CONVERGENCE OF REGULARIZED TIME-STEPPING METHODS FOR DIFFERENTIAL VARIATIONAL INEQUALITIES

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26 April 2012, revised 24 November 2012, 10 June 2013

Abstract. This paper provides convergence analysis of regularized time-stepping methods for the differential variational inequality (DVI), which consists of a system of ordinary differential equations and a parametric variational inequality (PVI) as the constraint. The PVI often has multiple solutions at each step of a time-stepping method and it is hard to choose an appropriate solution for guaranteeing the convergence. In [L. Han, A. Tiwari, M.K. Camlibel and J.-S. Pang, Convergence of time-stepping schemes for passive and extended linear complementarity systems, SIAM J. Numer. Anal., 47(2009) pp. 3768-3796], the authors proposed to use "least-norm solutions" of parametric linear complementarity problems at each step of the time-stepping method for the monotone linear complementarity system and showed the novelty and advantages of the use of the least-norm solutions. However, in numerical implementation, when the PVI is not monotone and its solution set is not convex, finding a least-norm solution is difficult. This paper extends the Tikhonov regularization approximation to the P_0-function DVI, which ensures that the PVI has a unique solution at each step of the regularized time-stepping method. We show the convergence of the regularized time-stepping method to a weak solution of the DVI and present numerical examples to illustrate the convergence theorems.

Keywords: differential variational inequalities, P_0-function, Tikhonov regularization, epi-convergence.

AMS Subject Classifications: 90C30, 90C33

1. Introduction. Let $\Omega \subseteq R^m$ be a nonempty closed and convex set, and $H : \Omega \to R^m$ be a continuous function. The (static) variational inequality, denoted by $\mathrm{VI}(\Omega, H)$, is to find a vector $y^* \in \Omega$ such that

$$(y - y^*)^T H(y^*) \geq 0, \qquad \forall y \in \Omega.$$

We denote by $\mathrm{SOL}(\Omega, H)$ the solution set of the $\mathrm{VI}(\Omega, H)$.

Let $F : R^{1+n+m} \to R^n$ and $G : R^{1+n+m} \to R^m$ be two continuous functions. In this paper we study the differential variational inequality (DVI), which consists of a system of ordinary differential equations and a parametric variational inequality as the constraint. Namely we consider

$$\begin{cases} \dot{x}(t) = F(t, x(t), y(t)) \\ y(t) \in \mathrm{SOL}(\Omega, G(t, x(t), \cdot)) \\ x(0) = x^0, \quad t \in [0, T]. \end{cases} \tag{1.1}$$

When $\Omega = R^m_+$, $F(t, x(t), y(t)) = Ax(t) + By(t) + f(t)$, and $G(t, x(t), y(t)) = Qx(t) + My(t) + g(t)$, the DVI (1.1) reduces to the following initial value linear complementarity system [17, 20]

$$\begin{cases} \dot{x}(t) = Ax(t) + By(t) + f(t) \\ 0 \leq y(t) \perp Qx(t) + My(t) + g(t) \geq 0 \\ x(0) = x^0, \quad t \in [0, T], \end{cases} \tag{1.2}$$

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where $A \in R^{n\times n}, B \in R^{n\times m}, Q \in R^{m\times n}, M \in R^{m\times m},\ f : [0,T] \to R^n$ and $g : [0,T] \to R^m$ are two given functions. The notation $\perp$ between two vectors means that they are perpendicular.

The DVI has many important applications in engineering and economics such as differential Nash games, electrical circuits, robotics, earthquake engineering and structural dynamics, see [3, 6, 9, 12, 17, 20, 24, 27].

In this paper we focus our study on the P_0-function DVI, in which the function $G(t,x,\cdot)$ is a P_0-function for any fixed t and x and the feasible set has the following form

$$\Omega = \prod_{\nu=1}^{N} \Omega_\nu, \tag{1.3}$$

where $\Omega_\nu \subseteq R^{m_\nu}$ is nonempty closed and convex, and $\sum_{\nu=1}^{N} m_\nu = m$. $H : R^m \to R^m$ is called a P_0-function [16] if for any $y, v \in \Omega$ we have

$$\max_{1\le\nu\le N} (y_\nu - v_\nu)^T (H_\nu(y) - H_\nu(v)) \ge 0.$$

The class of P_0-functions includes monotone functions as an important subclass. H is said to be monotone if for any $y, v \in R^m$,

$$(y - v)^T (H(y) - H(v)) \ge 0.$$

The P_0-function DVI with the feasible set (1.3) includes the monotone linear complementarity system (1.2) as a special case and has many applications in engineering. A typical example is the (1.2) with $M \equiv 0$. See Example 4.11 in [1], Example 10 in [28], Examples 8-9 in [2], Theorem 9.4 and Theorem 9.5 in [24] and subsection 7.3.2 in [18]. In section 4, we describe an example of the P_0-function DVI in modeling the electrical circuits with (ideal) diodes [25].

The DVI is to find a weak solution of (1.1), which is a pair of trajectories $(x(t), y(t))$ where x is absolutely continuous and y is integrable on $[0,T]$ such that

$$x(t) - x(s) = \int_s^t F(\tau, x(\tau), y(\tau))d\tau, \qquad \forall\ 0 \le s \le t \le T \tag{1.4}$$

and $y(t) \in \mathrm{SOL}(\Omega, G(t, x(t), \cdot))$ for almost all $t \in [0,T]$. The latter implies $y(t) \in \Omega$ holds almost everywhere and for any continuous functions $v : [0,T] \to \Omega$ it holds

$$\int_0^T [v(\tau) - y(\tau)]^T G(\tau, x(\tau), y(\tau))d\tau \ \ge\ 0. \tag{1.5}$$

A P_0-function DVI usually has multiple weak solutions. See the following example.

Example 1.1. *Consider the LCS (1.2) where $A = 1$, $B = (1,1)$, $Q = (1,0)^T$, $x^0 = 0$, $f(t) \equiv 0$, $g(t) \equiv (-c, 0)^T$, $0 < c < 1$ is a constant, and*

$$M = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$$

is a P_0-matrix. The LCS has infinitely many solutions $(x(t), y(t))$:

$$x(t) = \begin{cases} ct & \text{if } 0 \le t \le 1 \\ (c+y_2)e^{-1}e^t - y_2 & \text{if } t > 1, \end{cases} \qquad y(t) = \begin{cases} (c-ct, 0)^T & \text{if } 0 \le t \le 1 \\ (0, y_2)^T & \text{if } t > 1, \end{cases}$$

where $y_2 \geq 0$ is an arbitrary constant. This means that the solution set of the LCS is unbounded.

The time-stepping method is a popular numerical scheme for finding a weak solution of the DVI. It begins with the division of the time interval $[0, T]$ into N_h subintervals:

$$0 = t_{h,0} < t_{h,1} < \cdots < t_{h,N_h} = T. \tag{1.6}$$

Starting from x^0, it computes $y^0 \in \mathrm{SOL}(\Omega, G(0, x^0, \cdot))$ and two finite families of vectors

$$\left\{x^{h,1}, x^{h,2}, \cdots, x^{h,N_h}\right\} \subset R^n \quad \text{and} \quad \left\{y^{h,1}, y^{h,2}, \cdots, y^{h,N_h}\right\} \subset R^m$$

by the recursion: for $i = 0, 1, \cdots, N_h - 1$,

$$\begin{array}{l} x^{h,i+1} = x^{h,i} + hF(t_{h,i+1}, \sigma x^{h,i} + (1-\sigma)x^{h,i+1}, y^{h,i+1}) \\ y^{h,i+1} \in \mathrm{SOL}(\Omega, G(t_{h,i+1}, x^{h,i+1}, \cdot)), \end{array} \tag{1.7}$$

where $h > 0$ is the stepsize and $\sigma \in [0, 1]$ is a scalar defining an implicit ($\sigma = 0$), an explicit ($\sigma = 1$), or a semi-implicit ($\sigma \in (0, 1)$) scheme.

For the P_0-function DVI, the solution set $\mathrm{SOL}(\Omega, G(t, x, \cdot))$ is not necessarily bounded, convex or nonempty for any fixed t and x. When $\mathrm{SOL}(\Omega, G(t_{h,i+1}, x^{h,i+1}, \cdot))$ has more than one solution, selecting a "good" solution $y^{h,i+1}$ is essential. A wrong selection can cause the numerical method unstable or make the DVI unsolvable in the next step. By "good" we mean that the following conditions

$$\|x^{h,i+1}\| \;\leq\; c_1 + c_2\|x^0\| \quad \text{and} \quad \|y^{h,i+1}\| \;\leq\; c_3 + c_4\|x^0\| \tag{1.8}$$

are fulfilled, where c_1, c_2, c_3 and c_4 are positive constants, independent of h. It was shown in [24] under condition (1.8) that the piecewise linear interpolant of $\{x^{h,i}\}$ and the piecewise constant interpolant of $\{y^{h,i}\}$ converge in a certain sense to the weak solution x and y respectively.

In [17], Han et al proposed the following implicit time-stepping method using least-norm solutions for solving LCS (1.2)

$$\begin{array}{rcl} y^{h,i+1} & \in & \mathrm{argmin} \quad \|y\|^2 \\ & & \text{subject to}\, 0 \leq y \perp Q(I-hA)^{-1}[x^{h,i} + hf(t_{h,i+1})] + g(t_{h,i+1}) + M^h y \geq 0 \\ x^{h,i+1} & = & (I-hA)^{-1}(x^{h,i} + hBy^{h,i+1} + hf(t_{h,i+1})), \end{array} \tag{1.9}$$

where $M^h = M + hQ(I-hA)^{-1}B$. An elegant theorem was given in [17] to show that $\{y^{h,i}\}$ and $\{x^{h,i}\}$ generated by (1.9) satisfy (1.8). This technique can be extended to solve DVI as follows

$$\begin{array}{rcl} x^{h,i+1} & = & x^{h,i} + hF(t_{h,i+1}, x^{h,i+1}, y^{h,i+1}) \\ y^{h,i+1} & \in & \mathrm{argmin} \quad \|y\|^2 \\ & & \text{subject to}\, y \in \mathrm{SOL}(\Omega, G(t_{h,i+1}, x^{h,i+1}, \cdot)). \end{array} \tag{1.10}$$

However, the minimization problems in (1.9) and (1.10) are not easy to be solved in general, since their feasible sets are not convex and the standard constraint qualifications are not fulfilled. To our knowledge, there is not yet a practical algorithm available for computing the least-norm solution for DVI. Moreover, convergence results of the implicit time-stepping method in [17, 24] are not applicable for a P_0-matrix M

as the positive semidefiniteness of M^h is not guaranteed. In [2], Acary et al proposed an extended Moreau's time-stepping (EMTS) scheme for certain types of DVIs under the framework of Moreau's sweeping process. Using the EMTS, one may transform a DVI to a new canonical state space representation, and then apply the time-stepping method for the new system. Acary et al [2] presented promising numerical results of the EMTS scheme for some examples of P_0-matrix LCS, but did not derive convergence results.

In this paper we consider approximating the DVI (1.1) by the following regularized DVI:

$$\begin{cases} \dot{x}(t) = F(t, x(t), y(t)) \\ y(t) \in \mathrm{SOL}(\Omega, G(t, x(t), \cdot) + \mu I) \\ x(0) = x^0, \end{cases} \tag{1.11}$$

where I stands for the identity mapping and $\mu > 0$ is a regularization parameter. If $G(t, x, \cdot)$ is a P_0-function for any fixed t and x, and Ω has the form (1.3), then $G(t, x, \cdot) + \mu I$ with $\mu > 0$ is a uniform P-function for any fixed t and x and $\mathrm{SOL}(\Omega, G(t, x(t), \cdot) + \mu I)$ has a unique solution $y_\mu(t)$ for any fixed t and x which is Lipschitz continuous with respect to t and $x(t)$ (see Theorem 3.5.15 in [16] and pp.255–p.256 of [14]). Hence the regularized DVI (1.11) reduces to

$$\begin{cases} \dot{x}(t) = F(t, x(t), y(t)) \\ y(t) = \mathrm{SOL}(\Omega, G(t, x(t), \cdot) + \mu I) \\ x(0) = x^0, \end{cases} \tag{1.12}$$

and the regularized implicit time-stepping scheme has a simple version

$$\begin{aligned} x_\mu^{h,i+1} &= x_\mu^{h,i} + hF(t_{h,i+1}, x_\mu^{h,i+1}, y_\mu^{h,i+1}) \\ y_\mu^{h,i+1} &= \mathrm{SOL}(\Omega, G(t_{h,i+1}, x_\mu^{h,i+1}, \cdot) + \mu I). \end{aligned} \tag{1.13}$$

Moreover, (1.12) locally has a unique solution (x_μ, y_μ) over a time span $[0, T_\mu]$, where x_μ is continuously differentiable and y_μ is Lipschitz continuous. The aim of this paper is to prove the convergence of $\{(x_\mu, y_\mu)\}_{\mu>0}$ and certain approximations defined by the family $\{(x_\mu^{h,i}, y_\mu^{h,i})\}$ generated by the time-stepping method for (1.11).

The remainder of this paper is organized as follows. In section 2, we study the convergence of the solution (x_μ, y_μ) of (1.12) to a weak solution (x, y) of DVI (1.1). In section 3, we study the convergence of the regularized time-stepping scheme (1.13). In section 4, we use numerical examples to illustrate the convergence of the regularized time-stepping scheme. Numerical results show that the regularized time-stepping method is promising for the DVI.

2. Convergence of regularization approximation.

2.1. Epigraphical convergence. Let X and Y denote the spaces of the n-dimensional vector-valued continuous functions and the m-dimensional vector-valued square integrable functions over $[0, T]$, respectively. Denote

$$\|x\|_C := \sup_{t\in[0,T]} \|x(t)\|_2$$

for $x \in X$, and denote

$$\|y\|_{L_2} := \langle y, y\rangle^{1/2}$$

for $y \in Y$, where for any $y, u \in Y$

$$\langle y, u \rangle := \int_0^T y(t)^T u(t) dt.$$

We define the norm for $(x, y) \in X \times Y$:

$$\|(x, y)\|_{X \times Y} = \|x\|_C + \|y\|_{L^2}. \tag{2.1}$$

Let Z denote the space of the m-dimensional vector-valued continuous functions. We denote

$$\|(x, y)\|_{X \times Z} := \sup_{t \in [0,T]} \|(x(t), y(t))\|_2 \tag{2.2}$$

for $(x, y) \in X \times Z$. It is clear that $X \times Z \subset X \times Y$, and $X \times Y$ and $X \times Z$ are Banach spaces under the norms (2.1) and (2.2), respectively. In the remaining part, $\|(x, y)\|$ is always meant $\|(x, y)\|_{X \times Y}$ or $\|(x, y)\|_{X \times Z}$, and it is self-evident that we have $(x, y) \in X \times Y$ for the former case and $(x, y) \in X \times Z$ for the latter case.

It is known that $y^* \in \mathrm{SOL}(\Omega, H)$ if and only if it is a solution of the following system:

$$y - \Pi_\Omega(y - H(y)) = 0,$$

where $\Pi_\Omega(\cdot)$ denotes the projection onto the set Ω with respect to the norm $\| \cdot \|_2$. Define

$$\Phi(x, y)(t) = \begin{pmatrix} x(t) - \displaystyle\int_0^t F(\tau, x(\tau), y(\tau)) d\tau - x^0 \\ y(t) - \Pi_\Omega(y(t) - G(t, x(t), y(t))) \end{pmatrix}. \tag{2.3}$$

Obviously, $\Phi(x, y) \in X \times Y$ for an $(x, y) \in X \times Y$, and $\Phi(x, y) \in X \times Z$ if moreover $(x, y) \in X \times Z$. Then we can reformulate the DVI (1.1) as a minimization problem over $X \times Y$:

$$\min_{(x,y) \in X \times Y} \|\Phi(x, y)\|_{X \times Y}.$$

Obviously, $\|\Phi(x, y)\|_{X \times Y} = 0$ implies that (x, y) is a weak solution of the DVI (1.1). For a continuous function y, $\|\Phi(x, y)\|_{X \times Z} = 0$ implies that (x, y) is a classic solution of (1.1).

Similarly, we define the mapping for the regularized DVI (1.11):

$$\Phi_\mu(x, y)(t) := \begin{pmatrix} x(t) - \displaystyle\int_0^t F(\tau, x(\tau), y(\tau)) d\tau - x^0 \\ y(t) - \Pi_\Omega(y(t) - G(t, x(t), y(t)) - \mu y(t)) \end{pmatrix}. \tag{2.4}$$

Remind us that (1.11) locally has a unique classic solution $(x_\mu, y_\mu) \in X \times Z$ with respect to the topology given by $\| \cdot \|_{X \times Z}$. Then (x_μ, y_μ) is the only minimizer of

$$\min_{(x,y) \in X \times Z} \|\Phi_\mu(x, y)\|_{X \times Z}, \tag{2.5}$$

and is a minimizer of

$$\min_{(x,y) \in X \times Y} \|\Phi_\mu(x, y)\|_{X \times Y}, \tag{2.6}$$

where we have $\|\Phi_\mu(x_\mu, y_\mu)\|_{X\times Y} = \|\Phi_\mu(x_\mu, y_\mu)\|_{X\times Z} = 0$.

Let U be the space taken either as $U = X \times Y$ or $U = X \times Z$. For investigating the convergence of $\{(x_\mu, y_\mu)\}$ in U w.r.t. the norm $\|\cdot\|_U$, we need to show that the mapping $\|\Phi_\mu\|_U$ is epigraphically convergent to $\|\Phi\|_U$ when $\mu \downarrow 0$, which is closely related to the convergence of the function family $\{(x_\mu, y_\mu)\}_{\mu>0}$. The epigraph of a functional θ over U is defined as the set

$$\mathrm{eip}\,\theta \;:=\; \{(x, y, \gamma) \in U \times R \mid \theta(x, y) \leq \gamma\}.$$

Let $\{\theta^k\}_{k=1}^\infty$ be a given sequence of functionals over U. A sequence $\{\theta^k\}_{k=1}^\infty$ is said to be epigraphically convergent to a functional θ, denoted by

$$\theta^k \;\to^{\mathrm{epi}}\; \theta,$$

if the epigraph sequence $\{\mathrm{epi}\ \theta^k\}$ converges to the epigraph epi θ in the sense of Kuratowski, or equivalently, if

(a) for any sequence $\{(x^k, y^k)\}_{k=1}^\infty \subset U$ with $(x^k, y^k) \to (x, y)$ we have

$$\liminf_{k\to\infty} \theta^k(x^k, y^k) \;\geq\; \theta(x, y); \tag{2.7}$$

(b) and if there is a sequence $\{(x^k, y^k)\}_{k=1}^\infty \subset U$ with $(x^k, y^k) \to (x, y)$ and

$$\limsup_{k\to\infty} \theta^k(x^k, y^k) \;\leq\; \theta(x, y). \tag{2.8}$$

Here the convergence $(x^k, y^k) \to (x, y)$ is characterized by the norm $\|\cdot\|_U$. On the epigraphical convergence and the Kuratowski convergence we refer to [21, 26], for example.

Lemma 2.1. *Let $\{\mu_k\}_{k=1}^\infty \downarrow 0$ be given, and let $\Phi^k = \Phi_{\mu_k}$ be defined as in (2.4). Then for any sequence $\{(x^k, y^k)\}_{k=1}^\infty \subset U$ with $(x^k, y^k) \to (x, y)$ in U, we have $\|\Phi^k(x^k, y^k)\|_U \to \|\Phi(x, y)\|_U$, and*

$$\|\Phi^k\|_U \;\to^{\mathrm{epi}}\; \|\Phi\|_U. \tag{2.9}$$

Proof. Let $(x^k, y^k) \to (x, y)$ in U. Taking $U = X \times Y$, we can see

$$\begin{aligned}
& \|\Phi^k(x^k, y^k) - \Phi(x^k, y^k)\|_{X\times Y} \\
= \;& \left(\int_0^T \left\|\Pi_\Omega(y^k(t) - G(t, x^k(t), y^k(t)) - \mu_k y^k(t)) - \Pi_\Omega(y^k(t) - G(t, x^k(t), y^k(t)))\right\|_2^2 dt\right)^{1/2} \\
\leq \;& \left(\int_0^T \left\|\mu_k y^k(t)\right\|_2^2 dt\right)^{1/2} \;=\; \mu_k \|y^k\|_{L^2};
\end{aligned}$$

taking $U = X \times Z$, we have

$$\begin{aligned}
& \|\Phi^k(x^k, y^k) - \Phi(x^k, y^k)\|_{X\times Z} \\
= \;& \sup_{t\in[0,T]} \|\Pi_\Omega(y^k(t) - G(t, x^k(t), y^k(t)) - \mu_k y^k(t)) - \Pi_\Omega(y^k(t) - G(t, x^k(t), y^k(t)))\|_2 \\
\leq \;& \sup_{t\in[0,T]} \|\mu_k y^k(t)\|_2 \;=\; \mu_k \|y^k\|_C.
\end{aligned}$$

Now we have $\|\Phi^k(x^k, y^k)\|_U - \|\Phi(x^k, y^k)\|_U \to 0$.

On the other hand we know $\|\Phi(x^k, y^k)\|_U - \|\Phi(x, y)\|_U \to 0$ since $\|\Phi\|_U$ is continuous. Therefore we can conclude $\|\Phi^k(x^k, y^k)\|_U \to \|\Phi(x, y)\|_U$, which implies the epigraphical convergence (2.9). □

Now we study the convergence of $\{(x_{\mu_k}, y_{\mu_k})\}$ by using the epigraphical convergence (2.9). As stated before, (x_{μ_k}, y_{μ_k}) is a minimizer of (2.5) and (2.6) where we take $\mu = \mu_k$. However, in practice only a so-called ϵ-minimizer $(x^{\epsilon}_{\mu_k}, y^{\epsilon}_{\mu_k})$, instead of a true minimizer, is available, for which we have

$$\|\Phi_{\mu_k}(x^{\epsilon}_{\mu_k}, y^{\epsilon}_{\mu_k})\|_U \ \leq \min_{(x,y)\in U} \|\Phi_{\mu_k}(x, y)\|_U + \epsilon,$$

where $\epsilon > 0$ can be regarded as an error tolerance. For a functional θ over U, we denote its set of ϵ-minimizers by

$$\epsilon - \text{argmin } \theta := \{(\bar{x}, \bar{y}) \in U | \ \theta(\bar{x}, \bar{y}) \leq \inf_{(x,y)\in U} \theta(x, y) + \epsilon\}.$$

Then we have the following results, which are a direct consequence of Proposition 7.18 of [21].

Lemma 2.2. *Let $\{\theta^k\}_{k=1}^{\infty}$ be a given sequence of functionals over U that is epigraphically convergent to θ. Then we have*

$$\text{argmin } \theta \ \supseteq \ \bigcap_{\epsilon>0} \limsup_{k\to\infty} \left(\epsilon - \text{argmin } \theta^k\right) \ \supseteq \ \limsup_{k\to\infty} \left(\text{argmin } \theta^k\right),$$

where $\limsup_{k\to\infty}(\epsilon - \text{argmin}\theta^k)$ is the outer limit of the set sequence $\{\epsilon - \text{argmin}\theta^k\}$ defined in the sense of Kuratowski [21, 26]. If, in addition,

$$\bigcap_{\epsilon>0} \limsup_{k\to\infty} \left(\epsilon - \text{argmin } \theta^k\right) \ \neq \ \emptyset,$$

then

$$\text{argmin } \theta \neq \emptyset, \qquad \textit{and} \qquad \min_{(x,y)\in U} \theta(x, y) = \limsup_{k\to\infty} \inf_{(x,y)\in U} \theta^k(x, y).$$

Corollary 2.3. *Let $\{\theta^k\}_{k=1}^{\infty}$ be a given sequence of functionals over U that is epigraphically convergent to θ, and let (x^k, y^k) be an ϵ_k-minimizer of θ^k over U for every k, where $\epsilon_k \downarrow 0$. If $\{(x^k, y^k)\}_{k=1}^{\infty}$ has a cluster point (x, y), then it is a minimizer of θ in U, and*

$$\theta(x, y) = \limsup_{k\to\infty} \theta^k(x^k, y^k).$$

By using Lemma 2.1, Lemma 2.2 and Corollary 2.3, we have the following convergence result for the regularization approximation of the DVI.

Theorem 2.4. *Let $\{\mu_k\}_{k=1}^{\infty} \downarrow 0$ and $\{\epsilon_k\}_{k=1}^{\infty} \downarrow 0$ be given, and let Φ_{μ_k} be defined as in (2.4).*
(1) Let $(x^{\epsilon_k}_{\mu_k}, y^{\epsilon_k}_{\mu_k}) \in X \times Z$ be an approximation of the classic solution (x_{μ_k}, y_{μ_k}) of the regularized DVI (1.11) with $\mu = \mu_k$ such that

$$\|\Phi_{\mu_k}(x^{\epsilon_k}_{\mu_k}, y^{\epsilon_k}_{\mu_k})\|_{X\times Z} \ \leq \ \epsilon_k,$$

then any cluster point of $\{(x^{\epsilon_k}_{\mu_k}, y^{\epsilon_k}_{\mu_k})\}_{k=1}^{\infty}$ *is a classic solution of the DVI (1.1).*
(2) Let $(x^{\epsilon_k}_{\mu_k}, y^{\epsilon_k}_{\mu_k}) \in X \times Y$ *be an approximation of the weak solution* (x_{μ_k}, y_{μ_k}) *of the regularized DVI (1.11) with* $\mu = \mu_k$ *such that*

$$\|\Phi_{\mu_k}(x^{\epsilon_k}_{\mu_k}, y^{\epsilon_k}_{\mu_k})\|_{X\times Y} \ \leq \ \epsilon_k,$$

then any cluster point of $\{(x^{\epsilon_k}_{\mu_k}, y^{\epsilon_k}_{\mu_k})\}_{k=1}^{\infty}$ *is a weak solution of the DVI (1.1).*

Proof. From Lemma 2.1 it follows that $\{\|\Phi_{\mu_k}\|_{X\times Z}\}$ is epigraphically convergent to $\|\Phi\|_{X\times Z}$. Since (x_{μ_k}, y_{μ_k}) is the classic solution of (1.11), we have $\|\Phi_{\mu_k}(x_{\mu_k}, y_{\mu_k})\|_{X\times Z} = 0$, therefore $(x^{\epsilon_k}_{\mu_k}, y^{\epsilon_k}_{\mu_k})$ is an ϵ_k minimizer of the functional $\|\Phi_{\mu_k}\|_{X\times Z}$. Hence by Corollary 2.3 we know that a cluster point (x^*, y^*) must fulfill

$$\|\Phi(x^*, y^*)\|_{X\times Z} = 0 = \limsup_{k\to\infty} \|\Phi_{\mu_k}(x^{\epsilon_k}_{\mu_k}, y^{\epsilon_k}_{\mu_k})\|_{X\times Z},$$

which means that (x^*, y^*) is a classic solution of the DVI (1.1). The second part of the theorem can be shown in a very similar way. ◻

COROLLARY 2.5. *Let* $\{\mu_k\}_{k=1}^{\infty} \downarrow 0$ *and let* (x_{μ_k}, y_{μ_k}) *be the classic solution of the regularized DVI (1.11) for every* $\mu = \mu_k$*. If* $\{(x_{\mu_k}, y_{\mu_k})\}_{k=1}^{\infty}$ *has a cluster point* (x^*, y^*) *in the norm* $\|\cdot\|_{X\times Z}$*, then* (x^*, y^*) *is a classic solution of the DVI (1.1); if* $\{(x_{\mu_k}, y_{\mu_k})\}_{k=1}^{\infty}$ *has a cluster point* (x^*, y^*) *in the norm* $\|\cdot\|_{X\times Y}$*, then* (x^*, y^*) *is a weak solution of (1.1).*

2.2. Convergence analysis. Now we study the existence of the cluster point of the function family $\{(x_\mu, y_\mu)\}_{\mu>0}$. The following lemma is needed for showing the boundedness of $\{(x_\mu, y_\mu)\}_{\mu>0}$.

LEMMA 2.6. *Let* $T > 0$*,* $\alpha, \gamma \geq 0$ *and* $\beta > 0$*, and let* $\psi : [0, T] \to R_+$ *be (Lebesgue) integrable. If*

$$\psi(t) \leq \alpha + \int_0^t [\beta\psi(s) + \gamma] ds \tag{2.10}$$

then

$$\psi(t) \leq \alpha \exp(\beta t) + \frac{\gamma}{\beta} (\exp(\beta t) - 1) . \tag{2.11}$$

Proof. Let

$$\tilde{\psi}(t) = \psi(t) + \frac{\gamma}{\beta}.$$

Then we can write (2.10) as

$$\tilde{\psi}(t) - \frac{\gamma}{\beta} \leq \alpha + \int_0^t \left[\beta\left(\tilde{\psi}(s) - \frac{\gamma}{\beta}\right) + \gamma\right] ds \leq \alpha + \int_0^t \beta\tilde{\psi}(s) ds,$$

which yields

$$\tilde{\psi}(t) \leq \alpha + \frac{\gamma}{\beta} + \int_0^t \beta\tilde{\psi}(s) ds.$$

This, with the well known Gronwall Inequality (see [13, pp.146], for example), implies

$$\tilde{\psi}(t) \leq \left(\alpha + \frac{\gamma}{\beta}\right) \exp\left(\int_0^t \beta ds\right) = \left(\alpha + \frac{\gamma}{\beta}\right) \exp(\beta t) .$$

This yields (2.11) when we replace $\tilde{\psi}(t)$ by $\psi(t)$. $\square$

In the following theorem, we show that the uniform boundedness of $\{y_\mu\}_{\mu>0}$ implies the uniform boundedness of $\{x_\mu\}_{\mu>0}$ under the Lispchitz continuity of F.

THEOREM 2.7. *Assume that there are nonnegative constants κ_1 and κ_2 such that for any $s \in [0,T]$, $x^1, x^2 \in R^n$ and $y^1, y^2 \in R^m$,*

$$\|F(s,x^1,y^1) - F(s,x^2,y^2)\|_2 \;\le\; \kappa_1\|x^1 - x^2\|_2 + \kappa_2\|y^1 - y^2\|_2 \tag{2.12}$$

holds and assume that there is a positive constant α_2 independent of μ such that $\|y_\mu\|_{L^2} \le \alpha_2$ for any $\mu \in (0,\bar{\mu}]$. Then we have

$$\|x_\mu\|_C \;\le\; \alpha_1 := (\|x^0\|_2 + \kappa_2\alpha_2\sqrt{T})\exp(\kappa_1 T) + \frac{\|f_0\|_C}{\kappa_1}\left[\exp(\kappa_1 T) - 1\right]$$

for any $\mu \in (0,\bar{\mu}]$, where $f_0(s) = F(s,0,0)$ for $s \in [0,T]$.

Proof. For $t \in [0,T]$ we write

$$x_\mu(t) = x^0 + \int_0^t [F(s,x_\mu(s),y_\mu(s))]\,ds.$$

Noting that

$$\int_0^t \|y_\mu(s)\|_2 ds \;\le\; \sqrt{T\int_0^t \|y_\mu(s)\|_2^2 ds} \;\le\; \sqrt{T}\|y_\mu\|_{L^2} \;\le\; \sqrt{T}\alpha_2,$$

then from the Lipschitz condition (2.12) we have

$$\begin{aligned}
\|x_\mu(t)\|_2 &\le \|x^0\|_2 + \int_0^t \|F(s,x_\mu(s),y_\mu(s)) - F(s,0,0) + F(s,0,0)\|_2 ds \\
&\le \|x^0\|_2 + \int_0^t (\kappa_1\|x_\mu(s)\|_2 + \kappa_2\|y_\mu(s)\|_2 + \|f_0(s)\|_2)\,ds \\
&\le \|x^0\|_2 + \kappa_2\alpha_2\sqrt{T} + \int_0^t (\kappa_1\|x_\mu(s)\|_2 + \|f_0\|_C)\,ds.
\end{aligned}$$

This, together with Lemma 2.6, implies the conclusion. $\square$

Now we show the convergence of the function family $\{x_\mu, y_\mu\}_{\mu>0}$.

THEOREM 2.8. *In the setting of Theorem 2.7, there are a $\{\mu_k\}_{k=1}^\infty \downarrow 0$, an $x^* \in X$ and a $y^* \in Y$ such that $x_{\mu_k} \to x^*$ uniformly and $y_{\mu_k} \to y^*$ weakly. In addition, we have:*
(1) if $y_{\mu_k} \to y^*$ w.r.t. $\|\cdot\|_{L^2}$, then (x^*, y^*) is a weak solution of the DVI (1.1);
(2) if $y_{\mu_k} \to y^*$ uniformly, then (x^*, y^*) is a classic solution of the DVI (1.1).

Proof. By Theorem 2.7, we know that $\|y_{\mu_k}\|_{L^2} \le \alpha_2$ implies that $\{x_{\mu_k}\}$ is uniformly bounded. From this and the Lipschitz condition (2.12) it follows that $\{\dot{x}_{\mu_k}\}$ is uniformly bounded, and thus $\{x_{\mu_k}\}$ is equicontinuous on $[0,T]$. By the Arzelá-Ascoli theorem [23], we know that there is a sequence $\{\mu_k\}_{k=1}^\infty \downarrow 0$ such that $\{x_{\mu_k}\}$ converges to an $x^* \in X$ uniformly. Since Y is reflexive, and $\{y_{\mu_k}\}$ is uniformly bounded, by the Alaoglu theorem [23], there is a subsequence of $\{y_{\mu_k}\}$ that is weakly convergent to $y^* \in Y$.

Statements (1) and (2) can be shown by a direct application of Lemma 2.2. $\square$

Theorem 2.8 assumes the uniform boundedness of $\{y_\mu\}$ for $\mu \in (0, \bar{\mu}]$. The following theorem gives a sufficient condition to ensure that this assumption holds.

Let $\Omega_\epsilon = \{\, y \,|\, \mathrm{dist}(y, \mathrm{SOL}(\Omega, G(0, \eta, \cdot))) \le \epsilon \,\}$ for a positive number $\epsilon > 0$.

THEOREM 2.9. *Suppose* $\mathrm{SOL}(\Omega, G(0, \eta, \cdot))$ *is nonempty and bounded. If* $G(t, x, y)$ *is Lipschitzian with respect to* (t, x) *near* $(0, \eta)$ *for any* $y \in \Omega_\epsilon$ *with modular* L_G *and* $G(t, x(t), \cdot)$ *is a continuous* P_0*-function, then there are* $\bar{\mu} > 0$*,* $T_0 > 0$ *and an* α_2 *independent of* μ *such that* $\|y_\mu\|_{L^2} \le \alpha_2$ *for any* $\mu \in (0, \bar{\mu}]$ *over* $[0, T_0]$*.*

Proof. Denote

$$\Psi(t, x, y) = y - \Pi_\Omega(y - G(t, x, y)) \quad \text{and} \quad \Psi_\mu(t, x, y) = y - \Pi_\Omega(y - G(t, x, y) - \mu y).$$

Let $\mathcal{S}_\mu(t, x) \subseteq R^m$ be the solution set of $\Psi_\mu(t, x, y) = 0$ for fixed μ, t, x.

By Corollary 3.6.2 and Definition 3.6.3 of [16], we know that the function $\Psi(t, x, \cdot)$ and $\Psi_\mu(t, x, \cdot)$ are weakly univalent since $G(t, x, \cdot)$ is a continuous P_0-function, which follows that there exists $\delta_1 > 0$ such that for fixed t, x and μ, if

$$\sup\{\|\Psi_\mu(t, x, y) - \Psi(0, \eta, y)\|_2 : y \in \Omega_\epsilon\} \;<\; \delta_1,$$

then we have

$$\emptyset \;\ne\; \mathcal{S}_\mu(t, x) \;\subseteq\; \Omega_\epsilon. \tag{2.13}$$

Denote $\zeta_0 = \sup\{\|v\|_2 : v \in \Omega_\epsilon\}$. Choose $\bar{\delta}$, $\bar{T}$ and $\bar{\mu}$ such that $L_G(\bar{\delta} + \bar{T}) < \frac{1}{2}\delta_1$ and $\bar{\mu}\zeta_0 < \frac{1}{2}\delta_1$. Let $\mathcal{N}(\eta, \delta) = \{u : \|u - \eta\|_2 < \delta\}$. Then for any $(t, x) \in [0, \bar{T}] \times \mathcal{N}(\eta, \bar{\delta})$, and any $y \in \Omega_\epsilon$, we have

$$\begin{aligned} \|\Psi_\mu(t, x, y) - \Psi(0, \eta, y)\|_2 &\le \|\Pi_\Omega(y - G(t, x, y) - \mu y) - \Pi_\Omega(y - G(0, \eta, y))\|_2 \\ &\le \|G_\mu(t, x, y) - G(0, \eta, y)\|_2 + \mu\|y\|_2 \\ &\le L_G(t + \|x - \eta\|_2) + \bar{\mu}\zeta_0 < \delta_1. \end{aligned}$$

Denote

$$\zeta = \sup\{\|F(t, x, y)\|_2 : \; (t, x, y) \in [0, \bar{T}] \times \mathcal{N}(\eta, \bar{\delta}) \times \Omega_\epsilon\}.$$

Note that for any fixed $\mu > 0$, $\mathcal{S}_\mu(t, x)$ is a singleton set. Let $\mathcal{F}_\mu(t, x) = F(t, x, \mathcal{S}_\mu(t, x))$. By taking $\delta_0 > 0$ and $T_0 > 0$ such that $\delta_0 + \zeta T_0 < \bar{\delta}$, we can see that $\mathcal{F}_\mu(\cdot, \cdot)$ is continuous and maps $[0, T_0] \times \mathcal{N}(\eta, \delta_0 + \zeta T_0)$ into $\mathcal{N}(0, \zeta)$. By applying the Peano existence theorem to

$$\begin{cases} \dot{x}(t) = \mathcal{F}_\mu(t, x(t)) \\ x(0) = \eta, \end{cases}$$

we know that (1.11) has a solution (x_μ, y_μ) over $[0, T_0]$ in which x_μ is continuously differentiable and y_μ is continuous. Noting

$$x_\mu(t) = \eta + \int_0^t \mathcal{F}_\mu(s, x_\mu(s))ds,$$

clearly, $x_\mu(t) \in \mathcal{N}(\eta, \delta_0 + \zeta T_0)$ for any $t \in [0, T_0]$. Because $\delta_0 + \zeta T_0 < \bar{\delta}$, from (2.13), we obtain the uniform boundedness of $\{y_\mu\}$. $\square$

THEOREM 2.10. *Let* $G(t, x, \cdot)$ *be monotone for any fixed* t *and* x*, and let for any fixed* t *and* y*:*

$$\|G(t, x^1, y) - G(t, x^2, y)\|_2 \;\le\; \omega_2\|x^1 - x^2\|_2.$$

Let $x_{\mu_k} \to x^*$ uniformly with

$$\lim_{k\to\infty} \frac{\|x_{\mu_k} - x^*\|_C}{\mu_k} = \varsigma, \tag{2.14}$$

and let $y_{\mu_k} \to y^*$ w.r.t. $\|\cdot\|_{L^2}$. Then for any weak solution $(x^*, \tilde{y})$ of (1.1), we have

$$\|y^*\|_{L_2}^2 \le \|\tilde{y}\|_{L_2}\|y^*\|_{L_2} + (\varsigma\omega_2 T)\|y^* - \tilde{y}\|_{L_2}. \tag{2.15}$$

Proof. Since $y_\mu : [0,T] \to \Omega$ is continuous and $\tilde{y}$ solves $\mathrm{VI}(\Omega, G(t, x(t), \cdot))$ for almost every $t \in [0,T]$, we have

$$\langle y_\mu - \tilde{y}, G(t, x^*, \tilde{y})\rangle \ge 0 \tag{2.16}$$

and

$$\langle \tilde{y} - y_\mu, G(t, x_\mu, y_\mu) + \mu y_\mu\rangle \ \ge \ 0. \tag{2.17}$$

Adding (2.16) and (2.17) we obtain

$$0 \ \ge \ \langle y_\mu - \tilde{y}, G(t, x_\mu, y_\mu) + \mu y_\mu - G(t, x^*, \tilde{y})\rangle\,.$$

Using (2.16)-(2.17) again, we find

$$\begin{array}{rcl}\langle y_\mu - \tilde{y}, y_\mu\rangle & \le & -\frac{1}{\mu}\langle y_\mu - \tilde{y}, G(t, x_\mu, y_\mu) - G(t, x^*, \tilde{y})\rangle \\ & = & -\frac{1}{\mu}\langle y_\mu - \tilde{y}, G(t, x_\mu, y_\mu) - G(t, x_\mu, \tilde{y})\rangle - \frac{1}{\mu}\langle y_\mu - \tilde{y}, G(t, x_\mu, \tilde{y}) - G(t, x^*, \tilde{y})\rangle \\ & \le & -\frac{1}{\mu}\langle y_\mu - \tilde{y}, G(t, x_\mu, \tilde{y}) - G(t, x^*, \tilde{y})\rangle\,.\end{array}$$

Taking the sequence (x_{μ_k}, y_{μ_k}) converging to (x^*, y^*) with (2.14) fulfilled, from the Lipschitz continuity of $G(t,\cdot,y)$, we have

$$\begin{array}{rl} & -\frac{1}{\mu_k}\langle y_{\mu_k} - \tilde{y}, G(t, x_{\mu_k}, \tilde{y}) - G(t, x^*, \tilde{y})\rangle \\ \le & \frac{1}{\mu_k}\|y_{\mu_k} - \tilde{y}\|_{L^2}\|G(t, x_{\mu_k}, \tilde{y}) - G(t, x^*, \tilde{y})\|_{L^2} \ \le \ \frac{1}{\mu_k}\|y_{\mu_k} - \tilde{y}\|_{L^2} T\omega_2 \|x_{\mu_k} - x^*\|_C\,.\end{array}$$

Then we obtain

$$\langle y_{\mu_k} - \tilde{y}, y_{\mu_k}\rangle \ \le \ \frac{\|x_{\mu_k} - x^*\|_C}{\mu_k}\omega_2 T\|y_{\mu_k} - \tilde{y}\|_{L^2}\,,$$

and then

$$\langle y_{\mu_k}, y_{\mu_k}\rangle \ \le \ \langle \tilde{y}, y_{\mu_k}\rangle + \frac{\|x_{\mu_k} - x^*\|_C}{\mu_k}\omega_2 T\|y_{\mu_k} - \tilde{y}\|_{L^2}\,,$$

which yields the conclusion (2.15) when we take $k \to \infty$. This completes the proof. □

2.3. Linear complementarity system. In this subsection, we consider the LCS (1.2) with a P_0-matrix M and global Lipschitz continuous functions f and g. In such case, the global Lipschitz property (2.12) of $F(t, x(t), y(t)) = Ax(t) + By(t) + f(t)$ in Theorem 2.7 holds with $\kappa_1 = \|A\|_2$ and $\kappa_2 = \|B\|_2$, and $G(t, x(t), y(t)) = Qx(t) + My(t) + g(t)$ is a globally Lipschitzian continuous function with respect to (t, x) in

Theorem 2.9. Moreover, it is known that for any fixed $\mu > 0$ and $q \in R^n$, the P-matrix linear complementary problem

$$0 \le v \perp (M + \mu I)v + q \ge 0 \tag{2.18}$$

has a unique solution $v(M + \mu I, q)$ and there is a constant $c_{(M+\mu I)}$ such that

$$\|v(M + \mu I, q^1) - v(M + \mu I, q^2)\| \le c_{(M+\mu I)} \|q^1 - q^2\|.$$

See [10, 15]. Hence, the regularized LCS:

$$\begin{cases} \dot{x}(t) = Ax(t) + By(t) + f(t) \\ 0 \le y(t) \perp (M + \mu I)y(t) + Qx(t) + g(t) \ge 0 \\ x(0) = x^0 \in R^n \end{cases} \tag{2.19}$$

reduces to a standard ordinary differential equation with a globally Lipschitzian continuous right-hand function as the following

$$\begin{cases} \dot{x}(t) = Ax(t) + By(M + \mu I, Qx(t) + g(t)) + f(t) \\ x(0) = x^0 \in R^n. \end{cases} \tag{2.20}$$

By the well-known Picard-Lindelöf theorem, for any $T > 0$, (2.20) has a unique continuously differentiable solution x_μ in the interval $[0, T]$. Let $y_\mu = y(M+\mu I, Qx_\mu + g)$. Then (x_μ, y_μ) is the unique classic solution of (2.19) for any $\mu > 0$.

Recall that M is a Z-matrix if all of its off-diagonal elements are non-positive. In contrast with Theorem 2.9, without the boundedness of $\mathrm{SOL}(\Omega, G(0, \eta, \cdot))$, the following theorem shows the uniform boundedness of $\|y_\mu\|_{L^2}$ for $\mu \to 0$.

Theorem 2.11. *Suppose that M is a P_0-matrix and Z-matrix and the LCS (1.2) has a weak solution $(\tilde{x}, \tilde{y})$ in $[0, T]$. Then $\|y_\mu\|_{L^2} \le \|\tilde{y}\|_{L^2}$ for any $\mu > 0$ over $[0, T]$.*

Proof. If M is a Z-matrix and the solution set $\mathcal{S}$ of the linear complementarity problem (2.18) with $\mu = 0$ is nonempty, then there is a unique least-element solution $\bar{v}$ in $\mathcal{S}$ which satisfies $\bar{v} \le v$ for all $v \in \mathcal{S}$ [15]. Moreover, it is shown in [11] that if M is a P_0-matrix and Z-matrix, then for any $\mu_1 > \mu_2 > 0$, the solutions of (2.18) satisfy

$$v_{\mu_1} \le v_{\mu_2} \le \bar{v} \quad \text{and} \quad \lim_{\mu \downarrow 0} v_\mu = \bar{v}. \tag{2.21}$$

By the definition of the least-element solution, for any $\tilde{v} \in \mathcal{S}$, we have

$$v_{\mu_1} \le v_{\mu_2} \le \bar{v} \le \tilde{v} \quad \text{and} \quad \lim_{\mu \downarrow 0} v_\mu = \bar{v} \le \tilde{v}.$$

Hence, by the assumption of this theorem, we have $0 \le y_\mu(t) \le \tilde{y}(t)$, for almost all $t \in [0, T]$ which implies $\|y_\mu\|_{L^2} \le \|\tilde{y}\|_{L^2}$ for any $\mu > 0$ over $[0, T]$. □

The LCS (1.2) with $M = 0$ frequently appears in realistic settings [1, 2, 24]. Obviously, $M = 0$ is a P_0-matrix and Z-matrix. For $Qx(t) + g(t) \ge 0$, any vector $y(t) \ge 0$ with $y_i(t)(Qx(t) + g(t))_i = 0$ is a solution of the LCP: $0 \le y(t) \perp Qx(t) + g(t) \ge 0$, and $y_\mu(t) = 0$ is the unique solution of its regularized problem. Hence, we can see that even the solution set of the LCP is unbounded, the sequence of the unique solution $y_\mu(t)$ of the regularized LCP is uniformly bounded.

Example 2.1. *Consider the LCS (1.2) where $A = 1$, $B = (1, 1)$, $Q = (1, 0)^T$, $x^0 = 1$, $f(t) \equiv 0$, $g(t) \equiv 0$, and $M = 0$. The LCS has infinitely many solutions $(x(t), y(t))$:*

$$x(t) = (1 + y_2)e^t - y_2, \quad y_1(t) \equiv 0, \quad y_2(t) \equiv y_2,$$

where $y_2 \geq 0$ is an arbitrary constant. The solution set $\{(0, y_2)^T : y_2 \in R, y_2 \geq 0\}$ of the LCS at $t = 0$ is unbounded.

It is easy to verify that the regularized LCS

$$\begin{cases} \dot{x}(t) = Ax(t) + By(t) \\ 0 \leq y(t) \perp Qx(t) + \mu y(t) \geq 0 \\ x(0) = x^0 \quad t \in [0, T] \end{cases}$$

has the unique solution $(x_\mu(t), y_\mu(t))$:

$$x_\mu(t) = e^t, \quad y_\mu(t) \equiv (0, 0)^T.$$

Obviously, $\{y_\mu\}_{\mu>0}$ is uniformly bounded. It is worth noting that the limit function $(e^t, (0, 0)^T)$ is a least-element solution of the LCS, and the limit function of $x_\mu(t)$ is the so-called shortest path of the system.

LEMMA 2.12. *Let M be a P_0-matrix and a Z-matrix and let $D = diag(d_i)$, $d_i \in [0, 1]$. Then for any $\mu > 0$, the matrix $I - D + D(M + \mu I)$ is an M-matrix and $[I - D + D(M + \mu I)]^{-1}D \leq (M + \mu I)^{-1}$ componentwise. Moreover, if M is diagonalizable, then $\mu(M + \mu I)^{-1}$ is convergent when $\mu \to 0$.*

Proof. For the first statement of this lemma, we refer to [10, 15]. Let M be diagonalizable, that is, there are a nonsingular matrix $P \in R^{m\times m}$ and a diagonal matrix Λ =diag(λ_i) such that $M = P^{-1}\Lambda P$. Hence, we have

$$\mu(M+\mu I)^{-1} = P^{-1}\text{diag}\left(\frac{\mu}{\lambda_i + \mu}\right)P \qquad \text{and} \qquad \frac{\mu}{\lambda_i + \mu} \to \begin{cases} 1 & \text{if } \lambda_i = 0 \\ 0 & \text{if } \lambda_i \neq 0 \end{cases} \qquad \text{as } \mu \to 0.$$

□

The limit of $\mu(M + \mu I)^{-1}$ does not necessarily exist if M is not diagonalizable. Consider

$$M = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}.$$

It is clear that M is a P_0 matrix and a Z matrix, but not diagonalizable. We see that

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \leq [I - D + D(M + \mu I)]^{-1}D \leq (M + \mu I)^{-1} = \frac{1}{\mu}\begin{pmatrix} 1 & \frac{1}{\mu} \\ 0 & 1 \end{pmatrix}$$

and $\mu(M + \mu I)^{-1}$ is not convergent as $\mu \to 0$.

THEOREM 2.13. *Let M be a P_0-matrix and a Z-matrix and be diagonalizable. Let $x_{\mu_k} \to x^*$ uniformly with (2.14) fulfilled, and let $y_{\mu_k} \to y^*$ weakly in Y. Denote*

$$\tilde{M} \;=\; \lim_{\mu\downarrow 0} \mu(M + \mu I)^{-1}$$

and denote $e = (1, 1, \cdots, 1)^T$. Then for any weak solution $(x^*, \tilde{y})$ of (1.2), we have

$$\int_0^T y^*(t)dt \;\leq\; \int_0^T \tilde{y}(t)dt + (\varsigma T)\tilde{M}|Q|e, \tag{2.22}$$

where $|Q| = (|Q_{ij}|)$.

Proof. Note that $\tilde{y}(t)$ and $y_\mu(t)$ satisfy the complementarity conditions in (1.2) and (2.19) for almost every $t \in [0, T]$ and for any $t \in [0, T]$, respectively. We have

$$\min\{\tilde{y}(t), M\tilde{y}(t) + Qx^*(t) + g(t)\} = 0 \tag{2.23}$$

for almost every $t \in [0,T]$, and

$$\min\{y_\mu(t), (M+\mu I)y_\mu(t) + Qx_\mu(t) + g(t)\} = 0 \tag{2.24}$$

for every $t \in [0,T]$. Subtracting (2.24) by (2.23), we obtain

$$(I-D)(y_\mu(t)-\tilde{y}(t))+D[(M+\mu I)(y_\mu(t)-\tilde{y}(t))+\mu y(t)+Q(x_\mu(t)-x^*(t))] = 0, \tag{2.25}$$

where $D = diag(d_i)$, $d_i \in [0,1]$, $i = 1,\dots,m$, see [10]. From (1) of Lemma 2.12 we know that $I-D+D(M+\mu I)$ has a nonnegative inverse. Then by rearranging (2.25), we obtain

$$y_\mu(t) - \tilde{y}(t) = -[I-D+D(M+\mu I)]^{-1}D[\mu\tilde{y}(t) + Q(x_\mu(t) - x^*(t))]. \tag{2.26}$$

Considering moreover that $\tilde{y}(t) \geq 0$ holds for almost every $t \in [0,T]$, we have

$$\int_0^T [I-D+D(M+\mu I)]^{-1}D\tilde{y}(t)dt \;\geq\; 0. \tag{2.27}$$

The inequalities (2.26) and (2.27) yield

$$\int_0^T (y_\mu(t) - \tilde{y}(t))\,dt \;\leq\; -\int_0^T [I-D+D(M+\mu I)]^{-1}DQ(x_\mu(t) - x^*(t))dt. \tag{2.28}$$

From Lemma 2.12 we know that

$$\begin{array}{ll} & \left|[I-D+D(M+\mu I)]^{-1}DQ(x_\mu(t)-x^*(t))\right| \\ \leq & \left|[I-D+D(M+\mu I)]^{-1}D\right| |Q|\,|x_\mu(t)-x^*(t)| \\ \leq & (M+\mu I)^{-1}|Q|\,|x_\mu(t)-x^*(t)| \;\leq\; \|x_\mu - x^*\|_C\,(M+\mu I)^{-1}|Q|e. \end{array}$$

Consequently, from (2.28) it follows that

$$\int_0^T (y_\mu(t) - \tilde{y}(t))\,dt \;\leq\; \|x_\mu - x^*\|_C\,T(M+\mu I)^{-1}|Q|e.$$

Taking a subsequence (x_{μ_k}, y_{μ_k}) converging to (x^*, y^*) with (2.14) fulfilled, namely

$$\lim_{k\to\infty} \frac{\|x_{\mu_k} - x^*\|_C}{\mu_k} = \varsigma,$$

we have

$$\int_0^T (y_{\mu_k}(t) - \tilde{y}(t))\,dt \;\leq\; \frac{\|x_{\mu_k} - x^*\|_C}{\mu_k} T\mu_k(M+\mu_k I)^{-1}|Q|e,$$

and so

$$\int_0^T y_{\mu_k}(t)dt \;\leq\; \int_0^T \tilde{y}(t)dt + \frac{\|x_{\mu_k} - x^*\|_C}{\mu_k} T\mu_k(M+\mu_k I)^{-1}|Q|e,$$

which yields the conclusion (2.22) when we take $k \to \infty$. This completes the proof. □

REMARK 2.1. *From (2.15) and (2.22), we can derive that for $T \to 0$, y is a least norm solution and a least element solution of (1.1) and (1.2), respectively. Hence*

Theorem 2.10 and Theorem 2.13 can be respectively regarded as the generalizations of the regularization results for the VIs and LCPs. Refer to [11, 16].

REMARK 2.2. *Applying Theorem 2.8 to the LCS (1.2) where M is a P_0-matrix and Z-matrix, we can derive the existence of a weak solution of (1.2). Theorem 2.11, together with Theorem 2.8, shows that the solution (x_μ, y_μ) of the regularized LCS (2.19) is uniformly bounded for any $\mu > 0$. Moreover, Theorem 2.13 gives the limit properties (x_μ, y_μ) as $\mu \to 0$ comparing with any weak solution of the LCS (1.2).*

We end this section with the following example, for an illustration of the convergence results.

EXAMPLE 2.2. *Consider the LCS in Example 1.1. The regularized LCS has the following form*

$$\begin{cases} \dot{x}(t) = Ax(t) + By(t) + f(t) \\ 0 \le y(t) \perp Qx(t) + (M + \mu I)y(t) + g(t) \ge 0 \\ x(0) = (0,0)^T, \end{cases}$$

which has the unique solution $(x_\mu(t), y_\mu(t))$:

$$x_\mu(t) = \begin{cases} \frac{c}{\mu}(e^{\frac{\mu}{1+\mu}t} - 1) & \text{if } 0 \le t \le t^* \\ (1+\mu)^{-\frac{1+\mu}{\mu}} \cdot c \cdot e^t & \text{if } t > t^*, \end{cases} \qquad y_\mu(t) = \begin{cases} (\frac{c - x_\mu(t)}{1+\mu}, 0)^T & \text{if } 0 \le t \le t^* \\ (0,0)^T & \text{if } t > t^*, \end{cases}$$

where

$$t^* = \frac{(1+\mu)\log(1+\mu)}{\mu}.$$

Let $T \ge t^*$. By simple calculation, we can see that $\|y_\mu\|_{L^2}$ is convergent. From Theorem 2.8 it follows that $x_\mu \to x^*$ uniformly and $y_\mu \to y^*$ weakly, where

$$x^*(t) = \begin{cases} ct & \text{if } 0 \le t \le 1 \\ ce^{-1}e^t & \text{if } t > 1, \end{cases}$$

and

$$y^*(t) = \begin{cases} (c - ct, 0)^T & \text{if } 0 \le t \le 1 \\ (0,0)^T & \text{if } t > 1. \end{cases}$$

Moreover, we have the following convergence order:

$$\|x_\mu - x^*\|_C \le ce^T \left(e^{-1} - (1+\mu)^{-\frac{1+\mu}{\mu}}\right) = O(\mu)$$

and

$$\|y_\mu - y^*\|_{L^2} \le ct^* \max\left\{\frac{\mu}{1+\mu}, \left(\frac{1+\mu}{\mu}\log(1+\mu) - 1\right)\right\} = O(\mu).$$

It is worth noting that the limit function (x^*, y^*) is a least-element solution of the LCS, although the matrix M is not a Z-matrix.

3. Regularized time-stepping method. We study in this section a new numerical method for solving the DVI (1.1) by combining the regularization method and the time stepping method.

Given a division

$$0 = t_{h,0} < t_{h,1} < \cdots < t_{h,N_h} = T,$$

where $t_{h,i+1} - t_{h,i} = h = T/N_h$, $i = 0, \ldots, N_h - 1$, the regularized time stepping method computes the sequences

$$\{x_\mu^{h,0}, x_\mu^{h,1}, \cdots, x_\mu^{h,N_h}\} \qquad \text{and} \qquad \{y_\mu^{h,0}, y_\mu^{h,1}, \cdots, y_\mu^{h,N_h}\}$$

in the following manner:

$$\begin{array}{l} x_\mu^{h,i+1} = x_\mu^{h,i} + hF(t_{h,i+1}, \sigma x_\mu^{h,i} + (1-\sigma)x_\mu^{h,i+1}, y_\mu^{h,i+1}), \\ y_\mu^{h,i+1} \in \mathrm{SOL}(\Omega, G(t_{h,i+1}, x_\mu^{h,i+1}, \cdot) + \mu I), \end{array} \tag{3.1}$$

where $\mu > 0$, and $\sigma \in [0,1]$. It is easy to see that $(x_\mu^{h,i+1}, y_\mu^{h,i+1})$ is a solution of the variational inequality $\mathrm{VI}(R^n \times \Omega, R)$, where

$$R(x,y) = \begin{pmatrix} x - x_\mu^{h,i} - hF(t_{h,i+1}, \sigma x_\mu^{h,i} + (1-\sigma)x, y) \\ G(t_{h,i+1}, x, y) + \mu y \end{pmatrix}.$$

Under the assumption that $G(t,x,\cdot)$ is a P_0 function for any $x \in R^m$ and the Lipschitz continuous condition (2.12), $\mathrm{VI}(R^n \times \Omega, R)$ has a unique solution for any fixed $\mu > 0$ and h small enough. Hence, there is no need to find the least-norm solution over the solution set in the regularized time-stepping method.

Define a piecewise linear function $x^h(t)$ and a piecewise constant function $y_\mu^h(t)$ as follows:

$$\begin{array}{lcll} x_\mu^h(t) & := & x_\mu^{h,i} + \dfrac{t - t_{h,i}}{h}(x_\mu^{h,i+1} - x_\mu^{h,i}) & \forall t \in [t_{h,i}, t_{h,i+1}] \\ y_\mu^h(t) & := & y_\mu^{h,i+1} & \forall t \in (t_{h,i}, t_{h,i+1}]. \end{array} \tag{3.2}$$

We give a result on the uniform boundedness of the function families $x_\mu^h(t)$ and $y_\mu^h(t)$ under the assumption that $\|y_\mu^{h,i}\|_2$ is uniformly bounded. Such assumption holds when the feasible set is bounded or $G(t,x,y) = Qx(t) + My(t) + g(t)$ with M being a P_0-matrix and Z-matrix. Moreover, from Theorem 2.9, we can show that such assumption holds when the solution set $\mathrm{SOL}(\Omega, G(0,\eta,\cdot))$ is nonempty and bounded.

THEOREM 3.1. *Assume that condition (2.12) holds. If there is an α_2 independent of h and μ such that $\|y_\mu^{h,i}\|_2 \le \alpha_2$ for any $h \in (0,\tilde{h}]$, $\mu \in (0,\tilde{\mu}]$ and $i = 1, \ldots, N_h$. Then we have*

$$\|x_\mu^{h,i}\|_2 \le \alpha_1 := \|x^0\|_2 \exp\left(\frac{\kappa_2 T}{1 - \tilde{h}\kappa_2}\right) + \frac{\kappa_3\alpha_2 + \|f_0\|_C}{\kappa_2}\left[\exp\left(\frac{\kappa_2 T}{1 - \tilde{h}\kappa_2}\right) - 1\right]$$

for $h \in (0,\tilde{h}]$ and $\mu \in (0,\tilde{\mu}]$, where $f_0(s) = F(s,0,0)$ for $s \in [0,T]$ and $\tilde{h} < 1/\kappa_2$.

Proof. From (3.1) we can write

$$\begin{array}{rcl} \|x_\mu^{h,i+1}\|_2 & \le & \|x_\mu^{h,i}\|_2 + h\|F(t_{h,i+1}, \sigma x_\mu^{h,i} + (1-\sigma)x_\mu^{h,i+1}, y_\mu^{h,i+1})\|_2 \\ & \le & \|x_\mu^{h,i}\|_2 + h\|F(t_{h,i+1}, \sigma x_\mu^{h,i} + (1-\sigma)x_\mu^{h,i+1}, y_\mu^{h,i+1}) - F(t_{h,i+1},0,0)\|_2 \\ & & +h\|F(t_{h,i+1},0,0)\|_2 \\ & \le & \|x_\mu^{h,i}\|_2 + h\left(\kappa_2\|\sigma x_\mu^{h,i} + (1-\sigma)x_\mu^{h,i+1}\|_2 + \kappa_3\|y_\mu^{h,i+1}\|_2\right) + h\|f_0\|_C \\ & \le & \|x_\mu^{h,i}\|_2 + h\kappa_2\sigma\|x_\mu^{h,i}\|_2 + h\kappa_2(1-\sigma)\|x_\mu^{h,i+1}\|_2 + h\kappa_3\alpha_2 + h\|f_0\|_C, \end{array}$$

which yields

$$\|x_\mu^{h,i+1}\|_2 \;\leq\; \frac{1+h\kappa_2\sigma}{1-h\kappa_2(1-\sigma)}\|x_\mu^{h,i}\|_2 + \frac{\kappa_3\alpha_2+\|f_0\|_C}{1-h\kappa_2(1-\sigma)}h,$$

and so

$$\|x_\mu^{h,i}\|_2 \;\leq\; c_1^i\|x^0\|_2 + c_2\frac{c_1^i-1}{c_1-1}, \tag{3.3}$$

where

$$c_1 = \frac{1+h\kappa_2\sigma}{1-h\kappa_2(1-\sigma)}, \quad c_2 = \frac{\kappa_3\alpha_2+\|f_0\|_C}{1-h\kappa_2(1-\sigma)}h.$$

Noting that $ih \leq N_h h = T$ and $c_1 \leq \frac{1}{1-h\kappa_2}$, we can see

$$c_1^i \;=\; (c_1-1+1)^i \;\leq\; \exp(i(c_1-1)) \;\leq \exp\left(\frac{ih\kappa_2}{1-h\kappa_2}\right) \leq \exp\left(\frac{\kappa_2 T}{1-\tilde{h}\kappa_2}\right).$$

From inequality (3.3), by simple calculation, we derive the conclusion. ∎

We have the following convergence result on the regularized time-stepping method.

Theorem 3.2. *Suppose that (2.12) holds and*

$$\big\|G(s_1,x^1,y^1)-G(s_2,x^2,y^2)\big\|_2 \;\leq\; \omega_1|s_1-s_2|+\omega_2\|x^1-x^2\|_2+\omega_3\|y^1-y^2\|_2. \tag{3.4}$$

Assume that there is an α_2 independent of h and μ such that $\|y_\mu^{h,i}\|_2 \leq \alpha_2$ for any $h \in (0,\tilde{h}]$, $\mu \in (0,\tilde{\mu}]$ and $i = 1,\dots,N_h$. Then there are sequences $\{h_k\} \downarrow 0$ and $\{\mu_k\} \downarrow 0$ such that $x_{\mu_k}^{h_k} \to x^*$ uniformly and $y_{\mu_k}^{h_k} \to y^*$ weakly in Y. Furthermore, if $y_{\mu_k}^{h_k} \to y^*$ w.r.t. $\|\cdot\|_{L^2}$, then (x^*,y^*) is a weak solution of the DVI (1.1).

Proof. From Theorem 3.1 it follows that the family of functions $\{x_\mu^h(t)\}$ is uniformly bounded. We show $\{x_\mu^h(t)\}$ is equicontinuous. By the similar way in the proof of Theorem 3.1 we have

$$\begin{aligned}\|x_\mu^{h,i+1}-x_\mu^{h,i}\|_2 \;&\leq\; h\kappa_2\sigma\|x_\mu^{h,i}\|_2 + h\kappa_2(1-\sigma)\|x_\mu^{h,i+1}\|_2 + h\kappa_3\alpha_2 + h\|f_0\|_C \\ &\leq\; h(\kappa_2\alpha_1+\kappa_3\alpha_2+\|f_0\|_C).\end{aligned} \tag{3.5}$$

For any $s,t \in [0,T]$, we consider without loss of generality that $s \in [t_{h,k},t_{h,k+1}]$ and $t \in [t_{h,k+p},t_{h,k+p+1}]$. Then we have

$$\begin{aligned}\big\|x_\mu^h(t)-x_\mu^h(s)\big\|_2 \;&\leq\; \left\|(x_\mu^h(t)-x_\mu^{h,k+p}) + \sum_{j=1}^{p-1}(x_\mu^{h,k+j+1}-x_\mu^{h,k+j}) + (x_\mu^{h,k+1}-x_\mu^h(s))\right\|_2 \\ &\leq\; \big\|x_\mu^h(t)-x_\mu^{h,k+p}\big\|_2 + \sum_{j=1}^{p-1}\big\|x_\mu^{h,k+j+1}-x_\mu^{h,k+j}\big\|_2 + \big\|x_\mu^{h,k+1}-x_\mu^h(s)\big\|_2 \\ &\leq\; [(t-t_{h,k+p}) + \sum_{j=1}^{p-1} h + (t_{h,k+1}-s)](\kappa_2\alpha_1+\kappa_3\alpha_2+\|f_0\|_C)h \\ &\leq\; |t-s|(\kappa_2\alpha_1+\kappa_3\alpha_2+\|f_0\|_C)h,\end{aligned}$$

this implies that $\{x_\mu^h(t)\}$ is equicontinuous. So from the Arzelá-Ascoli theorem it follows that there are $\{h_k\} \downarrow 0$ and $\{\mu_k\} \downarrow 0$ such that $\{x_{\mu_k}^{h_k}\}$ converges uniformly to an x^*.

Since $\{y_\mu^{h,i}\}$ is assumed to be bounded, the piecewise constant function family $\{y_\mu^h\}$ is uniformly bounded, then by the Alaoglu theorem, we know that there is a subsequence of $\{y_{\mu_k}^{h_k}\}$, without loss of generality we may assume it to be $\{y_{\mu_k}^{h_k}\}$ itself, has a weak limit y^*.

Now we prove that (x^*, y^*) is a weak solution of the DVI (1.1). For $\tau \in [t_{h,i}, t_{h,i+1}]$, from (3.5) it follows that

$$\begin{aligned}
&\left\|F\left(t_{h,i+1}, \sigma x_\mu^{h,i} + (1-\sigma)x_\mu^{h,i+1}, y_\mu^{h,i+1}\right) - F\left(\tau, x_\mu^h(\tau), y_\mu^h(\tau)\right)\right\| \\
\leq\ & \kappa_1(t_{h,i+1} - \tau) + \kappa_2 \left|\frac{t_{h,i+1} - \tau}{h} - (1-\sigma)\right| \left\|x_\mu^{h,i+1} - x_\mu^{h,i}\right\|_2 \\
\leq\ & \kappa_1 h + 2\kappa_2(\kappa_2\alpha_1 + \kappa_3\alpha_2 + \|f_0\|_C)h
\end{aligned}$$

and

$$\begin{aligned}
&\left\|G\left(t_{h,i+1}, x_\mu^{h,i+1}, y_\mu^{h,i+1}\right) - G\left(\tau, x_\mu^h(\tau), y_\mu^h(\tau)\right)\right\|_2 \\
\leq\ & \omega_1(t_{h,i+1} - \tau) + \omega_2 \left|\frac{t_{h,i+1} - \tau}{h}\right| \left\|x_\mu^{h,i+1} - x_\mu^{h,i}\right\|_2 \\
\leq\ & \omega_1 h + \omega_2(\kappa_2\alpha_1 + \kappa_3\alpha_2 + \|f_0\|_C)h.
\end{aligned}$$

Hence, we obtain

$$F\left(t_{h,i+1}, \sigma x_\mu^{h,i} + (1-\sigma)x_\mu^{h,i+1}, y_\mu^{h,i+1}\right) = F\left(t, x_\mu^h(t), y_\mu^h(t)\right) + O(h)$$

and

$$G\left(t_{h,i+1}, x_\mu^{h,i+1}, y_\mu^{h,i+1}\right) = G\left(t, x_\mu^h(t), y_\mu^h(t)\right) + O(h).$$

This, together with $x_\mu^{h,0} = x^0$, follows for $t \in (t_{h,i}, t_{h,i+1}]$,

$$\begin{aligned}
&\left\|x_\mu^h(t) - x^0 - \int_0^t F\left(\tau, x_\mu^h(\tau), y_\mu^h(\tau)\right) d\tau\right\|_2 \\
=\ & \left\|x_\mu^h(t) - x^0 - \sum_{j=1}^{i} hF\left(t_{h,j}, \sigma x_\mu^{h,j} + (1-\sigma)x_\mu^{h,j}, y_\mu^{h,j}\right)\right. \\
&\quad \left. + (t - t_{h,i})F\left(t_{h,i+1}, \sigma x_\mu^{h,i} + (1-\sigma)x_\mu^{h,i+1}, y_\mu^{h,i+1}\right)\right\|_2 + O(h) \\
=\ & \left\|x_\mu^{h,i} + \frac{t - t_{h,i}}{h}(x_\mu^{h,i+1} - x_\mu^{h,i}) - x^0 - \sum_{j=1}^{i}\left(x_\mu^{h,j} - x_\mu^{h,j-1}\right)\right. \\
&\quad \left. + \frac{t - t_{h,i}}{h}\left(x_\mu^{h,i+1} - x_\mu^{h,i}\right)\right\|_2 + O(h) \\
=\ & 2\frac{t - t_{h,i}}{h}\left\|x_\mu^{h,i+1} - x_\mu^{h,i}\right\|_2 + O(h) \ =\ O(h)
\end{aligned}$$

and

$$\begin{aligned}
&\left\|y_\mu^h(t) - \Pi_\Omega\left(y_\mu^h(t) - G\left(t, x_\mu^h(t), y_\mu^h(t)\right) - \mu y_\mu^h(t)\right)\right\|_2 \\
=\ & \left\|y_\mu^{h,i+1} - \Pi_\Omega\left(y_\mu^{h,i+1} - G\left(t, x_\mu^h(t), y_\mu^h(t)\right) - \mu y_\mu^{h,i+1}\right)\right\|_2 \\
=\ & \left\|\Pi_\Omega\left(y_\mu^{h,i+1} - G\left(t_{h,i+1}, x_\mu^{h,i+1}, y_\mu^{h,i+1}\right) - \mu y_\mu^{h,i+1}\right)\right. \\
&\quad \left. - \Pi_\Omega\left(y_\mu^{h,i+1} - G\left(t, x_\mu^h(t), y_\mu^h(t)\right) - \mu y_\mu^{h,i+1}\right)\right\|_2 \\
\leq\ & \left\|G\left(t_{h,i+1}, x_\mu^{h,i+1}, y_\mu^{h,i+1}\right) - G\left(t, x_\mu^h(t), y_\mu^h(t)\right)\right\|_2 \ =\ O(h),
\end{aligned}$$

since $y_\mu^{h,i+1}$ is the solution of $\mathrm{VI}(\Omega, G(t_{h,i+1}, x^{h,i+1}, \cdot) + \mu I)$. Then

$$\left\|\Phi_{\mu_k}(x_{\mu_k}^{h_k}, y_{\mu_k}^{h_k})\right\|_{X\times Y} = O(h_k),$$

where Φ_{μ_k} is defined by (2.4) for $\mu = \mu_k$. Namely, $(x_{\mu_k}^{h_k}, y_{\mu_k}^{h_k})$ is an ϵ_k-minimizer of $\|\Phi_{\mu_k}\|_{X\times Y}$, where $\epsilon_k = O(h_k)$. Consequently, by using Lemma 2.2, we conclude that (x^*, y^*) is a minimizer of $\|\Phi(x,y)\|_{X\times Y}$, which is a weak solution of the DVI (1.1). The proof is completed. $\square$

THEOREM 3.3. *Let $G(t,x,\cdot)$ be monotone for any fixed t and x. In the setting of Theorem 3.2, we let $\{x_{\mu_k}^{h_k}\} \to x^*$ uniformly with*

$$\lim_{k\to\infty} \frac{\|x_{\mu_k}^{h_k} - x^*\|_C}{\mu_k} = \varsigma, \tag{3.6}$$

and let $\{y_{\mu_k}^{h_k}\} \to y^*$ w.r.t. $\|\cdot\|_{L^2}$. Then for any weak solution $(x^*, \tilde{y})$ of (1.1), we have

$$\|y^*\|_{L_2}^2 \le \|\tilde{y}\|_{L_2}\|y^*\|_{L_2} + (\varsigma\omega_2 T)\|y^* - \tilde{y}\|_{L_2}. \tag{3.7}$$

Proof. It can be shown by the similar manner adopted in the proof of Theorem 2.10. $\square$

4. Numerical experiments. In this section we illustrate the applicability and the numerical performance of the regularized time-stepping method proposed in this paper. We consider two examples of the linear DVI where the matrix M is a P_0-matrix and the set Ω is bounded. Hence $G(t,x(t),\cdot)$ is a P_0-function for any t and x. Clearly, both conditions (2.12) and (3.4) are fulfilled. Moreover, the assumption that $\mathrm{SOL}(\Omega, G(0,\eta,\cdot))$ is nonempty and bounded holds since Ω is bounded.

4.1. A P_0-matrix linear DVI. We use the following P_0-matrix linear DVI (1.1) to illustrate that the regularized time-stepping method works well, but the one based on the least norm solution cannot be applied. The DVI (1.1) has the following data:

$$F(t,x,y) = Ax + By \quad \text{and} \quad G(t,x,y) = Qx + My,$$

where $A = 1$, $B = (0,0,0)$, $Q = (-1,0,0)^T$,

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \Omega = \{y \in R^3 : 0 \le y_1, y_2, y_3 \le \gamma\} \quad (\gamma > 1).$$

Set the initial point $x(0) = 1$. It is known that the parameterized $\mathrm{VI}(\Omega, M^h(\cdot) + q^{h,i})$, where $q^{h,i} = (-\frac{1}{1-h}x^{h,i}, 0, 0)^T$ and $M^h = M$ is equivalent to

$$y_i = \mathrm{mid}(0, \gamma, (y - M^h y - q^{h,i})_i), \quad i = 1,2,3,$$

where $\mathrm{mid}(\cdot)$ is the median operator [16]. It is easy to compute for any given $0 \le -q_1^{h,i} \le \gamma$ that the parameterized $\mathrm{VI}(\Omega, M^h(\cdot) + q^{h,i})$ has the solution set:

$$\begin{aligned} \mathrm{SOL}(\Omega, G(t_{h,i}, x^{h,i}, \cdot)) \;=\; & \{(0,0,y_3)^T : y_3 \in [-q_1^{h,i}, \gamma]\} \;\bigcup\; \{(y_1, 0, -q_1^{h,i})^T : y_1 \in [0,\gamma]\} \\ \bigcup\; & \{(\gamma, 0, y_3)^T : y_3 \in [0, -q_1^{h,i}]\} \;\bigcup\; \{(0, y_2, 0)^T : y_2 \in [-q_1^{h,i}, \gamma]\} \\ \bigcup\; & \{(y_1, -q_1^{h,i}, 0)^T : y_1 \in [0,\gamma]\} \;\bigcup\; \{(\gamma, y_2, 0)^T : y_2 \in [0, -q_1^{h,i}]\}. \end{aligned}$$

We plot the solution set with $-q_1^{h,i} = 1$ and $\gamma = 2$ in Figure 4.1, where $\odot$ indicates the solution $(\gamma, 0, 0)$ found by the regularized method.

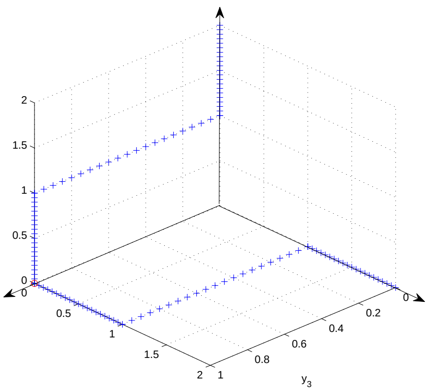

Fig. 4.1: Solution set of the linear DVI

Notice that for any $0 \leq -q_1^{h,i} \leq \gamma$, the solution set $\mathrm{SOL}(\Omega, M^h(\cdot) + q^{h,i})$ has two least norm elements: $(0, 0, -q_1^{h,i})^T$ and $(0, -q_1^{h,i}, 0)^T$. This leads a difficulty of the implementation of (1.9). However, by the regularized time-stepping method we can find the finite families

$$\left\{x_\mu^{h,1}, x_\mu^{h,2}, \cdots, x_\mu^{h,N_h}\right\} \subset R \quad \text{and} \quad \left\{y_\mu^{h,1}, y_\mu^{h,2}, \cdots, y_\mu^{h,N_h}\right\} \subset R^3, \quad \mu \in (0, 1/\gamma)$$

where $x_\mu^{h,i} = (\frac{1}{1-h})^i$ and $y_\mu^{h,i} = (\gamma, 0, 0)^T$. This yields a numerical solution of the DVI which converges to a classic solution $(e^t, (\gamma, 0, 0)^T)$ when $h \downarrow 0$.

4.2. An Example from Electrical Circuit Model. We illustrate the numerical performance of the regularized time-stepping method by a DVI arising from modeling the electrical circuits with (ideal) diodes [25]. The DVI has the following data:

$$F(t, x, y) = Ax + By + f(t) \quad \text{and} \quad G(t, x, y) = Qx + My,$$

where

$$A = \begin{pmatrix} -\frac{2}{3} & 0 \\ 0 & -\frac{1}{5} \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & \frac{1}{3} & -\frac{1}{3} & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad M = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$f(t) = 2\sin(3t - \frac{\pi}{3}),$$

and

$$\Omega = \{y \in R^4 : -10 \leq y_1, y_2 \leq 10,\ 0 \leq y_3, y_4 \leq 20\}.$$

For this example, we compute

$$\{x_\mu^{h,0}, x_\mu^{h,1}, \cdots, x_\mu^{h,N_h}\} \qquad \text{and} \qquad \{y_\mu^{h,0}, y_\mu^{h,1}, \cdots, y_\mu^{h,N_h}\}$$

in the implicit manner:

$$\begin{aligned} x_\mu^{h,i+1} &= x_\mu^{h,i} + hF(t_{h,i+1}, x_\mu^{h,i+1}, y_\mu^{h,i+1}) \\ y_\mu^{h,i+1} &= \mathrm{SOL}(\Omega, G(t_{h,i+1}, x_\mu^{h,i+1}, \cdot) + \mu I), \end{aligned}$$

where $\mu > 0$ is fixed. As stated in the last section, $(x_\mu^{h,i+1}, y_\mu^{h,i+1})$ is a solution of the variational inequality $\mathrm{VI}(R^n \times \Omega, R)$, where

$$R(x,y) = \begin{pmatrix} x - x_\mu^{h,i} - hF(t_{h,i+1}, x, y) \\ G(t_{h,i+1}, x, y) + \mu y \end{pmatrix}.$$

Since this is a linear DVI, the regularized time-stepping method has a simple version for implementation

$$\begin{aligned} y_\mu^{h,i+1} &= \mathrm{SOL}(\Omega, Q(I-hA)^{-1}[x^{h,i} + hf(t_{h,i+1})] + g(t_{h,i+1}) + (M^h + \mu I)(\cdot)) \\ x_\mu^{h,i+1} &= (I-hA)^{-1}(x^{h,i} + hBy_\mu^{h,i+1} + hf(t_{h,i+1})), \end{aligned} \tag{4.1}$$

where

$$M^h = M + hQ(I-hA)^{-1}B = \begin{pmatrix} \alpha & 0 & -1 & \alpha \\ 0 & \beta & -\beta & 1 \\ 1 & -\beta & \beta & 0 \\ \alpha & -1 & 0 & \alpha \end{pmatrix}, \quad \text{where} \quad \alpha = \frac{5h}{5+h},\ \beta = \frac{h}{3+2h}.$$

Since $M = -M^T$ and $Q(I - hA)^{-1}B$ is positive definite, we have $u^T M^h u \geq 0$ for any $u \in R^4$. Hence, M^h is a P_0-matrix and thus the solution set $\mathrm{SOL}(\Omega, Q(I - hA)^{-1}[x^{h,i} + hf(t_{h,i+1})] + g(t_{h,i+1}) + (M^h + \mu I)(\cdot))$ has a unique solution for any $\mu > 0$.

We use the semismooth Newton method in [22] to find $y_\mu^{h,i+1}$. In the implementation of this method we adopt all the parameters used therein. Let a numerical approximation $(\tilde{x}_\mu^{h,i}, \tilde{y}_\mu^{h,i})$ to $(x_\mu^{h,i}, y_\mu^{h,i})$ be available. We start the semismooth Newton method with $(\tilde{x}_\mu^{h,i}, \tilde{y}_\mu^{h,i})$ as it is usually close to the solution of the variational inequality $\mathrm{VI}(R^n \times \Omega, \tilde{R})$, where

$$\tilde{R}(x,y) = \begin{pmatrix} x - \tilde{x}_\mu^{h,i} - hF(t_{h,i+1}, x, y) \\ G(t_{h,i+1}, x, y) + \mu y \end{pmatrix}.$$

We compute the numerical solution of $x(t)$ and $y(t)$ with the initial state $x^0 = (-1, 0)^T$ for different values of μ. In Figure 4.2 we plot the numerical results, where the solid line indicates the exact solution of the DVI. Here we take the stepsize $h = 3 \times 10^{-4}$. The components $y_2(t)$ and $y_3(t)$ of the exact solution fail to be continuous, however, are approximated by a family of continuous functions. We enlarge in Figure 4.3 the curves near the discontinuity for illustrating the convergence.

The error bounds of the numerical solution $e_x^h(\mu) = \|\tilde{x}_\mu^h - x\|_C$ and $e_y^h(\mu) = \|\tilde{y}_x^h - y\|_{L^2}$ with respect to the regularization parameter μ, and the error bounds $e_x^\mu(h) = \|\tilde{x}_\mu^h - x\|_C$ and $e_y^\mu(h) = \|\tilde{y}_\mu^h - y\|_{L^2}$ with respect to the stepsize h are plotted in Figure 4.4. Numerical results show that the error bounds are monotone decreasing when

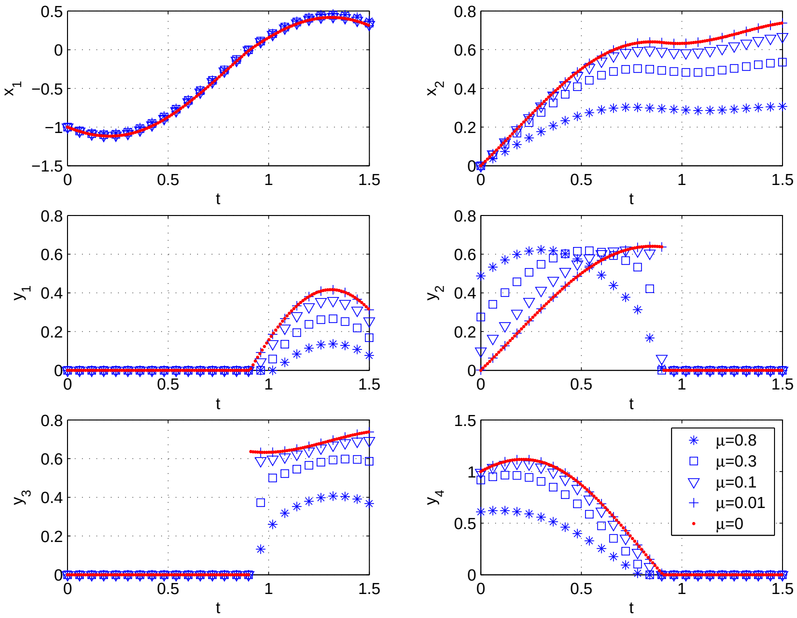

Fig. 4.2: Numerical results for $x(t)$ and $y(t)$

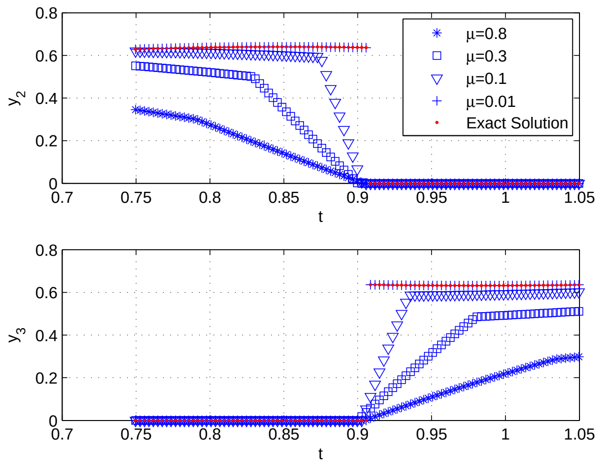

Fig. 4.3: Numerical results for $y_2(t)$ and $y_3(t)$

$\mu \downarrow 0$ and $h \downarrow 0$. However, comparing with $e^h_x(\mu)$ and $e^h_y(\mu)$, the error bounds $e^{\mu}_{x}(h)$ and $e^{\mu}_{x}(h)$ decrease slowly when $h \downarrow 0$. This is because of the low order convergence of the time-stepping method. Actually, since the solution of the DVI is at best piecewise differentiable, even the refined integrators (like the Runge-Kutta schemes) applied to DVI do not have the high order convergence [8].

Let the semismooth Newton method stop at $(\tilde{x}^{h,i+1}_{\mu}, \tilde{y}^{h,i+1}_{\mu})$, which is regarded as

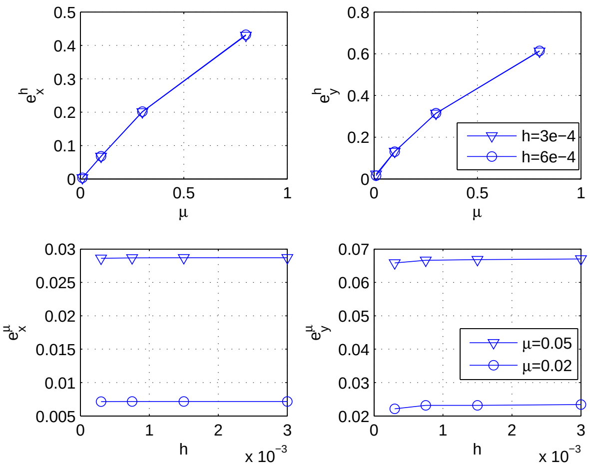

Fig. 4.4: : Errors with respect to regularization parameter

a numerical approximation of $(x_\mu^{h,i+1}, y_\mu^{h,i+1})$. We remind us that in Theorem 3.2 the boundedness of $\{y_\mu^{h,i}\}$ is imposed for guaranteeing the convergence. Such boundedness is ensured by the boundedness of $\Omega = \{y \in R^4 : -10 \leq y_1, y_2 \leq 10,\ 0 \leq y_3, y_4 \leq 20\}$. Here we compute the values

$$\beta_{\mu,h} := \max_{1 \leq i \leq N_h} \|\tilde{y}_\mu^{h,i}\|_2$$

for different choices of μ and h, and find that the values are all bounded by a constant 1.2.

This example is a passive system, and the implicit time-stepping method (1.9) using least-norm solutions can be applied. For the passivity property and the convergence of (1.9) we refer to [17]. We compare our regularized time-stepping method with the one using least-norm solutions, abbreviated respectively by “Reg.” and “LN.” in the same computational settings as mentioned above. We use the Matlab optimization solver “fmincon.m” to compute the least norm solution of the linear complementarity problem at each step of (1.9). The numerical results are plotted in Figure 4.5. The CPU time for the regularized time-stepping method (1.13) and the time-stepping method using least norm solutions (1.9) was about 0.8 (sec.) and 256 (sec.) respectively.

Preliminary numerical results indicate that the regularized time-stepping method is promising.

Acknowledgments: We would like to thank Mihai Anitescu and two anonymous referees for their valuable suggestions and comments. We are grateful to Roger Wets for his helpful comments on the epi-convergence in subsection 2.1.

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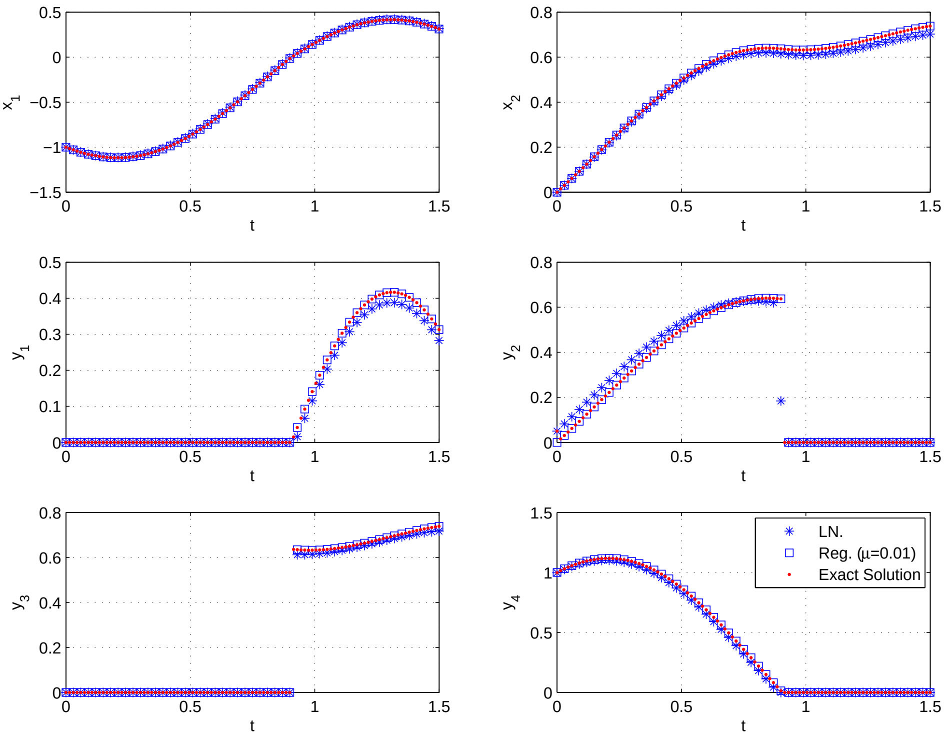

Fig. 4.5: Numerical results for $x(t)$ and $y(t)$

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