Math. Program., Ser. A (2011) 128:1–18
DOI 10.1007/s10107-009-0291-8

FULL LENGTH PAPER

Implicit solution function of P_0 and Z matrix linear complementarity constraints

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Received: 7 July 2008 / Accepted: 20 April 2009 / Published online: 28 May 2009

Abstract Using the least element solution of the P_0 and Z matrix linear complementarity problem (LCP), we define an implicit solution function for linear complementarity constraints (LCC). We show that the sequence of solution functions defined by the unique solution of the regularized LCP is monotonically increasing and converges to the implicit solution function as the regularization parameter goes down to zero. Moreover, each component of the implicit solution function is convex. We find that the solution set of the irreducible P_0 and Z matrix LCP can be represented by the least element solution and a Perron–Frobenius eigenvector. These results are applied to convex reformulation of mathematical programs with P_0 and Z matrix LCC. Preliminary numerical results show the effectiveness and the efficiency of the reformulation.

Keywords Linear complementarity problem · P_0-matrix · Z-matrix · Regularization technique · Perron–Frobenius theorem

Mathematics Subject Classification (2000) 90C30 · 90C33

X. Chen's work was supported partly by the Research Grants Council of Hong Kong and S. Xiang's work was supported partly by NSF of China (No.10771218).

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1 Introduction

Many applied problems in engineering and economics [8,13,17,18] involve linear complementarity constraints (LCC): for a given $x \in \mathcal{D}$, find a vector $y \in R^m$ such that

$$0 \leq y \perp q(x) + My \geq 0 \tag{1.1}$$

or show that no such vector exists, where $\mathcal{D}$ is a subset of R^n, $q : R^n \to R^m$ is a continuous function and M is an $m \times m$ matrix. Here $a \geq b$ means $a_i \geq b_i$ for all components of a and b, and $a \perp b$ means $a^T b = 0$. For instance, the linear program with linear complementarity constraints (LPCC) [7,10]

$$\begin{aligned} \text{minimize} \quad & c^T x + d^T y \\ \text{subject to} \quad & Ax \leq b \\ & 0 \leq y \perp p + Nx + My \geq 0 \end{aligned} \tag{1.2}$$

and differential linear complementarity problems (DLCP)

$$\begin{aligned} &\dot{x} = g(t, x, y) \\ &0 \leq y \perp p + Nx + My \geq 0, \end{aligned} \tag{1.3}$$

where $p, d \in R^m, c \in R^n, b \in R^l, A \in R^{l \times n}, N \in R^{m \times n}$, and $g : R^{n+m+1} \to R^n$. See [5,8,13,17,18].

The linear complementarity constraints make the problems hard to solve. For instance, because of the LCC, the linear program with linear complementarity constraints (1.2) is not a convex optimization problem and there is no feasible solution satisfying all inequalities strictly. The usual mathematical programming constraint qualification such as Mangasarian–Fromovitz constraint qualification does not hold at any feasible solution [25].

For a fixed $x \in R^n$, the LCC is a linear complementarity problem (LCP), denoted by LCP$(q(x), M)$. The LCC can be considered as the multivariate parametric linear complementarity problem which has been studied for sensitivity analysis of the LCP with perturbed data [5]. In this paper, we present new results for solutions of LCC, which are not only contributions to the sensitivity analysis of LCP with perturbed data, but also to the study of problems involving LCC.

We assume that M is a P_0 and Z matrix. This implies that for any $\mu > 0$, $M + \mu I$ is a P-matrix, and thus an M-matrix [5]. A square matrix is called a P_0-matrix (P-matrix) if all its principal minors are nonnegative (positive). A square matrix is called a Z-matrix if its off-diagonal entries are non-positive. A square matrix is called an M-matrix if it is a nonsingular Z-matrix and the entries of its inverse are nonnegative. The P_0 and Z matrix LCP has many applications in engineering and physics sciences [5,6,21].

It is known that for the Z-matrix LCP$(q(x), M)$, the solution set

$$\mathcal{Y}(x) := \{\, y \,|\, 0 \leq y \perp q(x) + My \geq 0\}$$

is nonempty if the feasible set

$$\mathcal{F}(x) := \{\, y \,|\, 0 \le y,\ \ q(x) + My \ge 0\}$$

is nonempty. Moreover, if the feasible set $\mathcal{F}(x)$ is nonempty, then $\mathcal{F}(x)$ contains a least element $\hat{y}$, i.e., $\hat{y} \le y$, for all $y \in \mathcal{F}(x)$, which is a solution of LCP$(q(x), M)$ [5]. Obviously, if $\mathcal{F}(x)$ is nonempty, then the least element uniquely exists [5]. Let

$$\mathcal{X} = \{x \,|\, \exists\, y \in R^m_+,\ \text{ s.t. } q(x) \ge -My\}.$$

The set $\mathcal{X}$ depends on q and M in general. If M is an S-matrix then $\mathcal{X} = R^n$ [5]. (M is called an S-matrix if there exists $y \in R^n_+$ such that $My > 0$.)

In this paper, we first define an implicit solution function $y(\cdot)$ of LCC from $\mathcal{X}$ to R^n_+ as follows: for a vector $x \in \mathcal{X}$, $y(x)$ is the least element solution of the linear complementarity problem, LCP$(q(x), M)$. By the uniqueness of the least element solution, $y(\cdot)$ is a single-valued function.

Assuming that each component of q is a concave function, we show that $\mathcal{X}$ is a convex set and each component of $y(\cdot)$ is a convex function on $\mathcal{X}$, that is, for any $u, v \in \mathcal{X}$, we have $\lambda u + (1-\lambda)v \in \mathcal{X}$ and

$$y(\lambda u + (1-\lambda)v) \le \lambda y(u) + (1-\lambda)y(v)$$

for any $\lambda \in (0, 1)$. This interesting result is proved by using the regularization technique. Let $y_\mu(x)$ be the unique solution of the regularized LCP$(q(x), M + \mu I)$ for $\mu > 0$. We show that for any $x \in \mathcal{X}$, $\mu_1 > \mu_2 > 0$,

$$y_{\mu_1}(x) \le y_{\mu_2}(x) \le y(x) \quad \text{and} \quad \lim_{\mu \downarrow 0} y_\mu(x) = y(x).$$

This implies that for a nonnegative vector $d \in R^m$, we have

$$d^T y_{\mu_1}(x) \le d^T y_{\mu_2}(x) \le d^T y(x) \quad \text{and} \quad \lim_{\mu \downarrow 0} d^T y_\mu(x) = d^T y(x).$$

Moreover, $d^T y_\mu(\cdot)$ is convex and continuous for any $\mu > 0$. By Theorem 2.5 in [4]

$$\|y_\mu(u) - y_\mu(v)\| \le \|(M + \mu I)^{-1}\| \|q(u) - q(v)\|.$$

The regularization technique enables us to extend the definition of the implicit solution function on R^n as

$$y(x) = \lim_{\mu \downarrow 0} y_\mu(x).$$

We show that $y(\cdot)$ is finite-valued on $\mathcal{X}$, but $y_i(x) = +\infty$ for some components of $y(x)$ if $x \notin \mathcal{X}$. This implies that for any positive vector $h \in R^m$,

$$h^T y(x) = \begin{cases} \min\{h^T y \mid y \in \mathcal{F}(x)\} & \text{if } x \in \mathcal{X} \\ +\infty & \text{otherwise.} \end{cases}$$

Application of the implicit solution function $y(\cdot)$ leads to new results for feasibility analysis of LCC and development of efficient algorithms to solve problems involving linear complementarity constraints. For example, we can reformulate the linear program with linear complementarity constraints (1.2) with $d \geq 0$ and M being a P_0 and Z matrix as a convex program

$$\begin{array}{ll} \text{minimize} & c^T x + d^T y(x) \\ \text{subject to} & Ax \leq b, \quad x \in \mathcal{X} \\ & y(x) = \operatorname{argmin}\{h^T y \mid p + Nx + My \geq 0, \quad y \geq 0\}, \end{array} \tag{1.4}$$

where $h \in R^m$ is a positive vector. Note that $y(x)$ in (1.4) is independent of h. If $d > 0$ and M is an M matrix, then $\mathcal{X} = R^n$ and (1.4) is equivalent to the following linear programming problem

$$\begin{array}{ll} \text{minimize} & c^T x + d^T y \\ \text{subject to} & Ax \leq b \\ & p + Nx + My \geq 0, \quad y \geq 0. \end{array} \tag{1.5}$$

We will discuss the equivalence relation with numerical examples in Sect. 4.

The convex program reformulation gives a new look at LPCC which is useful for the study of LPCC, both from a theoretical and a numerical point of view. It is noteworthy that many optimization softwares and algorithms [9,11,22] are efficient for convex programs but have difficulties to find global solutions of nonconvex programs.

At the end of Sect. 3, we show that the least element solution $y(x)$ is the unique vector having zero entries in the solution set $\mathcal{Y}(x)$ of LCP$(q(x), M)$, if M is a singular irreducible P_0 and Z matrix. Moreover, the solution set has the form $\mathcal{Y}(x) = \{y(x) + \lambda r\}$ where $\lambda \geq 0$ and r is a Perron–Frobenius positive eigenvector of the nonnegative matrix $I - \frac{1}{\alpha}M$ with $\alpha > \max_i M_{ii}$.

2 *M* is an M-matrix

In this section, we consider that M is an M-matrix. It is known that an M-matrix is a P-matrix and the LCP$(q(x), M)$ has a unique solution for every $x \in R^n$ [5], which implies that $\mathcal{X} = R^n$ and the implicit function $y(\cdot)$ is finite-valued everywhere on R^n. Now we give a new property of the function $y(\cdot)$.

Theorem 2.1 *Assume that M is an M-matrix and each component of q is a concave function. Then each component of the solution function $y(\cdot)$ is a convex function.*

Proof For $u, v \in R^n$, let $y(u)$ and $y(v)$ be solutions of LCP$(q(u), M)$ and LCP $(q(v), M)$, respectively. Let $w = \lambda u + (1-\lambda)v$, and let $y(w)$ be the solution of LCP$(q(w), M)$, where $\lambda \in [0, 1]$. In the following, we show that

$$y(w) = y(\lambda u + (1-\lambda)v) \le \lambda y(u) + (1-\lambda)y(v). \tag{2.1}$$

Set $\mathcal{J} = \{i \mid y(w)_i > 0\}$. Obviously, (2.1) holds for the case $\mathcal{J} = \emptyset$ by the non-negative property of the solution function. We assume, without loss of generality, $\mathcal{J} = \{1, 2, \ldots, k\}(k \le n)$. Otherwise, there is a permutation matrix U such that $Uy(w) = (y(w)_{i_1}, \ldots, y(w)_{i_k}, 0, \ldots, 0)^T$ and

$$\begin{aligned}\min(Uy(w), (UMU^T)Uy(w) + Uq(w)) &= \min(Uy(w), UMy(w) + Uq(w)) \\ &= \min(y(w), My(w) + q(w)) = 0.\end{aligned}$$

Note that UMU^T is also an M-matrix. Hence, by the assumption, we have

$$(My(w) + q(w))_i = 0, \quad i = 1, \ldots, k \tag{2.2}$$

and

$$y_i(w) = 0 \le \lambda y_i(u) + (1-\lambda)y_i(v), \quad i = k+1, \ldots, n. \tag{2.3}$$

Since $y(u)$ and $y(v)$ are solutions of LCP$(q(u), M)$ and LCP$(q(v), M)$, respectively, we know that

$$My(u) + q(u) \ge 0, \quad My(v) + q(v) \ge 0.$$

This, together with the concavity of q, implies that for $i = 1, 2, \ldots, k$

$$\begin{aligned}0 &= (My(w) + q(w))_i \\ &\ge (My(w) + \lambda q(u) + (1-\lambda)q(v))_i \\ &\ge (My(w))_i - (\lambda My(u) + (1-\lambda)My(v))_i \\ &= (M(y(w) - \lambda y(u) - (1-\lambda)y(v)))_i.\end{aligned} \tag{2.4}$$

Define $M_{11} = (m_{ij})_{1\le i,j\le k}$, $M_{12} = (m_{ij})_{1\le i\le k, k+1\le j\le n}$ and let I be the $(n-k) \times (n-k)$ identity matrix. Then M_{11} is an M-matrix, and $M_{12} \le 0$. Moreover, (2.3) and (2.4) imply

$$\begin{pmatrix} M_{11} & M_{12} \\ 0 & I \end{pmatrix} (y(w) - \lambda y(u) - (1-\lambda)y(v)) \le 0.$$

It is easy to find that the inverse is a nonnegative matrix,

$$\begin{pmatrix} M_{11} & M_{12} \\ 0 & I \end{pmatrix}^{-1} = \begin{pmatrix} M_{11}^{-1} & -M_{11}^{-1}M_{12} \\ 0 & I \end{pmatrix} \ge 0.$$

Hence we obtain

$$y(w) - \lambda y(u) - (1-\lambda)y(v) \le 0$$

that is,

$$y(\lambda u + (1-\lambda)v) \le \lambda y(u) + (1-\lambda)y(v).$$

The proof is completed.

Theorem 2.2 *Let M be an M-matrix. Then* (1.4) *is a convex program and its objective function is piecewise linear and satisfies*

$$\|y(u) - y(v)\| \le \|M^{-1}\|\|N\|\|u - v\|, \quad for\ u, v \in R^n. \tag{2.5}$$

Proof It is known that the solution function $y(\cdot)$ is piecewise linear [13, p. 171]. By Theorem 2.1 and the assumption $d \ge 0$, we can easily verify that the objective function $c^T x + d^T y(x)$ is convex, and thus (1.4) is a convex program. Moreover from Theorem 2.5 in [4], we obtain that for any $u, v \in R^n$

$$\|y(u) - y(v)\| \le \|M^{-1}\|\|N(u - v)\| \le \|M^{-1}\|\|N\|\|u - v\|.$$

The following example shows that in the case M is an M-matrix, the feasible set of (1.2) is not necessarily convex, but (1.4) is a convex optimization problem.

Example 2.1 Let $n = m = 2$, N be the identity matrix and p be the zero vector. Choose

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

Then $q(x) = Nx + p = x$. Let

$$u = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 2 \\ -2 \end{pmatrix}.$$

Then $s = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the solution of LCP(u, M) and $t = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the solution of LCP(v, M). Hence $\begin{pmatrix} u \\ s \end{pmatrix}$ and $\begin{pmatrix} v \\ t \end{pmatrix}$ are feasible solutions of (1.2). However,

$$\frac{1}{2}\begin{pmatrix} u \\ s \end{pmatrix} + \frac{1}{2}\begin{pmatrix} v \\ t \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

is not a feasible solution of (1.2), since $\frac{1}{2}(s+t)$ is not a solution of LCP$(\frac{1}{2}(u+v), M)$.

On the other hand, for any $x \in \{x \mid x_1 + x_2 = 0\} = \{x \mid Ax \le b\}$, components of the solution function $y(x)$ of LCP(x, M) have the following form

$$y_1(x) = \begin{cases} 0 & \text{if } x_1 \ge 0, \quad x_2 \le 0 \\ -\frac{1}{2}x_1 & \text{if } x_1 \le 0, \quad x_2 \ge 0 \end{cases}$$

and

$$y_2(x) = \begin{cases} -\frac{1}{2}x_2 & \text{if } x_1 \ge 0, \quad x_2 \le 0 \\ 0 & \text{if } x_1 \le 0, \quad x_2 \ge 0. \end{cases}$$

Both components are piecewise linear and convex on the convex set $\{x \mid Ax \le b\}$. Moreover, the solution function $y(\cdot)$ is globally Lipschitz, and satisfies

$$\|y(u) - y(v)\| \le \|M^{-1}\| \|u - v\|.$$

From the definition of $y(x)$, finding an optimal solution of (1.4) is relatively easy. For example, consider $d = (1, 1)$ and $c = (c_1, c_2)$. If $c_1 = c_2$, then $x^* = (0, 0)$ is the unique solution of (1.4). If $c_1 > 1/2, c_2 = 0$, then (1.4) has no solution. If $c_1 = 1/2, c_2 = 0$, then (1.4) has an unbounded solution set $\{x \mid x_1 + x_2 = 0, x_1 \le 0\}$.

3 M is a $\mathbf{P_0}$ and Z matrix

In this section, we consider that M is a P_0 and Z matrix. It is known that if the feasible set $\mathcal{F}(x)$ of LCP$(q(x), M)$ is nonempty, then the solution set $\mathcal{Y}(x)$ of LCP$(q(x), M)$ contains a unique least element [5, p. 201]. Hence the implicit solution function is finite-valued on the set $\mathcal{X} = \{x \mid \exists\, y \in R^m_+, \ \text{s.t. } q(x) \ge -My\}$.

It is easy to show that $\mathcal{X}$ is a convex set if each component of q is a concave function as

$$q(\lambda u + (1-\lambda)v) \ge \lambda q(u) + (1-\lambda)q(v) \ge -M(\lambda y(u) + (1-\lambda)y(v)),$$

for $\lambda \in [0, 1]$ and $u, v \in \mathcal{X}$. The aim of this section is to show that the solution function $y(\cdot)$ is convex if each component of q is concave. To achieve the goal, we use the regularization technique and consider the linear complementarity problem LCP$(q(x), M + \mu I)$, where μ is a positive number. For any $\mu > 0$, we know that $M + \mu I$ is an M-matrix and LCP$(q(x), M + \mu I)$ has a unique solution [5]. We denote the solution by $y_\mu(x)$. The following theorem shows that y_μ is monotonically increasing and converges to the least element solution $y(x)$ of LCP$(q(x), M)$ as $\mu \downarrow 0$, for $x \in \mathcal{X}$.

Theorem 3.1 *Let M be a P_0 and Z matrix. Then for any $x \in \mathcal{X}$, $\mu_1 > \mu_2 > 0$,*

$$y_{\mu_1}(x) \le y_{\mu_2}(x) \le y(x) \quad \text{and} \quad \lim_{\mu \downarrow 0} y_\mu(x) = y(x). \tag{3.1}$$

Proof It was observed in [3] that for any vectors $u, v, s, t \in R^m$, there is a diagonal matrix D whose diagonal elements are in [0, 1] such that

$$\min(u, v) - \min(s, t) = (I - D)(u - s) + D(v - t).$$

Precisely, each diagonal element of D has the form

$$D_{ii} = \begin{cases} 1 & \text{if} \quad u_i \geq v_i, \quad s_i \geq t_i \\ 0 & \text{if} \quad u_i \leq v_i, \quad s_i \leq t_i \\ \dfrac{\min(u_i, v_i) - u_i + s_i - \min(s_i, t_i)}{v_i - u_i + s_i - t_i} & \text{otherwise.} \end{cases}$$

Hence, for any $x \in \mathcal{X}$, $\hat{y} \in \mathcal{Y}(x)$ and $\mu > 0$, there is a diagonal matrix D whose diagonal elements belong to [0, 1] such that

$$\begin{aligned} 0 &= \min(\hat{y}, M\hat{y} + q(x)) - \min(y_\mu(x), (M + \mu I)y_\mu(x) + q(x)) \\ &= (I - D)(\hat{y} - y_\mu(x)) + D(M\hat{y} - (M + \mu I)y_\mu(x)), \end{aligned} \tag{3.2}$$

which implies that

$$(I - D + D(M + \mu I))(\hat{y} - y_\mu(x)) = \mu D\hat{y}.$$

Since $M + \mu I$ is an M-matrix, we have that $(I - D + D(M + \mu I))$ is an M-matrix [5] and

$$\hat{y} - y_\mu(x) = \mu(I - D + D(M + \mu I))^{-1} D\hat{y} \geq 0. \tag{3.3}$$

Hence $\{y_\mu(x)\}$ is bounded for $\mu > 0$. Similarly, for $\mu_1 > \mu_2$, we can show that

$$y_{\mu_2}(x) - y_{\mu_1}(x) = (\mu_1 - \mu_2)(I - D + D(M + \mu_1 I))^{-1} D y_{\mu_2}(x) \geq 0.$$

Therefore $\{y_\mu(x)\}$ is a bounded and monotonically increasing sequence as $\mu \downarrow 0$. This implies that $\{y_\mu(x)\}$ has a limit as $\mu \downarrow 0$. Let $z^* = \lim_{\mu \downarrow 0} y_\mu(x)$. Then

$$\min(y_\mu(x), (M + \mu I)y_\mu(x) + q(x)) = 0$$

yields

$$\min(z^*, Mz^* + q(x)) = 0$$

and thus z^* is a solution of LCP$(q(x), M)$. Moreover, from (3.3), we have

$$z^* \leq \hat{y}, \quad \text{for } \hat{y} \in \mathcal{Y}(x).$$

Since $\mathcal{Y}(x)$ contains a unique least element, z^* must be the least element solution $y(x)$ of LCP$(q(x), M)$.

Remark 3.1 According to Theorem 5.6.2 in [5], if M is positive semi-definite and LCP($q(x), M$) is solvable, then the sequence $\{y_\mu(x)\}$ converges to the least l_2-norm solution of LCP($q(x), M$). It is clear that a least element solution is a least l_2-norm solution, but the reverse is not true. Theorem 3.1 establishes new convergence properties of the regularization algorithms for linear complementarity problems.

Corollary 3.1 *Let M be a P_0 and Z matrix. For any $x \in R^n$, the LCP($q(x), M$) has a solution if and only if $\{y_\mu(x)\}_{\mu\downarrow 0}$ is bounded.*

Proof The "only if" part follows from Theorem 3.1. We only need to show the "if" part. If $\{y_\mu(x)\}_{\mu\downarrow 0}$ is bounded, then there is a convergent subsequence $\{y_{\mu_k}(x)\}$. Let the limit be $\bar{y}$. Then

$$\min(y_{\mu_k}(x), (M + \mu_k I)y_{\mu_k}(x) + q(x)) = 0$$

yields

$$\min(\bar{y}, M\bar{y} + q(x)) = 0$$

and hence $\bar{y}$ is a solution of LCP($q(x), M$).

Theorem 3.2 *Let M be a P_0 and Z matrix, and let each component of q be a concave function. Then each component of the solution function $y(\cdot)$ is a convex function on R^n and finite-valued on $\mathcal{X}$.*

Proof By Theorem 2.1, for any positive number μ, each component of $y_\mu(\cdot)$ is a convex function, that is,

$$y_\mu(\lambda u + (1-\lambda)v) \le \lambda y_\mu(u) + (1-\lambda)y_\mu(v) \tag{3.4}$$

for $\lambda \in [0, 1]$, $u, v \in R^n$. Suppose $u, v \in \mathcal{X}$. Let $\mu \downarrow 0$ in (3.4), we get the boundness of $\{y_\mu(\lambda u + (1-\lambda)v)\}_{\mu\downarrow 0}$ and convexity of $y(\cdot)$ by Theorem 3.1. Hence, by Corollary 3.1, $y(\cdot)$ is a finite-valued convex function on $\mathcal{X}$.

In the general case, since $y(\cdot) \ge 0$ on R^n, the convexity of $y(\cdot)$ follows (3.4).

The following theorem shows that for any given $x \in \mathcal{X}$, the convergence rate of $\{y_\mu(x)\}$ to $y(x)$ as $\mu \downarrow 0$ is at least linear. This result is useful for convergence analysis of regularization methods for problems involving P_0 and Z matrix linear complementarity constraints.

Theorem 3.3 *Let M be a P_0 and Z matrix. Then for any $x \in \mathcal{X}$, there is a positive constant Γ such that*

$$\|y(x) - y_\mu(x)\| \le \mu\Gamma\|y(x)\|, \tag{3.5}$$

where

$$\Gamma = \max\{\, \|M^{-1}_{\mathcal{J}\mathcal{J}}\| \mid M_{\mathcal{J}\mathcal{J}} \textit{ is a nonsingular principal submatrix of } M\},$$

which is independent of x.

Proof For a fixed $x \in \mathcal{X}$, let $\mathcal{J} = \{i \mid y_i(x) > 0\}$. If $\mathcal{J} = \emptyset$, then from Theorem 3.1, $y_\mu(x) = y(x) = 0$ for all $\mu > 0$, and hence (3.5) holds.

Suppose $\mathcal{J} \neq \emptyset$. From (3.1) of Theorem 3.1, we can claim that there is $\bar{\mu} > 0$ such that for any $\mu \in (0, \bar{\mu}]$,

$$y_\mu(x)_i > 0 \quad \text{for } i \in \mathcal{J} \quad \text{and} \quad y_\mu(x)_i = 0 \quad \text{for } i \notin \mathcal{J}.$$

Hence from (3.2–3.3), we can easily verify

$$(I - D + D(M + \mu I))(y(x) - y_\mu(x)) = \mu D y(x),$$

where $D = \text{diag}(\alpha_1, \ldots, \alpha_n)$ and

$$\alpha_i = 1 \quad \text{for } i \in \mathcal{J} \quad \text{and} \quad \alpha_i = 0 \quad \text{for } i \notin \mathcal{J}.$$

This, together with $(y_\mu(x) - y(x))_i = 0, i \notin \mathcal{J}$, we find

$$(y(x) - y_\mu(x))_{\mathcal{J}} = \mu(M_{\mathcal{J}\mathcal{J}} + \mu I)^{-1} y(x)_{\mathcal{J}} > 0. \tag{3.6}$$

Letting $\mu \downarrow 0$ in (3.6), from $y(x) = \lim_{\mu \downarrow 0} y_\mu(x)$, $y(x)_{\mathcal{J}} > 0$ and $(M_{\mathcal{J}\mathcal{J}} + \mu I)^{-1} \geq 0$, we obtain

$$\lim_{\mu \downarrow 0} \mu(M_{\mathcal{J}\mathcal{J}} + \mu I)^{-1} = 0. \tag{3.7}$$

Now we show $M_{\mathcal{J}\mathcal{J}}$ is an M-matrix. Obviously, $M_{\mathcal{J}\mathcal{J}}$ is a Z-matrix. If $M_{\mathcal{J}\mathcal{J}}$ is singular, then it has a zero eigenvalue and the spectral radius satisfies

$$\rho((M_{\mathcal{J}\mathcal{J}} + \mu I)^{-1}) \geq \frac{1}{\mu}.$$

This implies that

$$\|\mu(M_{\mathcal{J}\mathcal{J}} + \mu I)^{-1}\| \geq \rho(\mu(M_{\mathcal{J}\mathcal{J}} + \mu I)^{-1}) = \mu\rho((M_{\mathcal{J}\mathcal{J}} + \mu I)^{-1}) \geq 1,$$

which is a contradiction to (3.7). Hence, $M_{\mathcal{J}\mathcal{J}}$ is nonsingular and thus an M-matrix.

Similarly, from (3.1–3.3) of Theorem 3.1, for all $\mu > 0$ we have

$$y_\mu(x)_i \geq 0 \quad \text{for } i \in \mathcal{J}, \quad y_\mu(x)_i = 0 \quad \text{for } i \notin \mathcal{J}$$

and

$$(I - D_\mu + D_\mu(M + \mu I))(y(x) - y_\mu(x)) = \mu D_\mu y(x),$$

where $D_\mu = \text{diag}(\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \in [0, 1]$ and $\alpha_i = 0$ for $i \notin J$. Therefore, we get

$$(y(x) - y_\mu(x))_{\mathcal{J}} = \mu(I - D_{\mu\mathcal{J}} + D_{\mu\mathcal{J}}(M_{\mathcal{J}\mathcal{J}} + \mu I))^{-1} D_{\mu\mathcal{J}} y(x)_{\mathcal{J}}. \tag{3.8}$$

Since $M_{\mathcal{J}\mathcal{J}}$ is an M-matrix, $M_{\mathcal{J}\mathcal{J}} + \mu I$ is also an M-matrix. By Theorem 2.5 in [4] we obtain

$$\begin{aligned}
&\|(I - D_{\mu\mathcal{J}} + D_{\mu\mathcal{J}}(M_{\mathcal{J}\mathcal{J}} + \mu I))^{-1} D_{\mu\mathcal{J}}\| \\
&\quad \le \max_{D_{\mathcal{J}}} \| \left(I - D_{\mathcal{J}} + D_{\mathcal{J}} \left(M_{\mathcal{J}\mathcal{J}} \ + \mu I\right)\right)^{-1} D_{\mathcal{J}}\| \\
&\quad \le \|(M_{\mathcal{J}\mathcal{J}} + \mu I)^{-1}\|,
\end{aligned}$$

where $D_{\mathcal{J}}$ is a $|\mathcal{J}| \times |\mathcal{J}|$ diagonal matrix whose diagonal entries are in [0, 1]. Here $|\mathcal{J}|$ is the cardinality of the set $\mathcal{J}$. Hence, we obtain

$$\|y(x) - y_\mu(x)\| = \|(y(x) - y_\mu(x))_{\mathcal{J}}\| \le \mu\|(M_{\mathcal{J}\mathcal{J}} + \mu I)^{-1}\|\|y(x)_{\mathcal{J}}\|. \tag{3.9}$$

Since $M_{\mathcal{J}\mathcal{J}}$ is an M-matrix and $\mu > 0$, by Theorem 2.4.11 in [16], $M_{\mathcal{J}\mathcal{J}} + \mu I$ is an M-matrix and

$$\|(M_{\mathcal{J}\mathcal{J}} + \mu I)^{-1}\| \le \|M_{\mathcal{J}\mathcal{J}}^{-1}\|.$$

This, together with (3.9), deduces (3.5).

Remark 3.2 From the proof of Theorem 3.3, we can see that if there is a least element solution $y(x) > 0$, then M must be an M-matrix. In other words, if M is a singular P_0 and Z matrix, then there is no positive least element solution for any $x \in \mathcal{X}$.

The following theorem shows that if M is an irreducible P_0 and Z matrix, then for any $x \in \mathcal{X}$, the solution set of LCP($q(x), M$) can be represented by the least element solution and a Perron–Frobenius positive eigenvector which is independent of x. In other words, if a solution of LCC($q(x), M$) has zero entries, then it must be the least element solution.

Theorem 3.4 *Let M be an irreducible P_0 and Z matrix. Then for any $x \in \mathcal{X}$*

$$\mathcal{Y}(x) = \begin{cases} \{y(x)\}, & \text{if } My(x) + q(x) \ne 0, \\ \{y(x) + \lambda r, \lambda \ge 0\}, & \text{if } \ My(x) + q(x) = 0, \end{cases}$$

where r is a Perron–Frobenius positive eigenvector of $B = I - \frac{1}{\alpha}M$, with $\alpha >$ $\max_i M_{ii}$.

Proof According to Proposition 3.11.12 in [5], the solution set has the following property

$$\mathcal{Y}(x) = y(x) + \mathcal{S}(x)$$

where $\mathcal{S}(x)$ is the solution set of the following LCP with constant column $q(x) + My(x)$,

$$0 \le y \perp q(x) + My(x) + My \ge 0.$$

It is clear that the solution set $\mathcal{S}(x)$ coincides with the solution set of the following LCP

$$0 \le y \perp \frac{1}{\alpha}(q(x) + My(x) + My) \ge 0,$$

where $\alpha > \max_i(M_{ii})$. Then we have a regular splitting

$$\frac{1}{\alpha}M = I - B,$$

where $B \ge 0$, with $B_{ij} = \frac{1}{\alpha}M_{ij}$, $i \ne j$ and $B_{ii} = 1 - \frac{1}{\alpha}M_{ii}$.

Since $q(x) + My(x) \ge 0$, the zero vector 0 is in $\mathcal{S}(x)$. Suppose that $\mathcal{S}$ has a vector $y \ge 0$, $y \ne 0$. From (3.2), we can easily verify that there is diagonal matrix D whose diagonal elements are in [0, 1] such that

$$0 = \left(I - D + D\frac{1}{\alpha}M\right)(y - 0) = (I - D + D(I - B))y = (I - DB)y = 0. \tag{3.10}$$

Since $\frac{1}{\alpha}M = I - B$ is a P_0 and Z matrix and a regular splitting, we have $(1+\mu)I - B$ is nonsingular for $\mu > 0$ and the Perron–Frobenius Theorem [1] yields

$$\rho\left(\frac{1}{1+\mu}B\right) \le \frac{\rho((\mu I + \frac{1}{\alpha}M)^{-1}B)}{1 + \rho((\mu I + \frac{1}{\alpha}M)^{-1}B)} < 1.$$

Let $\mu \downarrow 0$, we obtain $\rho(B) \le 1$. Since $D \le I$, we have $DB \le B$, and thus $\rho(DB) \le \rho(B) \le 1$. From (3.10) and $y \ne 0$, we find that $I - DB$ is singular, which implies that $\rho(DB) = \rho(B) = 1$.

From the Perron–Frobenius Theorem, we know that $\rho(B)$ is a simple eigenvalue of B, that is, the eigenspace associated to $\rho(B)$ is one-dimensional, and there is a positive eigenvector $r > 0$ associated to $\rho(B)$. Let $E = \mathrm{diag}(r)$. Then

$$E^{-1}BEe = E^{-1}Br = E^{-1}\rho(B)r = E^{-1}r = e,$$

where $e = (1, \ldots, 1)^T$.

Now we show that $D = I$. Assume to the contrary that D has a diagonal element, which is strictly less than one, that is $0 \le D \le I$ and $D \ne I$.

From $\rho(DB) = \rho(B)$, we find that D has at least one element which is equal to one. Moreover, from

$$DE^{-1}BEe = De \le e, \quad \text{and} \quad De \ne e,$$

we find that the minimum row sum of $DE^{-1}BE$ is strictly less than 1, and the maximum row sum is equal to 1. Since $E^{-1}BE$ is nonnegative and irreducible, $0 \le$

$DE^{-1}E \le E^{-1}BE$ and $DE^{-1}E \ne E^{-1}BE$, then by Corollary 2.2 in Chapter 2 [15] we have

$$\rho(DB) = \rho(E^{-1}DEE^{-1}BE) = \rho(DE^{-1}BE) < 1,$$

where we use $E^{-1}D = DE^{-1}$. This is a contradiction.

Hence, we deduce that $D = I$. Moreover, from (3.10), we deduce that y is an eigenvector corresponding to $\rho(B) = 1$. By the Perron–Frobenius Theorem [1] there is $\lambda > 0$ such that $y = \lambda r$. Hence if $\mathcal{S}(x) \ne \{0\}$, then we can present the solution set of LCP($q(x)$, M) as

$$\mathcal{Y}(x) = y(x) + \lambda r, \quad \lambda \ge 0.$$

Now we show that $\mathcal{S}(x) \ne \{0\}$ if and only if

$$My(x) + q(x) = 0.$$

If $\mathcal{S}(x) \ne \{0\}$, then from the proof above, $0 \ne y \in \mathcal{S}(x)$ implies that y is an eigenvector of B and $y > 0$. Hence, the complementarity condition yields

$$\begin{aligned} 0 &= \frac{1}{\alpha}(My + My(x) + q(x)) \\ &= (I - B)y + \frac{1}{\alpha}(My(x) + q(x)) \\ &= \frac{1}{\alpha}(My(x) + q(x)) = 0. \end{aligned}$$

Conversely, if $My(x) + q(x) = 0$, then for any eigenvector y of B corresponding to $\rho(B) = 1$, we have $y > 0$ and $My + My(x) + q(x) = 0$. Hence $y \in \mathcal{S}(x)$. The proof is completed.

Corollary 3.2 *If M is a singular irreducible P_0 and Z matrix, then for any $x \in \mathcal{X}$, $y(x)$ is the unique element having zero entries in the solution set $\mathcal{Y}(x)$ of LCP($q(x)$, M).*

Proof From the proof of Theorem 3.3, the singularity of M implies that the set $\mathcal{Y}(x)$ contains the least element solution $y(x)$ which has zero entries. The uniqueness follows from Theorem 3.4.

We use the following example to illustrate Theorems 3.3 and 3.4.

Example 3.1 Let $q(x) = x$,

$$\mathcal{D} = \{x \mid x_1 + x_2 = 0, x_1 \ge 0\} \quad \text{and} \quad M = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

For any $x \in \mathcal{D}$, we find that $y = (0, x_1)^T$ is a solution of LCP(x, M). Since M is an irreducible P_0 and Z matrix and y has a zero entry, by Theorem 3.4, we can deduce that

y is the least element solution of LCP(x, M). Let $y(x) := y$. From $My(x) + x = 0$, we know that the solution set $\mathcal{Y}(x)$ of LCP(x, M) is unbounded. Let

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad r = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

It is easy to see that r is a Perron–Frobenius eigenvector corresponding to $\rho(B) = 1$. Then the solution set of LCP(x, M) can be represented by

$$\mathcal{Y}(x) = \left\{ y(x) + \lambda r = \begin{pmatrix} \lambda \\ x_1 + \lambda \end{pmatrix}, \quad \lambda \geq 0 \right\}.$$

It is easy to verify that $y_\mu(x) = \left(0, \dfrac{x_1}{1+\mu}\right)^T$ is the unique solution of LCP$(x, M + \mu I)$. $y_\mu(x)$ is monotonically increasing and converges to the least element solution $y(x) = (0, x_1)^T$. Moreover, $\|y_\mu(x) - y(x)\| \leq \dfrac{\mu}{1+\mu}\|y(x)\| \leq \mu\Gamma\|y(x)\|$ with $\Gamma = 1$.

4 Applications

In this section, we apply the implicit solution function defined by the least element solution to the linear program with linear complementarity constraints (1.2). We assume that M is a P_0 and Z matrix and d is a nonnegative vector. Let

$$\mathcal{X} = \{x \mid \exists\, y \in R^m_+, \text{ s.t. } p + Nx + My \geq 0\}.$$

It is easy to show that the set $\mathcal{X}$ is convex. Suppose that $\hat{x}, \tilde{x} \in \mathcal{X}$. Then there are $\hat{y}, \tilde{y} \in R^m_+$ such that

$$p + N\hat{x} + M\hat{y} \geq 0 \quad \text{and} \quad p + N\tilde{x} + M\tilde{y} \geq 0.$$

This yields that

$$p + N(\lambda\hat{x} + (1-\lambda)\tilde{x}) + M(\lambda\hat{y} + (1-\lambda)\tilde{y}) \geq 0$$

for all $\lambda \in (0, 1)$. From $\lambda\hat{y} + (1-\lambda)\tilde{y} \in R^m_+$, we deduce $\lambda\hat{x} + (1-\lambda)\tilde{x} \in \mathcal{X}$.

Lemma 4.1 [5] *Let M be a Z-matrix and q an arbitrary vector. If the LCP(q, M) is feasible, then the feasible set $\{\, y \mid y \geq 0,\ My + q \geq 0\}$ contains a least element y^*. Moreover, y^* solves the LCP(q, M).*

Proposition 4.1 *Let M be a Z-matrix. Problem (1.2) is equivalent to the convex optimization problem (1.4). Furthermore, if M is an M-matrix, and $d > 0$, then (1.2) is equivalent to the linear program (1.5).*

Proof We first show the equivalence relation in the feasibility and solvability. Let

$$\mathcal{D} = \{ x \mid Ax \le b \} \quad \text{and} \quad f(x, y) = c^T x + d^T y.$$

Let

$$\mathcal{F}(x) = \{ y \mid y \ge 0, \;\; p + Nx + My \ge 0\}.$$

Suppose (1.2) has a feasible point $(\hat{x}, \hat{y})$. Then $\hat{x} \in \mathcal{D} \cap \mathcal{X}$. From $h > 0$ and Lemma 4.1, $\mathcal{F}(\hat{x})$ contains a least element $\bar{y}$, such that $y(\hat{x}) = \bar{y}$. Hence (1.4) is feasible. Conversely, if (1.4) has a feasible point $\hat{x}$, then $(\hat{x}, y(\hat{x}))$ is a feasible point of (1.2). Therefore, we claim that (1.2) is feasible if and only if (1.4) is feasible.

If there is a sequence $\{x^k\}$ in the feasible set of (1.4) such that $f(x^k, y(x^k)) \to -\infty$, then from that $\{(x^k, y(x^k))\}$ is in the feasible set of (1.2), we find that (1.2) has no solution. Conversely, if there is a sequence (x^k, y^k) in the feasible set of (1.2) such that $f(x^k, y^k) \to -\infty$, then from $y^k \in \mathcal{F}(x^k)$ and Lemma 4.1, $\mathcal{F}(x^k)$ contains a least element which is the minimum point $y(x^k)$ of $h^T y$ for $y \in \mathcal{F}(x^k)$. The nonnegativity of d makes $f(x^k, y^k) \ge f(x^k, y(x^k)) \to -\infty$. Therefore, we claim that (1.2) is solvable if and only if (1.4) is solvable.

Now, we show that if (x^*, y^*) is a solution of (1.2) then x^* is a solution of (1.4); if x^* is a solution of (1.4), then $(x^*, y(x^*))$ is a solution of (1.2).

Suppose that (x^*, y^*) is a solution of (1.2). Then from Lemma 4.1, $h > 0$ and $d \ge 0$, we have $y(x^*) \le y^*$, and $f(x^*, y(x^*)) \le f(x^*, y^*)$. Hence $(x^*, y(x^*))$ is a solution of (1.2) and x^* is a solution of (1.4).

Suppose that x^* is a solution of (1.4). From Lemma 4.1, and $h > 0$, $y(x^*)$ is a solution of the LCP$(p + Nx^*, M)$. Moreover, from $d \ge 0$, we have $f(x^*, y(x^*)) \le f(x^*, y^*)$, for all y^* in the solution set of the LCP$(p + Nx^*, M)$. Hence, $(x^*, y(x^*))$ is a solution of (1.2).

The convexity of (1.4) follows from Theorem 3.2.

In addition, if M is an M-matrix, then $\mathcal{X} = R^n$. If $d > 0$, we can set $h = d$ in (1.4). In such case, (1.4) reduces to (1.5).

Example 4.1 We consider a linear program with linear complementarity constraints (1.2). We set $n = 2, l = 1, b = 0$. Let $k > 0$ be an integer, $m = 2k$, p be a zero vector in R^m, N be an $m \times n$ matrix with all entries being 0 except $N_{k,1} = N_{k+1,2} = 1, d = (1, \dots, 1)^T \in R^m$. Let $A = (-1, -1)$, and M be an $m \times m$ tridiagonal matrix with $-1, 2, -1$ along its superdiagonal, main diagonal and subdiagonal, respectively, except $M_{11} = 1$, $M_{m,m} = 1$.

It is easy to find that $\mathcal{D} = \{x \mid Ax \le b\} = \{x \mid x_1 + x_2 \ge 0\}$, and M is a singular irreducible P_0 and Z matrix with rank$(M) = m - 1$. For $x \in \mathcal{D}$, we can show that

$$y(x) = \begin{cases} (0e^T, \; -x_2 e^T)^T & \text{if } x_1 \ge 0, \;\; x_2 \le 0 \\ (-x_1 e^T, \; 0e^T)^T & \text{if } x_1 \le 0, \;\; x_2 \ge 0 \\ (0e^T, \; 0e^T)^T & \text{if } x_1 \ge 0, \;\; x_2 \ge 0 \end{cases}$$

is the least element solution of LCP$(p + Nx, M)$, where $e = (1, \dots, 1)^T \in R^k$. Obviously, $y(x) \geq 0$, and for $x_1 \geq 0$, $x_2 \leq 0$

$$(My(x) + Nx + p)_i = \begin{cases} x_1 + x_2 & \text{if} \quad i = k \\ 0 & \text{otherwise,} \end{cases}$$

for $x_1 \leq 0$, $x_2 \geq 0$

$$(My(x) + Nx + p)_i = \begin{cases} x_1 + x_2 & \text{if} \quad i = k + 1 \\ 0 & \text{otherwise,} \end{cases}$$

for $x_1 \geq 0$, $x_2 \geq 0$, $My(x) + Nx + p = Nx$.

Hence, $y(x)$ is a solution of LCP$(p + Nx, M)$. Moreover, by Corollary 3.2, $y(x)$ is the least element solution, since $y(x)$ has zero entries. Therefore, the problem (1.2) with these data of Example 4.1 is equivalent to the convex program

$$\begin{aligned} &\text{minimize} \;\; c^T x + d^T y(x) \\ &\text{subject to} \;\; x_1 + x_2 \geq 0. \end{aligned} \tag{4.1}$$

If $0 < c = (c_1, c_2)^T$ and $\max(c_1, c_2) < k$, $x^* = (0, 0)$ is the unique solution of (4.1).
If $c_1 > k$, $c_2 = 0$, (4.1) has no solution.
If $c_1 = k$, $c_2 = 0$, (4.1) has an unbounded solution set $\{ x \,|x_1 + x_2 \geq 0, x_1 \leq 0\}$.

Now we consider other approach to find a solution of (1.2) by using the regularization problem of (1.2)

$$\begin{aligned} &\text{minimize} \;\; c^T x + d^T y \\ &\text{subject to} \;\; Ax \leq b \\ &\qquad\qquad\quad 0 \leq y \perp p + Nx + (M + \mu I)y \geq 0, \end{aligned} \tag{4.2}$$

where $\mu > 0$. Since $M + \mu I$ is an M-matrix, and $d > 0$, by Proposition 4.1, (4.2) is equivalent to the linear program

$$\begin{aligned} &\text{minimize} \;\; c^T x + d^T y \\ &\text{subject to} \;\; Ax \leq b \\ &\qquad\qquad\quad p + Nx + (M + \mu I)y \geq 0, \;\; y \geq 0. \end{aligned} \tag{4.3}$$

We choose $c_1 = k/2$, $c_2 = k/4$. Then (4.1) has a unique solution $(x^*, y^*) = (0, 0)$. Let (x^*_μ, y^*_μ) be a solution of (4.2). We expect $(x^*_\mu, y^*_\mu) \to (0, 0)$ as $\mu \to 0$. We test the regularization approach for $m = 20 : 10 : 1,000$ with the same starting regularization parameter $\mu = 0.005$ and the reducing step size $\Delta\mu = 10^{-5}$. We terminate the program when $\mu < 10^{-6}$ or $\|(x_\mu, y_\mu)\|_\infty \leq 10^{-6}$. Figure 1 presents numerical results of μ and $\|(x_\mu, y_\mu)\|_\infty$ when the program is terminated for different m. The total cpu time for generating and solving these problems for $m = 20 : 10 : 1,000$ is 244.8 s. Preliminary numerical results show that the regularization method is efficient for solving the linear program with linear complementarity constraints. Numerical

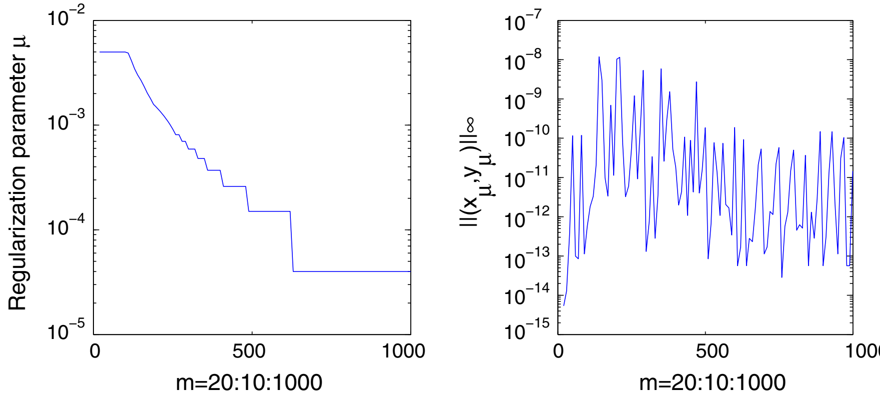

Fig. 1 Values of μ and $\|(x_\mu, y_\mu)\|_\infty$ terminated the algorithm for $m = 20 : 10 : 1000$ in Example 4.1

tests were carried out by using Matlab 7.4 with **linprog**, a linear programming code, on an IBM PC (2.39 GHz, 2 GB of RAM) with Windows XP operating system.

Discussion in this section can be extended to the mathematical program with linear complementarity constraints

$$\begin{array}{ll} \text{minimize} & f(x, y) \\ \text{subject to} & x \in \mathcal{D} \\ & 0 \le y \perp p + Nx + My \ge 0, \end{array} \tag{4.4}$$

where $\mathcal{D} \subseteq R^n$ is a convex set and $f : R^n \times R^m \to R$ is a convex function and nondecreasing in y, that is,

$$f(\lambda(x^1, y^1) + (1-\lambda)(x^2, y^2)) \le \lambda f(x^1, y^1) + (1-\lambda) f(x^2, y^2), \quad \text{for } \lambda \in [0, 1]$$

and

$$f(x, y^1) \le f(x, y^2), \quad \text{for } y^1 \le y^2. \tag{4.5}$$

5 Final remark

Using a solution function of complementarity problems in the constraints of mathematical programs has been studied in [2,12,19,23] under the assumption on the uniqueness of the solution of the complementarity problem. In this paper, we first use the least element solution to define a solution function of complementarity problems whose solution is not unique. We show that each component of the solution function defined by the least element in the solution set is convex if the involved matrix is a P_0 and Z matrix. Moreover, we present uniqueness of the least element solution for the irreducible P_0 and Z matrix LCP. These results can be applied to problems involving complementarity constraints. Numerical examples in Sect. 4 illustrate possible applications to the mathematical program with equilibrium constraints.

Acknowledgments The authors are grateful to the associated editor and two anonymous referees for their very helpful comments.

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