Examples of Perturbation Bounds of P-matrix Linear Complementarity Problems

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We use three examples to illustrate the perturbation bounds given in [CX].

We use the semi-smooth Newton method [13] to solve (1.8) in [CX] with stop criteria \(k_{r(x)} \leq 10^{-14}\) and computer precision \(\text{macheps} = 10^{-16}\). We report numerical results in Table 1 and Table 2 where the fourth column and the fifth column represent the measure \(\beta(M)\) for the LCP and the upper bounds (4.5), (4.6) of \(K(M)\) for the system of (1.8) in [CX], respectively. An exact error \(k_4x\) is computed as follows. First, we find approximation solutions \(\hat{x}\) and \(\hat{x} + 4x\) of \(\text{LCP}(M,q)\) and \(\text{LCP}(M + \Delta M, q + \Delta q)\), respectively. The perturbation bounds

\[
\text{bound} = \|\hat{M}^{-1}\|_\infty (\|\Delta M\|_\infty \|x(M + \Delta M, \hat{q} + \Delta \hat{q})\|_\infty + \|\Delta \hat{q}\|_\infty)
\]

are based on (2.4) and Theorem 2.5 in [CX].

**Example 1** We consider a problem which arises from finite difference approximation of free boundary problems for infinite journal bearings [4]. Here \(M\) is a tridiagonal M-matrix whose elements \(m_{ij}\) are defined

\[
m_{ij} = \begin{cases} 
-h_{i+\frac{1}{2}}^3, & j = i + 1, \\
h_{i-\frac{1}{2}}^3 - h_{i+\frac{1}{2}}^3, & j = i, \\
-h_{i-\frac{1}{2}}^3, & j = i - 1, \\
0, & \text{otherwise}
\end{cases}
\]

and the elements of vector \(q\) are defined

\[
q_i = \delta(h_{i+\frac{1}{2}} - h_{i-\frac{1}{2}}), \quad i = 1, 2, \ldots, n.
\]

In a common model for the infinitely long cylindrical bearing,

\[
\delta = \frac{2}{n + 1} \quad \text{and} \quad h_{i-\frac{1}{2}} = \frac{1 + \epsilon \cos(\pi(i - \frac{1}{2})\delta)}{\sqrt{\pi}}, \quad i = 1, 2, \ldots, n + 1.
\]

Following Cryer[4], we chose \(\epsilon = 0.8\). Reformulating the journal bearing problem to the \(\text{LCP}(M, q)\) causes truncation error and rounding error. One is interested in perturbation

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error in the solution of the LCP\((M, q)\) caused by small changes in \(M\) and \(q\). In this problem, \(M\) is ill-conditioned for large \(n\). Let

\[
\Delta M = \epsilon_M \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & \ddots \\ \ddots & \ddots & \ddots \\ 1 & 1 & 2 \end{pmatrix}, \quad \Delta q = \epsilon_q e. \quad (0.1)
\]

**Table 1. Perturbation bounds of Example 1**

<table>
<thead>
<tr>
<th>(n)</th>
<th>(\epsilon_M)</th>
<th>(\epsilon_q)</th>
<th>(\beta(M)|M|_\infty)</th>
<th>(\nu)</th>
<th>(|\Delta x|_\infty)</th>
<th>bound</th>
</tr>
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<tbody>
<tr>
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<td>498.0448</td>
<td>1.3085e3</td>
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<td>3.1546e5</td>
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<td>24.6533</td>
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</table>

**Example 2** \([1]\). We consider a tridiagonal H-matrix

\[
M = \begin{pmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ \ddots & \ddots & \ddots \\ 1 & -2 & 4 \end{pmatrix}, \quad \text{and} \quad q = -4e.
\]

Notice that \(M\) is well-conditioned for any \(n\). From Theorem 2.8 in \([CX]\), LCP\((M, q)\) is not sensitive to small changes in data. Let \(\Delta M\) and \(\Delta q\) be defined by (0.1).

**Table 2. Perturbation bounds of Example 2**

<table>
<thead>
<tr>
<th>(n)</th>
<th>(\epsilon_M)</th>
<th>(\epsilon_q)</th>
<th>(\beta(M)|M|_\infty)</th>
<th>(\nu)</th>
<th>(|\Delta x|_\infty)</th>
<th>bound</th>
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<tbody>
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**Example 3** Linear variational inequalities and complementarity problems have often been used to discuss formulation and solution of traffic equilibrium problems [5, 8]. Here, we use a simple traffic network in [5] to illustrate applications of perturbation bounds of the LCP with a positive definite matrix. This network consists of two nodes: \( w_1, w_2 \), and five paths: \( p_1, p_2, p_3, p_4, p_5 \). The two nodes are connected by two two-way streets and one one-way street. The paths \( p_1, p_2, p_3 \) are directed from \( w_1 \) to \( w_2 \), and \( p_4, p_5 \) are the returns of \( p_1 \) and \( p_2 \), respectively. (See Figure 1 in [5].) The travel demands are

\[
d_1 = 210 \quad \text{(from } w_1 \text{ to } w_2), \quad d_2 = 120 \quad \text{(from } w_2 \text{ to } w_1).
\]

Let \( x_i \) denote the flow on path \( p_i \), \( i = 1, 2, 3, 4, 5 \), and let \( X_d \) denote the set of \( x \) satisfying the travel demands \( d = (d_1, d_2) \), that is,

\[
X_d = \{ x \in \mathbb{R}^5 \mid x \geq 0, x_1 + x_2 + x_3 = d_1, x_4 + x_5 = d_2 \}.
\]

Furthermore, the personal travel cost function is given by

\[
c(x) := Mx + b
\]

where

\[
M = \begin{pmatrix}
10 & 0 & 0 & 5 & 0 \\
0 & 15 & 0 & 0 & 5 \\
0 & 0 & 20 & 0 & 0 \\
2 & 0 & 0 & 20 & 0 \\
0 & 1 & 0 & 0 & 25
\end{pmatrix}, \quad b = \begin{pmatrix}
1000 \\
950 \\
3000 \\
1000 \\
1300
\end{pmatrix}.
\]

By the Wardrop principle, a load pattern \( x^* \in X_d \) is user-optimized if and only if

\[
(Mx^* + b)^T(x - x^*) \geq 0, \quad \text{for all } x \in X_d.
\] (0.2)

This linear variational inequality problem is equivalent to the following constrained complementarity problem

\[
x \in X_d, \quad \tau \geq 0, \quad Mx + b - \tau \geq 0, \quad x^T(Mx + b - \tau) = 0,
\] (0.3)

where \( \tau = (\tau_1, \tau_1, \tau_1, \tau_2, \tau_2) \). Here \( \tau_1 \) and \( \tau_2 \) depict the minimum transportation cost on \( w_1 \) and \( w_2 \), respectively. The optimal solution of this example is

\[
x^* = (120, 90, 0, 70, 50)
\]

associated with \( \tau_1 = 2550 \) and \( \tau_2 = 2640 \).

In practical applications, the travel cost often contain errors, due to inaccurate data, uncertain weather, etc. Such errors affect travel demand and flow. In order for the optimal solution \( x^* \) to be of practical use, it is very important to have some sensitivity information of solution on the cost. Now we show that such information can be obtained by using perturbation bounds in Theorem 2.11 in [CX].

It is easy to verify that \( M \) is a positive definite matrix, and \( x^* \) is a solution of LCP\((M, q)\) with \( q = b - \tau \). Moreover, if \( \tilde{x} \) is a solution of LCP\((M + \Delta M, q + \Delta b)\), then \( \tilde{x} \) satisfies

\[
((M + \Delta M)\tilde{x} + q + \Delta b)^T(x - \tilde{x}) \geq 0, \quad \text{for all } x \in X_d,
\] (0.4)

where \( \tilde{d} = (\tilde{d}_1, \tilde{d}_2) \) with \( \tilde{d}_1 = \tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3 \) and \( \tilde{d}_2 = \tilde{x}_4 + \tilde{x}_5 \).
Suppose that the matrix and vector in the cost function \( c(x) \) have perturbations

\[
\Delta M = \epsilon_M \begin{pmatrix}
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1
\end{pmatrix}
\quad \text{and} \quad
\Delta b = \epsilon_b \begin{pmatrix}
1 \\
1 \\
0 \\
1 \\
1
\end{pmatrix}
\]

Since

\[
\| (\frac{M + M^T}{2})^{-1} \|_2 \leq 0.1124 \quad \text{and} \quad \| \Delta M \|_2 \leq \sqrt{5} \| M \|_\infty |\epsilon_M| \leq 2 \sqrt{5} |\epsilon_M|,
\]

by Theorem 2.11 in [CX], we obtain that for perturbation travel cost \( M + \Delta M \) and \( b + \Delta b \) with

\[
|\epsilon_M| < \frac{1}{0.1124 \times 2\sqrt{5}}
\]

the corresponding perturbation traffic flow \( \hat{x} \) satisfies

\[
\| \hat{x} - x^* \|_2 \leq \alpha_2(M)^2 \| (-q)_+ \|_2 \| \Delta M \|_2 + \alpha_2(M) \| \Delta b \|_2
\]

\[
\leq 2 \sqrt{5} \alpha_2(M)^2 \| (-q)_+ \|_2 |\epsilon_M| + \sqrt{5} \alpha_2(M) |\epsilon_b|
\]

\[
= \Delta x_{bd},
\]

where

\[
\alpha_2(M) = \frac{0.1124}{1 - 0.1124 \times 2\sqrt{5} |\epsilon_M|}
\]

and \( \| (\tau - b) \|_2 \leq 3073.7 \).

Moreover, the corresponding perturbation demand satisfies

\[
\| \tilde{d} - d \|_\infty \leq 3 \| \hat{x} - x^* \|_\infty \leq 3 \| \hat{x} - x^* \|_2 =: \Delta d_{bd}.
\]

Table 3. Perturbation bounds of Example 3

<table>
<thead>
<tr>
<th>(\epsilon_M)</th>
<th>(\epsilon_q)</th>
<th>(| \hat{x} - x^* |_2)</th>
<th>(\Delta x_{bd})</th>
<th>(\Delta d_{bd})</th>
</tr>
</thead>
<tbody>
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<td>1.7568</td>
<td>5.2704</td>
</tr>
</tbody>
</table>

Note that travel cost functions in many traffic equilibrium problems have the form: \( c(x) = Mx + b \), where \( M \) is a positive definite matrix. Analysis in this example can be extended to general cases. Moreover, the perturbation bounds are easy to compute.

**Remark** Theoretical analysis and numerical results show that the new perturbation bounds for the P-matrix LCP are rigorous, which improve previous perturbation bounds significantly. Moreover, \( \beta(M) \| M \| \) is the first measure closely related to the condition number \( \kappa(M) \) for the sensitivity of the solution of the P-matrix LCP.
References


