Examples of Perturbation Bounds of P-matrix Linear Complementarity Problems[1]

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We use three examples to illustrate the perturbation bounds given in

[CX] X. Chen and S. Xiang, Perturbation bounds of P-matrix linear complementarity problems, Technical Report, Department of Mathematical Sciences, Hirosaki University, February 2006, Revised January 2007.

We use the semi-smooth Newton method [13] to solve (1.8) in [CX] with stop criteria $\|r(x)\| \le 10^{-14}$ and computer precision $macheps = 10^{-16}$. We report numerical results in Table 1 and Table 2 where the fourth column and the fifth column represent the measure $\beta(M)\|M\|$ for the LCP and the upper bounds (4.5), (4.6) of $K(M)$ for the system of (1.8) in [CX], respectively. An exact error $\|\triangle x\|$ is computed as follows. First, we find approximation solutions $\hat{x} \ge 0$ and $\hat{x}+\triangle x \ge 0$ of LCP(M, q) and LCP$(M+\triangle M, q+\triangle q)$, respectively. Next, we define $\hat{q}$ and $\hat{q} + \triangle\hat{q}$ such that $\hat{x}$ and $\hat{x} + \triangle x$ are exact solutions of LCP$(M, \hat{q})$ and LCP$(M + \triangle M, \hat{q} + \triangle\hat{q})$, respectively. The perturbation bounds

$$\text{bound} = \|\tilde{M}^{-1}\|_\infty(\|\triangle M\|_\infty\|x(M + \triangle M, \hat{q} + \triangle\hat{q})\|_\infty + \|\triangle\hat{q}\|_\infty)$$

are based on (2.4) and Theorem 2.5 in [CX].

Example 1 We consider a problem which arises from finite difference approximation of free boundary problems for infinite journal bearings [4]. Here M is a tridiagonal M-matrix whose elements m_{ij} are defined

$$m_{ij} = \begin{cases} -h^3_{i+\frac{1}{2}}, & j = i + 1, \\ h^3_{i-\frac{1}{2}} + h^3_{i+\frac{1}{2}}, & j = i, \\ -h^3_{i-\frac{1}{2}}, & j = i - 1, \\ 0, & \text{otherwise} \end{cases} \qquad i, j = 1, 2, \cdots, n$$

and the elements of vector q are defined

$$q_i = \delta(h_{i+\frac{1}{2}} - h_{i-\frac{1}{2}}), \qquad i = 1, 2, \cdots, n.$$

In a common model for the infinitely long cylindrical bearing,

$$\delta = \frac{2}{n+1} \quad \text{and} \quad h_{i-\frac{1}{2}} = \frac{1 + \epsilon\cos(\pi(i - \frac{1}{2})\delta)}{\sqrt{\pi}}, \qquad i = 1, 2, \cdots, n + 1.$$

Following Cryer[4], we chose $\epsilon = 0.8$. Reformulating the journal bearing problem to the LCP(M, q) causes truncation error and rounding error. One is interested in perturbation

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error in the solution of the LCP(M, q) caused by small changes in M and q. In this problem, M is ill-conditioned for large n. Let

$$\triangle M = \epsilon_{\mathsf{M}} \begin{pmatrix} 2 & 1 & & & \\ 1 & 2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 2 & 1 \\ & & & 1 & 2 \end{pmatrix}, \qquad \triangle q = \epsilon_{\mathsf{q}} e. \tag{0.1}$$

Table 1. Perturbation bounds of Example 1
$\beta(M) = \|M^{-1}\|_\infty, \ \ \nu = \kappa_\infty(M)\| \max(\Lambda, I)\|_\infty$

n	ϵ_{M}	ϵ_{q}	$\beta(M)\|M\|_\infty$	ν	$\|\triangle x\|_\infty$	bound
10	0.0	-1.0e-3	498.0448	1.3085e3	0.1740	0.2201
	1.0e-3	1.0e-3	498.0448	1.3085e3	0.7768	2.1288
	-1.0e-3	-1.0e-3	498.0448	1.3085e3	1.2196	3.8864
100	0.0	-1.0e-3	1.0216e5	3.1546e5	23.4828	24.6533
	1.0e-5	1.0e-3	1.0216e5	3.1546e5	2.5249	24.6533
	-1.0e-5	-1.0e-3	1.0216e5	3.1546e5	51.4563	76.8289
1000	0.0	-1.0e-5	1.0168e7	3.1465e7	23.0367	24.2724
	1.0e-7	1.0e-5	1.0168e7	3.1465e7	2.5224	24.2724
	-1.0e-7	-1.0e-5	1.0168e7	3.1465e7	49.7432	73.9102

Example 2 [1]. We consider a tridiagonal H-matrix

$$M = \begin{pmatrix} 4 & -2 & & & \\ 1 & 4 & -2 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & -2 \\ & & & 1 & 4 \end{pmatrix}, \qquad \text{and} \qquad q = -4e.$$

Notice that M is well-conditioned for any n. From Theorem 2.8 in [CX], LCP(M, q) is not sensitive to small changes in data. Let $\triangle M$ and $\triangle q$ be defined by (0.1).

Table 2 Perturbation bounds of Example 2
$\beta(\tilde{M}) = \|\tilde{M}^{-1}\|_\infty, \quad \nu = \max(1, \|M\|_\infty)\|\tilde{M}^{-1}\max(\Lambda, I)\|_\infty$

n	ϵ_{M}	ϵ_{q}	$\beta(\tilde{M})\|M\|_\infty$	ν	$\|\triangle x\|_\infty$	bound
10	0.0	-1.0e-3	6.7828	27.1216	4.0812e-4	9.6899e-4
	1.0e-3	1.0e-3	6.7828	27.1216	2.4000e-3	7.3000e-3
	-1.0e-3	-1.0e-3	6.7828	27.1216	2.4000e-3	7.3000e-3
100	0.0	-1.0e-3	7.0000	28.0000	4.0825e-4	1.0000e-3
	1.0e-5	1.0e-3	7.0000	28.0000	4.2838e-4	1.1000e-3
	-1.0e-5	-1.0e-3	7.0000	28.0000	4.2838e-4	1.1000e-3
1000	0.0	-1.0e-5	7.0000	28.0000	4.0825e-6	1.0000e-5
	1.0e-7	1.0e-5	7.0000	28.0000	4.2839e-6	1.0653e-5
	-1.0e-7	-1.0e-5	7.0000	28.0000	4.2839e-6	1.0653e-5
10000	0.0	-1.0e-5	7.0000	28.0000	4.0825e-6	1.0000e-5
	1.0e-7	1.0e-5	7.0000	28.0000	4.2839e-6	1.0653e-5
	-1.0e-7	-1.0e-5	7.0000	28.0000	4.2839e-6	1.0653e-5

Example 3 Linear variational inequalities and complementarity problems have often been used to discuss formulation and solution of traffic equilibrium problems [5, 8]. Here, we use a simple traffic network in [5] to illustrate applications of perturbation bounds of the LCP with a positive definite matrix. This network consists of two nodes: w_1, w_2, and five paths: p_1, p_2, p_3, p_4, p_5. The two nodes are connected by two two-way streets and one one-way street. The paths p_1, p_2, p_3 are directed from w_1 to w_2, and p_4, p_5 are the returns of p_1 and p_2, respectively. (See Figure 1 in [5].) The travel demands are

$$d_1 = 210 \quad (\text{from } w_1 \text{ to } w_2), \qquad d_2 = 120 \quad (\text{from } w_2 \text{ to } w_1).$$

Let x_i denote the flow on path p_i, $i = 1, 2, 3, 4, 5$, and let X_d denote the set of x satisfying the travel demands $d = (d_1, d_2)$, that is,

$$X_d = \{x \in R^5 \,|\, x \geq 0, x_1 + x_2 + x_3 = d_1, x_4 + x_5 = d_2\}.$$

Furthermore, the personal travel cost function is given by

$$c(x) := Mx + b$$

where

$$M = \begin{pmatrix} 10 & 0 & 0 & 5 & 0 \\ 0 & 15 & 0 & 0 & 5 \\ 0 & 0 & 20 & 0 & 0 \\ 2 & 0 & 0 & 20 & 0 \\ 0 & 1 & 0 & 0 & 25 \end{pmatrix}, \qquad b = \begin{pmatrix} 1000 \\ 950 \\ 3000 \\ 1000 \\ 1300 \end{pmatrix}.$$

By the Wardrop principle, a load pattern $x^* \in X_d$ is user-optimized if and only if

$$(Mx^* + b)^T (x - x^*) \geq 0, \qquad \text{for all } x \in X_d. \tag{0.2}$$

This linear variational inequality problem is equivalent to the following constrained complementarity problem

$$x \in X_d, \;\; \tau \geq 0, \;\; Mx + b - \tau \geq 0, \;\; x^T (Mx + b - \tau) = 0, \tag{0.3}$$

where $\tau = (\tau_1, \tau_1, \tau_1, \tau_2, \tau_2)$. Here τ_1 and τ_2 depict the minimum transportation cost on w_1 and w_2, respectively. The optimal solution of this example is

$$x^* = (120, 90, 0, 70, 50)$$

associated with $\tau_1 = 2550$ and $\tau_2 = 2640$.

In practical applications, the travel cost often contain errors, due to inaccurate data, uncertain weather, etc. Such errors affect travel demand and flow. In order for the optimal solution x^* to be of practical use, it is very important to have some sensitivity information of solution on the cost. Now we show that such information can be obtained by using perturbation bounds in Theorem 2.11 in [CX].

It is easy to verify that M is a positive definite matrix, and x^* is a solution of LCP(M, q) with $q = b - \tau$. Moreover, if $\tilde{x}$ is a solution of LCP$(M + \triangle M, q + \triangle b)$, then $\tilde{x}$ satisfies

$$((M + \triangle M)\tilde{x} + q + \triangle b)^T (x - \tilde{x}) \geq 0, \qquad \text{for all } x \in X_{\tilde{d}}, \tag{0.4}$$

where $\tilde{d} = (\tilde{d}_1, \tilde{d}_2)$ with $\tilde{d}_1 = \tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3$ and $\tilde{d}_2 = \tilde{x}_4 + \tilde{x}_5$.

Suppose that the matrix and vector in the cost function $c(x)$ have perturbations

$$\triangle M = \epsilon_{\mathsf{M}} \begin{pmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \triangle b = \epsilon_{\mathsf{b}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Since

$$\|(\frac{M+M^{\mathsf{T}}}{2})^{-1}\|_2 \le 0.1124 \quad \text{and} \quad \|\triangle M\|_2 \le \sqrt{5}\|\triangle M\|_\infty |\epsilon_{\mathsf{M}}| \le 2\sqrt{5}|\epsilon_{\mathsf{M}}|,$$

by Theorem 2.11 in [CX], we obtain that for perturbation travel cost $M + \triangle M$ and $b + \triangle b$ with

$$|\epsilon_{\mathsf{M}}| < \frac{1}{0.1124 \times 2\sqrt{5}}$$

the corresponding perturbation traffic flow $\tilde{x}$ satisfies

$$\begin{aligned} \|\tilde{x} - x^*\|_2 &\le \alpha_2(M)^2 \|(-q)_+\|_2 \|\triangle M\|_2 + \alpha_2(M)\|\triangle b\|_2 \\ &\le 2\sqrt{5}\alpha_2(M)^2 \|(-q)_+\|_2 |\epsilon_{\mathsf{M}}| + \sqrt{5}\alpha_2(M)|\epsilon_{\mathsf{b}}| \\ &=: \triangle x_{\mathsf{bd}}, \end{aligned}$$

where

$$\alpha_2(M) = \frac{0.1124}{1 - 0.1124 \times 2\sqrt{5}|\epsilon_{\mathsf{M}}|} \quad \text{and} \quad \|(-q)_+\|_2 = \|\tau - b\|_2 \le 3073.7.$$

Moreover, the corresponding perturbation demand satisfies

$$\|\tilde{d} - d\|_\infty \le 3\|\tilde{x} - x^*\|_\infty \le 3\|\tilde{x} - x^*\|_2 =: \triangle d_{\mathsf{bd}}.$$

Table 3. Perturbation bounds of Example 3

ϵ_{M}	ϵ_{q}	$\Vert\tilde{x} - x^*\Vert_2$	$\triangle x_{\mathsf{bd}}$	$\triangle d_{\mathsf{bd}}$
0.0	1.0e-3	1.1117e-4	2.5134e-4	7.5401e-4
1.0e-3	0.0	0.0193	0.1738	0.5215
1.0e-3	1.0e-3	0.0195	0.1741	0.5223
-1.0e-2	-1.0e-2	0.1948	1.7568	5.2704

Note that travel cost functions in many traffic equilibrium problems have the form: $c(x) = Mx + b$, where M is a positive definite matrix. Analysis in this example can be extended to general cases. Moreover, the perturbation bounds are easy to compute.

Remark Theoretical analysis and numerical results show that the new perturbation bounds for the P-matrix LCP are rigorous, which improve previous perturbation bounds significantly. Moreover, $\beta(M)\|M\|$ is the first measure closely related to the condition number $\kappa(M)$ for the sensitivity of the solution of the P-matrix LCP.

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