Convergence of the reweighted $\ell_1$ minimization algorithm for $\ell_2$-$\ell_p$ minimization

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Abstract The iteratively reweighted $\ell_1$ minimization algorithm (IRL1) has been widely used for variable selection, signal reconstruction and image processing. In this paper, we show that any sequence generated by the IRL1 is bounded and any accumulation point is a stationary point of the $\ell_2$-$\ell_p$ minimization problem with $0 < p < 1$. Moreover, the stationary point is a global minimizer and the convergence rate is approximately linear under certain conditions. We derive posteriori error bounds which can be used to construct practical stopping rules for the algorithm.

Keywords $\ell_p$ minimization · stationary points · nonsmooth and nonconvex optimization · pseudo convex · global convergence

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1 Introduction

The nonsmooth, non-Lipschitz $\ell_p(0 < p < 1)$ regularization has advantages over smooth, convex regularization for restoring image with near edges, sparse

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signal reconstruction and variable selection. Iteratively reweighted $\ell_1$ minimization algorithms have been widely used for solving minimization problems with $\ell_p$ regularization

$$\min_{x \in \mathbb{R}^n} ||Ax - b||_2^2 + \lambda ||x||_p^p, \quad 0 < p < 1, \quad (1)$$

where $A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m, \lambda$ is a positive penalty parameter and

$$||x||^p_p = \sum_{i=1}^{n} |x_i|^p.$$ 

See [3,4,6–8,12,20,23,26]. A version of the IRL1 for solving the $\ell_2$-$\ell_p$ minimization problem (1) is as follows:

$$x^{k+1} \in \arg \min_{x \in \mathbb{R}^n} f_k(x, \varepsilon) := ||Ax - b||_2^2 + \lambda ||W^k x||_1 \quad (2)$$

where the weight $W^k = \text{diag}(w^k)$ is defined by the previous iterates and updated in each iteration as

$$w_i^k = \frac{p}{(|x_i^k| + \varepsilon)^{1-p}}, \quad i = 1, \ldots, n.$$ 

Here $\varepsilon$ is a positive parameter to ensure that the algorithm is well-defined.

At each iteration, the IRL1 (2) solves a convex $\ell_2$-$\ell_1$ minimization problem. There are many efficient algorithms for solving the $\ell_2$-$\ell_1$ minimization problem. In particular, new algorithms for solving large scale $\ell_2$-$\ell_1$ minimization problems have been developed in recent few years. Nocedal et al. proposed second-order methods for convex $\ell_1$ regularized optimization problems [2,24]. Fukushima [17] presented an SOR-type algorithm and a Jacobi-type algorithm that can effectively be applied to the $\ell_2$-$\ell_1$ problem by exploiting its special structure. The algorithms are globally convergent and can be implemented in a particularly simple manner. Moreover, the algorithms have close relations with coordinate minimization methods.

Extensive numerical experiments have shown that the IRL1 (2) is an efficient method for variable selection, signal reconstruction and image processing. In this paper, we prove that any sequence generated by the IRL1 (2) is bounded and any accumulation point is a stationary point $x^*$ of the following $\ell_2$-$\ell_p$ minimization problem.

$$\min_{x \in \mathbb{R}^n} f(x, \varepsilon) := ||Ax - b||_2^2 + \lambda \sum_{i=1}^{n} (|x_i| + \varepsilon)^p, \quad 0 < p < 1. \quad (3)$$

Moreover, the stationary point is a global minimizer of (3) in certain domain and the convergence rate is approximately linear under certain conditions. We derive posteriori error bounds

$$||x^k - x^*||_2 \leq \gamma ||x^{k+1} - x^k||_2$$
with a positive constant \( \gamma \), which can be used to construct practical stopping rules for the algorithm.

Note that the problem (2) may have multiple solutions since the objective is not strictly convex. However, convergence results in this paper hold for any choice \( x_{k+1} \) in the solution set \( \arg\min_{x \in \mathbb{R}^n} f_k(x, \varepsilon) \).

The model (3) can be considered as an approximation to the following constrained \( \ell_p \) optimization problem

\[
\min_{x \in \mathbb{R}^n} \sum_{i=1}^n (|x_i| + \varepsilon)^p, \quad \text{s.t.} \quad Ax = b, \quad (4)
\]

which is an approximation of the \( \ell_p \) minimization problem

\[
\min_{x \in \mathbb{R}^n} \|x\|_p, \quad \text{s.t.} \quad Ax = b. \quad (5)
\]

Problems (4) and (5) have been widely used \([3–6,8,16,20]\) when the vector \( b \) contains little or no noise. The models (1) and (3) are also called denoising models of (4) and (5). Recently, it has been proved that these four problems are NP-hard in \([9,18]\). An advantage of (4) and (3) is that their objective functions are Lipschitz continuous. However, relaxing non-Lipschitzian continuity to Lipschitzian continuity will not change the hardness of the problems.

We summarize some notations and results in nonsmooth optimization \([13]\), which will be used in this paper. It is known that a Lipschitz function \( g : \mathbb{R}^n \rightarrow \mathbb{R} \) is almost everywhere differentiable and its subgradient is defined by

\[
\partial g(y) = \text{co}\left\{ \lim_{y^k \rightarrow y} \nabla g(y^k) \right\},
\]

where "co" denotes the convex hull and \( D_g \) is the set of points at which \( g \) is differentiable.

We say \( x^* \) is a stationary point of \( g \) if \( 0 \in \partial g(x^*) \). If \( g \) is a convex function, then \( x^* \) is a global minimizer of \( g \) in \( \mathbb{R}^n \) if and only if \( x^* \) is a stationary point of \( g \).

A function \( g \) is convex if and only if \( \partial g \) is a monotone operator, that is,

\[
(y - x, \xi_y - \xi_x) \geq 0, \quad \forall \xi_y \in \partial g(y), \quad \forall \xi_x \in \partial g(x).
\]

We say a function \( g : \mathbb{R}^n \rightarrow \mathbb{R} \) is strongly pseudoconvex at \( x \) on \( D \) if for every \( \xi \in \partial g(x) \) and every \( y \in D \),

\[
\xi^T(y - x) \geq 0 \quad \Rightarrow \quad g(y) \geq g(x).
\]

We say a function \( g : \mathbb{R}^n \rightarrow \mathbb{R} \) is strongly pseudoconvex on \( D \) if \( g \) is strongly pseudoconvex at every point in \( D \) \([21, \text{Definition 5.1}]\).

Throughout this paper, \( \| \cdot \| \) denotes the \( \ell_2 \) norm and \( (x_i)_{1 \leq i \leq n} \) stands for the vector \( x \in \mathbb{R}^n \). \( |x| \) is the absolute value vector of \( x \), that is, \( |x| = (|x_1|, \ldots, |x_n|)^T \). The vector \( e_i \in \mathbb{R}^n \) is the ith column of the identity matrix. The vector \( a_i \in \mathbb{R}^m \) is the ith column of the matrix \( A \). The cardinality of a subset \( T \subset \{1, \ldots, n\} \) is denoted by \( |T| \), and its complement set is denoted by \( T^c \).
2 Convergence analysis

In this section, we give convergence analysis for the IRL1 (2). Note that both objective functions \( f \) and \( f_k \) are Lipschitz continuous for any fixed \( \varepsilon > 0 \). Hence we can define their subgradients in \( \mathbb{R}^n \). Moreover, both functions are nonnegative and satisfy
\[
f(x, \varepsilon) \to \infty, \quad f_k(x, \varepsilon) \to \infty \quad \text{as} \quad \|x\| \to \infty.
\]

Therefore, the solution sets of (2) and (3) are nonempty and bounded.

**Lemma 1** For any nonnegative constants \( \alpha, \beta \) and \( t \in (0, 1) \), we have
\[
\alpha^{1-t} \beta^t \leq (1-t)\alpha + t\beta,
\]
and equality holds if and only if \( \alpha = \beta \).

**Proof.** Young’s inequality states that for any nonnegative constants \( \mu \) and \( \nu \),
\[
\mu \nu \leq \frac{1}{q} \mu^q + \frac{1}{r} \nu^r, \quad \left( \frac{1}{q} + \frac{1}{r} = 1 \right)
\]
where equality holds if and only if \( \mu^q = \nu^r \). Set \( \frac{1}{q} = 1-t, \mu^q = \alpha \) and \( \nu^r = \beta \) in this inequality. We obtain (7) and equality holds if and only if \( \alpha = \beta \). □

**Lemma 2** Let \( \{x^k\} \) be the sequence generated by the IRL1 (2). Then we have
\[
f(x^{k+1}, \varepsilon) \leq f(x^k, \varepsilon) - \|A(x^{k+1} - x^k)\| - \delta(x^{k+1}, x^k),
\]
where
\[
\delta(x^{k+1}, x^k) = \frac{\lambda}{n} \sum_{i=1}^{n} (1 - p)(|x_i^k| + \varepsilon) + p(|x_i^{k+1}| + \varepsilon) - (|x_i^k| + \varepsilon)^{1-p}(|x_i^{k+1}| + \varepsilon)^p \geq 0
\]
and equality holds if and only if \( |x^{k+1}| = |x^k| \).

**Proof.** Since \( x^{k+1} \) is a solution of problem (2), the zero vector is contained in the generalized differential with respect to \( x \), that is,
\[
0 \in \partial f_k(x^{k+1}, \varepsilon).
\]
See [13]. The function \( f_k \) is the sum of \( n+1 \) convex functions, namely, \( \|Ax - b\|^2 \) and \( |x_i|, i = 1, \ldots, n \). By the addition rule of subgradient for the sum of convex functions [13, Corollary 3, p40], we have
\[
\partial f_k(x, \varepsilon) = \lambda \sum_{i=1}^{n} \frac{p \partial |x_i|}{(|x_i^k| + \varepsilon)^{1-p}} e_i + 2A^T(Ax - b).
\]

Hence, we find
\[
0 \in \partial f_k(x^{k+1}, \varepsilon) = \lambda \sum_{i=1}^{n} \frac{p}{(|x_i^k| + \varepsilon)^{1-p}} \partial |x_i^{k+1}| e_i + 2A^T(Ax^{k+1} - b),
\]
which means that there exist \( c_i^{k+1} \in \partial |x_i^{k+1}|, \) \( i = 1, \ldots, n \) such that

\[
\lambda \left( \frac{pc_i^{k+1}}{|x_i^k| + \epsilon} \right)_{1 \leq i \leq n} + 2A^T(Ax^{k+1} - b) = 0. \tag{11}
\]

By the definition of the subdifferential for \( |x_i| \), we have

\[
x_i^{k+1} = \begin{cases} 
1, & \text{if } x_i^{k+1} > 0, \\
-1, & \text{if } x_i^{k+1} < 0, \\
\alpha, & \text{if } x_i^{k+1} = 0, \quad \alpha \in [-1, 1]. \tag{12}
\end{cases}
\]

By (11), (12) and (7), we obtain

\[
f(x^k, \epsilon) - f(x^{k+1}, \epsilon) = \lambda \sum_{i=1}^n ((|x_i^k| + \epsilon)^p - (|x_i^{k+1}| + \epsilon)^p) + \|Ax^{k+1} - Ax^k\|^2 + 2(Ax^k - Ax^{k+1})^T(Ax^{k+1} - b) \geq \|Ax^{k+1} - Ax^k\|^2 + \lambda \sum_{i=1}^n \left( (|x_i^k| + \epsilon)^p - (|x_i^{k+1}| + \epsilon)^p + \frac{pc_i^{k+1}(|x_i^{k+1} - x_i^k|)}{(|x_i^k| + \epsilon)^{1-p}} \right) \tag{13}
\]

\[
= \|Ax^{k+1} - Ax^k\|^2 + \lambda \sum_{i=1}^n \left( (|x_i^k| + \epsilon) - (|x_i^{k+1}| + \epsilon)^{1-p}(|x_i^{k+1}| + \epsilon)^p + p(|x_i^{k+1}| - |x_i^k|) \right) \tag{13}
\]

\[
= \|Ax^{k+1} - Ax^k\|^2 + \lambda \sum_{i=1}^n \left( (1-p)(|x_i^k| + \epsilon) + p(|x_i^{k+1}| + \epsilon) - (|x_i^k| + \epsilon)^{1-p}(|x_i^{k+1}| + \epsilon)^p \right) \geq \|Ax^{k+1} - Ax^k\|^2.
\]

where the first inequality uses

\[
c_i^{k+1} x_i^{k+1} = |x_i^{k+1}| \quad \text{and} \quad |c_i^{k+1}| \leq 1
\]

and the last inequality uses Lemma 1 to claim that

\[
\delta_i(x^{k+1}, x^k) = \lambda \left( \frac{(1-p)(|x_i^k| + \epsilon) + p(|x_i^{k+1}| + \epsilon) - (|x_i^{k+1}| + \epsilon)^{1-p}(|x_i^{k+1}| + \epsilon)^p}{(|x_i^k| + \epsilon)^{1-p}} \right) \geq 0,
\]

and \( \delta(x^{k+1}, x^k) = \sum_{i=1}^n \delta_i(x^{k+1}, x^k) \geq 0. \]\n
\textbf{Lemma 3} Suppose that \( g_1 : \mathbb{R}^n \to \mathbb{R} \) and \( -g_2 : \mathbb{R}^n \to \mathbb{R} \) are convex on a closed convex set \( \Omega \), and \( g_1(x) \geq 0 \) and \( g_2(x) > 0 \), for all \( x \in \Omega \) then \( h(x) = \frac{g_1(x)}{g_2(x)} \) is strongly pseudoconvex on \( \Omega \).
Proof. This lemma is a simple generalization of [21, Proposition 5.2, p943], which proved that the condition number of a symmetric positive definite matrix is pseudoconvex. For completeness, we give a proof of this lemma.

From the convexity assumption, $g_1$ and $g_2$ are locally Lipschitz continuous and for any $x, y \in \Omega$ and $\xi_1 \in \partial g_1(x), \xi_2 \in \partial g_2(x)$, we have

$$g_1(y) - g_1(x) \geq \xi_1^T (y - x).$$

and

$$-g_2(y) + g_2(x) \geq -\xi_2^T (y - x).$$

Hence we obtain

$$g_1(y) - h(x) g_2(y) = g_1(y) - g_1(x) + h(x)(-g_2(y) + g_2(x))$$

$$\geq \xi_1^T (y - x) - h(x) \xi_2^T (y - x)$$

$$= (\xi_1 - h(x) \xi_2)^T (y - x)$$

$$= g_2(x) \left( \frac{\xi_1 g_2(x) - g_1(x) \xi_2}{g_2(x)^2} \right)^T (y - x).$$

By the quotient rule for the Clarke generalized gradient [13, Proposition 2.3.14], we find that $\frac{\xi_1 g_2(x) - g_1(x) \xi_2}{g_2(x)^2} \in \partial h(x)$, from that $g_2$ and $g_1$ are Clarke regular. Therefore we have $h(y) \geq h(x)$ if $\xi^T (y - x) \geq 0$ with $\xi \in \partial h(x)$. \qed

Lemma 4 For constants $\alpha > 0, \varepsilon > 0$ and $p \in (0, 1)$, let

$$\phi(t) = |t| + (\alpha t^2 + \beta t)(|t| + \varepsilon)^{1-p}.$$  

Then $\phi$ is convex in $[0, \infty)$ and $(-\infty, 0]$ if

$$|\beta| \leq \frac{\alpha \varepsilon}{1 - p}. \quad (14)$$

Proof. The function $\phi$ is differentiable in $R$ except $t = 0$. To show the convexity of $\phi$, we consider the second derivative of $\phi$ for $t \neq 0$.

First we consider $t > 0$. By simple calculation, we get

$$\phi''(t) = (t + \varepsilon)^{-1-p} (c_1 t^2 + c_2 t + c_3),$$

where

$$c_1 = \alpha(2 + (4-p)(1-p)),$$

$$c_2 = (2-p)((1-p)\beta + 4\alpha \varepsilon),$$

$$c_3 = 2 \varepsilon(\alpha \varepsilon + (1-p)\beta).$$

Obviously, $c_1 > 0$, $i = 1, 2$ and $c_3 \geq 0$. This implies that $\phi$ is convex for $t > 0$.

Now, we consider $t < 0$. In this case,

$$\phi(t) = -t + (\alpha t^2 + \beta t)(-t + \varepsilon)^{1-p}.$$
Similarly, we can find that for \( t < 0 \),
\[
\phi''(t) = (-t + \varepsilon)^{-1-p}(c_1t^2 + c_4t + c_5)
\]
where
\[
c_4 = (2 - p)((1 - p)\beta - 4\varepsilon), \\
c_5 = 2\varepsilon(\alpha\varepsilon - (1 - p)\beta).
\]
Obviously, \( c_4 < 0 \) and \( c_5 \geq 0 \). This implies that \( \phi''(t) \geq 0 \) and thus \( \phi \) is convex for \( t < 0 \). By the continuity of \( \phi \) and that for \( t_1 t_2 > 0 \)
\[
\phi(\mu t_1 + (1 - \mu)t_2) \leq \mu \phi(t_1) + (1 - \mu)\phi(t_2), \quad \text{for} \quad 0 \leq \mu \leq 1,
\]
we can take \( t_1 \to 0 \) or \( t_2 \to 0 \), and claim that \( \phi \) is convex in \([0, \infty)\) and \((-\infty, 0]\).

**Theorem 1** Let \( \{x^k\} \) be a sequence generated by the IRL1 (2). Then the sequence \( \{x^k\} \) is bounded and \( \lim_{k \to \infty} (x^{k+1} - x^k) = 0 \). Moreover, any accumulation point of \( \{x^k\} \) is a stationary point \( x^* \) of (3).

**Proof.** By Lemma 2, the sequence \( \{f(x^k, \varepsilon)\} \) is monotonically decreasing and bounded below. Hence it converges. It is clear that the sequence \( \{x^k\} \) is contained in the level set
\[
\mathcal{L}(x^0) = \{ x \mid f(x, \varepsilon) \leq f(x^0, \varepsilon) \}.
\]
Obviously, \( \mathcal{L}(x^0) \) is bounded from (6).

By (8), we have \( \delta(x^{k+1}, x^k) \to 0 \), as \( k \to \infty \). From Lemma 2 and \( \delta_i(x^{k+1}, x^k) \geq 0 \), we have
\[
\lim_{k \to \infty} f(x^k, \varepsilon) - f(x^{k+1}, \varepsilon) = \lim_{k \to \infty} \|A(x^{k+1} - x^k)\| = \lim_{k \to \infty} (|x^k| - |x^{k+1}|) = 0.
\]
This, together with (13), implies
\[
\lim_{k \to \infty} c_{i}^{k+1}(x_{i}^{k+1} - x_{i}^{k}) = 0, \quad i = 1, \ldots, n,
\]
where \( c_{i}^{k+1} \) is defined in (12). Note that \( c_{i}^{k+1} = 0 \) only if \( x_{i}^{k+1} = 0 \), and \( |c_{i}^{k+1}| = 1 \) if \( x_{i}^{k+1} \neq 0 \). For a fixed \( i \), suppose that there is a subsequence \( \{x_{i}^{k_{j}}\} \) such that \( x_{i}^{k_{j}+1} = 0 \), then from (15) we have \( \lim_{k \to \infty} x_{i}^{k} = 0 \). Otherwise, we have \( |c_{i}^{k+1}| = 1 \) for sufficiently large \( k \), which, together with (16), we have
\[
\lim_{k \to \infty} (x_{i}^{k+1} - x_{i}^{k}) = 0, \quad i = 1, \ldots, n.
\]

Let \( \{x^{n_k}\} \) be a subsequence of \( \{x^k\} \) which converges to \( x^* \). By (11) and (17), there exist \( c_i \in \partial|x_{i}^{k}| \), \( i = 1, \ldots, n \) such that
\[
0 = \lim_{k \to \infty} \lambda \left( \frac{p_{i}^{n_k}}{(|x_{i}^{n_k} + x_{i}^{n_k} - x_{i}^{n_k}| + \varepsilon)^{1-p}} \right)_{1 \leq i \leq n} + 2A^T(Ax^{n_k} - b)
\]
\[
= \lambda \left( \frac{p_{i}^{n_k}}{|x_{i}^{n_k} + \varepsilon|^{1-p}} \right)_{1 \leq i \leq n} + 2A^T(Ax^{*} - b) \in \partial f(x^*, \varepsilon).
\]
Hence \( x^* \) is a stationary point of (3).

\[ \square \]
Theorem 2

By (18), we have 0
\[
φ(x^*) = \lim_{n \to \infty} φ(x_n) = 0
\]

Proof. (1) Let
\[
(1) \quad \text{if for some } i, \epsilon \geq \left( \frac{\lambda(1-p)p}{2\|a_i\|^2} \right)^{\frac{1}{p}} \text{ holds, then}
\]
\[
f(x^*, \epsilon) \leq f(x^* + t e_i, \epsilon), \quad \text{for } t \in \left[ -x_i^*, \infty \right) \text{ if } x_i^* \geq 0,
\]
\[
\quad \left( -\infty, -x_i^* \right] \text{ if } x_i^* \leq 0.
\]

(2) If for some i, \( x_i^* \geq 0 \), then (20) holds. Moreover, if for some i, \( a_i^T(A_1x_i^* - b) = 0 \) holds, then \( x_i^* = 0 \) and
\[
f(x^*, \epsilon) \leq f(x^* + t e_i, \epsilon), \quad \text{for } t \in \mathbb{R}.
\]

Proof. (1) Let
\[
φ(t) = \lambda \| x^* + t e_i + \epsilon \|_p^p + \| A(x^* + t e_i) - b \|^2.
\]

The subdifferential of \( φ \) is
\[
\partial φ(t) = \lambda \frac{p \text{sign}(x_i^* + t) \| x_i^* + t + \epsilon \|^{1-p}}{\| x_i^* + t + \epsilon \|^{2-p}} + 2 \lambda a_i^T A(x^* + t e_i) - b.
\]

By (18), we have 0 \( \in \partial φ(0) \), that is, 0 is a stationary point of \( φ \). For \( t_1 \) and \( t_2 \) satisfying \( (x_i^* + t_1)(x_i^* + t_2) > 0 \), \( φ \) is continuously twice differentiable on \( [t_1, t_2] \). Thus there is \( t_0 \) between \( t_1 \) and \( t_2 \) such that
\[
φ'(t_1) - φ'(t_2) = \left( -\frac{\lambda(1-p)p}{\| x_i^* + t_0 + \epsilon \|^{2-p}} + 2 \lambda a_i^T \right)(t_1 - t_2).
\]
Hence if \( \varepsilon \geq \left( \frac{\lambda(1-p)p}{2\|a_i\|^2} \right)^{\frac{1}{2-p}} \), then
\[
(t_1 - t_2)(\varphi'(t_1) - \varphi'(t_2)) = \left( -\frac{\lambda(1-p)p}{(x_i^* + t_0 + \varepsilon)2-p} + 2\|a_i\|^2 \right)(t_1 - t_2)^2 
\geq \left( -\frac{\lambda(1-p)p}{\varepsilon^{2-p}} + 2\|a_i\|^2 \right)(t_1 - t_2)^2 \geq 0.
\]
Hence \( \varphi \) is convex in \([ -x_i^*, \infty]\) if \( x_i^* \geq 0 \), and in \((-\infty, -x_i^*] \) if \( x_i^* \leq 0 \). This, together with \( 0 \in \partial \varphi(0) \) implies that 0 is the minimizer of \( \varphi \) in \((-x_i^*, \infty) \) if \( x_i^* \geq 0 \), and in \((-\infty, -x_i^*] \) if \( x_i^* \leq 0 \). This gives (20).

(2) To prove this part, we show \( \varphi \) defined in (22) is strongly pseudoconvex in \([ -x_i^*, \infty)\) and \((-\infty, -x_i^*] \). The function \( \varphi \) can be rewritten as
\[
\varphi(t) = \lambda(|x_i^* + t| + \varepsilon)^p + \|a_i\|^2(x_i^* + t)^2 + 2a_i^T(A_ix_i^* - b)(x_i^* + t) + c_0,
\]
\[
\varphi(t) = \lambda\left(\frac{|x_i^* + t| + \varepsilon + \left( \frac{|a_i|^2}{\lambda} (x_i^* + t)^2 + \frac{2a_i^T(A_ix_i^* - b)}{\lambda} (x_i^* + t) \right) (|x_i^* + t| + \varepsilon)^{1-p}}{(x_i^* + t) + \varepsilon} \right) + c_0,
\]
where \( c_0 \) is a constant. Using Lemma 4, with
\[
\alpha = \frac{\|a_i\|^2}{\lambda} \quad \text{and} \quad \beta = \frac{2a_i^T(A_ix_i^* - b)}{\lambda},
\]
we find that the function
\[
|x_i^* + t| + \varepsilon + \left( \frac{|a_i|^2}{\lambda} (x_i^* + t)^2 + \frac{2a_i^T(A_ix_i^* - b)}{\lambda} (x_i^* + t) \right) (|x_i^* + t| + \varepsilon)^{1-p}
\]
is convex. Since \( (|x_i^* + t| + \varepsilon)^{1-p} \) is concave, we find that \( \varphi \) is strongly pseudoconvex by Lemma 3.

By the definition of the strong pseudoconvexity and (18), from \( \varphi(0) = f(x^*, \varepsilon) \) and \( \varphi(t) = f(x^* + te_i, \varepsilon) \), we obtain (20).

If \( a_i^T(A_ix_i^* - b) = 0 \), then (18) implies that
\[
0 = \lambda\left(\frac{pc_i^*}{(|x_i^*| + \varepsilon)^{1-p}} \right) + 2a_i^Ta_ix_i^*.
\]
(23)
Since \( c_i^* = 1 \) if \( x_i^* > 0 \) and \( c_i^* = -1 \) if \( x_i^* < 0 \), (23) only holds at \( x_i^* = 0 \). Moreover, it is easy to see that in such case with \( x_i^* = 0 \),
\[
\varphi(-x_i^*) = \varphi(0) \leq \varphi(t), \quad \text{for} \ t \in R,
\]
that is,
\[
f(x^* - x_i^*e_i, \varepsilon) = f(x^*, \varepsilon) \leq f(x^* + te_i, \varepsilon), \quad \text{for} \ t \in R.
\]
We obtain the desired results. \( \square \)

In [8], it was shown that any local minimizer \( x^* \) of (1) satisfies
\[
either \ |x_i^*| = 0 \ or \ |x_i^*| \geq L_i, \quad \forall \ i = 1, \cdots, n,
\]
(24)
where

\[ L_i := \left( \frac{\lambda p(1-p)}{2\|a_i\|^2} \right)^{\frac{1}{p}}. \]

This lower bound for absolute value of nonzero elements of any local minimizer of (1) can be easily extended to the model (3). We give the lower bound theory for (3) in the following theorem.

**Theorem 3** If \( 0 \leq \epsilon < L := \min_{1 \leq i \leq n} L_i \), then every local minimizer \( x^* \) of (3) satisfies

\[ \text{either } |x_i^*| = 0 \text{ or } |x_i^*| \geq L_i - \epsilon, \quad \forall i = 1, \ldots, n. \quad (25) \]

**Proof.** This Theorem is a simple generalization of Theorem 2.1 in [8]. For completeness, we give a brief proof.

Let \( x^* \) be a local minimizer of (3) with \( \|x^*\|_0 = k \), without loss of generality, we assume

\[ x^* = (x_{1}^*, \ldots, x_{k}^*, 0, \ldots, 0)^T. \]

Let \( z^* = (x_{1}^*, \ldots, x_{k}^*)^T \) and \( B \in \mathbb{R}^{m \times k} \) be the submatrix of \( A \), whose columns are the first \( k \) columns of \( A \). For a fixed \( \epsilon \geq 0 \), define a function \( g : \mathbb{R}^k \rightarrow \mathbb{R} \) by

\[ g(z, \epsilon) = \|Bz - b\|^2 + \lambda \sum_{i=1}^{k} (|z_i^*| + \epsilon)^p + (n - k)\epsilon^p. \]

We have

\[ f(x^*, \epsilon) = \|Ax^* - b\|^2 + \lambda \sum_{i=1}^{n} (|x_i^*| + \epsilon)^p = \|Bz^* - b\|^2 + \lambda \sum_{i=1}^{k} (|z_i^*| + \epsilon)^p + (n - k)\epsilon^p. \]

Since \( |z_i^*| > 0, i = 1, \ldots, k \), \( g \) is continuously differentiable at \( z^* \). Moreover, in a neighbourhood of \( z^* \),

\[ g(z^*, \epsilon) = f(x^*, \epsilon) \leq \min \{ f(x, \epsilon) | x_i = 0, i = k + 1, \ldots, n \} \]

\[ = \min \{ g(z, \epsilon) | z \in \mathbb{R}^k \}, \]

which implies that \( z^* \) is a local minimizer of the function \( g \). Hence the second order necessary condition for

\[ \min_{z \in \mathbb{R}^k} g(z, \epsilon) \quad (26) \]

holds at \( z^* \), which gives that the matrix

\[ 2B^TB + \lambda p(p-1)\text{diag}((|z_i^*| + \epsilon)^{p-2}) \]

is positive semi-definite. Therefore, we obtain

\[ 2e_i^TB^TB e_i + \lambda p(p-1)\text{diag}((|z_i^*| + \epsilon)^{p-2}) \geq 0, \quad i = 1, \ldots, k \]

where \( e_i \) is the \( i \)th column of the identity matrix of \( \mathbb{R}^{k \times k} \).
Note that \(|a_i|^2 = e_i^T B^T B e_i\). We find that
\[
(|x^*_i| + \varepsilon)^{p-2} \leq \frac{2||a_i||^2}{\lambda p(1-p)}, \quad i = 1, \ldots, k
\]
which implies that
\[
|x^*_i| \geq \left( \frac{\lambda p(1-p)}{2||a_i||^2} \right)^{\frac{1}{p-1}} - \varepsilon = L_i - \varepsilon, \quad i = 1, \ldots, k.
\]
Hence for any local minimizer \(x^*\) of (3) if \(x^*_i \neq 0\), then \(|x^*_i| \geq L_i - \varepsilon\).

In the proof of Theorem 3, we use the second order necessary optimality condition of a subproblem to derive the lower bound (25). Similarly, we can use the first order necessary optimality condition of the subproblem to derive other lower bound as Theorem 3.1 in [8]. Moreover, the lower bound theory of nonzero entries in local minimizers can be extended to vectors satisfying the first order and second order necessary optimality conditions.

For \(x \in \mathbb{R}^n\), let \(X = \text{diag}(x)\) and \(S = \{i \mid x_i \neq 0\}\). For problem (1), the necessary optimality conditions are as follows [8].

First order: \(2X A^T (Ax - b) + \lambda p|x|^p = 0\)

Second order: \(2X A^T A X + \lambda p(p - 1) \text{diag}(|x|^p)\) is positive semi-definite.

These two conditions can be equivalently written as

First order: \((2A^T (Ax - b))_S + \lambda p|x_S|^{p-1} = 0, \quad x_{SC} = 0\)

Second order: \(2A_S^T A_S + \lambda p(p - 1) \text{diag}(|x_S|^{p-2})\) is positive semi-definite, \(x_{SC} = 0\).

For problem (3), the necessary optimality conditions are as follows.

First order: \((2A^T (Ax - b))_S + \lambda p(|x| + \varepsilon)^{p-1} = 0, \quad x_{SC} = 0\)

Second order: \(2A_S^T A_S + \lambda p(p - 1) \text{diag}((|x_S| + \varepsilon)^{p-2})\) is positive semi-definite, \(x_{SC} = 0\).

It is easy to see that if \(x\) satisfies the second order necessary optimality condition of (1), then \(x\) satisfies the second order necessary optimality condition of (3), from the following inequality
\[
\lambda p(p - 1)|x_i|^{p-2} < \lambda p(p - 1)(|x_i| + \varepsilon)^{p-2}, \quad \text{for} \quad i \in S.
\]

Moreover, we have the following proposition for problems (1) and (3).

**Proposition 1** For any sequence \(\{\varepsilon_k\}\), which satisfies \(\varepsilon_k \to 0^+\), the following statements hold.

(i) Let \(\{x_{\varepsilon_k}\}\) be a sequence of vectors satisfying the first order necessary optimality condition of (3) with \(\varepsilon = \varepsilon_k\), then any accumulation point of \(\{x_{\varepsilon_k}\}\) satisfies the first order necessary optimality condition of (1).

(ii) Let \(\{x_{\varepsilon_k}\}\) be a sequence of vectors satisfying the second order necessary optimality condition of (3) with \(\varepsilon = \varepsilon_k\), then any accumulation point of \(\{x_{\varepsilon_k}\}\) satisfies the second order necessary optimality condition of (1).

(iii) Let \(\{x_{\varepsilon_k}\}\) be a sequence of global minimizers of (3) with \(\varepsilon = \varepsilon_k\), then any accumulation point of \(\{x_{\varepsilon_k}\}\) is a global minimizer of (1).
Then from 

\[ 0 \leq \|Ax_{\varepsilon_k} - b\|^2 + \lambda \|x_{\varepsilon_k}\|^p_p \leq f(x_{\varepsilon_k}, \varepsilon_k) \leq f(x^*, \varepsilon_k), \]

we have \( f(\bar{x}) \leq f(x^*) \) when \( \varepsilon_k \to 0^+ \). Hence \( \bar{x} \) is a global minimizer of \( (1) \). □ 

Now we derive the convergence rate of the IRL1 (2) and error bounds.

**Theorem 4** Assume that the sequence \( \{x^k\} \) generated by (2) converges to a local minimizer \( x^* \) of (3). Denote \( S = \{i \mid x^*_i \neq 0\} \) and \( \beta = \min_{i \in S} |x^*_i| \). If

\[
\frac{\lambda p(1-p)}{\beta^{2-p}} \leq 2\lambda_{\text{min}}(A^T_S A_S),
\]

then there exist positive constants \( \gamma_i, i = 1, 2, 3 \) and \( c \in (0,1) \) such that for all sufficiently large \( k \)

\[
\|x^k_S - x^*_S\| \leq \gamma_1 \|x^k_S - x^{k+1}_S\| + \gamma_2 \|x^{k+1}_S\|,
\]

and

\[
\|x^{k+1}_S - x^*_S\| \leq c\|x^k_S - x^*_S\| + \gamma_3 \|x^{k+1}_S\|.
\]

**Proof.** Denote \( S_k = \{ i \mid |x^k_i| \neq 0 \} \). Since \( x^k \to x^* \), by (25) for sufficiently large \( k \), we have \( S \subset S_k \) and there exists a small constant \( \delta \in (0, \varepsilon) \) such that \( |x^k_i| \geq \beta - \delta \), for \( i \in S \).

Consider the function

\[
g(z, \varepsilon) = \sum_{i \in S} \lambda(|z_i| + \varepsilon)^p + \|A_S z - b\|^2 + \lambda \sum_{i \in S^c} z_i^p, \quad z \in \mathbb{R}^{|S|}.
\]

From the proof of Theorem 3, we see that \( x^*_S \) is a local minimizer of \( g(z, \varepsilon) \). Therefore we have from the optimal condition for minimizing \( g(z, \varepsilon) \) that

\[
\left( \frac{\lambda p \text{sign}(x^*_i)}{(|x^*_i| + \varepsilon)^{2-p}} \right)_{i \in S} + 2A^T_S (A_S x^*_S - b) = 0,
\]

and the matrix

\[
\text{diag}\left( \left( \frac{\lambda p (p-1)}{(|x^*_i| + \varepsilon)^{2-p}} \right)_{i \in S} \right) + 2A^T_S A_S
\]

is positive semi-definite, which implies that the matrix \( A^T_S A_S \) is positive definite since \( p - 1 < 0 \).

Since \( x^{k+1} \) is a global minimizer of \( f_k(x, \varepsilon) \) and for sufficiently large \( k \),

\[
\text{sign}(x^{k+1}_i) = \text{sign}(x^k_i) = \text{sign}(x^*_i), \quad i \in S,
\]
we have
\[
\left( \frac{\lambda p \text{sign}(x_i^k)}{|x_i^k| + \epsilon} \right)_{i \in S} + 2 \left( \frac{A_S^T (A_S x^{k+1} - b)}{A_{Sc} (A_S x^{k+1} - b)} \right) = 0, \tag{29}
\]
where \(c_i^{k+1} \in \partial|x_i^{k+1}|\). By (28) and (29), we have
\[
B_S(x_S^k - x_S^*) = 2A_S^T A_S (x_S^k - x_S^{k+1}) - 2A_S^T A_{Sc} x_S^{k+1}, \tag{30}
\]
and
\[
x_S^{k+1} - x_S^* = -(2A_S^T A_S)^{-1} D_S (x_S^k - x_S^*) - (A_S^T A_S)^{-1} A_S^T A_{Sc} x_S^{k+1}, \tag{31}
\]
where \(\zeta_i\) is between \(x_i^k\) and \(x_i^*\) for any \(i \in S\), and
\[
D_S = \text{diag} \left( \left( \frac{\lambda p(1-p)}{|\zeta_i| + \epsilon} \right)_{i \in S}, \quad B_S = D_S + 2A_S^T A_S. \right.
\]
From \(\text{sign}(x_i^k) = \text{sign}(x_i^*)\), we have \(|\zeta_i| \geq \beta - \delta > 0\), for \(i \in S\). Moreover, from (27) and the following inequalities
\[
\frac{\lambda p(1-p)}{|\zeta_i| + \epsilon} < \frac{\lambda p(1-p)}{(\beta - \delta + \epsilon)^{2-p}} < \frac{\lambda p(1-p)}{(\beta)^{2-p}} < 2\lambda_{\text{min}}(A_S^T A_S), \tag{32}
\]
we obtain that \(B_S\) is nonsingular and we have from (30) and (31) that
\[
\|x_S^k - x_S^*\| \leq 2\|B_S^{-1}\| \|A_S^T A_S\| \|x_S^k - x_S^{k+1}\| + 2\|B_S^{-1}\| \|A_S^T A_{Sc}\| \|x_S^{k+1}\|,
\]
and
\[
\|x_S^{k+1} - x_S^*\| \leq \|(2A_S^T A_S)^{-1} D_S\| \|x_S^k - x_S^*\| + \|(A_S^T A_S)^{-1} A_S^T A_{Sc}\| \|x_S^{k+1}\|.
\]
By (27) and (32), we have \(\|(2A_S^T A_S)^{-1} D_S\| < 1\). Therefore, we complete the proof with \(\gamma_1 = 2\|B_S^{-1}\| \|A_S^T A_S\|\), \(\gamma_2 = 2\|B_S^{-1}\| \|A_S^T A_{Sc}\|\), \(\gamma_3 = \|(A_S^T A_S)^{-1} A_S^T A_{Sc}\|\) and \(c = \|(2A_S^T A_S)^{-1} D_S\|\).

Based on Theorem 3, for large \(k\), entries \(x_i^k\) satisfying \(|x_i^k| < L_\varepsilon\) very likely converge to zero. If we can guess the index set \(S\) of nonzero elements \(x^*\) correctly and set \(x_{Sc}^k = 0\) for all large \(k\), then from Theorem 4, we have
\[
\|x^k - x^*\| = \|x_S^k - x_S^*\| \leq \gamma_1 \|x_S^k - x_S^{k+1}\| = \gamma_1 \|x^k - x^{k+1}\|
\]
and
\[
\|x^{k+1} - x^*\| = \|x_S^{k+1} - x_S^*\| \leq c \|x_S^k - x_S^*\| = c \|x^k - x^*\|.
\]
3 Conclusion

Regularized minimization problems with $\ell_p$ regularization arise frequently in many fields such as finance, econometrics and signal processing. On the statistical side, the $\ell_p$ regularization is called the bridge penalty and minimizers of the minimization problem (1) with $\|x\|_p^p$ regularization are called bridge estimators [19]. Theoretical results show that the bridge estimators have various attractive features due to the concavity and non-Lipschitzian property of the regularization function $\|x\|_p^p$. However, the minimization problem (1) is nonconvex and non-Lipschitz. It is shown in [9] that (1) and its smoothed version (3) are strongly NP-hard. The reweighted $\ell_1$ minimization algorithm (IRL1) is developed to solve (1). The IRL1 has been widely used for variable selection, signal reconstruction and image processing. Moreover, extensive numerical experiments showed that the IRL1 is efficient for many applications.

We prove that any sequence generated by the IRL1 is bounded and any accumulation point is a stationary point of the minimization problem (3). In general, a stationary point of the minimization problem (3) is not a minimizer of (1). However, on the positive side, Theorem 3 shows any local minimizer of (3) has certain sparsity. These results are important for developing algorithms for solving the nonconvex and non-Lipschitz minimization problem (1) and applications in variable selection, signal reconstruction and image processing.

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