon(\xi^i, x) - b_t\|^2 - \hat{r}^2(\varepsilon, N))_+, \tag{4.6}
$$

using the closed form expression for $s_t^\varepsilon(\xi^i, x)$ derived in Lemma 2.2 and $\rho > 0$ is fixed. Note that since $s_t^\varepsilon(\xi, \cdot)$ is globally Lipschitz continuous with respect to x, there exists a sufficiently large ρ such that problem (4.4) and (4.6) are equivalent. Moreover, when the optimal value of problem (4.4) is 0, for $\rho > 0$, problem (4.4) and (4.6) are equivalent. In our algorithm, the subgradient of $s_t^\varepsilon(\xi^i, x)$ is approximated by the gradient sampling method in [6] and [13] with the subgradient formula given in Theorem 2.4.

In the following examples, we choose $r(\varepsilon) = 2\varepsilon$, $\hat{r}(\varepsilon, N) = r(\varepsilon) + N^{-\frac{2}{5}}$, $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) : \Omega \to \Xi \subseteq \mathbb{R}^2$, and $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ are independent with standard normal distribution Moreover, we use $\mathbb{D}(x_N^\varepsilon, X^*)$ to denote the distance between x_N^ε and X^*, and

$$
\text{error}(x_N^\varepsilon) = \sum_{i=1}^T \|\frac{1}{N_0}\sum_{i=1}^{N_0} s^*(\xi^i, x_N^\varepsilon) - b\|
$$

to measure the infeasibility of x_N^ε with a large sample size $N_0 = 10000 > N$, where $z^*(\xi, x_N^\varepsilon) = (s^*(\xi, x_N^\varepsilon), \gamma^*(\xi, x_N^\varepsilon))$ is the least norm solution of the LCP (1.5).

Example 4.1. We use a small example to highlight the convergence behavior with respect to sample size N and regularization parameter ε.

Let $T = 1$, $n = 2$, $K = 1$, $\nu = 5$. Since there is only one market in the example, for simplicity, we omit t here. Set $b = (\frac{1}{2}, \frac{1}{2})$, $c = (2, 3)$ and $p = (1, 2)$. We choose $X = \mathbb{R}^5$ and an initial point $x_0 = (1, 1, 1, 1, 1)$. It is easy to observe that $x^* = (x^1, 1, 0, 0)$ with $x^1 = (0, 0)$ is a solution and the optimal value of this problem at x^* is 0. Indeed, at x^*, we have

$$
\beta(\xi_1, x_2^*, x_3^*) = x_2^* + x_3^*\xi_1 = 1, \quad \alpha(\xi_2, x_4^*) = \exp(x_4^*\xi_2) = 1,
$$

and

$$u(\xi, x^*) = c\beta(\xi_1, x_2^*, x_3^*) - \alpha(\xi_2, x_4^*)p + x_1 = (1, 1).$$

The solution set of (1.5) is $S(\xi, x^*) = \{(\lambda, 1-\lambda) \,\big|\, 0 \le \lambda \le 1\}$ for all $\xi \in \Xi$. So the constraint $\mathbb{E}[\bar{s}(\boldsymbol{\xi}, x^*)] = b$ holds with the least norm solution

$$\bar{z}(\xi, x^*) = \big(\bar{s}(\xi, x^*),\, \gamma(\xi, x^*)\big) = (1/2, 1/2, 1).$$

Moreover, using this optimal solution x^* as a feasible solution for defining $\tilde{r}(\varepsilon)$ in (3.10), we obtain $\tilde{r}(\varepsilon) = 0$ and $r(\varepsilon) = 2\varepsilon$ in Theorem 3.5. Hence, by Theorem 3.5 and Proposition 3.6, problems (4.3) and (4.4) are solvable and Assumption 1 holds at x^*. The conditions of Theorem 3.7 are satisfied which means our convergence results hold for this problem.

We report numerical result for $\varepsilon = 0.05, 0.01$ and $N = 500, 800, 1100, 1400$. For each combination of ε and N, 20 independent test cases were carried out, each time solving the SAA regularized problem and obtaining an approximating solution x_N^ε and the associated "optimal" value f_N^ε. In our numerical tests, f_N^ε is always less than 1.0^{-5}. Table 4.1 presents the means of errors of the approximation solutions and the means of $\mathbb{D}(x_N^\varepsilon, X^*)$ when we solve problem (4.4). The table shows the downward trend of the errors when the value of ε gets smaller and the sample size N increases. In Figures 4.1-4.2, we use "boxplot" in Matlab to show the convergence trend of the error when the sample size N increases. Each box in the figures displays the range of errors of the computed solutions generated from 35 independent tests, where the central mark is the median and the edges of the box are the 25th and 75th percentiles.

TABLE 4.1
Values of error(x_N^ε) *and* $\mathbb{D}(x_N^\varepsilon, X^*)$ *with different* ε *and sample size*

N / ε	500		800		1100		1400	
	error	$\mathbb{D}$	error	$\mathbb{D}$	error	$\mathbb{D}$	error	$\mathbb{D}$
0.05	0.2170	0.1425	0.1855	0.1294	0.1667	0.1114	0.1159	0.1170
0.01	0.0473	0.0566	0.0339	0.0479	0.0251	0.0477	0.0209	0.0446

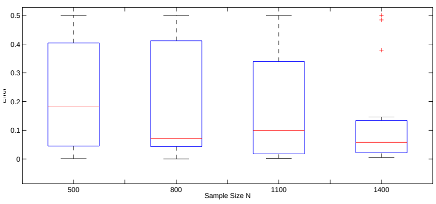

FIG. 4.1. *error*(x_N^ε) *when* $\varepsilon = 0.05$.

Example 4.2. We test two problems with $N = 500$ and $\varepsilon = 0.01$.

Test 1: $T = 2$, $n = 4$, $K = 1$, $\nu = 11$. Set

$$b = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix}, c = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \end{pmatrix}, p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}.$$

We choose $X = \mathbb{R}^{11}$ and an initial point $x_0 = (1, \ldots, 1) \in \mathbb{R}^{11}$. Similarly to Example 4.1, we can show that $x^* = (x^1, 1, 0, 0)\}$ with $x^1 = (0, \ldots, 0) \in \mathbb{R}^8$ is a solution and the optimal value at x^* is 0.

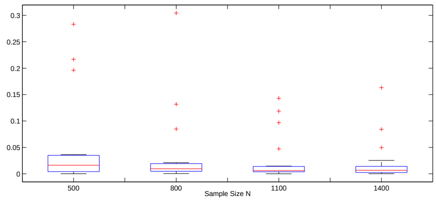

FIG. 4.2. *error*(x_N^ε) *when* $\varepsilon = 0.01$.

Test 2: $T = 3$, $n = 4$, $K = 1$, $\nu = 15$. Set

$$b = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix},\ c = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 \\ 9 & 10 & 11 & 12 \end{pmatrix},\ p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 10 & 11 & 12 & 13 \end{pmatrix}.$$

We choose $X = \mathbb{R}^{15}$ and an initial point $x_0 = (1, \ldots, 1) \in \mathbb{R}^{15}$. Similarly to Example 4.1, we can show that$x^* = (x^1, 1, 0, 0)$ with $x^1 = (0, \ldots, 0) \in \mathbb{R}^{12}$ is a solution and the optimal value at x^* is 0.

TABLE 4.2
Numerical results of Tests– Example 2

	$\mathbb{D}(x_N^\varepsilon, X^*)$	Optimal value	error(x_N^ε)
Test 1	0.0138	9.4590e-05	0.0378
Test 2	0.0775	0.0028	0.0216

The numerical examples in this paper are only preliminary numerical tests to illustrate the theoretical analysis. We plan to develop efficient algorithms for real applications in our future work.

5. Concluding remarks. Mathematical programs with set-valued stochastic equilibrium constraints (1.1) provide a powerful modeling paradigm for many important applications, in particular, in economics. For example, for the estimation of pure characteristics demand models with pricing. However, existing optimization methods with the sample average approximation become intractable for solving such problems. Recently, Pang et al. [18] proposed a mathematical programming with linear complementarity constraints (MPLCC) approach for the pure characteristics demand model with a finite number of observations. The MPLCC approach provides a promising computational method to estimate the consumer utility function. It is worth noting that if the objective function in the second stage is an arbitrary concave function $f(s, u_t(\xi, x))$ for any fixed (ξ, x), then the solution set $S_t(\xi, x)$ consists of all solutions to the following nonlinear complementarity problem

$$\begin{array}{lllll} 0 & \leq & s \perp & -\nabla_s f(s, u_t(\xi, x)) + \gamma e & \geq 0 \\ 0 & \leq & \gamma \perp & 1 - \langle e, s \rangle & \geq 0 \end{array} \tag{5.1}$$

for some γ. Since the matrix $M - \nabla_s^2 f(s, u_t(\xi, x))$ is positive semi-definite, the regularization approach can still be used to define a single valued solution function s_t^ε and $\lim_{\varepsilon \downarrow 0} d(s_t^\varepsilon(\xi, x), S_t(\xi, x)) = 0$, for any fixed (x, ξ). However, the proof for graphical convergence of s_t^ε may need stronger conditions than that of Theorem 3.2, as s_t^ε may not have a closed form and $-\nabla_s f(s, u_t(\xi, x))$ depends of (ξ, x, s).

Our main contribution is to develop the SAA regularized method. To handle the set-valued mapping in (1.1), we develop an efficient SAA regularized method using (1.8) and (1.9) which replaces the set-valued mapping by a single valued function. Problem (1.9) is a mathematical program with a convex quadratic objective function and globally Lipschitz continuous inequality constraints. Moreover, we derive a closed form of the solution of the regularized LCP(q, M^ε), which is useful for numerical computation and theoretical analysis. We show that a sequence of solutions $\{x_N^\varepsilon\}$ of the SAA regularized stochastic MPSLCC (1.9) converges to a solution of problem (1.4) as $\varepsilon \downarrow 0$ and $N \to \infty$.

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